p-adic Analogues of the Mandelbrot Set

by

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Jacqueline Anderson was born in Norwood, Massachusetts on July 16, 1986. She was raised by her parents, Michael and Theresa Anderson, with her brother Michael in Attleboro, Massachusetts, where she graduated from Bishop Feehan High School in 2004. She then attended Providence College, where she received a Bachelor of Arts degree in mathematics in 2008. Moving on to Brown for graduate school, she recieved a Master of Science degree in mathematics in 2010. During her time at Brown, she has been supported by research assistantships, a number of teaching assistantships and fellowships, and a GAANN fellowship.

In Memory of Michael Anderson

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CHAPTER 1

Introduction

Almost a century ago, Pierre Fatou and Gaston Julia laid the foundation for the study of complex dynamical systems. There was a resurgence of interest in the field in the early 1980s, with the development of computer imaging technology inspiring mathematicians such as Benoit Mandelbrot, Adrian Douady, John Hubbard, and Dennis Sullivan to make great strides in the study of complex dynamics. Complex dynamicists primarily concern themselves with examining the different behaviors that can result when one iterates a function ϕ defined on the Riemann sphere, and the theory is well-developed. Fifteen to twenty years ago, Robert Benedetto, Juan Rivera-Letelier, Liang-Chung Hsia, and others began studying p-adic dynamical systems, and have begun to build up a theory analogous to what one sees in the complex setting.

One of the most intriguing and inspiring objects in complex dynamics is the Mandelbrot set. The goal of this thesis is to define analogues of this object over p-adic fields and to explore some of their properties, in the hope that they will be as interesting and rich in information as the complex Mandebrot set has proven to be.

1. Complex Dynamics and the Mandelbrot Set

The classical Mandelbrot set is a subset of the moduli space of quadratic polynomials defined over the complex numbers. We say that two polynomials f and g are conjugates if there is an invertible affine linear transformation $\phi(z) = az+b$ such that $f = \phi^{-1} \circ g \circ \phi$. As $f^n = \phi^{-1} \circ g^n \circ \phi$, conjugate maps exhibit the same dynamical behavior, up to a change of coordinates. Every quadratic polynomial defined over the complex numbers can be conjugated in a unique way to be put in the following form:

$$f_c(z) = z^2 + c, \quad c \in \mathbb{C}.$$

Thus, we can identify the moduli space of quadratic polynomials in $\mathbb{C}[z]$ with the complex plane, parameterized by the coefficient c.

The Mandelbrot set is defined to be the following subset of this moduli space:

$$\mathcal{M}_{\mathbb{C}} = \{c \in \mathbb{C} : \text{the critical orbit of } f_c(z) = z^2 + c \text{ is bounded}\}. \tag{1.1.1}$$

The *critical orbit* of a quadratic polynomial f_c is the set of iterates of the critical point, $\{0, f_c(0), f_c(f_c(0)), \ldots\}$. The set $\mathcal{M}_{\mathbb{C}}$ is a complicated and fractal-like subset of the complex plane. Its alluring image and intriguing complexity, despite its simple definition, have inspired much research in complex dynamics in the thirty years since Mandelbrot first explored the set. During this time, it has become one of the most famous objects in mathematics.

Many of the Mandelbrot set's basic properties were established in the early 1980s by Douady and Hubbard in [**8, 9**]. Of particular interest to this thesis is the work done to explore Misiurewicz points on the Mandelbrot set. These are points that correspond to polynomials for which the critical orbit is strictly preperiodic. They appear on the boundary of the Mandelbrot set, which has been shown to be self-similar and to locally bear a resemblance to corresponding Julia sets [**15, 20**]. This is just one of many ways in which the Mandelbrot set encodes quite a bit of information about the dynamics of quadratic polynomials. There are still many open questions about the Mandelbrot set, which remains a topic of active research. For more information about the classical Mandelbrot set, see [**6, 7**].

