Ovals in the Desarguesian Plane of Order 16 (*) (**).

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Dedicated to Professor BENIAMINO SEGRE
on the occasion of his 70th birthday

Summary. – *All ovals in the Desarguesian plane of order* 16 *are determined. Up to equivalence under the collineation group of the plane there are exactly two classes of ovals. In the first class an oval consists of the points of a conic together with its nucleus. There is exactly one other class of ovals and the collineations fixing an oval in this class are transitive on the* 18 *points of the oval.*

1. – Introduction.

The literature on ovals in projective planes is extensive, but as the problem treated here is a specail one it will be sufficient to refer to Professor Segre's celebrated monograph « Introduction to Galois geometries » [2] for background material.

In a projective plane of even order an oval consists of $n+2$ points not hree on a line. In a Desarguesian plane PG(2^r), taking $n=2^r$, an irreducible conic contains $n+1$ points, and all of its tangents are concurrent in a single point called its nucleus. Then the conic together with its nucleus forms an oval of $n+2$ points. For $n=2$ and $n=4$ these are the only ovals and for $n=8$ the plane is necessarily Desarguesian [1] and it is left as an exercise for the reader to show that these are the only ovals. For $n=2^r$ with $r=5$ or $r\geqslant 7$ it is known that ovals exist in the Desarguesian plane PG$(2, n)$ which are not derived from conics with the adjunction of a nucleus.

In the present paper all ovals in the Desarguesian plane of order 16 PG$(2, n)$ are determined and it is shown that besides the conics and nucleus there is exactly one other class of ovals, where two ovals are in the same equivalence class if there is a collineation of the plane mapping one into the other.

2. – The Desarguesian plane of order 16. Notation.

We take the field GF(16) as the extension of GF(2) by an element a satisfying

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$a^4 + a + 1 = 0$. Here a is a primitive root and its powers are given as follows

$$
(2.1)\qquad
\begin{array}{llll}
a^0 = & \phantom{a^3 + a^2 + a +{}} 1 & a^8 = & a^2 \phantom{{}+a} + 1 \\
a = & \phantom{a^3 + a^2 +{}} a & a^9 = & a^3 \phantom{{}+a^2} + a \\
a^2 = & \phantom{a^3 +{}} a^2 & a^{10} = & a^2 + a + 1 \\
a^3 = & a^3 & a^{11} = & a^3 + a^2 + a \\
a^4 = & \phantom{a^3 + a^2 +{}} a + 1 & a^{12} = & a^3 + a^2 + a + 1 \\
a^5 = & \phantom{a^3 +{}} a^2 + a & a^{13} = & a^3 + a^2 \phantom{{}+a} + 1 \\
a^6 = & a^3 + a^2 & a^{14} = & a^3 \phantom{{}+a^2+a} + 1 \\
a^7 = & a^3 \phantom{{}+a^2} + a + 1 & a^{15} = & \phantom{a^3 + a^2 + a +{}} 1
\end{array}
$$

For compactness we represent the elements of GF(16) as 0 and a^i, $i = 0, \ldots, 14$.

A point P of the plane π in homogeneous form has coordinates x, y, z, $P = (x, y, z) \neq (0, 0, 0)$ and for any non zero scalar t, (x, y, z) and (tx, ty, tz) represent the same point. If $[a, b, c] \neq [0, 0, 0]$ the set of points (x, y, z) satisfying

$$
(2.2)\qquad ax + by + cz = 0
$$

is a line L, and if P is one of these points we say P is incident with L or belongs to L and write $P \in L$.

A collineation α is a one-to-one mapping of points onto points and lines onto lines preserving incidence, and so α is completely determined by its action on the points. The projective collineations are those given by linear transformations. If A is a non singular 3 by 3 matrix it determines a projective collineation α.

$$
(2.3)\qquad
\begin{aligned}
&\alpha\colon (x, y, z \to (x, y, z)\alpha \\
&(x, y, z)\alpha = (x, y, z)A = (x, y, z)\begin{bmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{bmatrix}.
\end{aligned}
$$

The projective collineations of our plane π form a linear group PGL(3, 16) of order $(q^2 + q + 1)(q^2 + q)(q^2)(q - 1)^2$ with $q = 16$

There is also a collineation t of order 4 given by the field automorphism $x \to x^2$ for $x \in \mathrm{GF}(16)$. This maps

$$
(2.4)\qquad (x, y, z) \to (x, y, z)t = (x^2, y^2, z^2)\,.
$$

The full collineation group G of π is generated by PGL(3, 16) and t and PGL(3, 16) is a normal subgroup of index 4 in G.

By Singer's theorem [2 p. 219] π has a projective collineation σ of order 273 which is cyclic on the 273 points of π and the 273 lines of π. We take the following particular choice for σ

$$\sigma = \begin{bmatrix} 1, & a^{14}, & a^3 \\ a^6, & 1, & 1 \\ 1, & a^3, & 0 \end{bmatrix}. \tag{2.5}$$

Using this collineation we may represent the points by the residues $0, \ldots, 272 \pmod{273}$ and lines by L_j, $j = 0, \ldots, 272 \pmod{273}$. The action of the collineation σ is given by

$$(i)\sigma = i + 1, \quad (L_j)\sigma = L_{j+1}. \tag{2.6}$$

Precisely, this correspondence is made by taking the point $(0, 0, 1)$ as the point 0 and the line $z = 0$ as L_1. In this notation the 17 points of L_1 are given as follows:

$$\begin{array}{ll} \text{Points of } L_1 & 239 = (1, a^6, 0) \\ 234 = (1, 0, 0) & 32 = (1, a^7, 0) \\ 117 = (0, 1, 0) & 128 = (1, a^8, 0) \\ 195 = (1, 1, 0) & 8 = (1, a^9, 0) \\ 16 = (1, a, 0) & 91 = (1, a^{10}, 0). \\ 2 = (1, a^2, 0) & 256 = (1, a^{11}, 0) \\ 1 = (1, a^3, 0) & 64 = (1, a^{12}, 0) \\ 205 = (1, a^4, 0) & 137 = (1, a^{16}, 0) \\ 182 = (1, a^5, 0) & 4 = (1, a^{14}, 0) \end{array} \tag{2.7}$$

The 17 points of L_1 are, of course, the following difference set modulo 273

(2.8) L_1: 1, 2, 4, 8, 16, 32, 64, 91, 117, 128, 137, 182, 195, 205, 234, 239, 256 (modulo 273).

