



# ANNALES

DE

# L'INSTITUT FOURIER

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**Overstability and resonance**

Tome 53, n° 1 (2003), p. 227-264.

[http://aif.cedram.org/item?id=AIF\\_2003\\_\\_53\\_1\\_227\\_0](http://aif.cedram.org/item?id=AIF_2003__53_1_227_0)

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## OVERSTABILITY AND RESONANCE

by A. FRUCHARD and R. SCHÄFKE

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### 1. Introduction.

In this article we consider two problems, the existence of nicely behaved solutions of certain nonlinear differential equations of first order and, as an application, the phenomenon of Ackerberg-O'Malley resonance for second order linear differential equations.

Consider first the analytic slow-fast equation

$$(1) \quad \varepsilon u' = \phi(x, u, \varepsilon)$$

where  $x \in [a, b] \subset \mathbb{R}$ ,  $u \in \mathbb{R}$ , and  $\varepsilon > 0$  is a small parameter (in this article, all derivatives are taken with respect to the variable  $x$ ).

We are interested in the asymptotic behavior as  $\varepsilon \rightarrow 0$  of families of solutions of (1) depending upon  $\varepsilon$ . In order to simplify notation and discussion, we talk of solutions instead of families of solutions and often, we omit the  $\varepsilon$ -dependence. In our context, a *solution bounded on an interval*  $I \subset [a, b]$  is a function defined and bounded on a rectangle of the form  $I \times ]0, \varepsilon_0]$ ,  $\varepsilon_0 > 0$  such that for every  $\varepsilon$ ,  $x \mapsto u(x, \varepsilon)$  is a solution of (1) for that value  $\varepsilon$  of the parameter. Note that no regularity condition with respect to  $\varepsilon$  is required. We call *slow curve* a solution  $u_0$  of the *reduced equation*  $0 = \phi(x, u, 0)$ .

We suppose that the interval  $]a, b[$  contains 0 and that equation (1) admits a slow curve that is attractive for  $x < 0$  and repulsive for  $x > 0$ . We will show that *canard* solutions [4] having bounded derivatives of any order

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*Keywords* : Resonance – Canard solution – Overstability – Singular perturbation.

*Math. classification* : 34E.

in a neighborhood of 0 exist if and only if the equation has a formal solution whose coefficients have no singularity at  $x = 0$ . To that purpose, we will use so-called overstable solutions of (1); here (1) is to be considered in the complex domain. More precisely, we will first show that a necessary and sufficient condition for (1) to have an overstable solution is the existence of a formal solution with coefficients analytic at  $x = 0$ , and then we will establish a correspondence between overstable solutions and (real valued) canard solutions.

The above results concern local solutions (i.e. solutions defined for  $x$  close to 0) as well as global solutions (defined on a neighborhood of the full interval  $[a, b]$ ). While several results obtained recently [5], [6], [9] permit to treat the local solutions (some of them will be presented briefly in Section 2), one main point of our work is the construction of global solutions using the local ones (cf. the proof of Theorem 6 in Section 3.3). It is essentially based on the Phragmen-Lindelöf theorem and thus on the analyticity of the equation.

We studied the above problem with a specific application in mind, as will be explained below. Originally, the problem of resonance posed by Ackerberg and O'Malley [1] concerned a boundary value problem

$$(2) \quad \varepsilon y'' + \varphi(x, \varepsilon)y' + \psi(x, \varepsilon)y = 0,$$

$$(3) \quad y(a) = 1, \quad y(b) = B > 0,$$

where  $y$  is a real valued function of a real variable  $x \in [a, b]$ ,  $\varphi$  and  $\psi$  are sufficiently smooth and  $\varepsilon > 0$  is a small parameter. This problem generally has a unique solution  $y(x, \varepsilon)$  and its asymptotic behavior as  $\varepsilon \rightarrow 0$  has to be studied.

In case the function  $\varphi_0 : x \mapsto \varphi(x, 0)$  has a constant sign on  $[a, b]$ , say positive, a boundary layer appears at  $x = a$  and on the rest of the interval,  $y(x, \varepsilon)$  tends to the solution of the so-called *reduced equation*

$$(4) \quad \varphi(x, 0)y' + \psi(x, 0)y = 0$$

satisfying the condition  $y_0(b) = B$ . If the sign of  $\varphi_0$  is negative, the boundary layer appears at  $x = b$ .

If the function  $\varphi_0$  vanishes at some point  $x_0 \in ]a, b[$  — for simplicity, we suppose that  $x_0 = 0$  — and if  $x\varphi(x, 0) \leq 0$ , then, *generically*, two boundary layers appear at  $a$  and  $b$  and the solution tends to 0 in the open

interval. It can happen, however, that there is only one boundary layer and  $y(x, \varepsilon)$  tends to a non trivial solution  $y_0$  of the reduced equation (4) in the remaining interval (in the sense of uniform convergence on compact subsets of the remaining half-open interval). In this case, we say that (2) exhibits *resonance in the sense of Ackerberg-O'Malley*.

N. Kopell [14] showed that this problem is equivalent to finding a solution of (2) tending to a non trivial solution of the reduced equation which might not necessarily satisfy the boundary conditions.

In the case that  $\varphi'_0(x) < 0$ , i.e.  $\varphi_0$  has a simple turning point in  $x = 0$ , W.D. Lakin [15] and L.P. Cook – W. Eckhaus [10] have shown independently that a necessary condition for resonance is the existence of a formal solution  $\hat{y}(x, \varepsilon) = \sum_{n \geq 0} y_n(x) \varepsilon^n$  of (2) having coefficients  $y_n$  that are analytic in a neighborhood of  $x = 0$ . This condition had been proposed by Matkowsky [17]; so it is called *Matkowsky condition*. We would also like to mention the work of Olver [18] and de Groen [11] in this context.

In [19], Y. Sibuya reduces the problem to  $\varphi_0(x) = -2x$  and shows that the Matkowsky condition is sufficient for resonance if  $\psi$  is analytic with respect to  $x$  in a disk with center 0 containing  $[a, b]$  (and analytic with respect to  $\varepsilon$  near  $\varepsilon = 0$ ). This condition was improved by C.H. Lin [16]: The Matkowsky condition is sufficient in the case that both  $\varphi$  and  $\psi$  are analytic in a neighborhood of  $[a, b] \times \{0\}$ .

We are mainly interested in the case of a turning point of higher order. Then the Matkowsky condition might not be necessary: In this context, we present a simple example of a linear first order equation whose formal solution has poles at  $x = 0$  but for which a canard solution exists nevertheless. This led us to strengthen the notion of resonance: We require that all derivatives of  $y$  are uniformly bounded. In the present work, we need the additional hypothesis that  $\psi_0$  has a zero at  $x = 0$ , too, and that its order is at least that of  $\varphi_0$  at  $x = 0$ . Then we show that the Matkowsky condition is necessary and sufficient for resonance.

Even if one of the results for a simple turning point is already known, we think that our method is simpler than C.H. Lin's and also we obtain a slightly better result (we show the existence of an analytic solution tending uniformly in neighborhood of  $[a, b]$  to a solution of the reduced problem (4) as  $\varepsilon \rightarrow 0$  in sectors of angular opening close to  $\pi$ ). The case of a simple turning point is also treated in a recent article [5], [6], where a control parameter is introduced. This method is somewhat related to Kopell's article [14].

Our method of proof is based on the study of a related Riccati equation. More precisely, the transformation  $y'/y = -\psi_0/\varphi_0 + u$  yields an equation of *slow-fast* type (1) and hence the above mentioned results apply. The precise relation between canard solutions of this Riccati equation and resonant solutions of (2) is presented in Section 4. This connection had already been established by J.-L. Callot in his thesis [7]. In order to simplify matters for readers unfamiliar with nonstandard analysis, we present statements and some proofs in “standard” notation.

At the end of the article, we present a construction of an analytic equation of the form (2) satisfying the Matkowsky condition such that all non zero formal solutions diverge. In fact, one could be led to believe that resonance appears only if the formal solutions are convergent (up to a multiplicative factor); there seems to be no simple example where the Matkowsky condition is satisfied, but the formal solutions diverge. Our construction also shows that the general result concerning the Gevrey-1 character of the formal solution is optimal. Thus, we hope to convince some readers, that the problem of resonance is not imaginary.

## 2. Overstability for a non linear first order equation.

Consider the analytic slow-fast equation (1) with the properties and notation introduced in the introduction. We suppose that the slow curve is  $u_0 \equiv 0$  and that the reduced equation is  $0 = f(x)u$ , where  $f(x)$  is the partial derivative  $f(x) = \partial\phi/\partial u(x, 0, 0)$ ; this can always be achieved by a simple change of variables  $u = u_0(x) + \varepsilon\tilde{u}(x)$ . In this way, we obtain an equation of the form

$$(5) \quad \varepsilon u' = f(x)u + \varepsilon P(x, u, \varepsilon),$$

where  $f$  and  $P$  satisfy the following hypotheses:

- The function  $f$  is analytic in a neighborhood (in  $\mathbb{C}$ ) of the real interval  $[a, b]$ ,  $a < 0 < b$ , and real valued on the real axis.
- We have  $xf(x) > 0$  for all  $x \in [a, b] \setminus \{0\}$ ; moreover there exist  $\lambda > 0$  and  $p \in 2\mathbb{N} + 1$  such that  $f(x) = \lambda x^p(1 + \mathcal{O}(x))$  as  $x \rightarrow 0$ .
- The function  $P$  is analytic in a neighborhood of  $[a, b] \times \{0\} \times \{0\}$  in  $\mathbb{C}^3$ .

### 2.1. Statement of the principal result.

Recall that we talk of solutions instead of families of solutions and that a *solution bounded on an interval*  $I \subset [a, b]$  is a function defined and bounded on a rectangle of the form  $I \times ]0, \varepsilon_0]$ ,  $\varepsilon_0 > 0$  such that for every  $\varepsilon$ ,  $x \mapsto u(x, \varepsilon)$  is a solution of (5) for that value  $\varepsilon$  of the parameter.

**DEFINITION 1.** — A local canard solution is a (real valued) solution of (5) that is defined and bounded on some interval  $] - \delta, \delta [$ ,  $\delta > 0$ .

A global canard solution is a solution of (5) defined and bounded on the interval  $[a, b]$ .

A solution  $u$  is called local, resp. global,  $C^\infty$ -canard solution if for any  $n \in \mathbb{N}$ , the derivative  $u^{(n)}$  exists and is bounded on some interval  $] - \delta, \delta [$ ,  $\delta > 0$ , resp. on  $[a, b]$ .

Note that a  $C^\infty$ -canard solution is more than a canard solution that is infinitely often differentiable.

**DEFINITION 2.** — Let  $D \subset \mathbb{C}$  be a simply connected domain containing 0 and  $S = S(\alpha, \beta) = \{\varepsilon \in \mathbb{C}^* ; \alpha < \arg(\varepsilon) < \beta\}$  be a sector. An analytic solution  $u = u(x, \varepsilon)$  is called overstable on  $D \times S$ , if for any domain  $D'$  whose closure is a compact subset of  $D$ , there exist some  $\varepsilon_0 > 0$  such that the solution  $u(x, \varepsilon)$  is defined and bounded on  $D' \times \{\varepsilon \in S, |\varepsilon| < \varepsilon_0\}$ .

We say that  $u$  is a local overstable solution in direction  $d$  if there exist a neighborhood  $D$  of 0 (in  $\mathbb{C}$ ) and a sector  $S$  containing  $e^{di}$  such that  $u$  is overstable on  $D \times S$ .

We say that  $u$  is a global overstable solution in direction  $d$  if there exist a neighborhood  $D$  of  $[a, b]$  (in  $\mathbb{C}$ ) and a sector  $S$  containing  $e^{di}$  such that  $u$  is overstable on  $D \times S$ .

*Remarks:*

1) The sector  $S$  in the definition is an infinite sector, but we only need  $\varepsilon$  sufficiently close to 0. Note that we do not define overstableness on a sector by requiring boundedness on all proper subsectors; this is useful when defining asymptotic expansions, etc. to achieve compatibility with differentiation and to simplify some statements, but is not needed here.

2) Note that the solution  $u$  might not be defined on the whole domain  $D$ . In the (original) context of nonstandard analysis, an overstable

solution was defined and “limited” on the “S-interior” of  $D$  for infinitely small  $\varepsilon$  in the sector  $S$ .

3) Taking the special form of (5) into account, it is natural only to consider canard solutions and overstable solutions close to  $u \equiv 0$  as we will do in the sequel.

4) As was the case for the canard solutions, no regularity with respect to  $\varepsilon$  is required. In particular, if for any direction  $d$  there is an overstable solution in  $D \times S$ , where  $S$  contains  $d$  then these solutions can simply be combined to an overstable solution on  $D \times \mathbb{C}^*$ . On the other hand, an analytic dependence of  $\varepsilon$  seems, in general, only possible in sectors of angular opening smaller than  $\pi$ .

The principal result for first order equations in this work is the following.

**THEOREM 3.** — *With the preceding hypotheses and notation, the following statements are equivalent:*

- 1) *There exist a formal solution  $\hat{u}(x, \varepsilon) = \sum_{n \geq 1} u_n(x) \varepsilon^n$  of (5) whose coefficients  $u_n$  are analytic near  $x = 0$ .*
- 2) *There exist a local overstable solution of (5) in direction  $d$  for every  $d$ .*
- 3) *There exist a global overstable solution of (5) in direction  $d = 0$ .*
- 4) *There exist a global  $C^\infty$ -canard solution of (5).*
- 5) *There exist a local  $C^\infty$ -canard solution of (5).*

*Remarks:*

1) The most interesting implications in our opinion are 1)  $\Leftrightarrow$  3), 1)  $\Leftrightarrow$  4) and 5)  $\Rightarrow$  4). It seems that the latter implication cannot be proved without using complex analysis. See also Example 1 below.