2. Non-Archimedean Dynamics

In the past two decades, much research has been done on dynamical systems in a non-Archimedean setting. For a survey of the subject, see [**4**] or [**18**]. A good deal of this work has been done with the intention of building up a theory analogous to what exists for dynamical systems over the complex numbers. People have studied Julia sets, Fatou components, periodic points, and other dynamical objects associated to dynamical systems over non-Archimedean fields; and researchers have sought analogues to famous theorems in complex dynamics, such as Sullivan's theorem stating that there are no wandering domains, or Fatou's theorem stating that every attracting cycle attracts a critical orbit [**3, 5, 12**]. Juan Rivera-Letelier and others have used the Berkovich projective line, a space that contains $\mathbb{C}_p$ but has the advantages of being compact and path-connected, to further develop the theory of p-adic dynamical systems, with much success. See, for example, [**1, 11, 16, 21**].

Non-Archimedean dynamical systems are studied in their own right for their unique and interesting properties, but their study also helps to shed light on the study of dynamical systems over global fields. In particular, understanding local dynamical systems is useful when studying canonical height functions, as a canonical height can be expressed as a sum of local heights. There is also much interest in studying post-critically finite rational maps, which can be identified as maps that are post-critically bounded at every place.

3. Summary of Results

While quite a bit of work has been done in recent years to begin building a theory of p-adic dynamics in the image of the well-established theory of complex dynamics, it is still a young field and much remains unexplored. One underdeveloped area is that of parameter spaces of rational functions, and in particular, Mandelbrot set analogues. There is a good reason for this, since the obvious analogue of the classical Mandelbrot set $\mathcal{M}_\mathbb{C}$ in a p-adic setting is uninspiring, as we will see in the next chapter. When one generalizes the notion of the Mandelbrot set to higher degree polynomials in a natural way, however, we find sets that are complicated and that seem to share many features with the famous complex Mandelbrot set. We will study these p-adic Mandelbrot sets associated to degree d polynomials, and we will find that there is a stark contrast in their structures depending on whether $p \geq d$ or $p < d$. The former situation is well-understood and easy to describe, while the latter is still highly mysterious. We will concern ourselves primarily with the latter case throughout this thesis.

In Chapter 2, we introduce definitions, notation, and tools that we will use throughout the thesis. We also survey known results regarding p-adic Mandelbrot sets and provide elementary proofs of these results and of other basic properties. We will define the p-adic Mandelbrot set for degree d polynomials as a set of normalized polynomials for which all critical points have bounded orbits, and we will describe these sets completely when $p > d$ or when d is a power of p. We will also establish these sets as closed subsets of $\mathbb{C}_p^{d-1}$ containing, and sometimes equal to, a polydisk centered at the origin.

In Chapter 3, we discuss the critical radius of the p-adic Mandelbrot set for polynomials of degree d. In this chapter, we introduce the quantity $r(d,p)$, which measures the maximum possible p-adic absolute value for critical points of polynomials of degree d in the corresponding p-adic Mandelbrot set. We seek to determine $r(d,p)$ for different values of d and p, and we prove results for $\frac{1}{2}d < p < d$ in Theorem 3.4 and for $d = 2p$ and $d = 3p$ in Propositions 3.7 and 3.9, respectively. For all other combinations of d and p, we provide examples that give lower bounds for $r(d,p)$ and we show that there is no uniform upper bound.

Finally, in Chapter 4, we examine the one-parameter family of cubic polynomials $f_t(z) = z^3 - \frac{3}{2}tz^2$ for $t \in \mathbb{C}_2$ to give a sense of the structure and complexity of these p-adic Mandelbrot sets. In particular, we define what it means to be a Misiurewicz point and examine the boundary of the Mandelbrot set associated to this family near a particular Misiurewicz point. In doing so, we find that the boundary is self-similar and that it resembles the Julia set for the corresponding polynomial. Both of these phenomena are analogous to what occurs for the classical Mandelbrot set in the complex quadratic setting, inspiring confidence that p-adic Mandelbrot sets for degree d polynomials may share some of the interesting properties of the classical Mandelbrot set when $p < d$.

CHAPTER 2

Notation, Tools, and Proofs of Known Results

In this chapter, we establish the definitions and notation that we use throughout this thesis. We also outline some key tools that are used in the proofs that follow. In the third section, we provide proofs of two known results, as they do not appear in this form in the literature. Finally, in Section 4, we prove basic facts about the structure of p-adic Mandelbrot sets.

