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Overall properties of heterogeneous material

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SUMMARY

Small-scale heterogeneity in crustal structure can be very complex and difficult to describe in detail and yet, at the same time, can be very important for the description of, for instance, tectonic stress and porosity. Statistical properties of such heterogeneity can be derived from the properties of waves of relatively large wavelength as they propagate through the crust. The differences between measured wavelengths and attenuation coefficients and those of compact rock and the variations of these quantities in space and time provide, in principle, a means of determining quantities like the density and orientations of microfractures and the nature of crack infill material. The theoretical basis for inferences of this kind is the concept of the 'effective' or 'equivalent' material based on an averaging process taken over the microstructure. A number of methods have been used to calculate the properties of the effective medium, several of which are described here.

Key words: anisotropy, composites, cracked materials, effective media, elastic waves.

1 INTRODUCTION

The large-scale structure of the Earth has been known now for several decades and as time has passed more and more of the detail has been filled in. Part of this advance has consisted of more accurate determination of seismic velocities, and of the shape and position of the Earth's major discontinuities. As the scale of variation of the details of structure to be investigated becomes smaller and smaller, the effort to provide an exact description steadily increases. In addition, the need for precise details eventually disappears and a statistical description is not only sufficient but also the most useful information that can be provided. This is particularly true in the exploration industry, where the nature and content of microcracks and intergranular pores are of major interest, and in the investigation of tectonic stress and failure, where once again small-scale fracturing and the flow of fluid into the fractures is important.

In these circumstances where, on the one hand, the scale-size of heterogeneities is too small for seismic investigation with viable wavelengths to be able to scan in detail and, on the other, a statistical description is what is being asked for, the method of attack changes from one based on the idea that, to 'see' structure, wavelengths less than the scale-size must be used, to one where wavelengths large compared with the scale-size are used so that the waves cannot 'see' details but only an average or smeared-out structure.

If the heterogeneous microstructure has certain preferred directions (if, say, the pores are preferentially aligned) then the average or mean response on the macroscopic level will be anisotropic. In particular, if the microstructure has certain symmetries, the macroscopic response will show the same symmetries. A random distribution of spherical inclusions will be, in the large, isotropic; a distribution of microcracks randomly orientated about normals fixed parallel to a given direction will give rise to transverse isotropy with the symmetry axis along the same direction. This principle has been observed, of course, for a very long time in the prediction of the symmetry classes of crystals from their bonding structure (Auld 1973) and, in the last 10 years or so, measured anisotropy in the Earth's crust has often been assigned to alignments in the microstructure (Crampin 1985). It is not usually clear whether the anisotropy is due to microstructure or whether it is 'intrinsic' (due to molecular structure). Further information is available from the attenuation which may also be anisotropic (Crampin 1981, 1984).

Attenuation may be due to 'intrinsic' effects—viscoelastic behaviour of the rock or pore infill—or scattering by the heterogeneity. The difference between these two mechanisms is that, in the second, the scattered waves may be recorded as signal coda and noise. If of course the scattering is significant, the waves are interacting with the structural detail whose scale-size is therefore no longer insignificant compared with a wavelength.

2 THE CONTINUUM HYPOTHESIS

In continuum mechanics the hypothesis is made that, if a real material with its atomic and molecular structure is replaced by an equivalent model continuum, which remains a continuum no matter how much subdivided, the macroscopic mechanical properties of the material remain unchanged. This is called the continuum hypothesis (Hunter 1976) and it enables theoretical concepts, like that of a material *point*, and the density at such a point as the limit of mass over volume as the volume tends to *zero*. So long as all spatial variations are on a scale much larger than the scale-length of the molecular structure this hypothesis works well. Although, in order to construct a pulse, a Fourier sum over all frequencies and therefore comprising infinitesimal wavelengths is necessary, it is never necessary to take account of the particulate structure of the material; the material can always be regraded as a perfect continuum.

In a similar way, microfractures, pores and other microstructure may, if the statistical distribution of structure is locally uniform, be replaced by an 'effective' or 'equivalent' material, which is a perfect continuum with properties such that, on a scale large compared with the scale-length of the microstructure, its mechanical properties are identical to those of the heterogeneous material. Thus a material which behaves as an anisotropic elastic solid for wavelengths larger than a certain scale may be porous, fractured or otherwise heterogeneous on a much smaller scale. However, for all disturbances with variations on the larger scale, the material may be regarded as a homogeneous continuum, even if a Fourier sum extending to infinite frequency is employed.