2) Statement 3) can be improved and made more precise: For every  $\delta > 0$  there is a neighborhood  $D$  of  $[a, b]$  and an overstable solution of (5) on  $D \times S(-\frac{1}{2}\pi + \delta, \frac{1}{2}\pi - \delta)$  analytic with respect to  $x$  and  $\varepsilon$  (cf. Theorem 6).

3) The existence of a local overstable solution in *one* direction  $d$  already implies statement 1. This could be proved similarly to 5)  $\Rightarrow$  1), but a shorter proof will be given at the end of Section 3.1.

2.2. Examples.

Before proving the theorem, we present two simple counterexamples (if the hypotheses are not satisfied). The first is an equation (non analytic with respect to  $\varepsilon$ ) having a formal solution without poles but no canard solution. The second example is an equation admitting a canard solution (not “ $C^\infty$ ”) but no formal solution.

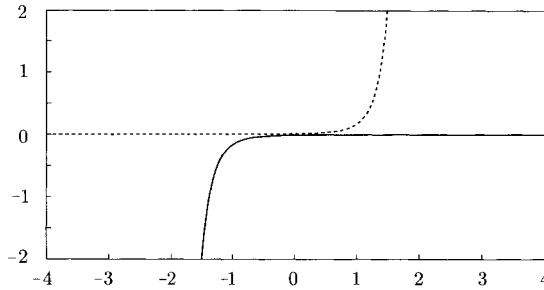


Figure 1. The two solutions of  $\varepsilon u' = xu + \alpha$  vanishing as  $x \rightarrow +\infty$  and  $x \rightarrow -\infty$  for  $\varepsilon = \frac{1}{4}$  and  $\alpha = \exp(-1/\varepsilon)$ .

Example 1. — Consider the equation

$$(6) \quad \varepsilon u' = xu + \alpha$$

where  $\alpha$  might depend upon  $\varepsilon$  but is constant with respect to  $x$ . The general solution can be written

$$(7) \quad u(x) = u(x_0) \exp\left(\frac{x^2 - x_0^2}{2\varepsilon}\right) + \int_{x_0}^x \frac{\alpha}{\varepsilon} \exp\left(\frac{x^2 - \xi^2}{2\varepsilon}\right) d\xi.$$

If the asymptotic expansion of  $\alpha = \alpha(\varepsilon)$  as  $\varepsilon$  tends to 0 is the zero series, then equation (6) has a formal solution without poles: the zero series. On the other hand, if  $\varepsilon \log \alpha(\varepsilon)$  tends to a limit  $\ell$  then the equation has a “boundary” (“*butée*” [4]) in the points  $x = \pm\sqrt{2\ell}$ . For example, if  $\alpha = \exp(-1/\sqrt{\varepsilon})$  then there are no canard solutions, if  $\alpha = \exp(-1/\varepsilon)$  then there are local canard solutions but no global canard solutions whenever  $a < -\sqrt{2}$  and  $\sqrt{2} < b$ . The above Theorem 3 states that there are no such “boundaries” for differential equations analytic in  $x$  and  $\varepsilon$  (if a formal solution without poles exists).

Example 2. — Consider

$$(8) \quad \varepsilon u' = 2x^3u - 2\varepsilon x.$$



This equation does not have a formal solution whose coefficients are analytic at  $x = 0$ ; its first term is  $u_1(x) = 1/x^2$ . On the other hand, we will see that it has a canard solution.

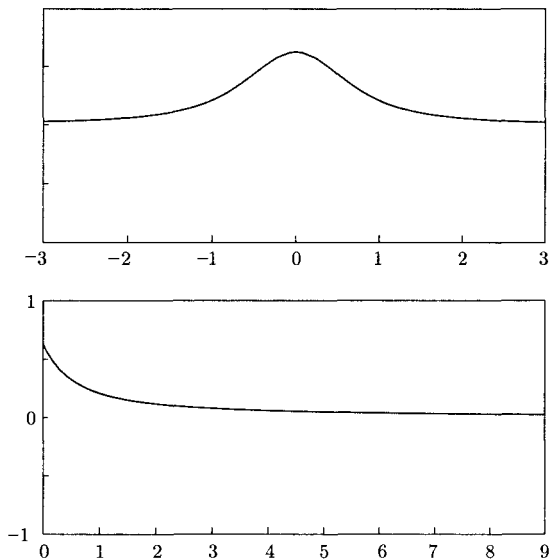


Figure 2. The solutions  $\tilde{u}$  and  $\tilde{v}$  for  $\varepsilon = \frac{1}{4}$ .

To see this, make a change of variables  $t = x^2$  and let  $v(t) = u(x)$ . This leads to the equation

$$(9) \quad \varepsilon \frac{dv}{dt} = tv - \varepsilon.$$

Consider now its solution  $\tilde{v}$  with initial condition  $\tilde{v}(+\infty) = 0$ . It is defined on  $\mathbb{R}$  and given again by variation of constants. Hence

$$\tilde{v}(t) = \int_0^{+\infty} \exp\left(-\frac{s^2}{2\varepsilon} - \frac{ts}{\varepsilon}\right) ds.$$

(One can also write  $\tilde{v}$  in terms of the normalised complementary error function  $\operatorname{erfcx}(\xi) := e^{\xi^2} \int_{\xi}^{+\infty} e^{-u^2} du$ ; namely  $\tilde{v}(t) = \sqrt{2\varepsilon} \operatorname{erfcx}(t/\sqrt{2\varepsilon})$ .)

For  $t \geq 0$ , we have  $|\tilde{v}(t)| \leq \tilde{v}(0) = \sqrt{\frac{1}{2}\pi\varepsilon}$ . Thus the solution  $\tilde{u}$  of (8) with initial condition  $\tilde{u}(+\infty) = 0$ , which is given by  $\tilde{u}(x) = \tilde{v}(x^2)$ , is bounded — and even tends to 0 uniformly — on  $\mathbb{R}$ .

Note, however, that this solution  $\tilde{u}$  has an unbounded fourth derivative and is therefore no  $C^\infty$ -canard solution. Indeed, by differentiating (8) we obtain

$$\varepsilon u^{(4)} = 2x^3 u^{(3)} + 18x^2 u'' + 36xu' + 12u,$$

hence  $\tilde{u}^{(4)}(0) = 6\sqrt{2\pi/\varepsilon}$ .

In the complex plane, the so-called “landscape” [5], [8] corresponding to (8) is the surface  $\mathbb{C} \times \mathbb{R} \ni (x, z)$  described by  $z = R(x) := \operatorname{Re}(\int_0^x 2\xi^3 d\xi)$ ; here it has four mountains containing each one of the half axes – positive or negative real or imaginary – and four valleys in between. The solution  $\tilde{u}$  has, as analytic continuation, a solution close to the slow curve on the two mountains containing the real axis, but on the other two mountains, it is unbounded. An overstable solution would be close to the slow curve on a complete neighborhood of 0.

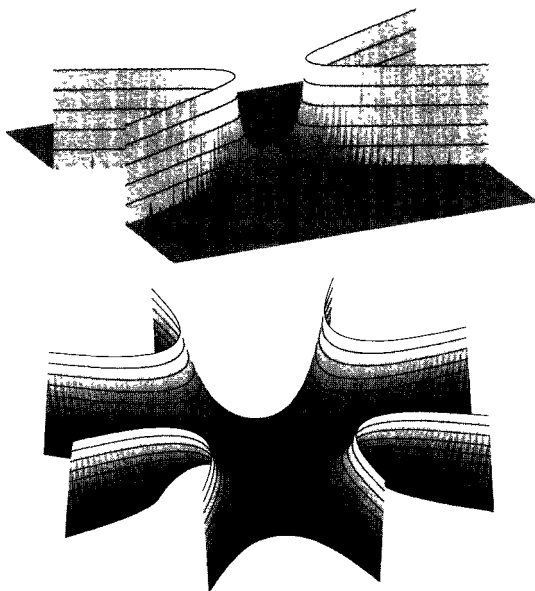


Figure 3. Top: absolute value of  $\tilde{u}$ . Bottom: landscape described by  $z = R(x) := \frac{1}{2} \operatorname{Re}(x^4)$ .

### 3. Proof of Theorem 3.

We will prove that each of the statements implies the following one and that 5)  $\Rightarrow$  1), but first, we describe the formal solution of (5).

These considerations are needed in Section 3.5, but might be of interest already here.

### 3.1. The formal solution of (5).

The computation of formal solutions  $\hat{u}(x, \varepsilon) = \sum_{n \geq 1} u_n(x) \varepsilon^n$  of (5) having coefficients analytic at  $x = 0$  yields the following equation:

$$(10) \quad \sum_{n \geq 1} u'_n(x) \varepsilon^{n+1} = f(x) \sum_{n \geq 1} u_n(x) \varepsilon^n + \varepsilon \sum_{k, \ell \geq 0} p_{k, \ell}(x) \left( \sum_{\nu \geq 1} u_\nu(x) \varepsilon^\nu \right)^k \varepsilon^\ell,$$

where  $p_{k, \ell}$  are defined by

$$P(x, u, \varepsilon) = \sum_{k, \ell \geq 0} p_{k, \ell}(x) u^k \varepsilon^\ell.$$

If  $\phi_n(x) = \phi_n(u_0, \dots, u_n)(x)$  denotes the coefficient of  $\varepsilon^n$  in the expression

$$\sum_{k, \ell \geq 0} p_{k, \ell}(x) \left( \sum_{\nu \geq 1} u_\nu \varepsilon^\nu \right)^k \varepsilon^\ell,$$

we obtain the following recursion formula (with  $u_0 \equiv 0$ ):

$$(11) \quad \forall n \geq 1, \quad u'_{n-1} = f u_n + \phi_{n-1}.$$

If  $u_1, \dots, u_{n-1}$  could already be calculated and have no poles at  $x = 0$  and if

$$(12) \quad u'_{n-1}(x) - \phi_{n-1}(x) = \mathcal{O}(x^p)$$

as  $x \rightarrow 0$  then (11) yields  $u_n := (1/f)(u'_{n-1} - \phi_{n-1})$  which is analytic near  $x = 0$ . Conversely, condition (12) is clearly a necessary condition for  $u_n$  to be analytic near  $x = 0$ . Thus we have shown

**PROPOSITION 4.** — *There is a formal solution of (5) whose coefficients  $(u_n)_{n \in \mathbb{N}}$  are analytic near  $x = 0$  if and only if the order of the zero  $x = 0$  of  $u'_n - \phi_n(u_0, \dots, u_n)$  is at least  $p$  for every integer  $n$ . In this case, the coefficients are given recursively by  $u_0 = 0$  and*

$$(13) \quad u_n = \frac{1}{f} (u'_{n-1} - \phi_{n-1}).$$

Observe that the coefficients  $u_n(x)$  of a formal solution of (5), if they are analytic at  $x = 0$ , are automatically analytic in a common domain; this follows immediately from the proposition.

Note that for every  $n \geq 1$ , if  $u_1, \dots, u_{n-1}$  could already be calculated and have no poles at  $x = 0$  then there is a function  $R_n(x, \varepsilon)$  analytic with respect to both variables and bounded in a neighborhood of  $x = 0$  (in the sense of the beginning of Section 2) such that

$$(14) \quad P(x, \sum_{\nu=1}^{n-1} u_\nu(x)\varepsilon^\nu, \varepsilon) = \sum_{\nu=0}^{n-1} \phi_\nu(x)\varepsilon^\nu + R_n(x, \varepsilon)\varepsilon^n.$$

This will be useful later.

On this occasion we give a proof of the statement in Remark 3 below Theorem 3. Suppose that  $u$  is a local overstable solution in one direction  $d$ . We show that  $u$  has an asymptotic expansion  $u(x, \varepsilon) \sim \sum_{n=0}^\infty u_n(x)\varepsilon^n$  as  $\varepsilon \rightarrow 0$  in direction  $d$ , i.e. for every  $n$  we show the existence of analytic  $u_1(x), \dots, u_{n-1}(x)$  and of  $v_n(x, \varepsilon)$  analytic and bounded in some neighborhood of  $x = 0$  and for  $\varepsilon$  in some sector containing direction  $d$ , such that

$$(15) \quad u(x, \varepsilon) = \sum_{\nu=1}^{n-1} u_\nu(x)\varepsilon^\nu + v_n(x, \varepsilon)\varepsilon^n.$$

This of course implies the existence of a formal solution.

We proceed by induction. For  $n = 1$  we use  $f(x)u(x, \varepsilon) = \varepsilon u'(x, \varepsilon) - \varepsilon P(x, u(x, \varepsilon), \varepsilon)$ ; first this shows that  $f(x)\varepsilon^{-1}u(x, \varepsilon)$  is bounded in some neighborhood of the origin (using Cauchy's formula) and hence also  $\varepsilon^{-1}u(x, \varepsilon)$  is bounded in some neighborhood of 0 if we exclude a small disk around 0. Then the boundedness of  $\frac{1}{\varepsilon}u(x, \varepsilon)$  on a full neighborhood of 0 follows by the maximum principle.