1. Background and Definitions

1.1. Foundations of Dynamical Systems. Generally speaking, a (discrete) dynamical system consists of a set and a function from that set to itself. Throughout this document, the sets in our dynamical systems will be either the complex numbers $\mathbb{C}$ or a non-Archimedean field, usually $\mathbb{Q}_p$ or $\mathbb{C}_p$, and the functions that we study will be polynomials. We let $\mathbb{C}_p$ denote the completion of the algebraic closure of the p-adic numbers $\mathbb{Q}_p$. These p-adic fields are metric spaces, with a metric induced by the p-adic absolute value $|\cdot|_p$. Our p-adic absolute value will be normalized in the usual way so that $|p|_p = p^{-1}$.

We will be primarily working over $\mathbb{C}_p$, which is totally disconnected and not locally compact. The property of $\mathbb{C}_p$ that we will use most is that the p-adic absolute value on $\mathbb{Q}_p$ extends to $\mathbb{C}_p$ and satisfies the ultrametric triangle inequality, which says that for all $\alpha, b \in \mathbb{C}_p$,

$$|a+b|_p \leq \max\{|a|_p, |b|_p\}, \text{ with equality if } |a|_p \neq |b|_p.$$

For more information about $\mathbb{Q}_p$ or $\mathbb{C}_p$ and their properties, see [**14**].

Let $f \in K[z]$ be a polynomial defined over a field K with a nontrivial absolute value. We will denote by f^n the n^{th} iterate of the map f, so that f^n is equal to f

composed with itself n times. We define the *orbit* of a point $z \in K$ as follows:

$$\mathcal{O}_f(z) = \{f^n(z) : n \in \mathbb{N}\}.$$

A point $z \in K$ is called *periodic* if $f^n(z) = z$ for some positive integer n. The smallest such n is called the *period* of z. We say z is *preperiodic* if it has a finite orbit. In other words, z is preperiodic if there exist positive integers n and m such that $f^m(z) = f^{m+n}(z)$. We say a point $z \in K$ is *strictly preperiodic* if it is preperiodic but not periodic.

We further classify periodic points as *attracting, repelling,* or *neutral*, depending on the absolute value of their multipliers. The *multiplier* λ of a periodic point z with period n is defined as follows:

$$\lambda = (f^n)'(z).$$

If $|\lambda| < 1$, we say that z is an attracting periodic point. If $|\lambda| > 1$, we say that z is repelling, and if $|\lambda| = 1$, we say that z is neutral. These terms reflect the dynamical behavior of points near z as one iterates the function f. If z is attracting, nearby points move toward the attracting cycle as one iterates f. If z is repelling, nearby points move away from the cycle as f is iterated.

For a dynamical system $f : K \to K$, each point $z \in K$ is classified as being in one of two sets, the Fatou set or the Julia set, depending on its orbit. Roughly speaking, there are some points whose orbits resemble those of all nearby points, and other points whose orbits cannot be predicted by the orbits of nearby points. These latter points exhibit chaotic dynamical behavior, and comprise the Julia set of f. The former set, the complement of the Julia set, is called the Fatou set.

More formally, the *Fatou set* associated to a dynamical system f is the largest open subset U of K on which the set of iterates $\{f^n : n \geq 1\}$ is equicontinuous. We can then define the *Julia set* to be the complement of the Fatou set in K.

There are a number of equivalent ways to define the Julia set of a polynomial $f \in \mathbb{C}[z]$. It is equal to the closure of the set of repelling periodic points, and it is also equal to the boundary of the set of points whose orbits are bounded [**6**, Theorem

III.3.1]. This latter characterization, which only holds for polynomials and not for rational functions in general, is also true for polynomials defined over p-adic fields [**4**, Proposition 4.37], and is the definition that is most convenient for our purposes.

1.2. Generalizing the Mandelbrot set to a p-adic setting. The complex Mandelbrot set, defined in (1.1.1), is a subset of the moduli space of quadratic polynomials in $\mathbb{C}[z]$, defined by the orbit of the critical point 0. The orbit of the critical point is of interest due to a classical theorem of Fatou, which states that every attracting periodic cycle for a dynamical system defined over the complex numbers attracts a critical orbit. This result was extended by Shishikura to include neutral cycles in [**17**]. Thus, critical orbits detect the presence of non-repelling cycles in a dynamical system. The complex Mandelbrot set can be equivalently defined as the set of parameters c for which the corresponding polynomial $f_c(z) = z^2 + c$ has a connected Julia set [**6**, Theorem VIII.1.1].