The main difference between these two applications of the continuum hypothesis is that, whereas derivations of the continuum properties from the known molecular structure have not been particularly successful, there are several theories which provide a means of calculating the parameters of the effective material from a presupposed microstructure.

3 MATHEMATICAL MODELS OF A HETEROGENEOUS MEDIUM

There are four main idealizations of materials with complex 3-D microstructures which are used to derive 'overall' properties of equivalent homogeneous media. The first is a continuous elastic material whose properties vary smoothly but randomly with position in space. The elastic constants $\boldsymbol{c}$ and density ρ vary with the position vector $\mathbf{x}$ in such a way that ρ and the spatial derivatives of $\boldsymbol{c}$ are continuous functions. The elastodynamic equations of motion for the displacement field,

$$\mathscr{L}_i(\mathbf{u}) \equiv \frac{\partial}{\partial x_j} c_{ijpq} \frac{\partial u_p}{\partial x_q} - \rho \frac{\partial^2 u_i}{\partial t^2} = -\rho b_i \tag{1}$$

where $\mathbf{b}$ is the body force density, hold everywhere.

The second model is of isolated inclusions within a homogeneous matrix material. The inclusions consist of material with different properties from those of the matrix, or may be empty, so that the equations of motion (1) are satisfied in the matrix and in the inclusions (if non-empty) with appropriate values of $\boldsymbol{c}$ and ρ, and solutions are linked by continuity conditions at the boundaries of the inclusions. Since almost all methods of analysis applied to this model involve the solution of the problem of a single inclusion within an unbounded matrix with an imposed stress field, most models of inclusions are very simple. The most popular shape is an ellipsoid on account of the fact that analytic solutions exist for a homogeneous ellipsoid, set in unbounded homogeneous material, and deformed by an arbitrary static stress field at infinity (Eshelby 1957). Fortunately, the ellipsoid is a very versatile shape, varying from a spherical ball to an elongated (elliptic) crack, dry or fluid-filled, with vanishing aspect ratio. However, there is nothing in principle to prevent the use of a numerical solution of an inclusion of arbitrarily complex shape.

The third model is of connected, fluid-filled pores within a homogeneous solid. The analysis, usually called Biot theory after its originator (Biot 1956), results in coupled equations for the motion of the solid and the flow of the fluid. In principle, as for model 2 above, the pores may have any shape but, since the calculation of the overall parameters requires the solution for a single pore with an arbitrary imposed stress field (Burridge & Keller 1981), economy of time and effort requires that the shape should be reasonably simple.

The last model is that of an aggregate or granular material and consists of separate compact solid grains in contact with and pressing on their neighbours and surrounded by an interstital fluid. In general the grains are taken to be homogeneous spheres (see, for example, Walton 1987).

Finally mention must be made of a model in which the material structure varies in one dimension only—finely layered material. Each layer may be isotropic or anisotropic elastic or viscoelastic, and even (Schoenberg & Douma 1988) a layer of parallel cracks.

4 STOCHASTIC MODELS AND THE AVERAGING PROCESS

The idea of an effective homogeneous material clearly implies that, if stress, strain or displacements, etc. are measured on a large enough scale, the values obtained and the relationships between them will be those of a homogeneous continuum. This in turn implies some kind of spatial averaging process. For instance, the average stress would be defined as

$$\bar{\sigma}_{ij}(\mathbf{x}) = \frac{1}{V} \int_V \sigma_{ij}\, dV, \tag{2}$$

where V is a region with a length-scale ℓ large compared with that of the microstructure and with centroid $\mathbf{x}$. The continuum hypothesis depends on the assumption that the values obtained for $\bar{\boldsymbol{\sigma}}$ do not depend on the exact shape of V. It may be that the stress field, or the properties of the effective material, will vary on a large scale L, and this will be consistent so long as

$$L \gg \ell \gg a, \tag{3}$$

where a is a scale-length of the microstructure.