Suppose now that statement (15) is true for some  $n$ . Using (14), insertion into (5) implies

$$f(x)v_n(x, \varepsilon) = (u'_{n-1} - \phi_{n-1})(x) + \varepsilon \tilde{R}_n(x, \varepsilon),$$

where  $\tilde{R}_n := v'_n - R_n - \varepsilon^{-n}(P(x, \Sigma u + \varepsilon^n v_n) - P(x, \Sigma u))$  is analytic at  $(0, 0)$ . As  $f(x)$  has a zero of order  $p$  at  $x = 0$ , Rouché's theorem applied to small circles around 0 (and for sufficiently small  $\varepsilon$ ) implies that  $u'_{n-1} - \phi_{n-1}$  also has a zero of order at least  $p$  at  $x = 0$  and hence  $u_n = (1/f)(u'_{n-1} - \phi_{n-1})$  is analytic in some full neighborhood of 0. Thus  $f(x)(v_n(x, \varepsilon) - u_n(x)) = \varepsilon \tilde{R}_n(x, \varepsilon)$  and as before  $v_{n+1}(x, \varepsilon) = \varepsilon^{-1}(v_n(x, \varepsilon) - u_n(x))$  is bounded in some neighborhood of 0. □

### 3.2. From the formal solution to overstable solutions.

In this section, we begin the proof of Theorem 3. Concerning the implication 1)  $\Rightarrow$  2), we use a general result of [9] (Proposition 8.3), which can be expressed in terms of the present article in the following way.

PROPOSITION 5. — *If equation (5) admits a formal solution whose coefficients are analytic near  $x = 0$  then it has a local overstable solution for any direction  $d$ .*

*Proof.* — For the convenience of the reader, we include a sketch of the proof. The proof presented here is somewhat simpler than in [9] because the equation is scalar and the existence of a formal solution is assumed.

We first prove that the formal solution is Gevrey of order one uniformly in  $x$  in some disk  $D(0, r)$ ,  $r > 0$ , i.e. it satisfies

$$\exists A, C > 0, \forall x \in D(0, r), \forall n \in \mathbb{N}, \quad |u_n(x)| \leq AC^n n!.$$

For that purpose, we rewrite (5) in the form

$$(16) \quad u(x, \varepsilon) = \frac{\varepsilon}{f(x)} (u'(x, \varepsilon) - P(x, u(x, \varepsilon), \varepsilon))$$

and assume that  $f$  and  $P$  are analytic and bounded for  $|x| < R$ ,  $|u| < u_1$ ,  $|\varepsilon| < \varepsilon_1$ , and that  $f$  vanishes only at  $x = 0$ . By assumption, we can write  $f(x) = x^p/g(x)$  with  $g$  analytic at  $x = 0$ ,  $g(0) \neq 0$ . The formal series  $\hat{u}(x, \varepsilon)$  satisfies the following equation using the shift operator  $S$  given by  $S\varphi : x \mapsto (\varphi(x) - \varphi(0))/x$ :

$$(17) \quad u(x, \varepsilon) = \varepsilon g(x) S^p (u'(x, \varepsilon) - P(x, u(x, \varepsilon), \varepsilon)).$$

This equation has a unique fixed point in the set of formal series  $\sum c_n(x)\varepsilon^n$  having coefficients analytic at  $x = 0$ ; this is, of course, the formal solution  $\hat{u}$  of (5). Using a “majorant equation”, we will prove that it is Gevrey of order 1.

At this point, we introduce the so-called Nagumo norms. Choose some  $\rho \in ]0, R[$  and define  $d(x) = R - |x|$  if  $|x| \geq \rho$ ,  $d(x) = R - \rho$  otherwise, and

$$\|\varphi\|_k = \sup_{|x| < R} |\varphi(x)| d(x)^k \in [0, +\infty].$$

These are norms, and they have several nice properties. The most useful ones are:

- (i)  $\|\varphi \cdot \psi\|_{k+l} \leq \|\varphi\|_k \cdot \|\psi\|_l,$
- (ii)  $\|\varphi'\|_{k+1} \leq e(k+1)\|\varphi\|_k,$
- (iii)  $\|S\varphi\|_k \leq (2/\rho)\|\varphi\|_k$

(the proofs are straightforward, see [9], Section 3).

Given two formal series  $u(x, \varepsilon) = \sum_{k \geq 0} u_k(x)\varepsilon^k, v(z) = \sum_{\ell \geq 0} v_\ell z^\ell,$  we write

$$u(x, \varepsilon) \ll v(z) \quad \text{if} \quad \|u_j\|_j \leq j! v_j \quad \text{for all } j \in \mathbb{N}.$$

The following statements are crucial (see [9], Lemma 4.4; we omit the arguments  $(x, \varepsilon)$  and  $(z)$  here):

$$\text{If } u \ll v \text{ and } \tilde{u} \ll \tilde{v} \text{ then } u\tilde{u} \ll v\tilde{v}, \quad \varepsilon \frac{du}{dx} \ll ezv \text{ and } \varepsilon u \ll Rzv.$$

With the notation  $P(x, u, \varepsilon) = \sum_{i,j \geq 0} p_{i,j}(x)u^i\varepsilon^j,$  let

$$q_{i,j} = \sup_{|x| < R} |p_{i,j}(x)|, \quad Q(U, z) = \sum_{i,j \geq 0} \frac{q_{i,j}}{j!} U^i z^j,$$

and analogously  $G = \sup_{|x| < R} |g(x)|.$  Consider now the ‘‘majorant’’ equation  $U = \Phi(U)$  with

$$\Phi(U)(z) = G\left(\frac{2}{\rho}\right)^p (ezU(z) + RzQ(U(z), z)).$$

This equation is of the form  $U(z) = zA(U(z), z)$  with  $A$  holomorphic in  $(0, 0) \in \mathbb{C}^2.$  By the implicit function theorem, it has a unique fixed point  $U = U^*(z)$  with  $U^*(0) = 0$  and  $U^*$  analytic in 0. Moreover the property

$$u(x, \varepsilon) \ll v(z) \implies \phi(u)(x, \varepsilon) \ll \Phi(v)(z)$$

where  $\phi(u) : (x, \varepsilon) \mapsto \varepsilon g(x)S^p(du/dx(x, \varepsilon) - P(x, u(x, \varepsilon), \varepsilon))$  implies that  $\phi^n(0)(x, \varepsilon) \ll U^*(z)$  for any  $n.$  Thus the fact that the first  $n$  coefficients of  $\phi^n(0)$  and the fixed point  $\hat{u}$  coincide, yields  $\hat{u} \ll U^*(z).$  This proves that the formal solution  $\hat{u}$  is Gevrey-1 on any disk  $|x| < r$  with  $r < R.$

The Gevrey property implies that the formal Borel transform  $v = \hat{B}\hat{u}$  of  $\hat{u},$  given by  $v(x, t) = \sum_{n \geq 0} u_n(x)t^n/(n-1)!$  is analytic on  $D_r \times D_\sigma$

for some  $\sigma > 0$ . Given  $T \in D_\sigma$ , the truncated Laplace transform  $u_T = \mathcal{L}_T v$  of  $v$  given by

$$u_T(x, \varepsilon) = \int_0^T e^{-t/\varepsilon} v(x, t) dt$$

is a *quasi-solution in the direction*  $\arg(T)$  in the sense that there are  $A, \varepsilon_0, \delta > 0$  such that for any  $\varepsilon \in S(\arg T - \delta, \arg T + \delta, \varepsilon_0)$  and any  $x \in D_r$ :

$$|\varepsilon u'_T(x, \varepsilon) - f(x)u_T(x, \varepsilon) - P(x, u_T(x, \varepsilon), \varepsilon)| \leq \exp(-A/|\varepsilon|).$$

This follows essentially from the fact that  $\mathcal{S}_T(\hat{a} \cdot \hat{b}) - \mathcal{S}_T(\hat{a}) \cdot \mathcal{S}_T(\hat{b})$ , where  $\mathcal{S}_T = \mathcal{L}_T \hat{\mathcal{B}}$ , is exponentially small for any two Gevrey-1 formal series and hence also  $\mathcal{S}_T(P(x, \hat{u}(x, \varepsilon), \varepsilon) - P(x, \mathcal{S}_T \hat{u}(x, \varepsilon), \varepsilon))$  is exponentially small (see [9] for the tedious, but straightforward proof.)

Finally Gronwall's lemma shows that the solution  $u_\varepsilon$  of (5) with initial condition  $u_\varepsilon(0) = u_T(0, \varepsilon)$  satisfies

$$|u_\varepsilon(x, \varepsilon) - u_T(x, \varepsilon)| \leq \exp(-(A - M|x|)/|\varepsilon|)$$

on the set of all  $|x| < r, |\varepsilon| < \varepsilon_1, \arg \varepsilon \in [T - \delta, T + \delta]$  satisfying  $|u_\varepsilon(x, \varepsilon)| < |u_T(x, \varepsilon)| + 1$ ; here  $M$  is some constant independent of  $x, \varepsilon$ . Thus  $u_\varepsilon(x, \varepsilon)$  remains bounded on the disk  $|x| < \min(r, A/M)$  and hence is a local overstable solution in the direction  $\arg T$ .

We refer to [9] for more details. □

Another proof consists of imbedding (5) in an equation with  $p$  additional parameters

$$(18) \quad \varepsilon u' = f(x)u + \varepsilon P(x, u, \varepsilon) + \sum_{k=1}^p \alpha_k x^{k-1}$$

and using the result 6 of [6]. Let  $F$  be the antiderivative of  $f$  vanishing at  $x = 0$  and let  $C_r$  be the connected component containing 0 of the set  $\{x \in \mathbb{C} \mid |F(x)| < r\}$ . For sufficiently small  $r$ , the functions  $f$  and  $P$  are analytic in a neighborhood of the closure of  $C_r$ , resp.  $C_r \times \{0\} \times \{0\}$ . By the cited result, for every direction  $d$  and  $\varepsilon$  with  $\arg \varepsilon = d$ , summation "to the least term"  $\bar{a} = \sum_{\nu=0}^{\lfloor r/|\varepsilon| \rfloor} a_\nu \varepsilon^\nu$  of the unique formal solution  $\hat{a} = \sum_{n \geq 0} a_n \varepsilon^n, \hat{u} = \sum_{n \geq 0} u_n(x) \varepsilon^n$  ( $u_n$  analytic on  $[a, b]$ ) provides a value of the parameter  $a = (\alpha_1, \dots, \alpha_p)$  for which (18) has an overstable solution in this direction. As it was supposed that (5) without parameter already has a formal solution with analytic coefficients,  $\hat{a}$  has to be the zero series; hence  $\bar{a} = 0$ .

Observe that the overstable solution provided by the theorem will in general not depend analytically upon  $\varepsilon$  in a full neighborhood of  $\varepsilon = 0$ ; this corresponds to the divergence of the formal solution. The first proof yields overstable solutions that are analytic with respect to  $\varepsilon$  in small sectors with vertices at the origin. The alternative proof yields solutions that are (infinitely) piecewise polynomial in  $\varepsilon$  on annuli.

### 3.3. Analytic continuation of an overstable solution.

We show here the following somewhat surprising statement concerning local and global overstable solutions.

**THEOREM 6.** — *If equation (5) has an overstable solution in some neighborhood of  $x = 0$  for every direction  $d$  then for any  $\delta > 0$  there exist a neighborhood  $D$  of  $[a, b]$  and an analytic overstable solution of (5) on  $D \times S(-\frac{1}{2}\pi + \delta, \frac{1}{2}\pi - \delta)$ .*

We again consider the antiderivative  $F$  of  $f$  vanishing at  $x = 0$ . By the hypotheses on  $f$ , the function  $F$  has real values on the real axis. Furthermore, there are  $\tilde{a}, \tilde{b} \in \mathbb{R}$  with  $\tilde{a} < a < b < \tilde{b}$  such that  $F(\tilde{a}) \neq F(\tilde{b})$  and the functions  $f$  and  $P$  of (5) are analytic with respect to  $x$  in a simply connected domain  $D$  containing  $[\tilde{a}, \tilde{b}]$ . It is therefore sufficient to show the existence of an overstable solution in a domain containing  $]\tilde{a}, \tilde{b}[$ . Without loss of generality, we can also assume that  $F(\tilde{a}) > F(\tilde{b})$ . For simplicity of notation we will write  $a, b$  instead of  $\tilde{a}, \tilde{b}$ .

Before going into details, we would like to give an overview of the proof and introduce some essential tools. To fix ideas, we suppose for the moment that  $\varepsilon$  is a positive real number. First, we consider the landscape described by  $z = R(x) := \operatorname{Re}(F(x))$ . It seems well-known that a solution of (5) with initial condition near the slow curve  $u = 0$  can be continued analytically along any path descending this landscape and remains close to the slow curve there. We present in Lemma 7 a statement adapted for use in this article.

Under our hypotheses, the landscape has a saddle point at  $x = 0$  where  $p + 1$  mountains and  $p + 1$  valleys are interlaced. Notice that, since the equation is real on the real axis,  $[a, b]$  is contained in the union of two crest lines. *A priori*, a solution continued analytically from some initial point on one mountain, say  $\mathbb{R}^-$ , will not remain bounded or may even cease to exist when continued on the other mountains. Here, we already supposed that a local solution exists, i.e. a solution remaining close to the slow curve



on a small part of each mountain. We have to show that there is a solution remaining bounded on some neighborhood of  $[a, b]$ , i.e. on *two* mountains.

For  $\varepsilon \in \mathbb{C} \setminus \mathbb{R}^+$ , the landscape function  $R$  has to be modified to

$$R_d(x) = \operatorname{Re}(e^{-id}F(x))$$

with  $d = \arg \varepsilon$ . As  $\arg \varepsilon$  approaches  $\frac{1}{2}\pi$  or  $-\frac{1}{2}\pi$ , the real axis becomes less steep; indeed it is a part of the separatrices of the saddle if  $|\arg \varepsilon| = \frac{1}{2}\pi$ . Hence if  $|\arg \varepsilon| = \frac{1}{2}\pi - \delta$ ,  $\delta$  small enough, then there is an overstable solution from  $a$  to  $b$  because the local overstable solution can be continued analytically.