Here the residues $a_1, \ldots, a_{17}$ listed in 2.8 have the property that every non zero residue d can be expressed in exactly one way in the form $a_i - a_j \equiv d \pmod{273}$, and this is why it is called a difference set. Certain computations are easily carried out in terms of the difference set. For example, to find the line joining two points r and s we find from (2.8) the a_i and a_j such that $a_i - a_j \equiv r - s \pmod{273}$ and then determine t by $r \equiv a_i + t$, $s \equiv a_j + t$. Then L_{t+1} contains both r and s. For example to find the line containing 10 and 19 we note that $19 - 10 \equiv 9 \equiv 137 - 128 \pmod{273}$. Here $137 + 155 \equiv 19 \pmod{273}$ $128 + 155 \equiv 10 \pmod{273}$. Hence L_{156} is the line containing 10 and 19 and from (2.8) we can immediately list the remaining points on L_{156}. A list, not given here, was made of all 273 points giving for each number the coordinates of the point. In fact two lists were made, one listing the points in order of the numbers $0, 1, \ldots, 272$, the other a systematic listing of the points by coordinates, first those with $z = 0$ as in 2.7) and for the remaining points $(x, y, 1)$ first those with $x = 0$ and then $(a^i, y, 1)$ $i = 0, \ldots, 14$. The construction of these lists was straight-forward but tedious. Once constructed it was easy to go from one form to the other according to the calculations desired.

In this paper enough points will be given simultaneously in both forms so that it should not be difficult for the interested reader to construct the rest by application of the collineation σ in 2.5).

3. – Construction of the ovals in the Desarguesian plane of order 16.

Following Professor Segre's usage [3] an oval in the Desarguesian plane of order 16 is a set of 18 points of the plane, no three on a line. An irreducible conic C contains 17 points and the tangests to C all concur in a single point N, the *nucleus* of the conic C. The points of a conic together with its nucleus form an oval. But there are also other ovals in this plane.

The problem attacked and solved in this paper is the determination of all ovals in the Desarguesian plane of order 16. The image of any oval under a collineation is again an oval, and we shall consider two such ovals as equivalent and it is clearly sufficient to find one oval in each equivalence class.

Under the projective group any four points, no three on a line, may be taken by a collineation into the following four points.

$$(3.1)\qquad \begin{aligned} 0 &= (0,\ 0,\ 1) \\ 117 &= (0,\ 1,\ 0)\,. \\ 234 &= (1,\ 0,\ 0) \\ 39 &= (1,\ 1,\ 1) \end{aligned}$$

Hence it is sufficient to find all ovals containing these four points. We shall refer to them as the base points.

These 4 points are permuted among themselves by a group of order 24 (the symmetric group on 4 points) generated by the following two pro'ective collineations

$$(3.2)\qquad r = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad s = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

On the points 0, 117, 234, 39 r has the action $r = (0, 117, 234, 39)$ and $s = (0, 234, 117)(39)$. In addition all four of the points 0, 117, 234, 39 are fixed by the field automorphism t where $(x, y, z)t = (x^2, y^2, z^2)$. The group $H = \langle r, s, t \rangle$ generated by r, s, t is of order 96.

An oval containing the base points of (3.1) will be equivalent to another oval containing these points under the action of the group H. The remaining 14 points of the oval must be taken from the 182 points not lying on any one of the six lines joining pairs of the four points of (3.1). These 182 points are permuted by H in 5 orbits of lengths 2, 12, 24, 48, 96 respectively. These 5 orbits are lettered A, B, C, D, E and their points in form of residues modulo 273 are listed here:
Oval includes 0 39 117 234

A 52 143

B 25 89 98 166 200 217 235 236 238 242 250 266

C 3 14 19 24 45 66 67 70 77 87 108 112 124 129 150 152 171 173
179 192 193 213 255 263

D 6 15 17 21 29 31 33 36 38 48 51 69 73 81 97 101 102 105
106 111 115 120 135 136 141 168 170 177 185 201 202 218 226 230 232 233
241 243 248 249 251 252 261 262 264 268 269 270

E 7 9 12 18 23 27 28 30 34 35 37 42 46 53 54 56 57 58
60 62 63 68 72 74 75 76 84 85 90 92 93 95 96 99 107 109
113 114 116 123 126 131 132 134 138 140 144 145 146 147 148 151 153 154
155 159 162 163 165 174 175 180 183 184 186 187 189 190 191 194 198 204
207 209 210 212 214 216 219 222 223 224 225 228 229 231 237 240 246 253
257 258 265 267 271 272

With respect to these orbits it is worth noting that $52 = (a^5, a^{10}, 1)$ and $143 = (a^{10}, a^5, 1)$ are the points which together with the base points of (3.1) form an oval of 6 points in the unique subplane of order 4 containing the base points. The line joining 52 and 143 has the equation $x + y + z = 0$ and the porbit B consists of the remaining permissible points on this, excluding the points $(1, 1, 0)$, $(1, 0, 1)(0, 1, 1)$ which lie on lines joining pairs of base points.

First case: Oval contains $0 = (0, 0, 1)$, $117 = (0, 1, 0)$, $234 = (1, 0, 0)$,
$39 = (1, 1, 1)$, $52 = (a^5, a^{10}, 1)$ and $143 = (a^{10}, a^5, 1)$.

Let us first consider ovals which besides the base points also include both points $52 = a^5, a^{10}, 1)$ and $143 = (a^{10}, a^5, 1)$. Since the set of 6 points 0, 39, 117, 234, 52, 143 are taken into themselves by the group H the points which do not lie on lines joining any two of them form complete orbits of H in (3.3). As already observed the points of orbit B lie entirely on the line $x+y+z=0$ joining 52 and 143. Since $0=0, 0, 1)$, $52=(a^5, a^{10}, 1)$ and $23=(a, a^6, 1)$ lie on the line $a^5x+y=0$ and 23 is in the orbit E, the complete orbit E must be excluded in choosing points to complete the 6 points 0, 39, 117, 234, 52, 143 to an oval. The remaining orbits C and D are both admissible. Hence a seventh point may be chosen either as an arbitrary point in the C orbit, say $67=(a, a^9, 1)$ or as an arbitrary point in the D orbit, say $136=(a^{11}, a^{14}, 1)$. But these choices are equivalent under the following collineation