The natural way to extend the definition of the Mandelbrot set to a p-adic setting is to retain the definition in (1.1.1), replacing $\mathbb{C}$ with $\mathbb{C}_p$. This definition gives the following set:

$$\{c \in \mathbb{C}_p : \mathcal{O}_{f_c}(0) \text{ is bounded}\}, \text{ where } f_c(z) = z^2 + c.$$

This set proves to be much less inspiring than the classical Mandelbrot set $\mathcal{M}_{\mathbb{C}}$, as it is simply the p-adic unit disk:

PROPOSITION 2.1. *Let p be a prime number and let*

$$\mathcal{M}_{2,p} = \{c \in \mathbb{C}_p : \mathcal{O}_{f_c}(0) \textit{ is bounded}\},$$

where $f_c = z^2 + c \in \mathbb{C}_p[z]$. Then, $\mathcal{M}_{2,p} = \{c \in \mathbb{C}_p : |c|_p \leq 1\}$.

PROOF. This is a straightforward consequence of the non-Archimedean absolute value on $\mathbb{C}_p$. First, suppose that $|c|_p > 1$. Then, due to the ultrametric triangle inequality,

$$|f_c^n(0)|_p = |c|_p^{2^{n-1}}.$$

Since $|c|_p > 1$, this quantity grows without bound as n grows, so f_c is not in $\mathcal{M}_{2,p}$.

Now consider $c \in \mathbb{C}_p$ with $|c|_p \leq 1$. Then, f_c will map the p-adic unit disk to itself. Therefore, the orbit of 0 stays within the unit disk, and $f_c \in \mathcal{M}_{2,p}$. $\square$

We now generalize the definition of $\mathcal{M}_{2,p}$ to higher degree polynomials. Just as quadratic polynomials may be conjugated to be put in the form $f_c(z) = z^2 + c$, we will choose a normal form for our degree d polynomials. Let

$$\mathcal{P}_{d,p} = \{f \in \mathbb{C}_p[z] : f \text{ has degree } d, \text{ is monic, and } f(0) = 0\}.$$

Note that any degree d polynomial in $\mathbb{C}_p[z]$ can be conjugated by an invertible affine linear transformation so that it has this form, so we may restrict ourselves to studying polynomials in $\mathcal{P}_{d,p}$ without loss of generality. Note also, however, that there is usually more than one way to conjugate a polynomial so that it has this form — in fact, there are up to $d^2 - d$ ways to do so. For a given polynomial of g of degree d, we first conjugate by $\phi_1(z) = az$, where a is chosen so the resulting conjugated polynomial $\phi_1^{-1} \circ g \circ \phi_1$ is monic. We have $d - 1$ choices for a, as we can multiply any appropriate choice by a $(d-1)^{st}$ root of unity, and the leading coefficient will be 1. We then conjugate by $\phi_2(z) = z + b$, where b is a fixed point. This translation ensures that the resulting polynomial $\phi_2^{-1} \circ \phi_1^{-1} \circ g \circ \phi_1 \circ \phi_2$ has a fixed point at 0. We have up to d choices for b, as our polynomial has up to d distinct fixed points. Thus, any given conjugacy class of degree d polynomials has up to $d(d-1)$ representatives in $\mathcal{P}_{d,p}$. Unlike the quadratic parameterization used in the definition of $\mathcal{M}_{\mathbb{C}}$, this normal form does not define a moduli space for degree d polynomials, but a finite-to-one parameter space. We choose this normal form because it allows us to parameterize a polynomial by its critical points.

Definition 2.2. Let $f \in \mathcal{P}_{d,p}$. We say that f is *post-critically bounded*, or PCB, if all of its critical orbits are bounded. In other words, f is PCB if and only if $\mathcal{O}_f(c)$ is a bounded set for all c such that $f'(c) = 0$.