One may construct a laboratory test, or mathematical model, by isolating V and imposing a static average stress and measuring average strain in order to determine the

overall elastic constants. The static stress field satisfies

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{4}$$

and so, by application of the divergence theorem, we have

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_S x_j \sigma_{ik} n_k \, dS \tag{5}$$

where S is the boundary of V and $\mathbf{n}$ the outward normal to S. We may, therefore, impose any average stress by using suitable surface tractions $\mathbf{t}$. For instance, if we put

$$t_i = \sigma^0_{ij} n_j \tag{6}$$

at each point of S, where $\boldsymbol{\sigma}^0$ is a constant, then

$$\bar{\sigma}_{ij} = \sigma^0_{ij}. \tag{7}$$

The average strain is

$$\bar{e}_{ij} = \frac{1}{V}\int_V e_{ij}\, dV = \frac{1}{2V}\int_S (u_i n_j + u_j n_i)\, dV, \tag{8}$$

and this too can be measured, or imposed, on the surface S. The problem is linear and so the displacements must be a linear function of the imposed stress which means, if we use the tractions given by equation (6), a linear function of $\bar{\boldsymbol{\sigma}}^0$. Substituting into equation (8), we get the linear relation

$$\bar{e}_{ij} = C_{ijpq}\sigma^0_{pq} = C_{ijpq}\bar{\sigma}_{pq} \tag{9}$$

for some $\boldsymbol{C}$. The continuum hypothesis implies that the relation between $\bar{\boldsymbol{\sigma}}$ and $\bar{\mathbf{e}}$ is independent of the method of loading and so the C_{ijpq} are the elastic constants of the effective material. In practical measurements, the averaging may be done explicitly or by the measuring instrument itself if its area of contact with the material is large compared with the scale-size of the heterogeneity.

The most explicit application of this averaging process to derive dynamical equations for heterogeneous material is the use of the two-space method of homogenization by Burridge & Keller (1981) to derive overall material parameters for the Biot model of a porous solid. In this method, the parameter ϵ is defined by

$$\epsilon = a/L, \tag{10}$$

where a and L are scale-lengths, respectively, of the microstructure and the macroscopic field. It is then assumed that every property of the medium is of the form $f(\mathbf{x}, \mathbf{y})$, where

$$\mathbf{y} = \mathbf{x}\epsilon^{-1}. \tag{11}$$

The first argument in f describes the relatively slow variation of some property, (for instance, the elastic constants) of the material, while the second argument generates the fast variation associated with the microstructure. The next step is to assume that each field quantity $u(\mathbf{x}, \mathbf{y}, \epsilon)$ can be written in a series expansion in ϵ:

$$u(\mathbf{x}, \mathbf{y}, \epsilon) = u_0(\mathbf{x}, \mathbf{y}) + \epsilon u_1(\mathbf{x}, \mathbf{y}) + \cdots \tag{12}$$

and the equations of motion in the solid and fluid phases are evaluated at the zeroth (ϵ^0) and first (ϵ^1) order level. The final step is to compute the averge $\bar{F}(\mathbf{x})$ of any function $\bar{F}(\mathbf{x}, \mathbf{y})$ by integrating $\mathbf{y}$ over a sphere of radius R and centre $\mathbf{x}$ and allowing R to tend to infinity. This leads to the coupled equations of poroelasticity which were derived by Biot (1956) on more heuristic arguments.

An alternative to spatial averaging is to take the mean or expectation over a statistical ensemble of material structures. The average stress $\bar{\boldsymbol{\sigma}}$, for instance, given by equation (2) is replaced by the expectation or mean stress $\langle \boldsymbol{\sigma} \rangle$ given by

$$\langle \boldsymbol{\sigma} \rangle(\mathbf{x}) = \int_\Omega \boldsymbol{\sigma}(\mathbf{x}, \omega) P(\omega)\, d\omega, \tag{13}$$

where Ω is the ensemble space of possible realizations ω with the probability measure $P(\omega)$.

The method of smoothing, applied by Keller (1964) to models 1 and 2 in the previous section, leads to an equation for $\langle \mathbf{u} \rangle$, the expectation of the displacement. In the case of model 1, the parameters of the equation depend on variances and covariances of the form

$$R_{12}(\mathbf{x}, \mathbf{y}) = \langle c_1(\mathbf{x}) c_2(\mathbf{y}) \rangle, \tag{14}$$

where c_1 and c_2 are any two of the elastic constants or density. In practice this quantity is identified with the spatial average

$$\int_V c_1(\boldsymbol{\xi} + \mathbf{x}) c_2(\boldsymbol{\xi} + \mathbf{y})\, dV_\xi, \tag{15}$$

on the assumption that the material is statistically homogeneous over the scale-length ℓ of V, and $\ell \gg a$ as before. Similarly, $\langle \mathbf{u} \rangle(\mathbf{x})$ is identified with the average