$$
(3.4)\qquad
\begin{array}{rcl}
 & (x,\ y,\ z) \rightarrow (a^{10}x,\ a^5y,\ z) & \\
0 = (0\ ,\ 0\ ,\ 1) & \rightarrow & (0\ ,\ 0\ ,\ 1) = 0 \\
117 = (0\ ,\ 1\ ,\ 0) & \rightarrow & (0\ ,\ a^5,\ 0) = 117 \\
234 = (1\ ,\ 0\ ,\ 0) & \rightarrow & (a^{10},\ 0\ ,\ 0) = 234 \\
39 = (1\ ,\ 1\ ,\ 1) & \rightarrow & (a^{10},\ a^5,\ 1) = 143 \\
52 = (a^5,\ a^{10},\ 1) & \rightarrow & (1\ ,\ 1\ ,\ 1) = 39 \\
143 = (a^{10},\ a^5,\ 1) & \rightarrow & (a^5,\ a^{10},\ 1) = 52 \\
67 = (a\ ,\ a^9,\ 1) & \rightarrow & (a^{11},\ a^{14},\ 1) = 136
\end{array}
$$

Hence using this collineation and the group H we may up to equivalence chance any point of either orbit C or D as our seventh point. It suits our convenience to choose this seventh point as the point $29=(a^4, a^8, 1)$ from the D orbit.

Having chosen seven points of the oval there remain only the following 41 points not on a line joining two of the seven points

(3.5)

C 14, 24, 45, 67, 77, 87, 112, 150, 173, 149, 192, 255

D 15, 17, 21, 31, 33, 36, 38, 48, 97, 102, 106, 135, 141, 170, 202, 218, 226, 230, 232, 233, 241, 251, 252, 261, 262, 264, 268, 269, 270

As there are 18 points on the oval, each of the 17 lines through one of the points of the oval must contain a further point of the oval. The line L_{203} whose equation is $y+a^2z=0$ goes through the base point $234=(1, 0, 0)$ but the only other point of the 41 points of (3.5) which is on this line is $218=(a, a^2, 1)$. It follows that $218=(a, a^2, 1)$ must be an eighth point of the oval. Now with the 8 points chosen there remain only 10 points not on any line joining two of the 8 and these 18 points taken together do indeed form an oval whose points are given here:

$$\text{Conic } x^2 + yz = 0 \text{ and its nucleus } (1, 0, 0)$$

$$(3.6)\qquad \begin{array}{ll} 0 = (0\ ,\ 0\ ,\ 1) & 233 = (a^3\ ,\ a^6\ ,\ 1) \\ 117 = (0\ ,\ 1\ ,\ 0) & 268 = (a^6\ ,\ a^{12},\ 1) \\ 234 = (1\ ,\ 0\ ,\ 0) & 202 = (a^7\ ,\ a^{14},\ 1) \\ 39 = (1\ ,\ 1\ ,\ 1) & 106 = (a^8\ ,\ a\ ,\ 1) \\ 52 = (a^5\ ,\ a^{10},\ 1) & 226 = (a^9\ ,\ a^3\ ,\ 1) \\ 143 = (a^{10},\ a^5\ ,\ 1) & 251 = (a^{11},\ a^7\ ,\ 1) \\ 29 = (a^4\ ,\ a^8\ ,\ 1) & 170 = (a^{12},\ a^9\ ,\ 1) \\ 218 = (a\ ,\ a^2\ ,\ 1) & 97 = (a^{16},\ a^{11},\ 1) \\ 232 = (a^2\ ,\ a^4\ ,\ 1) & 230 = (a^{14},\ a^{16},\ 1) \end{array}$$

The 18 points of this oval are the points of the irreducible conic $x^2 + yz = 0$ together with its nucleus $234 = (1, 0, 0)$. We have now shown that an oval containing the base points and both $52 = (a^5, a^{10}, 1)$ and $143 = (a^{10}, a^5, 1)$ is equivalent to the oval in (3.6).

If an oval contains besides the base points a point of the orbit D, we may suppose this to be the point 241. Then we have the following collineation:

$$(3.7)\qquad \begin{array}{rcl} (x, y, z) & \to & (ax, a^4y, z) \\ 0 = (0\ ,\ 0\ ,\ 1) & \to & (0,\ 0\ ,\ 1) = \ \ 0 \\ 117 = (0\ ,\ 1\ ,\ 0) & \to & (0,\ a^4,\ 0) = 117 \\ 234 = (1\ ,\ 0\ ,\ 0) & \to & (a,\ 0\ ,\ 0) = 234 \\ 39 = (1\ ,\ 1\ ,\ 1) & \to & (a,\ a^4,\ 1) = 238 \\ 241 = (a^{14},\ a^{11},\ 1) & \to & (1,\ 1\ ,\ 1) = \ \ 39 \end{array}$$

Thus such an oval is equivalent to an oval containing the base points and the point 238 of the orbit B.

Similarly if an oval contains a point of the orbit E we may suppose it to be the point 96. Then we have the following collineation

$$(3.8)\qquad \begin{array}{rcl} (x, y, z) & \to & (ax, a^9y, z) \\ 0 = (0,\ 0,\ 1) & \to & (0,\ 0\ ,\ 1) = \ \ 0 \\ 117 = (0,\ 1,\ 0) & \to & (0,\ a^9,\ 0) = 117 \\ 234 = (1,\ 0,\ 0) & \to & (a,\ 0\ ,\ 0) = 234 \\ 39 = (1,\ 1,\ 1) & \to & (a,\ a^9,\ 1) = \ \ 67 \\ 96 = (a,\ a,\ 1) & \to & (1,\ 1\ ,\ 1) = \ \ 39 \end{array}$$

Thus such an oval is equivalent to an oval containing the base points and the point 67 of the orbit C.

Since every oval containing the base points contains points from at least one of the orbits B, C, D or E we may take as our fifth point either $238 = (a, a^4, 1)$ from the orbit B or $67 = (a, a^9, 1)$. The choice of a fifth point gives us our second and third cases. We list the remaining points in these cases.