With this notion, we define a p-adic Mandelbrot set for degree d polynomials as follows:

$$\mathcal{M}_{d,p} = \{f \in \mathcal{P}_{d,p} : f \text{ is PCB}\}. \tag{2.1.2}$$

REMARK 2.3. We will often parameterize a polynomial $f \in \mathcal{P}_{d,p}$ by its set of critical points $\{c_1, \ldots c_{d-1}\}$, listed with multiplicity, which allows us to think of $\mathcal{M}_{d,p}$ as a subset of $\mathbb{C}_p^{d-1}$. Of course, since the set of critical points is an unordered set, and since the normal form used in $\mathcal{P}_{d,p}$ is not unique, there are multiple points in $\mathbb{C}_p^{d-1}$ that correspond to the same polynomial (or to polynomials in the same conjugacy class). However, as noted earlier, a given conjugacy class of polynomials corresponds to only finitely many points in $\mathbb{C}^{d-1}$. When we think of $\mathcal{M}_{d,p}$ as a subset of $\mathbb{C}_p^{d-1}$, it is important to keep in mind that $\mathcal{M}_{d,p}$ will have the appropriate redundancies and symmetries.

REMARK 2.4. One may wonder why we choose to parameterize $f \in \mathcal{P}_{d,p}$ by its critical points rather than by its coefficients. In some situations, such as the one described in Theorem 2.8, the two notions are interchangeable, and it is equally easy to describe the polynomials in $\mathcal{M}_{d,p}$ in terms of their coefficients as it is to describe them in terms of their critical points. In other situations, such as in Proposition 2.6, it is much easier and more natural to describe the polynomials in $\mathcal{M}_{d,p}$ in terms of their critical points. This is because there is a uniform bound on the absolute value of the critical points for polynomials in $\mathcal{M}_{d,p}$ when $d = p^k$, but the bounds on the coefficients a_i vary from coefficient to coefficient depending on the p-adic valuation of i.

2. Notation and Tools

2.1. Notation. Throughout this document, we fix a prime number p and we let

$$f(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z \in \mathcal{P}_{d,p}$$

be a degree d polynomial in $\mathbb{C}_p[z]$. In the interest of having less cluttered notation, we suppress the subscript p from our notation for absolute values and valuations. We denote the critical points of f by $c_1, \ldots, c_{d-1}$, not necessarily distinct, labeled so that

$$|c_1| \geq |c_2| \geq \cdots \geq |c_{d-1}|.$$

We denote the closed disk centered at a of radius s in $\mathbb{C}_p$ by

$$\bar{D}(a,s) = \big\{z \in \mathbb{C}_p : |z-a| \leq s\big\}.$$

The *filled Julia set of* f is the set

$$\mathcal{K}_f = \{z \in \mathbb{C}_p : \text{the } f\text{-orbit of } z \text{ is bounded}\}.$$

We let $R \geq 0$ be the smallest number such that

$$\mathcal{K}_f \subseteq \bar{D}(0, p^R). \tag{2.2.3}$$

Equivalently, we can define R as follows:

$$R = \max\left(\left\{\frac{-v(a_i)}{d-i} : 1 \leq i \leq d-1\right\} \cup \{0\}\right).$$

Here, R is chosen to be maximal such that if $|z| = p^R$ and $R > 0$, then $|z^d| = |a_i z^i|$ for some i. We can deduce that this definition is equivalent to the one given in (2.2.3) using [**4**, Theorem 3.9].

Writing $f(z) = \prod_{i=1}^{d}(z - z_i)$, where the z_i are the roots of f, one sees that there is yet another equivalent way to define R. We define R so that $R = 0$ if $|z_i| \leq 1$ for all roots z_i, and otherwise, R is chosen so that

$$p^R = \max_{f(z)=0} |z|.$$

We also set

$$r = -v(c_1). \tag{2.2.4}$$

The quantity r measures the size of the largest critical point of f. Equivalently, r is defined so that

$$p^r = \max_{f'(c)=0} |c|.$$

We will often use the fact that

$$a_i = (-1)^{d-i}\frac{d}{i}\sigma_{d-i}, \tag{2.2.5}$$

where σ_j denotes the j^{th} symmetric function of the critical points of f.