$$\int_V \mathbf{u}(\boldsymbol{\xi} + \mathbf{x})\, dV_\xi, \tag{16}$$

where now the scale of V must be large compared with the correlation length of $\langle \mathbf{u} \rangle$. In the absence of other information (which can in principle be found although with difficulty), this correlation length may be identified with the scale-length a of the microstructure. It is understood, of course, that the scale-length ℓ should be small compared with L, the spatial scale of variation of $\langle \mathbf{u} \rangle$ itself. This stochastic model is valid even if these identifications with spatial averages cannot be made. However, the interpretation of the results is different.

A word of caution must be added on the measurement of $\langle \mathbf{u} \rangle$ or $\bar{\mathbf{u}}$. In either case, if the size of the measuring instrument is not large compared with the small-scale variations in $\mathbf{u}$, then it is not the average which is being measured. In general,

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}', \tag{17}$$

where $\bar{\mathbf{u}}' = \mathbf{0}$, but $\mathbf{u}'$ may in fact dominate $\bar{\mathbf{u}}$ on the record. This additional component may be eliminated by averaging over a number of different measurements taken at points within an area of scale-size ℓ, or its magnitude may be estimated by calculating either

$$\overline{|\mathbf{u}'|^2} = \overline{(\mathbf{u} - \bar{\mathbf{u}})^2} = \overline{|\mathbf{u}|^2} - |\bar{\mathbf{u}}|^2 \tag{18}$$

or

$$\langle |\mathbf{u}|^2 \rangle - |\langle \mathbf{u} \rangle|^2.$$

Once again, this is possible in principle, but difficult since it

is necessary not only to solve the equations for $\langle \mathbf{u} \rangle$, but also to construct and solve a similar equation for $\langle |\mathbf{u}|^2 \rangle$.

One application of the averaging process is not particularly clear and that is in the justification of the use of simple models for grains, pores and inclusions. For instance, microfractures are not all elliptic in shape but, in averaging over a volume containing a larger number of them, it seems reasonable to replace each one by an 'average' crack. It is then assumed that this average crack is simple in structure and can be represented by an ellipsoidal cavity with vanishing aspect ratio.

5 METHODS OF ANALYSIS

If the model of a heterogeneous medium is taken to be granular, then the grains are typically assumed to be spheres and various modifications of Hertzian contact are used to analyse their interaction. An average may then be taken over a large number of grains with random points of contact (see Walton 1987) to obtain equations for the equivalent continuum.

If the model consists of interconnected pores then, as stated above, some variety of Biot theory is normally used. The coupled equations for solid and fluid motion predict the existence of three separate waves speeds in the effective continuum, even though the random arrangement of the pores means that the effective material properties are isotropic. The longitudinal and transverse plane waves of elastodynamic theory are predicted alongside a slower 'fluid' wave. One of the major successes of the theory has been the observation of this third wave (see, for instance, Berryman 1980). Calculation of the effective parameters on the basis of a presumed micro structure is not easy and proceeds according to a scheme like, for instance, that given by the two-space method of homogenization of Burridge & Keller (1981).

For a model structure consisting of inclusions, cavities or cracks embedded in a homogeneous matrix, methods of calculating the effective elastic parameters have generally concentrated on approximations valid for small concentrations of inclusions. A number of first-order expressions have been obtained by writing equation (2) as

$$\bar{\boldsymbol{\sigma}} = \sum_{n=0}^{N} \nu_n \boldsymbol{\sigma}^n, \tag{19}$$

where $\nu_n = V_n/V$

$$\boldsymbol{\sigma}^n = \frac{1}{V_n} \int_{V_n} \boldsymbol{\sigma}\, dV, \tag{20}$$

and V_n is the volume occupied by the nth inclusion, or set of inclusions of similar material. The matrix material is included in the sum as $n = 0$. On the assumption that the materials of the matrix and the inclusions are homogeneous, equation (19) becomes

$$\bar{\boldsymbol{\sigma}} = \sum_{n=0}^{N} \nu_n \boldsymbol{c}^n \mathbf{e}^n, \tag{21}$$