Second case: Oval contains $0 = (0, 0, 1)$, $117 = (0, 1, 0)$, $234 = (1, 0, 0)$
$39 = (1, 1, 1)$, $238 = (a, a^4, 1)$

(3.9)

A 52 143

B 25 89 98 166 200 217 235 236 242 250 266

C 14 19 24 45 66 70 77 87 108 112 124 129 150 171 179 263

D 6 29 36 38 51 69 73 81 97 101 106 111 115 135 141 168
170 177 185 201 202 226 230 241 243 248 249 252 264 268 269 270

E 9 12 18 27 28 30 34 35 37 42 46 53 54 56 68 75
76 92 93 95 96 99 107 113 114 123 126 131 132 143 138 140
144 145 146 148 151 153 154 155 159 163 174 175 180 183 186 187
189 190 198 207 209 210 212 214 216 219 222 223 224 225 228 237
240 253 257 258 265 267 271 242

Third case: Oval contains $0 = (0, 0, 1)$, $117 = (0, 1, 0)$, $234 = (1, 0, 0)$
$39 = (1, 1, 1)$, $67 = (a, a^9, 1)$

(3.10)

A 52 143

B 25 98 166 200 235 242 250 266

C 3 14 19 24 45 66 70 87 112 124 129 150 171 173 179 193
213 255 263

D 6 17 21 29 31 48 69 73 81 97 101 115 120 135 136 141
168 185 201 202 226 232 233 241 248 249 251 252 261 264 268 270

E 7 12 27 30 34 35 42 46 53 56 57 58 60 63 68 74
75 76 85 90 92 93 95 96 113 116 123 131 132 134 140 144
145 148 151 154 155 159 162 174 175 180 183 184 186 187 189 190
191 194 198 204 207 210 212 214 216 219 222 224 225 228 231 237
246 253 257 258 265 267 271 272

LEMMA. – *Every oval of the third case is equivalent to an oval of the first or second case.*

PROOF. – We start with the five points 0, 117, 234, 39, 67 and further points are among the 133 points listed in 3.10). Using the group H and the collineation of 3.7), if any of the points of the B or D orbits in (3.10) is a point of the oval then it is equivalent to an oval containing the base points and the point $238=(a, a^4, 1)$ of the B orbit and so is an oval of the second case. If the oval contains one of the points $52=(a^5, a^{10}, 1)$ or $143=(a^{10}, a^5, 1)$ then it contains one point on the line L_{235} which is $x+y+z=0$, and so must contain two points of this line. But then either the oval contains both points 52 and 143 and is an oval of the first case or as the only admissible points of $x+y+z=0$ are in the B orbit, it will contain a point of the B orbit and so be equivalent to an oval in the second case.

But many other choices of points in the third case can be shown to lead to ovals equivalent to those in the first or second case. For examply suppose we use the point $3=(a^4, a^7, 1)$ in the third case. Then we apply collineations as follows

$$(3.11)\qquad \begin{array}{rcccl} & & rt^{-1}(x,y,z) & \to & (a^{14}x, a^6y, z) \\ 0=(0,0,1) & \to & (0,1,0)=177 & \to & (0, a^6, 0)=117 \\ 117=(0,1,0) & \to & (1,0,0)=234 & \to & (a^{14}, 0, 0)=234 \\ 234=(1,0,0) & \to & (1,1,1)=39 & \to & (a^{14}, a^6, 1)=96\,. \\ 39=(1,1,1) & \to & (0,0,1)=0 & \to & (0,0,1)=0 \\ 67=(a, a^9, 1) & \to & (a, a^9, 1)=67 & \to & (1,1,1)=39 \\ 3=(a^4, a^7, 1) & \to & (a^7, a^6, 1)=87 & \to & (a^6, a^{12}, 1)=268 \end{array}$$

But here this is equivalent to an oval containing the base points and the point 268 of the D orbit of H. But we have shown that using the group H and the collineation of (3.7) this is equivalent to an oval of the second case.

Similar computations show that most choices of points in (3.10) lead to ovals of the second case. The only points remaining in the third case are

Oval contains 0, 39, 117, 234, 67

(3.12) C 179, 213, 255, 263

E 7, 56, 57, 63, 92, 95, 96, 113, 151, 174, 214, 225, 231, 267, 271, 242

But the line L_{18} through $0=(0,0,1)$ whose equation is $a^{11}x+y=0$ contains no one of the 20 C and E points listed in (3.12). Hence there is no oval involving only these points, and our lemma is proved.

In finding the ovals which arise in the second case and also in studying their equivalences considerable use is made of the group $H=\langle r, s, t\rangle$. The collineations r,

s, t are given here as permutations on the 152 points not on the 6 lines joining the base points.

$$(x, y, z) = (x + y, x + z, x)$$

$$r = (0, 117, 234, 39)$$

A (52, 143)

B (25, 235) (89, 266) (98, 236) (166, 217) (200, 242) (238, 250)

C (3, 45, 152, 112) (14, 171, 108, 19) (24, 87, 124, 77) (66, 255, 129, 213) (67, 179, 193, 263) (70, 192, 150, 173)

D (6, 270, 38, 97) (15, 177, 232, 264) (17, 185, 105, 226) (21, 170, 136, 115) (29, 33, 36, 261) (31, 230, 48, 249) (51, 218, 201, 120) (69, 102, 73, 251)

(3.13) (81, 248, 202, 111) (101, 168, 268, 269) (106, 243, 141, 135) (233, 241, 262, 252)

E (7, 204, 74, 231) (9, 147, 76, 23) (12, 99, 265, 151) (18, 258, 186, 224) (27, 271, 253, 155) (28, 207, 237, 228) (30, 126, 223, 107)(35, 162, 214, 63) (37, 240, 189, 146) (42, 175, 225, 95) (46, 210, 56, 267)(53, 138, 183, 272) (54, 131, 198, 212) (58, 159, 229, 219) (60, 132, 165, 140) (62, 184, 72, 84) (68, 134, 190, 174) (75, 187, 93, 163) (85, 92, 90, 145) (96, 246, 209, 116) (113, 148, 216, 257) (114, 191, 180, 194) (123, 154, 144, 153) (34, 109, 222, 57)

$$(x, y, z)\, s = (z, x, y)$$

$$s = (0, 234, 117)\ (39)$$

A (52) (143)

B (238, 166, 25) (98, 89, 242) (217, 250, 235) (200, 266, 236)

C (67, 112, 171) (179, 14, 192) (193, 70, 24) (262, 77, 3) (173, 129, 87) (124, 255, 45) (152, 66, 108) (19, 213, 150)

D (6, 17, 243) (15, 268, 38) (21, 262, 251) (29, 51, 249) (31, 120, 233) (33, 48, 136) (36, 170, 73) (69, 241, 201) (81, 232, 135) (97, 168, 185)

(3.14) (101, 202, 105) (102, 218, 261) (106, 177, 270) (269, 264) (115, 230, 252) (141, 226, 248)