Second, it is possible to “measure” the exponential closeness of solutions of (5); see Lemma 8 for a statement adapted to our needs – it is essentially based on the linearization of (5) around a solution. We say that two functions  $\alpha(\varepsilon)$  and  $\beta(\varepsilon)$  have a relative type  $A$  if for any  $\tilde{A} < A$ ,

$$(\alpha - \beta)(\varepsilon) = \mathcal{O}(\exp(-\tilde{A}/\varepsilon)).$$

Two solutions of (5) with initial conditions close to the slow curve at some point  $x_0$ , when continued analytically along some descending path, become exponentially close, with a type given by the loss of altitude. Conversely, two solutions with a relative type  $A$  in some point  $x_0$  remain exponentially close when they are continued analytically along a path not surpassing a relative altitude of  $A$ .

Consider now the solutions  $u_a, u_b$  of (5) vanishing at  $x = a$  (resp.  $x = b$ ). Using the local overstable solution, we will show that these solutions are overstable on some  $x$ -neighborhood of  $]a, 0]$  (resp.  $]0, b]$ ) and for  $\varepsilon$  in the sector  $S(-\frac{1}{2}\pi + \delta, \frac{1}{2}\pi - \delta)$  for small positive  $\delta$ .

As there is an overstable solution on  $[a, b]$  for the directions  $d = \pm(\frac{1}{2}\pi - \delta)$ ,  $\delta$  small, their values at  $x = 0$  are of relative type  $R_d(b)$  for these values of  $d$ . In other words, there is some  $\delta > 0$  such that for all  $\eta > 0$  the following estimate holds:

$$(19) \quad u_a(0, \varepsilon) - u_b(0, \varepsilon) = \mathcal{O}(\exp(-(F(b) - \eta)/\varepsilon))$$

as  $\varepsilon \rightarrow 0$  and  $|\arg \varepsilon| = \frac{1}{2}\pi - \delta$ . At this point, we use a third ingredient, the Phragmen-Lindelöf Theorem. It implies that the estimate (19) remains valid for arguments of  $\varepsilon$  between  $\pm(\frac{1}{2}\pi - \delta)$  and yields exactly the relative type needed (in Lemma 8) so that the solution  $u_a$  remains bounded on  $[a, b]$  — and hence is overstable on  $[a, b]$ .

Let us now give more details. First, we present a statement concerning analytic continuation of solutions of (5) (see e.g. Theorem 2 of [8] in a version using nonstandard analysis and in [21] in the linear case, see also Proposition D of [5]) For the convenience of the reader, we include a proof. We needed to modify the known result in the following way:

(a) We construct a solution on an entire  $x$ -domain and not only a restriction to a path.

(b) The solution we construct is analytic with respect to  $\varepsilon$  in a sector that can be chosen large if the domain permits this. For the sake of simplicity, we will only present the result for a particular kind of  $x$ -domain.

LEMMA 7. — *Let  $x_1 \in D$  and consider a  $C^1$  family  $\Gamma = \{\gamma_s; s \in [0, 1]\}$  of paths starting at  $x_1$ , i.e. a  $C^1$  mapping  $\Gamma : [0, 1]^2 \rightarrow D$  such that  $\Gamma(s, 0) = x_1$  for all  $s \in [0, 1]$ . Let  $d_1 < d_2$  be two directions in  $\mathbb{S}^1$ . Suppose that all the paths in the family  $\Gamma$  are downward for every direction in  $[d_1, d_2]$ ; in other words,*

$$(20) \quad \forall d \in [d_1, d_2], \forall (s, t) \in [0, 1]^2, \quad \frac{d}{dt} R_d(\gamma_s(t)) < 0,$$

where  $R_d(x) := \operatorname{Re}(F(x)e^{-id})$ . Denote by  $K$  the image  $\Gamma([0, 1]^2)$  of  $\Gamma$ . Finally, assume that  $u_1 = u_1(\varepsilon)$  is an initial condition analytic in  $\varepsilon$  and satisfies  $u_1 = o(1)$  as  $\varepsilon \rightarrow 0$ .

Then there exist  $\varepsilon_0 > 0$  such that the solution  $u$  of (5) with initial condition  $u(x_1, \varepsilon) = u_1(\varepsilon)$  is analytic in a neighborhood of  $S \times (K \setminus \{x_1\})$  where  $S := \{\varepsilon \in \mathbb{C}; 0 < |\varepsilon| \leq \varepsilon_0, d_1 \leq \arg \varepsilon \leq d_2\}$  and satisfies  $u(x, \varepsilon) = o(1)$  as  $S \ni \varepsilon \rightarrow 0$  uniformly on  $K$ .

*Proof.* — First of all, by compactness it is possible to enlarge the sector  $S$  and the family of paths  $\Gamma$  such that (20) remains valid. Therefore it suffices to show that  $u$  is analytic in the interior of  $S \times K$ . By the same compactness argument, there exist  $C, \delta > 0$  such that for all  $(s, t) \in [0, 1]^2$  and  $d \in [d_1, d_2]$  one has  $|\gamma'_s(t)| < C$  and  $d/dt R_d(\gamma_s(t)) < -\delta$ .

By shrinking  $D$  somewhat, if necessary, we can assume that for  $r, \varepsilon > 0$  sufficiently small, the absolute value  $|P(x, u, \varepsilon)|$  is bounded by some constant  $M$  for  $x \in D, |u| < r$  and  $|\varepsilon| < \varepsilon_0$ .

Let  $\rho \in ]0, r[$ . We show that for sufficiently small  $\rho$ , there exist  $\varepsilon_1 > 0$  such that for every  $\varepsilon \in S$  with  $|\varepsilon| < \varepsilon_1$  and every  $s \in [0, 1]$ , the function  $v_\varepsilon : t \mapsto u(\varepsilon, \gamma_s(t))$  is defined on  $[0, 1]$  and satisfies  $|v_\varepsilon(t)| < \rho$ . Since  $\varepsilon$  is

small we have  $|v_\varepsilon(0)| = |u_1(\varepsilon)| < \rho$ . It is sufficient to show that the set  $\{|v_\varepsilon| < \rho\}$  is invariant under (5), i.e. if  $|v_\varepsilon(t)| = \rho$  then  $d/dt |v_\varepsilon(t)| < 0$  for sufficiently small  $\varepsilon$ . We calculate

$$\begin{aligned} \frac{d}{dt} v_\varepsilon(t) \bar{v}_\varepsilon(t) &= 2 \operatorname{Re}(u'(\gamma_s(t)) \gamma'_s(t) \bar{u}(\gamma_s(t))) \\ &= \operatorname{Re} \left( \left( \frac{2}{\varepsilon} f(\gamma_s(t)) v_\varepsilon(t) + 2P(\gamma_s(t), v_\varepsilon(t), \varepsilon) \right) \gamma'_s(t) \bar{v}_\varepsilon(t) \right) \\ &= \frac{2}{|\varepsilon|} |v_\varepsilon(t)|^2 \frac{d}{dt} R_d(\gamma_s(t)) + p(t, v_\varepsilon(t), \varepsilon) \end{aligned}$$

where  $d := \arg \varepsilon$  and  $p(t, v, \varepsilon) := 2 \operatorname{Re}(P(\gamma_s(t), v, \varepsilon) \gamma'_s(t) \bar{v})$ .

If  $|v_\varepsilon(t)| = \rho$  then we find

$$\frac{d}{dt} |v_\varepsilon(t)|^2 \leq -\frac{2\delta\rho^2}{|\varepsilon|} + 2MC\rho,$$

hence  $d/dt |v_\varepsilon(t)|^2 < 0$  provided  $|\varepsilon| < \varepsilon_1 := \min(\varepsilon_0, \delta\rho/MC)$ . This proves the existence of  $u$  along every path of  $\Gamma$  and also that  $u(x, \varepsilon) = o(1)$  as  $S \ni \varepsilon \rightarrow 0$ . The analyticity with respect to  $\varepsilon$  follows from the theorem on analytic dependence upon parameters.  $\square$

Our second auxiliary statement corresponds essentially to Propositions 1 and 2 and Theorem 1 of [8] (cf. also the commentary ‘relation entrée-sortie’ below Proposition E in [5]). Here again we needed to change the latter statements somewhat for our needs and to give them a more quantitative meaning than in [8].

LEMMA 8. — *Let  $u_0$  be a solution of (5) defined in some simply connected  $x$ -domain  $\Omega$  and for all sufficiently small  $\varepsilon$  with  $\arg \varepsilon = d$ . Let  $x_1$  and  $x_2$  be two points of  $\Omega$  and  $\gamma$  a path in  $\Omega$  connecting them. Let  $u$  be another solution of (5) defined in some neighborhood of  $x_1$  and let finally  $\alpha \geq 0$  such that*

$$u(x_1, \varepsilon) - u_0(x_1, \varepsilon) = o(e^{-\alpha/|\varepsilon|}).$$

*Suppose that for  $x \in \gamma([0, 1])$  we have  $R_d(x) - R_d(x_1) \leq \alpha$ .*

*Then the solution  $u$  can be continued analytically along  $\gamma$  and the estimate*

$$u(x_2, \varepsilon) - u_0(x_2, \varepsilon) = o(e^{-\beta/|\varepsilon|}) \text{ holds with } \beta = \alpha + R_d(x_1) - R_d(x_2).$$

*Proof.* — We use the fact that the difference  $y_\varepsilon = u(\cdot, \varepsilon) - u_0(\cdot, \varepsilon)$  satisfies an equation of the form

$$\varepsilon y'_\varepsilon = f(x)y_\varepsilon + \varepsilon P_1(x, u_0(x, \varepsilon), y_\varepsilon, \varepsilon)y_\varepsilon$$

where  $P_1$  is some function analytic in a neighborhood of  $\Omega \times \{0\}^3$  satisfying  $P(x, u + y, \varepsilon) - P(x, u, \varepsilon) = P_1(x, u, y, \varepsilon)y$  there.

For any path  $\mathcal{C}$  containing  $x_1$ , where  $y_\varepsilon$  is defined and satisfies  $|y_\varepsilon(x)| \leq \delta$  with some sufficiently small  $\delta$ , we conclude that  $y_\varepsilon$  satisfies

$$\begin{aligned} y_\varepsilon(x) &= y_\varepsilon(x_1) \exp\left(\frac{1}{\varepsilon} (F(x) - F(x_1) + \mathcal{O}(\varepsilon))\right) \\ &= \mathcal{O}\left(y_\varepsilon(x_1) \exp\frac{1}{|\varepsilon|} (R_d(x) - R_d(x_1))\right) \end{aligned}$$

uniformly on  $\mathcal{C}$  and hence, by the hypotheses of the lemma,  $y_\varepsilon(x) = o(1)$  as  $\varepsilon \rightarrow 0$ .

By a classical theorem on the existence of solutions of ordinary differential equations,  $y_\varepsilon$  can be continued analytically along  $\gamma$  and satisfies the estimate

$$y_\varepsilon(x) = o\left(\exp\left(\frac{1}{|\varepsilon|} (R_d(x) - R_d(x_1) - \alpha)\right)\right)$$

on it. □

**Continuation of the proof of Theorem 6.** — Remember that the meaning of  $a$  and  $b$  has changed since the beginning of the proof: It is sufficient, at this point, to prove the existence of an overstable solution in some domain containing  $]a, b[$ .

For  $r \in ]0, F(b)[$ , we denote by  $C_r$  the connected component of the set  $\{x \in D; |F(x)| \leq r\}$  containing 0. If  $r$  is sufficiently small, then  $C_r$  is compact and for every direction  $d \in \mathbb{S}^1$ , there exist an overstable solution  $u_0^d$  in some domain containing  $C_r$ .

Denote by  $\frac{1}{2}\pi > \delta > \delta' > 0$  the numbers satisfying  $r = F(b) \sin \delta = F(a) \sin \delta'$  (see Figure 4 below). Consider the domain  $\Delta$  enclosed by the arcs  $\{x \in D; \arg(F(a) - F(x)) = \pm\delta', |F(a) - F(x)| \leq F(a) \cos \delta'\}$  and  $\{x \in D; \arg(F(b) - F(x)) = \pm\delta, |F(b) - F(x)| \leq F(b) \cos \delta\}$  and by the two parts of the boundary of  $C_r$  making these into a Jordan curve, which is  $C^1$  except in  $a$  and  $b$ . For sufficiently small  $r > 0$  the closure of

this domain is contained in  $D$ . We have  $\Delta = \Delta_a \cup \Delta_b \cup \tilde{C}_r$ , where  $\Delta_a$  and  $\Delta_b$  are the domains, containing  $a$  (resp.  $b$ ) on their boundary, whose  $F$ -images are the triangles with vertices  $F(a), p_a, q_a$  (resp.  $F(b), p_b, q_b$ ), where  $p_a = r \exp i(\frac{1}{2}\pi - \delta')$ ,  $q_a = r \exp i(\delta' - \frac{1}{2}\pi)$ ,  $p_b = r \exp i(\frac{1}{2}\pi - \delta)$ ,  $q_b = r \exp i(\delta - \frac{1}{2}\pi)$ .