where $\boldsymbol{c}^n$ is the elasticity tensor for the nth material, and

$$\mathbf{e}^n = \frac{1}{V_n} \int_{V_n} \mathbf{e}\, dV. \tag{22}$$

Similarly, we have

$$\bar{\mathbf{e}} = \sum_{n=0}^{N} \nu_n \mathbf{e}^n \tag{23}$$

and, operating on this equation with $\boldsymbol{c}^0$ and subtracting from (21), we eliminate the average over the matrix material:

$$(\boldsymbol{I} - \boldsymbol{c}^0 \boldsymbol{S})\bar{\boldsymbol{\sigma}} = \sum_{n=1}^{N} \nu_n (\boldsymbol{c}^n - \boldsymbol{c}^0)\mathbf{e}^n, \tag{24}$$

where $\boldsymbol{I}$ is the identity and $\boldsymbol{S}$ the effective compliance tensor ($\bar{\mathbf{e}} = \boldsymbol{S}\bar{\boldsymbol{\sigma}}$).

If an average stress $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^0$ is imposed on a representative volume V by use of tractions

$$t_i = \sigma^0_{ij} n_j$$

as given in equation (6), the linearity of the problem means that there exists a tensor $\boldsymbol{K}^n$ such that

$$\mathbf{e}^n = \boldsymbol{K}^n \boldsymbol{\sigma}^0 = \boldsymbol{K}^n \bar{\boldsymbol{\sigma}}, \qquad 1 \le n \le N. \tag{25}$$

Substituting into equation (24), we get

$$(\boldsymbol{I} - \boldsymbol{c}^0 \boldsymbol{S})\bar{\boldsymbol{\sigma}} = \left\{ \sum_{n=1}^{N} \nu_n (\boldsymbol{c}^n - \boldsymbol{c}^0)\boldsymbol{K}^n \right\} \bar{\boldsymbol{\sigma}}. \tag{26}$$

Since $\bar{\boldsymbol{\sigma}}$ is arbitrary, it follows that

$$\boldsymbol{I} - \boldsymbol{c}^0 \boldsymbol{S} = \sum_{n=1}^{N} \nu_n (\boldsymbol{c}^n - \boldsymbol{c}^0)\boldsymbol{K}^n$$

or

$$\boldsymbol{S} = \boldsymbol{s}^0 - \sum_{n=1}^{N} \nu_n (\boldsymbol{s}^0 \boldsymbol{c}^n - \boldsymbol{I})\boldsymbol{K}^n, \tag{27}$$

where $\boldsymbol{s}^0$ is the compliance tensor for the matrix material.

It remains to calculate $\boldsymbol{K}^n$ and this, clearly, is not easy. The simplest method, based on the assumption that the density of inclusions is small ($\nu_n \ll 1$), is to approximate $\boldsymbol{K}^n$ by its equivalent for the case of a single inclusion embedded in matrix material which is either unbounded, or bounded in such a way that the volume ratio ν_n is maintained. The results, which ignore all interactions between inclusions, are referred to as first order.

A straightforward improvement on first order approximations is given by the *self-consistent method* in which $\boldsymbol{K}^n$ is calculated for a single inclusion embedded in *effective* material. This means that $\boldsymbol{K}^n$ depends on $\boldsymbol{S}$ and equation (27) becomes an implicit relation for $\boldsymbol{S}$ which no longer depends on ν_n in a linear way. This turns out to give reasonable looking values for $\boldsymbol{S}$ as ν_n increases to values which are not small, whereas the linear dependence of first-order results quickly become unacceptable as ν_n increases. The self-consistent result can be expanded in powers of ν_n and the term in $(\nu_n)^1$ appears always to agree with first-order theory. However, in two cases where comparison have been made, the term in $(\nu_n)^2$ does not agree with second-order theory (Chatterjee, Mal & Knopoff 1978; Hudson 1980).

Second-order theory can be constructed by calculating $\boldsymbol{K}^n$ for an inclusion embedded in the matrix in the presence of one other inclusion (Batchelor & Green 1972; Willis & Acton 1976; Chatterjee *et al.* 1978). In general, however, the averaging process which must be applied to the position

of the second inclusion (which extends to infinity unless boundaries are introduced, thus complicating things in another way) leads to divergent integrals. A technique called 'renormalization' has to be applied, and this is sufficiently difficult to discourage any second-order solution of this type other than for spherical inclusions.