E (7, 224, 99) (9, 190, 53) (12, 253, 204) (18, 231, 198) (23, 75, 174) (27, 184, 54) (28, 183, 35) (30, 148, 116) (34, 165, 189) (37, 180, 90) (42, 114, 60) (46, 257, 58) (56, 154, 246) (57, 175, 140) (62, 258, 131)

(63, 93, 207) (68, 163, 214) (72, 155, 151) (74, 271, 212) (76, 237, 187)
(84, 265, 186) (85, 225, 222) (92, 109, 240) (95, 145, 19) (96, 113, 267)
(107, 159, 216) (123, 126, 209) (132, 194, 146) (134, 162, 138)
(144, 210, 219) (147, 272, 228) (153, 229, 223)

$$(x, y, z)t = (x^2, y^2, z^2)$$
$$t = (0)\,(39)\,(117)\,(234)$$

A (52, 143)

B (238, 98, 217, 266) (25, 242, 235, 200) (250, 236, 166, 89)

C (3, 171, 192, 24) (14, 70, 77, 112) (19, 173, 124, 152) (45, 108, 150, 87)
(66, 213, 129, 255) (67, 179, 193, 263)

D (6, 69, 111, 48) (15, 36, 141, 120) (17, 241, 269, 136) (21, 105, 252, 168)
(29, 106, 218, 232) (31, 38, 73, 248) (33, 243, 201, 264) (51, 177, 261, 135)
(51, 249, 270, 102) (97, 251, 202, 230) (101, 115, 185, 262)
(3.15) (170, 226, 233, 268)

E (7, 35, 175, 56) (9, 240, 30, 72) (12, 138, 222, 96) (18, 207, 60, 144)
(23, 37, 107, 184) (27, 174, 90, 216) (28, 140, 154, 224) (34, 209, 265, 272)
(42, 210, 231, 63) (46, 74, 214, 95) (53, 109, 116, 151) (54, 75, 180, 159)
(57, 246, 99, 183) (58, 212, 163, 191) (62, 76, 146, 223) (68, 145, 257, 271)
(84, 147, 189, 126) (85, 113, 253, 134) (92, 148, 155, 190) (93, 114, 219, 198)
(123, 186, 228, 165) (131, 187, 194, 229) (132, 153, 258, 237)
(162, 225, 267, 204)

The subgroup of $H = \langle r, s, t\rangle$ fixing the point $238 = (a, a^4, 1)$ is elementary Abelian of order 8 and is generated by the three elements srt^2, r^2, and $u = s^{-1} r^2 s$. On the *A* and *B* orbits the action of srt^2 is

srt^2 (0, 39) (177) (234)

(3.16) *A* (52, 143)

B (238) (217) (98) (266) (25, 166) (89, 242) (200, 236) (235, 250)

r^2 and u are the identy on the *A* and *B* orbits.

In the second case the oval contains the points $0 = (0, 0, 1)$, $177 = (0, 1, 0)$, $234 = (1, 0, 0)$ $39 = (1, 1, 1)$, and $238 = (a, a^4, 1)$. Since $(a, a^4, 1)$ lies on the line $x + y + z = 0$ which is L_{203} the oval must contain a second point on this line. But these are precisely the points of the *A* and *B* orbits except for the points $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ which are collinear with two base points. We may now divide the second case into subcases depending on which further point on $x+y+z=0$ is taken. From (3.16) we see that choosing 238 and 143 is equivalent to choosing 238 and 52. The 66 unordered pairs of B points are moved by H in five orbits of which (238, 25), (238, 89), (238, 98), (238, 217) and (238, 235) are representatives.

Hence we may divide the second case into six subcases

		Oval contains
	First subcase	0, 39, 117, 234, 238, 52
	Second	0, 39, 117, 234, 238, 25
(3.17)	Third	0, 39, 117, 234, 238, 89
	Fourth	0, 39, 117, 234, 238, 98
	Fifth	0, 39, 117, 234, 238, 217
	Sixth	0, 39, 117, 234, 238, 235

All ovals arising in these six subcases were found by direct search, with the assistance of a computer, adding a point at a time from the points remaining. The basis argument used is that a line containing one point of an oval must contain a second point. Thus if we find that a line through 238 has only three other admissible points on it we can subdivide our case by taking each of these points in turn as a point of the oval. In almost every instance when eight points had been chosen on the oval the rest were uniquely determined if a completion was possible.

Although the ovals were calculated independently they will be listed in sets of one two or four under the action of the collineations 1, r^2, u, r^2u. This gives a partial equivalence and also is a check on the correctness of the calculations.

First subcase ovals

I

0, 117, 234, 39, 52, 238, 179, 77, 187, 6, 202, 270, 18, 42, 198, 179, 185, 45 = I
234, 39, 0, 117, 52, 238, 263, 87, 163, 38, 81, 97, 186, 225, 54, 115, 226, 112 = Ir^2
117, 0, 39, 234, 52, 238, 66, 108, 53, 202, 6, 248, 253, 37, 265, 201, 177, 70 = Ir^2u
39, 234, 117, 0, 52, 238, 129, 14, 183, 81, 38, 111, 27, 189, 12, 51, 264, 150 = Iu

II

0, 117, 234, 39, 52, 238, 243, 148, 201, 153, 101, 263, 18, 27, 66, 248, 97, 115 = II
(3.18) 234, 39, 0, 117, 52, 238, 135, 257, 51, 154, 268, 179, 186, 253, 129, 111, 270, 170 = IIr^2

IIu = II

III

0, 117, 234, 39, 52, 238, 243, 148, 189, 73, 87, 106, 237, 34, 66, 19, 269, 45 = III
234, 39, 1, 117, 52, 238, 135, 257, 37, 69, 77, 141, 28, 222, 129, 171, 168, 112 = IIIr^2
117, 0, 39, 234, 52, 238, 268, 154, 225, 249, 14, 269, 68, 114, 179, 124, 106, 70 = IIIr^2u
39, 234, 117, 0, 52, 238, 101, 153, 42, 230, 108, 168, 190, 180, 263, 24, 141, 150 = IIIu

We note that any oval containing the base points and a point of the A orbit must contain a further point of the line $x + y + z = 0$ and so either contains both A points and is equivalent to the oval (3.6) of the first case or contains one A point and one B point which can be taken to be 52 and 238, and so is in the first subcase (3.18).