Whenever we count critical points, roots, or periodic points for f, we do so with multiplicity.

2.2. The Newton Polygon. The Newton polygon is a useful object in p-adic analysis that we will use frequently. Consider a polynomial

$$g(z) = \sum_{i=0}^{d} b_i z^i.$$

The Newton polygon for g is the lower convex hull of the set of points $\{(i, v(b_i))\}$. If any $b_i = 0$, that point is omitted. (One can think of that point as being at infinity.) The Newton polygon of g encodes information about the roots of g. In particular, it tells us that if the Newton polygon for g has a segment of horizontal length n and slope m, then g has n roots of absolute value p^m, counting with multiplicity. For proofs of these facts, see [**14**, Section IV.3].

One consequence of these facts is that for polynomials, or more generally, for power series over $\mathbb{C}_p$, a disk in $\mathbb{C}_p$ is mapped everywhere n-to-1 (counting with multiplicity) onto its image, which is also a disk. The following proposition, whose proof can be found in [**4**, Corollary 3.11], will prove useful.

PROPOSITION 2.5. *Let $f(z) = \sum_{i=0}^{d} b_i(z-a)^i \in \mathbb{C}_p[z]$ be a polynomial of degree d and let $D = \bar{D}(a, p^s)$ be a disk in $\mathbb{C}_p$. Then $f(D) = \bar{D}(f(a), p^r)$, where*

$$r = \max_{1\le i\le d}\{si - v(b_i)\}.$$

Moreover, $f : D \to f(D)$ is everywhere m-to-1 for some positive integer m, counting with multiplicity.

3. Proofs of Known Results

In this section, we summarize all previously known results about $\mathcal{M}_{d,p}$ of which we are aware for various combinations of d and p. We provide elementary proofs for the two main results stated here, as they do not appear in the literature in this form.

First, we treat the case that $d = p^k$, for some positive integer k. A proof of this result can be found in [**10**], but we present a proof here that is more straightforward and tailored to the normal form that we are using.

Proposition 2.6. *Let $d = p^k$ for some positive integer k, and let $f \in \mathcal{P}_{d,p}$. Then f is PCB if and only if $|c| \leq 1$ for all critical points c of f. Hence, $\mathcal{M}_{d,p}$ is simply a product of closed unit disks in $\mathbb{C}_p^{d-1}$.*

Proof. First, suppose that all of the critical points for f lie in the unit disk. Then, using (2.2.5), all of the coefficients of f are p-integral, and so

$$f(\bar{D}(0,1)) \subseteq \bar{D}(0,1).$$

Therefore, f is PCB.

Now, suppose f is PCB. By comparing the Newton polygons for f and f', one sees that the slope of the rightmost segment of the Newton polygon for f' is greater than the slope of the rightmost segment of the Newton polygon for f. This is because the leading coefficient of f' is p^k while the leading coefficient of f is 1, so the rightmost point on the Newton polygon for f' is k units above the rightmost point of the Newton polygon for f. Comparing all of the other coefficients, we see that any point $(i-1, v(ia_i))$ on the Newton polygon for f' is at most $k-1$ units above the corresponding point $(i, v(a_i))$ on the Newton polygon for f. Therefore, the largest critical point of f is strictly larger than the largest root of f. If this critical point c were outside the unit disk, then $|f(c)| = |c|^d > R$, and f would not be PCB. Therefore, f is PCB if and only if all critical points lie in the unit disk. $\square$

Now, we turn to the case that $p > d$. This is also a known result, but as it does not appear to be in the literature, we present an elementary proof. We will make use of the following lemma.