For static problems the calculation of $\boldsymbol{S}$ (or its inverse $\boldsymbol{C}$) is sufficient for all purposes and, if the inclusions are taken to be ellipsoidal in shape, solutions for $\boldsymbol{K}^n$ for an isolated ellipsoid are provided by Eshelby (1957). For dynamic problems, the imposed field is no longer a static stress $\boldsymbol{\sigma}^0$ but is usually taken to be the stress associated with a plane harmonic wave

$$\mathbf{u} = \mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{28}$$

However, in the long-wavelength limit,

$$ka \ll 1, \qquad k = |\mathbf{k}|, \tag{29}$$

the static solution for $\boldsymbol{K}^n$ is appropriate.

If dynamic solutions for $\boldsymbol{K}^n$ are used, depending on ka, then the values for the effective parameters are, in general, complex and frequency dependent, implying a decay of the mean wave due to scattering. This also implies, of course, an increase in the incoherent signal. As mentioned before, it is possible for the incoherent part to dominate the signal. A second complication with such dynamic solutions for $\boldsymbol{K}^n$ is that, unless the effective material is isotropic, the values calculated for the effective parameters depend on the direction of the wave and are therefore no longer material constants. The continuum hypothesis is no longer valid (since ka is no longer vanishingly small) and there is no 'effective material'. However, the expressions describing the 'mean' plane wave are still valid; they are just not so easy to use.

The effective stress–strain relation is not, however, sufficient for the solution of dynamic problems. We need to use the averaged momentum equation

$$\frac{\partial}{\partial x_j}\bar{\sigma}_{ij} = \frac{\partial}{\partial t}\bar{p}_i, \tag{30}$$

where

$$\mathbf{p} = \rho\frac{\partial \mathbf{u}}{\partial t}.$$

The left-hand side of equation (30) is related to mean strain, and hence to the mean displacement $\mathbf{u}$, by the stress–strain relation. It is necessary to calculate the right-hand side from

$$\bar{\mathbf{p}} = \frac{1}{V}\sum_{n=0}^{N}\rho_n\int_{V_n}\frac{\partial \mathbf{u}}{\partial t}\,dV, \tag{31}$$

where ρ_n is the density of the nth material.

We may eliminate the average over the matrix material as before:

$$\bar{\mathbf{p}} - \rho_0\frac{\partial \bar{\mathbf{u}}}{\partial t} = \sum_{n=1}^{N} v_n(\rho_n - \rho_0)\frac{1}{V_n}\int_{V_n}\frac{\partial \mathbf{u}}{\partial t}\,dV, \tag{32}$$

and we may now use a first-order or self-consistent approach to find

$$\frac{1}{V_n}\int_{V_n}\left(\frac{\partial \mathbf{u}}{\partial T}\right)dV$$

in terms of the imposed mean field $\partial\bar{\mathbf{u}}/\partial t$. The long-wavelength approximation gives

$$\bar{\mathbf{p}} = \bar{\rho}\frac{\partial \bar{\mathbf{u}}}{\partial t}, \tag{33}$$

where $\bar{\rho}$ is the average density (Sabina & Willis 1988) as might be expected. Solutions involving ka lead to expressions in which $\bar{\mathbf{p}}$ depends on $\mathbf{k}$ in both magnitude and direction. Once again, although valid, this shorter wavelength formulation does not lead to the characterization of an effective elastic material.

A method of analysing methods of smoothly varying material parameters or isolated inclusions (models 1 and 2, respectively) which is workable up to second order and leads directly to an equation of motion for the mean displacement is the method of smoothing (Keller 1964). This technique has the advantage of dealing quite painlessly with the problem of renormalization. For continuously varying material (model 1), the elastic parameters and density are viewed as varying slightly about their spatially averaged values. Using the spatial average as a reference medium the first non-zero perturbation term in the expression for the effective parameters is in fact second order in the variation of the actual parameters about their means. The effective static stress–strain relation corresponds to 'non-local elasticity' (Beran & McCoy 1970):

$$\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) = \int_{V_n}\boldsymbol{C}(\mathbf{x},\boldsymbol{\xi})\langle\mathbf{e}\rangle(\boldsymbol{\xi})\,dV_{\xi} \tag{34}$$

(this method uses the stochastic average, denoted by $\langle\rangle$) where $\boldsymbol{C}(\mathbf{x},\boldsymbol{\xi})$ is a tensor of elastic moduli which is negligibly small when $|\mathbf{x}-\boldsymbol{\xi}|/a$ is large, and a must now be taken to be the correlation distance for variations in the microstructure. Thus stress depends on a weighted average of the strain within, roughly, a correlation distance. Derivation of similar results for a dilute distribution of cracks is given by Hudson & Knopoff (1989).