The ovals of the second through sixth subcases are as follows:

Ovals of second through sixth subcases

1

0, 39, 117, 234, 238, 25, 93, 9, 53, 66, 97, 113, 115, 159, 187, 190, 240, 243 = 1
234, 117, 39, 0, 238, 25, 75, 76, 183, 129, 270, 216, 170, 219, 163, 68, 146, 135 = $1r^2$
117, 234, 0, 39, 238, 25, 272, 214, 187, 179, 111, 144, 51, 96, 53, 28, 95, 268 = $1r^2u$
39, 0, 234, 117, 238, 25, 138, 35, 163, 263, 248, 123, 201, 209, 183, 237, 175, 101 = $1u$

2

0, 39, 117, 234, 238, 25, 93, 9, 29, 35, 183, 210, 237, 243, 258, 264, 269, 271 = 2
234, 117, 39, 0, 238, 25, 75, 76, 36, 214, 53, 267, 28, 135, 224, 177, 168, 155 = $2r^2$
117, 234, 0, 39, 238, 25, 272, 214, 241, 76, 163, 107, 68, 268, 271, 226, 106, 258 = $2r^2u$
39, 0, 234, 117, 238, 25, 138, 35, 252, 9, 187, 126, 190, 101, 155, 185, 141, 224 = $2u$

3

0, 39, 117, 234, 238, 25, 107, 76, 46, 53, 113, 115, 144, 150, 209, 265, 268, 269 = 3
234, 117, 39, 0, 238, 25, 126, 9, 56, 183, 216, 170, 123, 70, 96, 12, 101, 168 = $3r^2$
117, 234, 0, 39, 238, 25, 210, 35, 30, 187, 144, 51, 113, 112, 219, 198, 243, 106 = $3r^2u$
39, 0, 234, 117, 238, 25, 267, 214, 223, 163, 123, 201, 216, 45, 159, 54, 135, 141 = $3u$

4

0, 39, 117, 234, 238, 25, 107, 237, 37, 42, 56, 73, 96, 101, 144, 185, 209, 270 = 4
(3.19) 234, 117, 39, 0, 238, 25, 126, 28, 189, 225, 46, 69, 209, 268, 123, 226, 96, 97 = $4r^2$
117, 234, 0, 39, 238, 25, 210, 68, 42, 37, 223, 249, 159, 135, 113, 177, 219, 248 = $4r^2u$
39, 0, 234, 117, 238, 25, 267, 190, 225, 189, 30, 230, 219, 243, 216, 264, 159, 111 = $4u$

5

0, 39, 117, 234, 238, 89, 92, 148, 12, 24, 29, 42, 53, 134, 144, 150, 177, 240 = 5
234, 117, 39, 0, 238, 89, 145, 257, 265, 124, 36, 225, 183, 174, 123, 70, 264, 246 = $5r^2$
117, 234, 0, 39, 238, 89, 140, 154, 54, 171, 241, 37, 187, 207, 113, 112, 185, 95 = $5r^2u$
39, 0, 234, 117, 238, 89, 132, 153, 198, 19, 252, 189, 163, 228, 216, 45, 226, 175 = $5u$

6

0, 39, 117, 234, 238, 89, 92, 148, 27, 36, 76, 97, 106, 129, 134, 189, 224, 271 = 6
234, 117, 39, 0, 238, 89, 145, 257, 253, 29, 9, 270, 141, 66, 174, 37, 258, 155 = $6r^2$
117, 234, 0, 39, 238, 89, 140, 154, 186, 252, 35, 111, 269, 263, 207, 225, 155, 258 = $6r^2u$
39, 0, 234, 117, 238, 89, 132, 153, 18, 241, 214, 248, 168, 179, 228, 42, 271, 224 = $6r^2u$

7

0, 39, 117, 234, 238, 89, 99, 198, 12, 27, 35, 70, 106, 134, 144, 174, 179, 249 = 7
234, 117, 39, 0, 238, 89, 151, 54, 265, 253, 214, 150, 141, 174, 123, 134, 263, 230 = $7r^2$
117, 234, 0, 39, 238, 89, 212, 265, 54, 186, 76, 45, 269, 207, 113, 228, 66, 73 = $7r^2u$
39, 0, 234, 117, 238, 89, 131, 12, 198, 18, 9, 112, 168, 228, 216, 207, 129, 69 = $7u$

(3.19)

8

0, 39, 117, 234, 238, 89, 155, 189, 19, 27, 37, 96, 171, 185, 209, 226, 253, 271 = 8 = $8r^2$

39, 0, 234, 117, 238, 89, 258, 42, 24, 18, 225, 219, 124, 264, 159, 177, 186, 224 = $8u$ = $8r^2u$

9

0, 39, 117, 234, 238, 98, 92, 134, 30, 68, 69, 77, 114, 150, 175, 187, 240, 270 = 9

234, 117, 39, 0, 238, 98, 154, 174, 223, 190, 73, 87, 180, 70, 95, 163, 146, 97 = $9r^2$

117, 234, 0, 39, 238, 98, 140, 207, 46, 237, 230, 108, 34, 112, 146, 53, 95, 248 = $9r^2u$

39, 0, 234, 117, 238, 98, 132, 228, 56, 28, 249, 14, 222, 45, 240, 183, 175, 111 = $9u$

10

0, 39, 117, 234, 238, 98, 92, 183, 45, 54, 87, 108, 126, 132, 150, 187, 210, 265 = 10 = $10u$

234, 117, 39, 0, 238, 98, 145, 53, 112, 198, 77, 14, 107, 140, 70, 163, 267, 12 = $10r^2$ = $10r^2u$

11

0, 39, 117, 234, 238, 98, 107, 264, 12, 18, 27, 46, 77, 115, 150, 159, 174, 258 = 11

234, 117, 39, 0, 238, 98, 126, 177, 265, 186, 253, 56, 87, 170, 70, 219, 134, 224 = $11r^2$

117, 234, 0, 39, 238, 98, 210, 226, 54, 253, 186, 30, 108, 51, 112, 96, 228, 271 = $11r^2u$

39, 0, 234, 117, 238, 98, 267, 185, 198, 27, 18, 223, 14, 201, 45, 209, 207, 155 = $11u$

12

0, 39, 117, 234, 238, 98, 155, 114, 34, 106, 141, 168, 180, 222, 224, 258, 269, 271 = 12 = $12r^2$ = $12u$

13

0, 39, 117, 234, 238, 217, 92, 223, 9, 19, 24, 68, 96, 148, 185, 268, 271, 272 = 13

234, 117, 39, 0, 238, 217, 145, 30, 76, 171, 124, 190, 209, 257, 226, 101, 155, 138 = $13r^2$