LEMMA 2.7. *Let $f \in \mathbb{C}_p[x]$ be a degree d polynomial, let $\bar{D}(a, s)$ be a disk in $\mathbb{C}_p$, and let m be an integer with $p \nmid m$. If f maps $\bar{D}(a, s)$ m-to-1 onto its image, then $\bar{D}(a, s)$ contains exactly $m - 1$ critical points of f, counted with multiplicity.*

PROOF. Without loss of generality, we may replace f with a conjugate so that $\bar{D}(a, s) = \bar{D}(0, 1)$. Let

$$f = \sum_{i=0}^{d} b_i z^i.$$

Then, counting with multiplicity, $f(z) - f(0)$ has m roots in the unit disk. Using properties of the Newton polygon for $f(z) - f(0)$, this implies that m is the largest positive integer such that

$$v(b_m) = \min_{1 \le i \le d} v(b_i).$$

Now consider the Newton polygon for f'. Since m is the largest integer such that $v(b_m)$ is minimal and $p \nmid m$, we see that m is also the largest integer such that $v(mb_m)$ is minimal among all $v(ib_i)$. Therefore, the Newton polygon for f' has exactly $m - 1$ non-positive slopes, which implies that there are $m - 1$ critical points, counted with multiplicity, in $\bar{D}(0, 1)$. $\square$

THEOREM 2.8. *Let $p > d$. Then $f \in \mathcal{P}_{d,p}$ is PCB if and only if $|c_i| \le 1$ for all critical points c_i of f. Hence, $\mathcal{M}_{d,p}$ is a product of closed unit disks in $\mathbb{C}_p^{d-1}$.*

PROOF. First, suppose that $|c_i| \le 1$ for all i. Then $|a_i| = |\frac{d}{i}\sigma_{d-i}| \le 1$ for all i, and so $f(\bar{D}(0, 1)) \subseteq \bar{D}(0, 1)$. Therefore, f is PCB.

Now let f be PCB and suppose, for contradiction, that $-v(c_1) = r > 0$. Let m be maximal such that $-v(c_m) = r$. (In other words, there are exactly m critical points with absolute value p^r.) First, we show that there are exactly m roots $z_1, z_2, \ldots, z_m$ of f such that $-v(z_i) = r$.

Since $p > d$, the Newton polygons for f and f' are the same, up to horizontal translation. Thus, the rightmost segment of the Newton polygon for f has the same slope and horizontal length as the rightmost segment of the Newton polygon for f', and therefore, f has exactly m roots z_i such that $-v(z_i) = r$.

Next, we use Lemma 2.7 to reach a contradiction. Consider $f^{-1}(\bar{D}(0, p^r))$. This is a union of at most d smaller disks $\bar{D}(z_i, p^{s_i})$, where the z_i are the roots of f. Note that, since f is PCB, each critical point must lie in one of these disks. By Proposition 2.5, we know that each $s_i \leq r/d$.

So, each of the m large critical points $c_1, \ldots, c_m$ must lie in the following set:

$$V = \bigcup_{i=1}^{m} \bar{D}(z_i, p^{s_i}).$$

The set V is a disjoint union of $n \leq m$ disks. Relabel the subscripts so that we can express V as follows:

$$V = \coprod_{i=1}^{n} \bar{D}(z_i, p^{s_i}).$$

Let $\bar{D}(z_i, p^{s_i})$ map d_i-to-1 onto $\bar{D}(0, r)$. Then, since V contains exactly m preimages of 0, counted with multiplicity, we have

$$\sum_{i=1}^{n} d_i = m.$$

Let b_i be the number of critical points in $\bar{D}(z_i, p^{s_i})$. Then, Lemma 2.7 tells us that $b_i = d_i - 1$, and so the number of critical points, counted with multiplicity, in V is

$$\sum_{i=1}^{n} b_i = m - n < m.$$

This is a contradiction. Thus, if f is PCB, all of its critical points lie in the unit disk. □

4. Basic Properties of $\mathcal{M}_{d,p}$ as a subset of $\mathbb{C}_p^{d-1}$

In this section, we parameterize a polynomial $f \in \mathcal{P}_{d,p}$ by its set of critical points $(c_1, c_2, \ldots, c_{d-1})$, as in Remark 2.3. In doing so, we treat $\mathcal{M}_{d,p}$ as a subset of $\mathbb{C}_p^{d-1}$. We have seen in Theorem 2.8 and Proposition 2.6 that if $p > d$ or if $d = p^k$, then

$\mathcal{M}_{d,p}$ is the unit polydisk in $\mathbb{C}_p^{d-1}$. In general, if $p < d$, the set $\mathcal{M}_{d,p}$ is more difficult to describe. It is no longer a polydisk, but it is a closed set which contains a polydisk centered at the origin.