In general, the kernel $\boldsymbol{C}$ appears as an infinite series of terms of which the nth represents multiple interactions of order n between scattering centres, and has not been evaluated in full. The highest order calculations have been for $n=2$, involving pair-wise interactions between scatterers. The results for continuously varying material were given by Karal & Keller (1964) and, for discrete scatterers, by Hudson (1980). For wavelengths large compared with the correlation distance/inclusion scale-length a, equation (34) reduces to

$$\langle\boldsymbol{\sigma}\rangle = \boldsymbol{C}\langle\mathbf{e}\rangle \tag{35}$$

as in equation (9), where $\boldsymbol{C}$ is a tensor of material constants.

For materials with discrete inclusions cavities or cracks (model 2), the first-order term (involving the reaction of isolated scatterers only) is non-zero and agrees with other first-order methods. The second-order term, involving pair-wise scatterer interactions, depends in general on the statistics of the scatterer distribution, whereas the self-consistent method, which is claimed to be more accurate than first-order results, does not. Not surprisingly then, the two theories are not in agreement at second order (Hudson 1980). It appears that theories which involve scatterer–scatterer interactions are more accurate than the self-

consistent method, and that the method of smoothing is the simplest and most straightforward way of arriving at second-order accuracy.

Results for a distribution of circular cracks with normals aligned in a given direction are given in the form (Hudson 1980)

$$\boldsymbol{C} = \boldsymbol{c}^0 + \epsilon \boldsymbol{c}^1 + \epsilon^2 \boldsymbol{c}^2 + O(\epsilon^3), \tag{36}$$

where this time the small parameter ϵ is given by

$$\epsilon = \nu a^3, \tag{37}$$

ν being the number density of cracks and a the crack radius; $\boldsymbol{c}^0$ is tensor of elastic moduli of the uncracked matrix material and the first-order term $\boldsymbol{c}^1$ is given by

$$c^1_{ijpq} = -\left[\frac{\lambda^2}{\mu}\delta_{ij}\delta_{pq} + 2\lambda(\delta_{ij}n_p n_q + \delta_{pq}n_i n_j) + 4\mu n_i n_j n_p n_q\right]\bar{U}_{33} - \mu(n_i n_p \delta_{jq} + n_i n_q \delta_{jp} + n_j n_p \delta_{iq} + n_j n_q \delta_{ip} - 4n_i n_j n_p n_q)\bar{U}_{11}, \tag{38}$$

where λ and μ are the Lamé moduli of the uncracked material, $\mathbf{n}$ is the unit normal to the cracks, and $\bar{U}_{11}$ and $\bar{U}_{33}$ represent the response of a single crack to shear and normal stresses respectively. If the material contains cracks orientated in more than one direction, the expression for $\boldsymbol{c}^1$ is obtained by summing or integrating the formula (38) for a single direction over the appropriate range (Hudson 1986). A variety of different crack responses—dry, fluid-filled, partially filled—can be accommodated through the quantities $\bar{U}_{11}$ and $\bar{U}_{33}$ (Hudson 1981, 1988).

Once $\boldsymbol{c}^1$ is determined, $\boldsymbol{c}^2$ is found from

$$c^2_{ijpq} = \frac{1}{\mu} c^1_{ijrs}\chi_{rskl}c^1_{klpq}, \tag{39}$$

where

$$\chi_{ijpq} = \{\delta_{ip}\delta_{jq}[4 + \mu/(\lambda + 2\mu)] - (\delta_{iq}\delta_{jp} + \delta_{ij}\delta_{pq})[1 - \mu/(\lambda + 2\mu)]\}/15. \tag{40}$$

A wide range of crack distributions and conditions can be accommodated in this way and it appears that, with parameter values appropriate to conditions in the Earth's crust, the second-order formulae above are reasonably accurate for values of ϵ up to 0.1 (Crampin 1984).

In the long-wavelength approximation, as described above, the stress–strain relation is exactly the same as for the static problem except for cases where the inclusions or cracks contain a viscous fluid, in which case the viscosity enters in association with the strain-rate. The equation of motion is

$$C_{ijpq}\frac{\partial^2\langle u_p\rangle}{\partial x_j \partial x_q} = \rho^* \frac{\partial^2\langle u_i\rangle}{\partial t^2}, \tag{41}$$

using the stress–strain relation (35) on the left and an equivalent density ρ^* on the right. In the long-wavelength approximation, ρ^* becomes $\bar{\rho}$, the spatially averaged density both for continuously varying material (McCoy 1973) and for distributions of inclusions (Hudson, unpublished manuscript). In the case of cracks of vanishing aspect ratio, ρ^* is just the density ρ of the matrix material (Hudson 1980).