117, 234, 0, 39, 238, 217, 140, 56, 214, 124, 171, 237, 159, 154, 177, 243, 258, 93 = $13r^2u$

39, 0, 234, 117, 238, 217, 132, 46, 35, 24, 19, 28, 219, 153, 264, 135, 224, 75 = $13u$

14

0, 39, 117, 234, 238, 217, 92, 270, 6, 9, 38, 46, 56, 76, 97, 145, 224, 258 = 14 = $14r^2$

39, 0, 234, 117, 238, 217, 13, 111, 81, 35, 202, 223, 30, 214, 248, 140, 271, 155 = $14u$ = $14r^2u$

(3.19)

15

0, 39, 117, 234, 238, 217, 93, 217, 34, 38, 129, 134, 144, 179, 198, 209, 257, 270 = 15

234, 117, 39, 0, 238, 217, 75, 28, 222, 6, 66, 174, 123, 263, 54, 96, 148, 97 = $15r^2$

117, 234, 0, 39, 238, 217, 272, 68, 114, 81, 263, 207, 113, 66, 265, 219, 153, 248 = $15r^2u$

39, 0, 234, 117, 238, 217, 138, 190, 180, 202, 179, 228, 216, 129, 12, 159, 154, 111 = $15u$

16

0, 39, 117, 234, 238, 217, 123, 207, 12, 34, 54, 101, 114, 134, 135, 216, 226, 264 = 16 = $16r^2u$

234, 117, 39, 0, 238, 217, 144, 228, 265, 222, 198, 268, 180, 174, 143, 113, 185, 177 = $16r^2$

17

0, 39, 117, 234, 238, 235, 92, 177, 126, 140, 148, 154, 185, 189, 225, 243, 267, 268 = 17 = $17r^2u$

234, 117, 39, 0, 238, 235, 145, 264, 107, 132, 257, 153, 226, 37, 42, 135, 210, 101 = $17r^2$

18

0, 39, 117, 234, 238, 235, 93, 53, 34, 134, 135, 151, 177, 214, 226, 240, 243, 253 = 18

234, 117, 39, 0, 238, 235, 75, 183, 222, 174, 243, 99, 264, 35, 185, 146, 135, 27 = $18r^2$

117, 234, 0, 39, 238, 235, 272, 187, 114, 207, 101, 131, 185, 9, 264, 95, 268, 18 = $18r^2u$

39, 0, 234, 117, 238, 235, 138, 163, 180, 228, 268, 212, 226, 76, 177, 175, 101, 186 = 18u

4. – Equivalence of the ovals.

At this stage it has been shown that every oval in the plane is equivalent under the collineation group to the oval of (3.6) which consists of the points of the conic $x^2 + yz = 0$ together with its nucleus $(1, 0, 0)$, or to one of the ovals in (3.18) where we have three classes or in (3.19) where we have eighteen classes. Thus we have at most 22 classes of ovals in the plane.

In this section further equivalences will be given explicitly on these 22 classes and we will achieve the objective of this paper in showing that, in the Desarguesian plane of order 16, under the collineation group there are exactly two classes of ovals, one the class of irreducible conics together with their nuclei, as represented by (3.6) and one other class, whose representative we shall choose to be II in (3.18). First we show that all ovals in (3.19) are equivalent to those in (3.6) and (3.18) and then we show that all ovals in (3.18) are equivalent to II.

In (3.19) there are further equivalences under the group $H = \langle r, s, t \rangle$. We have the following specific equivalences:

$$\begin{aligned} 1\, rst^2 &= 3 \\ 2\, rst^2 &= 4r^2 \\ 9\, rs^{-1}t^2 &= 11r^2u \\ 13\, rs^{-1} &= 15u \\ 14\, rs^{-1} &= 16 \end{aligned} \tag{4.1}$$

Hence representatives of classes of ovals in (3.19) may be taken as 1, 2, 5, 6, 7, 8, 9, 10, 12, 13, 14, 17, and 18. We now show that these are equivalent to the ovals in (3.6) and (3.18).

To the oval $1u$ we apply the collineation $(x, y, z) \to (a^4x, a^6y, z)$ and then $tr^{-1}s$

$$\begin{array}{rclcl} 1u(x, y, z) & \to & (a^4x, a^6y, z) & & tr^{-1}s \\ 39 = (1, 1, 1) & \to & (a^4, a^6, 1) = 197 & \to & 70 \\ 0 = (0, 0, 1) & \to & (0, 0, 1) = 0 & \to & 39 \\ 234 = (1, 0, 0) & \to & (a^4, 0, 0) = 234 & \to & 0 \\ 117 = (0, 1, 0) & \to & (0, a^6, 0) = 117 & \to & 234 \\ 238 = (a, a^4, 1) & \to & (a^5, a^1{}_1, 1) = 52 & \to & 52 \\ 25 = (a^3, a^{14}, 1) & \to & (a^7, a^5, 1) = 68 & \to & 37 \\ 138 = (a^{11}, a^9, 1) & \to & (1, 1, 1) = 39 & \to & 117 \\ 35 = (a^{14}, a^5, 1) & \to & (a^3, a^{11}, 1) = 77 & \to & 66 \\ 163 = (a^{10}, a^2, 1) & \to & (a^{14}, a^8, 1) = 129 & \to & 108 \end{array} \tag{4.2}$$

$$
\begin{array}{rlcll}
263 = (a^8, & a^{12}, 1) & \to & (a^{12}, a^3, 1) = 36 & \to 6 \\
248 = (a^{12}, & a^6, 1) & \to & (a, a^{12}, 1) = 33 & \to 177 \\
123 = (a^2, & a^7, 1) & \to & (a^6, a^{16}, 1) = 242 & \to 238 \\
201 = (a^4, & a^3, 1) & \to & (a^8, a^9, 1) = 223 & \to 265 \\
209 = (a^5, & a^{16}, 1) & \to & (a^9, a^4, 1) = 246 & \to 253 \\
183 = (a^6, & a, 1) & \to & (a^{10}, a^7, 1) = 27 & \to 53 \\
237 = (a^9, & a^{11}, 1) & \to & (a^{13}, a^2, 1) = 185 & \to 201 \\
175 = (a^{13}, & a^{10}, 1) & \to & (a^2, a, 1) = 252 & \to 202 \\
101 = (a^7, & a^8, 1) & \to & (a^{11}, a^{14}, 1) = 136 & \to 248
\end{array}
\tag{4.2}
$$

Here 4.2 gives a collineation which maps the oval $1u$ into the oval $\mathrm{I}r^2u$, as may be checked from 3.18.