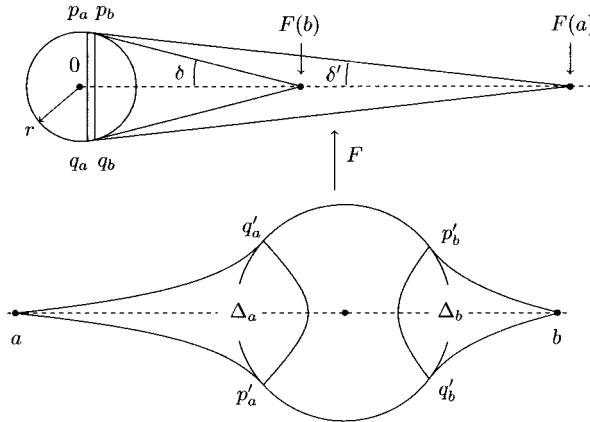


Figure 4. The domain  $\Delta$  and its  $F$ -image (here  $F(x) = x^2$ ). Primes denote the points whose  $F$ -images are the points without primes.

We consider now the “closed” sector  $S_\delta = \{\varepsilon \in \mathbb{C}^* ; |\arg(\varepsilon)| \leq \frac{1}{2}\pi - \delta\}$  and similarly  $S_{\delta'}$ ; we have  $S_\delta \subset S_{\delta'}$ . We denote by  $u_a$  resp.  $u_b$  the solutions of (5) with initial conditions  $u_a(a, \varepsilon) = 0$  resp.  $u_b(b, \varepsilon) = 0$ .

We claim that these solutions are overstable in  $\Delta_a \times S_{\delta'}$  resp.  $\Delta_b \times S_\delta$ .

We prove here the claim for  $u_a$ . Given  $D'$  relatively compact in the interior of  $\Delta_a$ , we have to show that  $u_a$  can be analytically continued on  $D' \times S_{\delta'}(\varepsilon_0)$  for  $\varepsilon_0$  small enough and satisfies  $u_a \rightarrow 0$  as  $|\varepsilon| \rightarrow 0$  uniformly on  $D'$ . Here  $S_{\delta'}(\varepsilon_0) = \{\varepsilon \in S_{\delta'} ; |\varepsilon| < \varepsilon_0\}$ . It suffices to prove that for any  $\tilde{\delta} < \delta'$  there exist  $\varepsilon_0$  such that  $u_a$  can be continued on  $\tilde{\Delta}_a \times S_{\delta'}(\varepsilon_0)$ , where  $\tilde{\Delta}_a$  is defined as  $\Delta_a$  with  $\tilde{\delta}$  instead of  $\delta'$ .

Consider the family  $\Gamma$  of paths starting from  $a$  whose images by  $F$  are the segments joining  $F(a)$  to each point of  $[\tilde{p}_a, \tilde{q}_a]$ , where  $\tilde{p}_a = r \exp i(\frac{1}{2}\pi - \tilde{\delta})$ ,  $\tilde{q}_a = r \exp i(\tilde{\delta} - \frac{1}{2}\pi)$ . Parametrizing these segments

linearly, we have for any  $d \in [-\frac{1}{2}\pi + \delta', \frac{1}{2}\pi - \delta']$  and any path  $\gamma$  of  $\Gamma$ :

$$\begin{aligned} \forall t \in [0, 1], \quad \frac{d}{dt} R_d(\gamma(t)) &= R_d(\gamma(1)) - R_d(\gamma(0)) \\ &\leq -(F(a) - r) \sin(\delta' - \tilde{\delta}) < 0. \end{aligned}$$

Hence Lemma 7 applies and yields the claim.

As we already have an overstable solution  $u_0^d$  in a neighborhood of  $C_r$  for every direction  $d \in \mathbb{S}^1$ , Lemma 8 yields that both  $u_a$  and  $u_b$  can be continued analytically to overstable solutions on  $C_r \times S_{\delta'}$ , resp.  $C_r \times S_{\delta}$ . In particular, they are both defined in  $x = 0$  for any  $\varepsilon$  small enough in  $S_{\delta}$ . Moreover, by Lemma 8, we have

$$u_a(0, \varepsilon) - u_b(0, \varepsilon) = \mathcal{O}(\exp(-\tilde{r}/\varepsilon))$$

for any  $\tilde{r} < r$  and for  $\varepsilon$  sufficiently small with  $|\arg(\varepsilon)| \leq \frac{1}{2}\pi - \delta$ . For convenience, we write it in the form

$$u_a(0, \varepsilon) - u_b(0, \varepsilon) = \mathcal{O}\left(\exp\left(-\frac{r + o(1)}{\varepsilon}\right)\right).$$

Consider now the function

$$\varphi : S_{\delta}(\varepsilon_0) \longrightarrow \mathbb{C}, \quad \varepsilon \longmapsto u_a(0, \varepsilon) - u_b(0, \varepsilon)$$

with  $S_{\delta}(\varepsilon_0) := \{\varepsilon \in \mathbb{C}; 0 < |\varepsilon| < \varepsilon_0, |\arg(\varepsilon)| \leq \frac{1}{2}\pi - \delta\}$ . This function is analytic and bounded in the sector  $S_{\delta}(\varepsilon_0)$  and satisfies, for small  $\varepsilon$  with  $|\arg(\varepsilon)| = \frac{1}{2}\pi - \delta$ , the estimate

$$|\varphi(\varepsilon)| = \mathcal{O}\left(\exp\left(-\frac{r + o(1)}{|\varepsilon|}\right)\right) = \mathcal{O}\left(\exp\left(-\frac{F(b) + o(1)}{\varepsilon}\right)\right).$$

The classical Phragmen-Lindelöf theorem implies that this estimate holds for all sufficiently small  $\varepsilon$  in  $S_{\delta}(\varepsilon_0)$ . Using again Lemma 8 (this time with  $u_0 = u_b$ ,  $u = u_a$ ), we conclude that for  $\arg \varepsilon \in [-\frac{1}{2}\pi + \delta, \frac{1}{2}\pi - \delta]$ , the solution  $u_a$  is overstable on the entire domain  $\Delta$ . As  $\delta$  becomes small with  $r \in ]0, F(b)[$  chosen above, the proof is complete.  $\square$

*Remark.* — We restricted the theorem to the case that  $f$  is real valued on the real axis and  $[a, b]$  is a real interval, but Theorem 6 can easily be adapted to more “complex” situations.

### 3.4. From overstable solutions to canard solutions.

PROPOSITION 9. — *If there exist a local (resp. global) overstable solution of (5) then there also exist a local (resp. global)  $C^\infty$ -canard solution.*

*Proof.* — Given a local (resp. global) overstable solution we construct one having real values on the real axis, namely the solution having a certain real initial condition on some point of the real axis and then we use Lemma 8. For example, using the notation of the preceding section, if  $F(a) \geq F(b)$ , the solution vanishing at some point  $\tilde{a} < a$  sufficiently close to  $a$  is overstable on some neighborhood of  $[a, b]$ . (By the way, this is the solution called  $u_{\tilde{a}}$  of the above proof.)

The restriction of such an overstable solution to the real axis is obviously a canard solution. It is a  $C^\infty$ -canard solution, because by Cauchy's formula, the derivatives will also be bounded on  $[a, b]$ .  $\square$

### 3.5. From canard solutions to the formal solution.

We have already proved all the implications  $i) \Rightarrow i+1)$  of Theorem 3 (the last one is trivial). To complete its proof, it is sufficient to show that the last statement implies the first.

THEOREM 10. — *If equation (5) has a local  $C^\infty$ -canard solution, then it also has a formal solution  $\hat{u}(\varepsilon, x) = \sum_{n \geq 1} u_n(x)\varepsilon^n$  having coefficients analytic in a neighborhood of  $x = 0$ . These are determined recursively by (13).*

*Proof.* — We give an indirect proof; so suppose that (5) has a  $C^\infty$ -canard solution  $u = u(x, \varepsilon)$  on some interval  $[-\delta, \delta]$  independent of  $\varepsilon$  but there is no formal solution of (5) without poles at  $x = 0$ . Using Proposition 4, we can suppose that there is some  $n \geq 1$  such that

$$(21) \quad u'_{k-1} - \phi_{k-1} = f u_k \quad \text{for } k \in \{1, \dots, n-1\}$$

(if  $n > 1$ ) and

$$(22) \quad u'_{n-1}(x) - \phi_{n-1}(x) = -a_q x^q + x^{q+1} g(x)$$

with  $q < p$ ,  $a_q \neq 0$  and  $g$  analytic in some neighborhood of  $x = 0$ . We will show that some derivative  $u^{(m)}(x, \varepsilon)$  cannot be bounded as  $\varepsilon \rightarrow 0$  and thus obtain a contradiction.

Let us first give a brief overview of the proof. Consider the difference between  $u$  and the  $(n-1)$ -th partial sum of the formal solution. Seen under

some adequate “magnifying glass”, i.e. using an appropriate change of the dependent and independent variables, this difference is a bounded solution of a certain regularly perturbed differential equation. Hence its dominant term satisfies a certain differential equation which is shown to have no polynomial solution. Taking into account that the change of variables in some sense “flattens” the derivatives, we deduce that some derivative of sufficiently high order must be unbounded as  $\varepsilon \rightarrow 0$ . We now give a detailed proof.

We consider the following change of the dependent variable (in the case  $n = 1$ , replace the sum by 0):

$$(23) \quad u = \sum_{k=1}^{n-1} u_k \varepsilon^k + \frac{\varepsilon^n}{\eta} v \quad \text{with} \quad \eta = \varepsilon^{\frac{p-q}{p+1}}.$$

LEMMA 11. — *If  $u$  is a solution of (5), then the function  $v$  defined by (23) satisfies an ordinary differential equation of the form*

$$(24) \quad \varepsilon v' = (f(x) + \varepsilon Q(x, v))v + \eta(a_q x^q - x^{q+1}g(x)) + \varepsilon \eta R(x)$$

where  $g, Q$  and  $R$  are analytic for  $x$  in some small interval  $[-\delta, \delta]$ , small  $\varepsilon$  and  $v$  in some disk, and remain bounded as  $\varepsilon \rightarrow 0$ .

*Proof.* — Insertion of (23) into (5) yields

$$(25) \quad \sum_{k=1}^{n-1} u'_k \varepsilon^{k+1} + \frac{\varepsilon^{n+1}}{\eta} v' = \sum_{k=1}^{n-1} f(x) u_k \varepsilon^k + \frac{\varepsilon^n}{\eta} f(x) v + \varepsilon P\left(x, \Sigma u + \frac{\varepsilon^n}{\eta} v, \varepsilon\right),$$

where we write  $\Sigma u$  for  $\sum_{k=1}^{n-1} u_k \varepsilon^k$ . By using (21) for  $k = 1, \dots, n - 1$  and multiplying by  $\eta/\varepsilon^n$ , we obtain

$$\begin{aligned} \varepsilon v' &= f(x)v - \eta(u'_{n-1}(x) - \phi_{n-1}(x)) \\ &\quad + \frac{\eta}{\varepsilon^{n-1}} \left( P\left(x, \Sigma u + \frac{\varepsilon^n}{\eta} v, \varepsilon\right) - \sum_{k=0}^{n-1} \phi_k(x) \varepsilon^k \right). \end{aligned}$$

Now

$$P\left(x, \Sigma u + \frac{\varepsilon^n}{\eta} v, \varepsilon\right) = P(x, \Sigma u, \varepsilon) + \frac{\varepsilon^n}{\eta} v Q(x, v)$$

with

$$Q(x, v) := \int_0^1 \frac{\partial P}{\partial u} \left( x, \Sigma u + \frac{\varepsilon^n}{\eta} v \theta, \varepsilon \right) d\theta.$$

Moreover,  $P(x, \Sigma u, \varepsilon) = \sum_{k=0}^{n-1} \phi_k(x) \varepsilon^k + \varepsilon^n R(x)$  with  $R := R_n$  by Formula (14) in Section 3.1. This implies the lemma together with Formula (22). □



We will use now the attractivity of the slow curve to show that  $v$  remains bounded as  $\varepsilon \rightarrow 0$  on  $[-\delta, 0[$ . For fixed  $x < 0$  this follows simply from the existence of an asymptotic expansion away from 0 (see [12] or [W], Theorem 40.1). We first show that this persists for some  $x = x(\varepsilon)$  of order  $\varepsilon^{1/p+1}$ .

LEMMA 12. — *For all  $r > 0$  there exist  $\varepsilon_1 > 0$  and  $Z > 0$  such that  $|v(\varepsilon, x)| \leq r$  for all  $x < -Z\varepsilon^{1/(p+1)}$  and  $0 < \varepsilon \leq \varepsilon_1$ .*

It is sufficient to show this for sufficiently small  $r$ . We already know that  $v(x, \varepsilon) = \mathcal{O}(\eta)$  on every interval of the form  $[a, -\delta]$  with  $\delta > 0$  independent of  $\varepsilon$ . If  $\varepsilon$  is sufficiently small, it remains to show the statement on  $[-\delta, -Z\varepsilon^{1/(p+1)}[$  for some properly chosen  $Z$ .

By the hypotheses, for  $r > 0$  sufficiently small, there is  $K > 0$  such that  $|Q(\varepsilon, x, v)| \leq K$  for all  $x \leq 0$ ,  $|v| \leq r$  and small  $\varepsilon$ . Moreover, there exist  $\mu > 0$  such that  $f(x) \leq -\mu|x|^p$  for  $-\delta \leq x \leq 0$ .

It is sufficient to show that the set  $\{|v| \leq r\}$  is an invariant set for (24) on the interval  $[-\delta, -Z\varepsilon^{1/(p+1)}[$ , if  $Z$  has been chosen properly. It is hence sufficient to show that for  $x < -Z\varepsilon^{1/(p+1)}$ , we have  $v'(\varepsilon, x) < 0$  if  $v(\varepsilon, x) = r$  (and similarly  $v'(\varepsilon, x) > 0$  if  $v(\varepsilon, x) = -r$ ).