Analysis of a 1-D, finely layered structure is usually along the lines given by Backus (1962), of a fairly empirical spatial average. The result is simple and very general. Alternatively, the theory of stochastic differential equations may be used (see de Hoop, Chang & Burridge 1991). Results are presented as rigorously derived asymptotic expressions.

6 ATTENUATION

The formulae above apply to systems where the wavelengths of a disturbance of the material is long compared with the scale-length of the microstructure; for instance with the size and spacing of cracks. These conditions are in accord with those on which the continuum hypothesis depends and the resulting effective material is perfectly elastic unless there is intrinsic damping in the materials of the composite. That is, there is no attuation due to scattering at this level, only viscous absorption by, for instance, viscous fluid infill of crack (Hudson 1981).

In order to obtain expressions for effective damping due to scattering out of the mean wave, it is necessary to consider the next higher terms in the long-wavelength approximation. If such terms are taken from only the first-order terms in the stress–strain relation (ignoring interactions between scatterers) the result is in accord with that given by Waterman & Truell (1961) for the acoustic problem; that is the attenuation coefficient λ for a plane wave is given by (Hudson 1981)

$$\gamma = \frac{\nu\Sigma}{2}, \tag{42}$$

where Σ is the scattering cross-section for a single scatterer. This is a physically appealing result although, for parameter values appropriate to the crust, the value of γ is generally extremely small (Peacock & Hudson 1990). However, since nearby scatterers lie within a wavelength of each other, their motion must be correlated, and the scattered energy can hardly be expected to be the simple sum implied by equation (42). An investigation of the interaction (second-order) terms (Hudson 1990) shows that this doubt is justified and the attenuation given by (42) is exactly cancelled by the lowest order scattering terms in $\boldsymbol{c}^2$. Attenuation due to scattering is therefore associated with even higher order (and presumably even smaller) terms in the expansion in powers of the ratio of scale-length to wavelength. Such expressions may be required where scattering is sufficiently large to show up on the records, but in this case attention must be paid to the meaning of the stochastic average and the mean wave; that is, the appropriate average of the observations must be used for a suitable comparison with theory. However, it does seem that the prime candidate as the source of attenuation in crustal materials is viscous infill of cracks and inclusions.

7 CONCLUSIONS

For the analysis of the dynamic response of materials with heterogeneities whose scale-length is small compared with a wavelength, two methods stand out as the most effective and versatile; they are Biot's equations for porous media, and the method of smoothing. The former has the advantage of being able to deal with medium and high porosities.

However, the connection between the macroscopic overall properties and the structure of the pores is complicated; it is not easy to construct a 'simple' model of an individual pore. Without this connection, inferences from macroscopic properties to a description of the microstructure are impossible. In addition, Biot theory is based on an averaging rather than a stochastic process and is only useful when wavelengths are so large that scattered signals can be neglected. The method of smoothing, on the other hand, can be used for medium or even short wavelengths so long as the 'mean wave' is interpreted correctly. Different types and shapes of inclusions can be handled in a simple way; all that is required is the solution analytic or numerical, for a single inclusion in an unbounded matrix. Inclusions are isolated in the model, a serious restriction, although an empirical method of allowing flow between inclusions has been given by O'Connell & Budiansky (1977). However, the greatest constraint on the range of application of the method of smoothing is that the porosity, or number density of inclusions, must be small.

If the porosity approaches unity, then it is questionable whether the Biot model or that of a granular material is more appropriate. In general, it seems that the theory available for the Biot model is rather more flexible and to be preferred unless the material really is an aggregate of loose, fairly regular, grains.

Although the methods for 3-D inclusions—the self-consistent method, the method of smoothing—can in principle, be applied to 1-D structures there do not appear to have been any comparison between such results and those of 1-D averaging except in applications to aligned cracks (Schoenberg & Douma 1988) where the 3-D results are used to provide the appropriate parameters for the equivalent layer in the stack. Comparison in this case shows that the 1-D average remains a first-order theory in the sense that interactions between cracks are ignored.

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