Similar collineations are as follows

$$
\begin{array}{ll}
(2u) & \alpha\, s\, tr^{-1} = \mathrm{III}r^2 \text{ with } \alpha\colon (x, y, z) \to (a^4x, a^6y, z) \\
(5u) & \alpha\, tr^{-1} = \mathrm{III}r^2u \text{ with } \alpha\colon (x, y, z) \to (a^{14}x, a^{11}y, z) \\
(6u) & \alpha\, tr^{-1} = \mathrm{II}r^2 \text{ with } \alpha\colon (x, y, z) \to (a^{14}x, a^{11}y, z) \\
(7u) & \alpha\, s^{-1}\, rt^{-1} = Ir^2u \text{ with } \alpha\colon (x, y, z) \to (a^{13}x, ay, z) \\
(9u) & \alpha\, tr^{-1} = \mathrm{II}r^2 \text{ with } \alpha\colon (x, y, z) \to (a^{13}x, a^7y, z) \\
(11) & \alpha\, s^{-1}\, tr^{-1} = \mathrm{III}r^2 \text{ with } \alpha\colon (x, y, z) \to (a^3x, a^2y, z) \\
\multicolumn{2}{l}{\alpha\, r^{-1} = \text{conic } x^2 + yz = 0 \text{ and nucleus } (1, 0, 0)} \\
 & \qquad \text{with } \alpha\colon (x, y, z) \to (a^{13}x, a^7y, z) \\
(13) & \alpha\, s^{-1}\, tr^{-1} = \mathrm{I} \text{ with } \alpha\colon (x, y, z) \to (a^{14}x, a^{11}y, z) \\
(14) & \alpha\, t = 1r^2u \text{ with } \alpha\colon (x, y, z) \to (a^9x, a^7y, z) \\
(17r^2) & \alpha rt^{-1} = \mathrm{I} \text{ with } \alpha\colon (x, y, z) \to a^{14}x, a^{11}y, z) \\
(18u) & \alpha rt^{-1} = \mathrm{I}u \text{ with } \alpha\colon (x, y, z) \to (a^3x, a^4y, z)
\end{array}
\tag{4.3}
$$

In (4.3) if we compose the map of 14 into 1 with that of (4.2) mapping $1u$ onto $\mathrm{I}r^2p$ we have now shown that every oval is equivalent to the oval in (3.6) or to one of those in (3.18).

The points of the oval II are:

Oval II

$$
\begin{array}{ll}
0 = (0, 0, 1) & 153 = (a^6, a^{14}, 1) \\
117 = (0, 1, 0) & 101 = (a^7, a^8, 1) \\
234 = (1, 0, 0) & 263 = (a^8, a^{12}, 1)
\end{array}
\tag{4.4}
$$

$$\begin{array}{llll} & 39 = (1\,,\ 1\ ,\ 1) & & 18 = (a^9\ ,\ a^5\ ,\ 1) \\ & 52 = (a^5,\ a^1{}_1,\ 1) & & 27 = (a^{10},\ a^7\ ,\ 1) \\ & 238 = (a\ ,\ a^4\ ,\ 1) & & 66 = (a^{11},\ a^2\ ,\ 1) \\ (4.4) & 243 = (a^2,\ a^9\ ,\ 1) & & 248 = (a^{12},\ a^6\ ,\ 1) \\ & 148 = (a^3,\ a^{16},\ 1) & & 97 = (a^{16},\ a^{11},\ 1) \\ & 201 = (a^4,\ a^3\ ,\ 1) & & 115 = (a^{14},\ a\ ,\ 1) \end{array}$$

If we apply the collineation $(x, y, z) \to (a^{10}x, a^5y, z)$ to II and follow this by $s^{-1} r^{-1} s\ t^2$ we obtain oval IIIr^2u. Similarly if we apply the collineation $(x, y, z) \to$ $\to (a^{14}x, a^{11}y, z)$ to II and follows this by s we obtain oval Ir^2u. Thus all ovals in (3.18) are equivalent to each other.

We have now proved the result which was the objective of this paper.

MAIN THEOREM. – *In the Desarguesian plane of order* 16 *there are under equivalence by collineations exactly two classes of ovals. One of these classes consists of the irreducible conics, each with its nucleus adjoined to form the oval. The other class consists of ovals equivalent to oval* II.

We have shown that every oval is either equivalent to that in (3.6) which is the conic $x^2 + yz = 0$ together with its nucleus $(1, 0, 0)$ or to the oval II in (4.4). It is easily seen that oval II is not a conic together with its nucleus. As five points of a conic determine it uniquely, we can easily show that the conic determined by say five out of the first six does not contain all but one point of the oval.

The oval II has as automorphisms the automorphism u, b, and w where

$$(x, y, z) \to (x, y, z)\,u = (x, y, z)\begin{bmatrix} 0, & 1, & 0 \\ 1, & 0, & 0 \\ 1, & 1, & 1 \end{bmatrix}$$

$$(x, y, z) \to (x, y, z)\,b = (x^2, y^2, z^2)\begin{bmatrix} a^{11}, & a^4, & a^5 \\ 0\ , & 0\ , & a^{10} \\ a^6\ , & 0\ , & 0 \end{bmatrix}$$

$$w\colon (x, y, z) \to (x, y, z)\,w = (x^2, y^2, z^2)\begin{bmatrix} a\ , & 1\ , & a^{12} \\ a^3, & 0\ , & 0 \\ a\ , & a^6, & a^{11} \end{bmatrix}$$

As permutations on the points of the oval u, b, and w take the form

$u = (0, 39)\ (117, 234)\ (52)\ (238)\ (101, 243)\ (148, 153)\ (115, 201)\ (66, 263)\ (18, 27)$
$(97, 248)$

$b = (0, 234, 153, 115, 27, 97, 52, 117)\,(18, 201, 39, 238, 66, 248, 263, 101)\,(48, 243)$

$w = (0, 52, 153, 66, 263, 27, 39, 148)\,(18)\,(101, 117, 234, 201, 97, 243, 248, 238)\,(115)$

These generate a group transitive on the points of the oval II. The collineation group of the oval in (3.6) fixes the nucleus and is triply transitive on the remaining points of the conic $x^2 + yz = 0$. In this respect the oval II does not have a nucleus.

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