For  $x = -z\varepsilon^{1/(p+1)}$  such that  $v(\varepsilon, x) = r$ , we find using (24)

$$\varepsilon v'(\varepsilon, x) \leq -\mu r \varepsilon^{\frac{p}{p+1}} z^p + |a_q| \varepsilon^{\frac{p}{p+1}} z^q + \mathcal{O}(\varepsilon)$$

and hence  $v'(\varepsilon, x) < 0$  if  $z > Z$  where  $Z$  is chosen greater than  $(|a_q|/\mu r)^{1/(p-q)}$ , and if  $\varepsilon$  is sufficiently small.  $\square$

We now make the following change of variables:

$$x = \varepsilon^{\frac{p}{p+1}} X, \quad V(X, \varepsilon) = v(x, \varepsilon) \quad \text{and} \quad f(x) = \varepsilon^{\frac{p}{p+1}} F(X, \varepsilon).$$

Thus we have  $F(X, \varepsilon) = X^p(\lambda + o(1))$  uniformly for  $X$  in bounded subsets of the real axis with some  $\lambda \neq 0$  independent of  $\varepsilon$ .

As  $v$  satisfies equation (24), the new function  $V$  satisfies

$$(26) \quad \dot{V} = \lambda X^p V + a_q X^q + \alpha(\varepsilon) \tilde{P}(\varepsilon, X, V),$$

where the dot denotes differentiation with respect to  $X$ , where  $\alpha(\varepsilon)$  tends to 0 as  $\varepsilon$  tends to 0 and  $\tilde{P}$  denotes a function analytic and bounded (as  $\varepsilon \rightarrow 0$ ) on all bounded  $X, V$ -sets. Lemma 12 provides some number  $X_0 < 0$  such that  $V(X_0, \varepsilon)$  remains bounded. By compactness, there is a sequence  $\varepsilon_\nu \rightarrow 0$  such that  $V(X_0, \varepsilon_\nu)$  has a limit, say  $\check{V}$ . For the sequel, fix such a sequence  $\varepsilon_\nu$ .

LEMMA 13. — Suppose that  $W$  is the solution of

$$(27) \quad \dot{W} = \lambda X^p W + a_q X^q$$

with initial condition  $W(\check{X}) = \check{V}$ . Then for every integer  $m \geq 0$  and for every real  $X$  we have

$$V^{(m)}(X, \varepsilon_\nu) \longrightarrow W^{(m)}(X), \quad \nu \longrightarrow \infty.$$

*Proof.* — Consider (26) and (27) in the complex domain. As  $W$  is a solution of a linear ordinary differential equation without finite singularity,  $W$  is an entire function. By the classical theorem on the dependence upon parameters, it follows that  $V(X, \varepsilon_\nu)$  is defined for every point  $X$  and sufficiently large  $\nu$ . Moreover, we have  $V(X, \varepsilon_\nu) \rightarrow W(X)$ , as  $\nu \rightarrow 0$ , uniformly for  $X$  in any bounded subset of  $\mathbb{C}$ . For the derivatives this follows using Cauchy’s formula or the differential equation.  $\square$

The proof of the following lemma is immediate (compare degrees and use  $p > q$ ).

LEMMA 14. — Equation (27) has no polynomial solution.

We return now to the proof of Theorem 10. Lemmas 13 and 14 imply that for every positive integer  $m$  there are a  $X_m \in \mathbb{R}$  and  $a_m > 0$  independent of  $\nu$  such that  $|V^{(m)}(X_m, \varepsilon_\nu)| \geq a_m$  for sufficiently large  $\nu$  (one could choose  $a_m := \frac{1}{2} W^{(m)}(X_m)$  with  $W^{(m)}(X_m) \neq 0$ ). We conclude that (with  $\delta_\nu = \varepsilon_\nu^{1/(p+1)}$ ) we have

$$|v^{(m)}(X_m \delta_\nu, \varepsilon_\nu)| = \delta_\nu^{-m} |V^{(m)}(X_m, \varepsilon_\nu)| \geq \delta_\nu^{-m} a_m$$

for sufficiently large  $\nu$  and hence also

$$|u^{(m)}(X_m \delta_\nu, \varepsilon_\nu)| \geq \varepsilon_\nu^{n + \frac{q-p-m}{p+1}} a_m.$$

Thus, for  $m > n(p + 1)$ , this absolute value tends to infinity as  $\nu \rightarrow \infty$ , contradicting the hypothesis of the theorem.  $\square$

### 4. Application to linear differential equations of second order.

In this section, we return to equation (2)

$$\varepsilon y'' + \varphi(x, \varepsilon) y' + \psi(x, \varepsilon) y = 0$$

where  $\varphi$  and  $\psi$  are analytic with respect to  $(x, \varepsilon)$  in a neighborhood of  $[a, b] \times \{0\}$  in  $\mathbb{C}^2$ . As in the introduction,  $\varphi_0$  and  $\psi_0$  are obtained from  $\varphi$  and  $\psi$  by putting  $\varepsilon = 0$ . We suppose that  $\varphi_0$  has a unique zero in  $[a, b]$  and denote by  $p$  its order. Moreover, we suppose that  $\varphi_0$  is positive on  $[a, 0[$  and negative on  $]0, b]$  (and hence  $p$  is odd).

As before, the derivatives are taken with respect to  $x$  and we often omit to indicate the dependence upon  $\varepsilon$ . The word “bounded” always denotes “bounded uniformly with respect to  $\varepsilon$ ”. Again the dependence of the solutions upon  $\varepsilon$  needs not be analytic unless explicitly stated.

**DEFINITION 15.** — *Let  $I \subset [a, b]$  be an interval containing 0 in its interior. Then a  $C^\infty$ -resonant solution on  $I$  is a solution of (2) that is bounded on  $I$  (in the above sense) that does not tend to the zero function as  $\varepsilon \rightarrow 0$  and whose derivatives of any order are bounded on  $I$ . A local  $C^\infty$ -resonant solution is a  $C^\infty$ -resonant solution on some open interval containing 0. A global  $C^\infty$ -resonant solution is a  $C^\infty$ -resonant solution on  $[a, b]$ .*

A local analytic resonant solution in direction  $d$  is a solution of (2) analytic in some neighborhood of  $x = 0$  that does not tend to the zero function as  $\varepsilon \rightarrow 0$  in some sector containing  $d$ . A global analytic resonant solution in direction  $d$  is a solution of (2) analytic on some domain containing  $[a, b]$  that does not tend to the zero function as  $\varepsilon \rightarrow 0$  in some sector containing  $d$ .

From this point on, we have to make a rather restrictive hypothesis.

**HYPOTHESIS (H).** — *The function  $\psi_0$  has a zero of order  $p$  or higher at  $x = 0$ .*

If this hypothesis is not satisfied, it seems possible to show that the existence of a resonant solution implies that  $\psi_0$  has a zero of order  $\geq p - 1$  and that  $-x\psi_0(x)/\varphi_0(x)$  tends to a nonnegative integer congruent to 0 or 1 mod  $p + 1$  as  $x \rightarrow 0$ . We hope that we can deal with this more general setting in a future article.

**THEOREM 16.** — *Under the preceding hypotheses and with the above notation, the following statements are equivalent :*

1) *There exists a formal solution  $\hat{y}(x, \varepsilon) = \sum_{n \geq 0} y_n(x)\varepsilon^n$  of (2) whose coefficients  $y_n$  are analytic in a neighborhood of 0.*

- 2) *There exists a local  $C^\infty$ -resonant solution.*
- 3) *There exists a global  $C^\infty$ -resonant solution.*
- 4) *There exists local analytic resonant solutions for every direction  $d$ .*
- 5) *There exists a global analytic resonant solution for the direction  $d = 0$ .*

*Remarks:*

1) The implication 1)  $\Rightarrow$  3) generalizes C.H. Lin's Theorem [16].

2) Statement 5) can be improved as below Theorem 3: for every  $\delta > 0$  there is a neighborhood  $D$  of  $[a, b]$  and a global analytic resonant solution of (2) on  $D \times S(-\frac{1}{2}\pi + \delta, \frac{1}{2}\pi - \delta)$  analytic with respect to  $x$  and  $\varepsilon$ . This slightly improves Lin's Theorem even in the case of a simple turning point (i.e.  $\varphi_0$  has a simple zero at  $x = 0$ ;  $p = 1$ ).

3) If  $p = 1$  then Hypothesis (H) is not needed: It is possible to reduce all situations such that it is satisfied by going over from (2) to a second order equation for  $y'$ , etc.

*Proof.* — As indicated in the introduction, we use the corresponding Riccati equation: We put  $v = y'/y$  and obtain the following equation:

$$(28) \quad \varepsilon v' = -\varphi(x, \varepsilon)v - \psi(x, \varepsilon) - \varepsilon v^2$$

which has the slow curve  $v_0 = -\psi_0/\varphi_0$ . It is at this point that Hypothesis (H) is needed; it guarantees that the slow curve is analytic on  $[a, b]$ . The change of variables  $v = v_0 + u$  yields an equation of the form

$$(29) \quad \varepsilon u' = f(x)u + \varepsilon Q(x, u, \varepsilon)$$

with  $f = -\varphi_0$  and  $Q = -\varepsilon^{-1}(\varphi - \varphi_0)(v_0 + u) - \varepsilon^{-1}(\psi - \psi_0) - (v_0 + u)^2 - v_0'$ . This equation satisfies the hypotheses of Section 2 and Theorem 3 can be applied. It is hence sufficient to establish a correspondence between the formal solution of (2) and that of (29) as well as one between  $C^\infty$ -resonant solutions of (2) and canard solutions of (29) resp. analytic resonant solutions of (2) and overstable solutions of (29). In other words, Theorem 16 relies on the following statement. □

PROPOSITION 17. — 1) *Equation (2) has a non trivial formal solution  $\hat{y}(x, \varepsilon) = \sum_{n \geq 0} y_n(x) \varepsilon^n$  with coefficients analytic near  $x = 0$  if and only*

if (29) has a formal solution  $\hat{u}(x, \varepsilon) = \sum_{n \geq 1} u_n(x) \varepsilon^n$  with coefficients analytic near  $x = 0$ .

2) Let  $y$  be a solution of (2) and let  $u = y'/y - v_0 = y'/y + \psi_0/\varphi_0$  be the corresponding solution of (29). Then  $y$  is a local (resp. global)  $C^\infty$ -resonant solution of (2) if and only if  $u$  is a local (resp. global)  $C^\infty$ -canard solution of (29). Furthermore  $y$  is a local (resp. global) analytic resonant solution of (2) if and only if  $u$  is a local (resp. global) overstable solution of (29).

*Proof.* — 1) If equation (2) has a non trivial formal solution  $\hat{y}(x, \varepsilon) = \sum_{n \geq 0} y_n(x) \varepsilon^n$  with coefficients analytic on  $[a, b]$  then its first term  $y_0$  is a nontrivial solution of a non singular linear first order equation (divide the reduced equation by  $x^p$ ) and hence does not vanish on  $[a, b]$ . It follows that (29) also has a formal solution  $\hat{u} = \hat{y}'/\hat{y} + \psi_0/\varphi_0$ , whose coefficients are analytic on  $[a, b]$ . Conversely, if  $\hat{u}(x, \varepsilon) = \sum_{n \geq 1} u_n(x) \varepsilon^n$  is a formal solution of (29) then the expression  $\exp(\int_0^x v_0(\xi) d\xi) \exp(\int_0^x \hat{u}(\xi, \varepsilon) d\xi)$  is a formal series that (formally) satisfies (2).

2) The equivalence between Ackerberg-O'Malley resonance (not in the  $C^\infty$  sense) of the linear equation (2) and the presence of canard solutions of the associated Riccati equation (28) had been established in [7]. These results of Jean-Louis Callot can also be found in [3].

Here,  $u = y'/y - v_0$  is equivalent to  $y = \exp(\int^x (u + v_0) d\xi)$  and clearly this establishes a correspondance between analytic solutions.

As the solutions  $u$  of (29) and  $v$  of (28) are related by  $u = v - v_0$ , the statements concerning  $u$  are equivalent to those concerning  $v$ . It remains (for the equivalence of  $C^\infty$ -resonance and  $C^\infty$ -canards) to show that  $y^{(n)}$  is bounded for every  $n$  if and only if  $v^{(n)}$  is bounded for every  $n$ . To show this, induction yields

$$(30) \quad v^{(n-1)} = \frac{y^{(n)}}{y} - \frac{Q_n(y, y', \dots, y^{(n-1)})}{y^n}$$

where  $Q_n$  is a polynomial. As  $y_0$  does not vanish on  $[a, b]$  this shows that “ $y^{(n)}$  are bounded for all  $n$ ” implies “ $v^{(n)}$  are bounded for all  $n$ ”. Conversely, by (30),  $y^{(n)}$  can be written in terms of  $v^{(n-1)}$  and  $y, y', \dots, y^{(n-1)}$ . Thus by induction, “ $v^{(n)}$  are bounded for all  $n$ ” implies “ $y^{(n)}$  are bounded for all  $n$ ”.  $\square$

## 5. Construction of equations exhibiting resonance.

We want to construct equations of the form

$$(31) \quad \varepsilon y'' - 2xy' + g(x, \varepsilon)y = 0$$

where  $g$  is analytic with respect to  $x, \varepsilon$  in some neighborhood of  $0 \in \mathbb{C}^2$  such that (31) has resonant solutions and all its formal solutions diverge. It seems that there is no simple example of such an equation. For simplicity, we only treat the case  $y_0(x) \equiv 1$ , which is equivalent to  $g(x, 0) \equiv 0$ . We will always suppose that  $|\varepsilon| < \varepsilon_0$  with some fixed  $\varepsilon_0 \leq 1$ . The advantage of having a fixed coefficient of  $y'$  is the following: If  $y_1$  is a solution of (31), then a second solution is given by  $y_2 = Ty_1$ , where

$$(32) \quad (Ty)(x, \varepsilon) = y(x, \varepsilon) \int_0^x e^{t^2/\varepsilon} y(t, \varepsilon)^{-2} dt.$$

The method of construction is as follows:

1) We construct a function  $z : D \times S \rightarrow \mathbb{C}$ , where  $D = D(0, 1) \subset \mathbb{C}$  is the open unit disk and  $S$  is the open sector  $S(-\frac{5}{4}\pi, \frac{5}{4}\pi, \varepsilon_0)$  on the Riemann surface of the logarithm, that has the following properties:

$$(33) \quad z(0, \varepsilon) \equiv 1 \text{ and } z(x, \varepsilon) \rightarrow 1 \text{ as } S \ni \varepsilon \rightarrow 0$$

uniformly on  $D$  and

$$(34) \quad z(x, \varepsilon e^{2\pi i}) = z(x, \varepsilon) + e^{2/\varepsilon} (Tz)(x, \varepsilon) \quad \text{for } |\arg \varepsilon + \pi| < \frac{1}{4}\pi,$$

where  $T$  is the operator defined by (32).

2) We show that the equation (31) satisfied by  $z$ , i.e. the one with

$$(35) \quad g(x, \varepsilon) = -\varepsilon \frac{z''}{z}(x, \varepsilon) + 2x \frac{z'}{z}(x, \varepsilon),$$

is a non trivial example of an equation having an analytic resonant solution, in the sense that the only convergent formal solution is the zero solution.

Briefly, we construct an equation (31) such that its analytic resonant solution has a prescribed monodromy when  $\varepsilon$  surrounds the origin once.

*Remarks:*

1) The constant function  $y_0(x) \equiv 1$  can be replaced by any function that does not vanish on the closure of  $D$ . The coefficient  $2x$  of  $y'$  can be replaced by other functions of  $x$  or  $(x, \varepsilon)$ ; this is particularly interesting in the case  $x^p$  with odd  $p$ .

2) The function  $e^{2/\varepsilon}$  can be replaced by any function analytic in the sector  $|\arg \varepsilon + \pi| < \frac{1}{4}\pi$  that is exponentially small as  $\varepsilon \rightarrow 0$ . The sector  $S$  can be replaced by any good covering. The size of the  $x$ -neighborhood on which  $z$  is constructed depends on the constant (here 2) in the exponent and on the good covering. This allows the construction of resonant equations on any convex  $x$ -domain containing 0.

3) The above function  $z$  has not necessarily real values for real  $x, \varepsilon$ , but this can be accomplished by using a more "symmetric" monodromy relation, for example

$$z(x, \varepsilon e^{\pi i}) - i e^{-2/\varepsilon} (Tz)(x, \varepsilon e^{\pi i}) = z(x, \varepsilon e^{-\pi i}) + i e^{-2/\varepsilon} (Tz)(x, \varepsilon e^{-\pi i})$$

for  $|\arg \varepsilon| < \frac{1}{4}\pi$ .

The first part of the above outline is rather technical and involved. We will present the details at the end of this section (Theorem 19). In Part 2, we have to show that  $g$  mentioned there is analytic in  $x, \varepsilon$  and that all non zero formal solutions of the corresponding equation diverge. The fact that  $z$  is not a single valued function of  $\varepsilon$  because of (34) already indicates that if it has an asymptotic expansion as  $\varepsilon \rightarrow 0$  it will not be convergent.

Suppose we have already shown the existence of a function  $z$  satisfying (33) and (34). Reducing  $\varepsilon_0$  if necessary, we can define a function  $g: D \times S \rightarrow \mathbb{C}$  by (35). As  $g$  remains bounded as  $\varepsilon \rightarrow 0$  the analyticity of  $g$  follows from

**PROPOSITION 18.** — *The function  $g$  defined by (35) is single valued, i.e.  $g(x, \varepsilon e^{2\pi i}) = g(x, \varepsilon)$  for  $x \in D, \varepsilon \in S, |\arg \varepsilon + \pi| < \frac{1}{4}\pi$ .*

*Proof.* — This could be verified by a short calculation, but there is another way of proof. Let us define  $g$  by (35). Then  $z$  satisfies (31) with this function  $g$ . We have seen that  $Tz$  satisfies the same equation  $z$  does, hence the right hand side of (34) also does. Using (34), we obtain that  $z(x, \varepsilon)$  and  $z(x, \varepsilon e^{2\pi i})$  satisfy the same equation (31) for  $x \in D$  and  $|\arg \varepsilon + \pi| < \frac{1}{4}\pi$ . Using again (35), the proof is complete.  $\square$

As  $|(Tz)(x, \varepsilon)| \leq C e^{|x|^2/|\varepsilon|}$  for  $x \in D$ ,  $|\arg \varepsilon + \pi| < \frac{1}{4}\pi$ , the difference  $z(x, \varepsilon e^{2\pi i}) - z(x, \varepsilon)$  is exponentially small for  $x \in D$ . By a theorem of Ramis–Sibuya (see below, too),  $z(x, \varepsilon)$  has an asymptotic expansion of Gevrey order 1 as  $\varepsilon \rightarrow 0$ . We denote by  $\hat{z}(x, \varepsilon) = \sum_{n=0}^{\infty} z_n(x)\varepsilon^n$  the corresponding formal series.

As  $z$  is not single valued, the series  $\hat{z}$  necessarily diverges for some values of  $x \neq 0$ . In the sequel, we will even determine the asymptotic behavior of  $z_n(x)$  as  $n \rightarrow \infty$  for  $x \neq 0$ ,  $\arg x \notin \frac{1}{4}\pi\mathbb{Z}$ .

Because of (33), we have as  $\arg \varepsilon = \pm\pi, \varepsilon \rightarrow 0$ ,

$$Tz(x, \varepsilon) \sim \frac{1}{2}\sqrt{-\pi\varepsilon} \quad \text{if } |\arg x| < \frac{1}{4}\pi \text{ or } |\arg x - \pi| < \frac{1}{4}\pi,$$

and

$$Tz(x, \varepsilon) \sim \frac{\varepsilon}{2x} e^{x^2/\varepsilon} \quad \text{if } \left| \arg x \pm \frac{1}{2}\pi \right| < \frac{1}{4}\pi.$$

Of course, these approximations are only uniform with respect to  $x$  in compact subsets of the above sectors. This implies

$$(36) \quad \begin{cases} z(x, \varepsilon e^{2\pi i}) - z(x, \varepsilon) \sim \frac{1}{2}\sqrt{-\pi\varepsilon} e^{2/\varepsilon} \\ \text{resp. } z(x, \varepsilon e^{2\pi i}) - z(x, \varepsilon) \sim \frac{\varepsilon}{2x} \exp\left(\frac{2+x^2}{\varepsilon}\right) \end{cases}$$

in the above cases. Using the Cauchy-Heine formula [19]

$$z(x, \varepsilon) = \frac{1}{2\pi i} \int_{\varepsilon_0 e^{-\pi i}}^{\varepsilon_0 e^{\pi i}} z(x, \mu) \frac{d\mu}{\mu - \varepsilon} + \frac{1}{2\pi i} \int_0^{\varepsilon_0 e^{-\pi i}} (z(x, \mu) - z(x, \mu e^{2\pi i})) \frac{d\mu}{\mu - \varepsilon}$$

( $x \in D, |\varepsilon| < \varepsilon_0, |\arg \varepsilon| < \pi$ ) one obtains

$$z_n(x) = \frac{1}{2\pi i} \int_{\varepsilon_0 e^{-\pi i}}^{\varepsilon_0 e^{\pi i}} z(x, \mu) \mu^{-n} \frac{d\mu}{\mu} + \frac{1}{2\pi i} \int_0^{\varepsilon_0 e^{-\pi i}} (z(x, \mu) - z(x, \mu e^{2\pi i})) \mu^{-n} \frac{d\mu}{\mu}.$$

Using (36), this yields the existence of some constant  $C$  and some analytic function  $G \neq 0$  such that

$$z_n(x) \sim C \cdot 2^{-n} \Gamma\left(n - \frac{1}{2}\right) \quad (\text{resp. } z_n(x) \sim G(x)(2+x^2)^{-n} \Gamma(n-1))$$

as  $n \rightarrow \infty$  in the above cases.



This confirms the divergent Gevrey 1 character of  $\hat{z}$ . *A priori*, it might be possible to construct a convergent series solution by multiplying  $\hat{z}$  by a certain formal series independent of  $x$ . This is not the case, because  $\hat{z}(x, \varepsilon)$  is already convergent for  $x = 0$ . Observe that every formal series solution of (31) is necessarily a product of  $\hat{z}$  and some formal series independent of  $x$ . In fact, the Wronskian of two solutions is a multiple of  $e^{x^2/\varepsilon}$  and hence there cannot exist two linearly independent formal solutions.

Therefore, if we have found a function  $z$  satisfying (33) and (34), we obtain an example of an analytic equation (31) having an analytic resonant solution such that all non zero formal solutions diverge.

One might think that some related series  $\hat{y}(t, \varepsilon^{1/s}) = \hat{z}(t\varepsilon^{r/s}, \varepsilon)$  converges for certain values of  $r, s$ . This is not the case, however. The derivative  $d\hat{y}/dt(t, \varepsilon^{1/s}) = d\hat{z}/dx(t\varepsilon^{r/s}, \varepsilon)\varepsilon^{r/s}$  would have to be convergent, too, but  $d\hat{z}/dx(0, \varepsilon)$  diverges. In fact, by (33) and (34), the derivative  $dz/dx(0, \varepsilon)$  of  $z$  satisfies

$$\frac{dz}{dx}(0, \varepsilon e^{2\pi i}) = \frac{dz}{dx}(0, \varepsilon) + e^{2/\varepsilon}$$

for  $|\arg \varepsilon + \pi| < \frac{1}{4}\pi$ . As before, this implies that the coefficients  $d_n$  of its asymptotic series as  $\varepsilon \rightarrow 0$  satisfy  $d_n \sim C \cdot 2^{-n}(n-1)!$  as  $n \rightarrow \infty$ , hence it is divergent.

Beginning with the above equation and replacing  $e^{2/\varepsilon}$  by any given exponentially small function (and replacing our covering by other “good coverings”), it is possible to construct analytic equations (31) such that, for its unique formal solution  $\hat{z}$  satisfying  $\hat{z}(0, \varepsilon) = 1$ , the derivative  $d\hat{z}/dx(0, \varepsilon)$  is a *prescribed* Gevrey-1 series; this will be further discussed in a subsequent article.

It remains to show

**THEOREM 19.** — *For  $\varepsilon_0$  sufficiently small, there exist a function  $z$  satisfying (33) and (34).*

*Proof.* — We put  $z(x, \varepsilon) = 1 + u(x, \varepsilon)$  and it remains to be shown that there exist a function  $u : D \times S \rightarrow \mathbb{C}$  such that  $u(0, \varepsilon) \equiv 0$ ,  $u = \mathcal{O}(\varepsilon)$  as  $\varepsilon \rightarrow 0$  uniformly for  $x \in D$  and

$$(37) \quad u(x, \varepsilon e^{2\pi i}) = u(x, \varepsilon) + e^{2/\varepsilon} [T(1 + u)](x, \varepsilon)$$

for  $x \in D$ ,  $\varepsilon \in S$ ,  $|\arg \varepsilon + \pi| < \frac{1}{4}\pi$ . This will be done by rewriting (37) as a fixed point equation. Thus we introduce two Banach spaces:

- $\mathcal{B}(D)$ , the vector space of all functions  $v(x, \varepsilon)$  analytic on  $D \times V$ ,  $V = S(-\frac{5}{4}\pi, -\frac{3}{4}\pi, \varepsilon_0)$ , for which there exist some constant  $C$  such that  $|v(x, \varepsilon)| \leq C|\varepsilon|^2$  for  $x \in D, \varepsilon \in V$ . As norm  $\| \cdot \|_2$  on  $\mathcal{B}(D)$ , we choose the infimum of these constants.

- $\mathcal{E}(D)$ , the vector space of all functions  $w(x, \varepsilon)$  analytic on  $D \times S$ ,  $S = S(-\frac{5}{4}\pi, \frac{5}{4}\pi, \varepsilon_0)$ , for which there exist some constant  $K$  such that  $|w(x, \varepsilon)| \leq K|\varepsilon|$  for  $x \in D, \varepsilon \in S$ . As norm  $\| \cdot \|_1$  on  $\mathcal{E}(D)$ , we choose the infimum of these constants.

For functions  $v : V \rightarrow \mathbb{C}$  and  $w : S \rightarrow \mathbb{C}$  independent of  $x$ , we introduce analogously the Banach spaces  $\mathcal{B}$  and  $\mathcal{E}$ .

We use the following lemma that will be proved later.

LEMMA 20. — *There is a continuous linear operator  $\Sigma : \mathcal{B} \rightarrow \mathcal{E}$  such that for every  $v \in \mathcal{B}$  its image  $w = \Sigma v$  satisfies*

$$(33) \quad w(\varepsilon e^{2\pi i}) = w(\varepsilon) + v(\varepsilon) \quad \text{for } \varepsilon \in V.$$

We define another operator  $\Sigma_D : \mathcal{B}(D) \rightarrow \mathcal{E}(D)$  by

$$(\Sigma_D v)(x, \cdot) = \Sigma[v(x, \cdot)] \quad \text{for all } x \in D.$$

Then it is sufficient to show the existence of a fixed point  $u$  of  $\Sigma_D \tilde{T}$  close to  $0 \in \mathcal{E}(D)$  satisfying  $u(0, \varepsilon) \equiv 0$ . Here  $\tilde{T} : \mathcal{E}(D) \supset D_{\tilde{T}} \rightarrow \mathcal{B}(D)$  is defined by

$$(\tilde{T}w)(x, \varepsilon) = e^{2/\varepsilon} [T(1 + w)](x, \varepsilon) \quad \text{for } x \in D, \varepsilon \in V$$

and  $D_{\tilde{T}}$  denotes the subset of  $\mathcal{E}(D)$  containing all functions with  $|w(x, \varepsilon)| \leq \frac{1}{2}$ . The operator  $\tilde{T}$  is well-defined because  $e^{2/\varepsilon}$  is exponentially small.

As usually, it is shown that  $\Sigma_D \tilde{T}$  is a contraction for sufficiently small  $\varepsilon_0$ . Thus, it has a fixed point  $u \in \mathcal{E}(D)$ . Using the definitions of  $\tilde{T}$  and  $\Sigma_D$ , we show that  $u(0, \varepsilon) \equiv 0$ . □

*Proof of Lemma 20.* — The construction of a holomorphic function with prescribed monodromy is classical. Given  $v$  analytic on  $V$ , the function  $w : S(-\pi, \pi, \varepsilon_0) \rightarrow \mathbb{C}$  defined by

$$(39) \quad w(\varepsilon) = -\frac{1}{2\pi i} \int_0^{-\varepsilon_0} v(\mu) \frac{d\mu}{\mu - \varepsilon}$$

can be continued analytically to an analytic function, also called  $w$ , on all of  $S$  that satisfies (38). Unfortunately,  $w$  has a logarithmic singularity at  $\varepsilon = -\varepsilon_0$  and is not in  $\mathcal{E}$ . To overcome this difficulty, we consider some average of integrals similar to (39), roughly speaking, we integrate with respect to  $\varepsilon_0$ . A factor  $\varepsilon/\mu$  is introduced to accomodate the norms of  $\mathcal{B}$  and  $\mathcal{E}$ .

For  $v \in \mathcal{B}$ , we define

$$(\Sigma v)(\varepsilon) = \frac{4}{\pi} \int_{-\frac{1}{8}\pi}^{+\frac{1}{8}\pi} w_\theta(\varepsilon) \, d\theta,$$

where  $w_\theta : S \rightarrow \mathbb{C}$  is defined by

$$(40) \quad w_\theta(\varepsilon) = -\frac{1}{2\pi i} \int_0^{\varepsilon_\theta} \frac{\varepsilon}{\mu} v(\mu) \frac{d\mu}{\mu - \varepsilon} \quad \text{for } \varepsilon \in S \text{ and } \theta \in \left[-\frac{1}{8}\pi, \frac{1}{8}\pi\right].$$

Here  $\varepsilon_\theta = -\varepsilon_0 \exp(\theta i)$ . If  $-\frac{3}{4}\pi \leq \arg \varepsilon \leq \frac{3}{4}\pi$  then the path of integration is the line from 0 to  $\varepsilon_\theta$ , if  $\arg \varepsilon < -\frac{3}{4}\pi$ , the path of integration is above  $\varepsilon$ , if  $\arg \varepsilon > \frac{3}{4}\pi$ , the path of integration is below  $\varepsilon$  (see Figure 5).

Cauchy’s formula shows that  $w_\theta$  satisfies (38) for every  $\theta$  and hence so does  $w = \Sigma v$ . It remains to show that  $w \in \mathcal{E}$  and that there exist  $K > 0$  such that  $\|\Sigma v\|_1 \leq K\|v\|_2$  for all  $v \in \mathcal{B}$ .

Let us first estimate  $w_\theta(\varepsilon)$ . For  $-\frac{3}{4}\pi \leq \arg \varepsilon \leq \frac{3}{4}\pi$ , we have  $|\mu - \varepsilon| \geq |\mu| \sin(\frac{1}{8}\pi)$ , hence  $|\varepsilon^{-1} w_\theta(\varepsilon)| \leq C_1 \|v\|_2$  with  $C_1 = (\varepsilon_0/2\pi)(1/\sin(\frac{1}{8}\pi))$ .

For  $-\frac{5}{4}\pi < \arg \varepsilon < -\frac{3}{4}\pi$ , we will show that

$$(41) \quad \left| \frac{1}{\varepsilon} w_\theta(\varepsilon) \right| \leq \left( C_1 + \varepsilon_0 + \frac{\varepsilon_0}{2\pi} \int_{\gamma(\theta, \arg \varepsilon + \pi)} \frac{1}{|\mu - \varepsilon|} |d\mu| \right) \|v\|_2$$

with the above  $C_1$ , where  $\gamma(\theta, \psi)$  (see Figure 5) is the path described by  $\mu = \varepsilon_0 e^{\phi i}$  and  $\phi \in [-\frac{5}{4}\pi, \theta - \pi]$  in the case  $\theta < \psi$ , but  $\phi \in [\theta - \pi, -\frac{3}{4}\pi]$  in the case  $\psi \leq \theta$ . In fact, if  $\theta < \psi = \arg \varepsilon + \pi < \frac{1}{4}\pi$  then we can use paths of integration  $\tilde{\gamma}$  from 0 to  $\varepsilon_\theta$  arbitrarily close to  $\gamma_1 \cup \gamma(\theta, \psi)$  (where  $\gamma_1$  is the line from 0 to  $\varepsilon_0 e^{-5\pi i/4}$ ) that are still in the interior of  $V$ . The estimate

$$\left| \int_{\tilde{\gamma}} \frac{1}{\mu(\mu - \varepsilon)} v(\mu) \, d\mu \right| \leq \int_{\tilde{\gamma}} \left| \frac{\mu}{(\mu - \varepsilon)} \right| \cdot |d\mu| \cdot \|v\|_2$$

for all such  $\tilde{\gamma}$  and the limit  $\tilde{\gamma} \rightarrow \gamma_1 \cup \gamma(\theta, \psi)$  yield

$$\left| \frac{1}{\varepsilon} w_\theta(\varepsilon) \right| \leq \frac{1}{2\pi} \int_{\gamma_1 \cup \gamma(\theta, \psi)} \frac{|\mu|}{|\mu - \varepsilon|} |d\mu| \cdot \|v\|_2.$$

As in the former case, we obtain  $\int_{\gamma_1} \frac{|\mu|}{|\mu - \varepsilon|} |d\mu| \leq \varepsilon_0/\sin(\frac{1}{8}\pi)$ . For the remaining integral, it is sufficient to use  $|\mu| \leq \varepsilon_0$  on  $\gamma(\theta, \psi)$ . By combining the two integrals, we obtain (41) for  $\theta < \arg \varepsilon + \pi$  (without the term  $\varepsilon_0$ ).

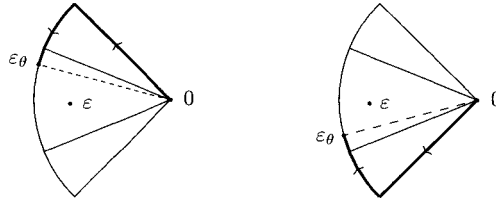


Figure 5. The path  $\gamma_1 \cup \gamma(\theta, \psi)$ , on the left in the case  $\theta < \psi$ , on the right if  $\psi \leq \theta$ .

In the case  $\arg \varepsilon + \pi \leq \theta$ , we first use Cauchy’s formula which shows

$$w_\theta(\varepsilon) = v(\varepsilon) + \frac{1}{2\pi i} \int_0^{\varepsilon\theta} \frac{\varepsilon}{\mu(\mu - \varepsilon)} v(\mu) d\mu,$$

where now the path of integration from 0 to  $\varepsilon\theta$  is above  $\varepsilon$ . The estimates of the integral are identical to the above subcase, only there is another term to estimate  $\frac{1}{\varepsilon} v(\varepsilon) \leq \varepsilon_0 \|v\|_2$ . This proves (41) in the second subcase, too.

For  $\frac{3}{4}\pi \leq \arg \varepsilon \leq \frac{5}{4}\pi$ , we show analogously

$$(42) \quad \left| \frac{1}{\varepsilon} w_\theta(\varepsilon) \right| \leq \left( C_1 + \varepsilon_0 + \frac{\varepsilon_0}{2\pi} \int_{\gamma(\theta, \arg \varepsilon - \pi)} \frac{1}{|\mu - \varepsilon|} |d\mu| \right) \|v\|_2.$$

Recall that  $(\Sigma v)(\varepsilon) = (4/\pi) \int_{-\frac{1}{8}\pi}^{+\frac{1}{8}\pi} w_\theta(\varepsilon) d\theta$ . In order to show that  $\Sigma : \mathcal{B} \rightarrow \mathcal{E}$  is continuous, it is hence sufficient to show that

$$(43) \quad \int_{-\frac{1}{8}\pi}^{+\frac{1}{8}\pi} d\theta \int_{\gamma(\theta, \arg \varepsilon + \pi)} \frac{1}{|\mu - \varepsilon|} |d\mu| \leq K$$

for every  $\varepsilon$  with  $-\frac{5}{4}\pi \leq \arg \varepsilon \leq -\frac{3}{4}\pi$ ,  $|\varepsilon| < \varepsilon_0$ . By splitting the exterior integral in two parts at  $\theta = \arg \varepsilon + \pi$ , if this value is in  $[-\frac{1}{8}\pi, \frac{1}{8}\pi]$ , it can be seen that the double integral assumes its maximum for  $\arg \varepsilon = -\pi$ . Thus, it is sufficient to estimate  $2 \int_0^{\frac{1}{8}\pi} d\theta \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \frac{1}{|e^{s^2} + \varepsilon/\varepsilon_0|} ds$  in this case. As the denominator is at least  $\sin s$ , the theorem of Fubini-Tonelli yields (43) with  $K = 2 \int_0^{\frac{1}{4}\pi} \frac{s}{\sin s} ds$  (by integrating over the triangle  $0 \leq \theta \leq s \leq \frac{1}{4}\pi$  instead of the set  $0 \leq \theta \leq \frac{1}{8}\pi$ ,  $\theta \leq s \leq \frac{1}{4}\pi$ ).  $\square$

*Remark.* — Bernard Malgrange, whom we would like to thank here, suggested another proof of Lemma 20 using the  $\bar{\partial}$  operator. This proof is outlined below.

For convenience, we now think of sectors as subsets of  $\mathbb{C}$ ; thus the directions  $d$  and  $d + 2\pi$  are identified. We do not indicate the radius of the sectors — it is always  $\varepsilon_0$ . For simplicity, we introduce a second sector  $S(-\frac{1}{4}\pi, \frac{1}{4}\pi)$ .

It is sufficient to show the existence of two analytic functions  $w_1$  on  $S(-\frac{1}{4}\pi, \frac{5}{4}\pi)$  and  $w_2$  on  $S(-\frac{5}{4}\pi, \frac{1}{4}\pi)$  such that

$$(44) \quad w_1 - w_2 \equiv \begin{cases} v & \text{on } S(\frac{3}{4}\pi, \frac{5}{4}\pi) = S(-\frac{5}{4}\pi, -\frac{3}{4}\pi) \quad \text{and} \\ 0 & \text{on } S(-\frac{1}{4}\pi, \frac{1}{4}\pi). \end{cases}$$

The idea is to first solve the problem in the class  $C^\infty$  and then to add a correction in order to obtain an analytic solution.

Let  $\alpha$  be  $C^\infty$  on  $S(\frac{3}{4}\pi, \frac{5}{4}\pi)$  such that  $\alpha \equiv 0$  on  $S(\frac{3}{4}\pi, \frac{7}{8}\pi)$  and  $\alpha \equiv 1$  on  $S(\frac{9}{8}\pi, \frac{5}{4}\pi)$ . Then define  $\tilde{w}_1, \tilde{w}_2$  by

- $\tilde{w}_1 := \alpha v, \tilde{w}_2 := (\alpha - 1)v$  on  $S(\frac{3}{4}\pi, \frac{5}{4}\pi)$  and
- $\tilde{w}_1, \tilde{w}_2 := 0$  on  $S(-\frac{\pi}{4}, \frac{3}{4}\pi)$ , resp.  $S(-\frac{3}{4}\pi, \frac{\pi}{4})$ .

Then conditions (44) are satisfied, but  $\tilde{w}_1, \tilde{w}_2$  are not analytic: We have  $\bar{\partial}\tilde{w}_1 = \bar{\partial}\tilde{w}_2 = g$  where  $g := \bar{\partial}(\alpha v) = v\bar{\partial}\alpha$  is  $C^\infty$  on the punctured disk  $D(0, \varepsilon_0) \setminus \{0\}$ .

Then we construct a  $C^\infty$  correction  $h$  defined on the punctured disk such that  $\bar{\partial}h \equiv g$ . This can be done using the classical formula [hor]

$$h(z) = \frac{1}{2\pi i} \iint_{S(\frac{7}{8}\pi, \frac{9}{8}\pi)} \frac{g(\zeta)}{z - \zeta} d\zeta \wedge d\bar{\zeta}.$$

The initial problem (44) is then solved by the analytic functions  $w_1 := \tilde{w}_1 - h, w_2 := \tilde{w}_2 - h$ .

If  $\alpha$  is chosen as a function of the angle, say  $\alpha(re^{\theta i}) = \varphi(\theta)$ , then  $g$  (and hence  $w_1, w_2$ ) is bounded as soon as  $v(z)/z$  is, because  $g(z) = v(z)\partial\alpha(z)/\partial\bar{z} = v(z)i/\bar{z}\varphi'(\arg z)$ . This shows that a bounded linear operator from  $\mathcal{B}$  to  $\mathcal{E}$  can be constructed in this way.

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Manuscrit reçu le 11 mars 2002,  
accepté le 16 mai 2002.

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