PROPOSITION 2.9. *Let $f \in \mathcal{P}_{d,p}$ with $d \neq p^k$ for any positive integer k. Define s as follows:*

$$s = \min_{1 \leq i \leq d-1} \frac{v(d) - v(i)}{d - i}.$$

Then, $\mathcal{M}_{d,p}$ contains the polydisk $\bar{D}(0, p^s)^{d-1}$. In other words, if $-v(c) \leq s$ for all critical points c of f, then f is PCB.

Note that if $d = p^k$, this proposition does not apply, but we have already seen in Proposition 2.6 that $\mathcal{M}_{d,p}$ is equal to the unit polydisk in this case. We now proceed to prove Proposition 2.9:

PROOF. If $d \neq p^k$, we note that $s \leq 0$, as $v(d)$ is not strictly larger than $v(i)$ for all i between 1 and $d-1$. Recall that, if $f(z) = \sum_{i=1}^{d} a_i z^i \in \mathcal{P}_{d,p}$, then the coefficients a_i can be obtained from the critical points c_i as follows:

$$a_i = (-1)^{d-i} \frac{d}{i} \sigma_{d-i},$$

where σ_n denots the n^{th} symmetric function on the set of critical points. By this definition, if $-v(c_i) \leq s$ for all critical points c_i of f, we see that

$$v(a_i) \geq v(d) - v(i) - (d-i)s \geq v(d) - v(i) - (v(d) - v(i)) = 0.$$

Thus, all coefficients of f are p-integral, and f maps the unit disk to itself. Since $s \leq 0$, all critical points are in the unit disk, and therefore have bounded orbits. $\square$

If $p < d$, it is possible for $\mathcal{M}_{d,p}$ to contain points beyond the polydisk defined in Proposition 2.9. These sets can be complicated and difficult to describe, but they are always closed sets.

PROPOSITION 2.10. *Equate $f \in \mathcal{P}_{d,p}$ with the point $(c_1, \ldots, c_{d-1}) \in \mathbb{C}_p^{d-1}$, where the c_i are the critical points of f, as in Remark 2.3. Then, $\mathcal{M}_{d,p}$ is a closed subset of $\mathbb{C}_p^{d-1}$.*

Proof. We will prove that the set of non-PCB maps is an open subset of $\mathbb{C}_p^{d-1}$. Let $f \in \mathcal{P}_{d,p}$ with critical points $c, c_2, \ldots, c_{d-1}$, and suppose, without loss of generality, that the orbit of c is unbounded. Let R be as defined in (2.2.3). Let n be minimal such that $|f^n(c)| > p^R$. Now suppose $g \in \mathcal{P}_{d,p}$ with critical points $\gamma_i = c_i + \delta_i$, where $|\delta_i| \leq \epsilon$. We will show that, if ϵ is sufficiently small, the g-orbit of $\gamma = c + \delta_1$ is unbounded.

Suppose that $f(z) = \sum_{i=1}^{d} A_i(c, c_2, \ldots, c_{d-1})z^i$ and $g(z) = \sum_{i=1}^{d} A_i(\gamma, \gamma_2, \ldots, \gamma_{d-1})z^i$, where A_i is the symmetric polynomial defined in (2.2.5). We want to choose ϵ to satisfy the following three conditions:

- ϵ is small enough that $v(A_i(c, c_2, \ldots, c_{d-1})) = v(A_i(\gamma, \gamma_2, \ldots \gamma_{d-1}))$,
- ϵ is small enough that $|f^n(c) - f^n(\gamma)| \leq p^R$,
- ϵ is small enough that $|f^n(\gamma) - g^n(\gamma)| \leq p^R$.

If these three conditions are satisfied, then we claim that $\mathcal{O}_g(\gamma)$ is unbounded. The first condition guarantees that the radius of the smallest disk containing the filled Julia set $\mathcal{K}_g$ is the same as that for $\mathcal{K}_f$, namely, p^R. Thus, to show that the g-orbit of γ is unbounded, we need to show that $|g^m(\gamma)| > p^R$ for some m. The next two conditions combine to show that $|g^n(\gamma)| = |f^n(c)| > p^R$. We know that $|f^n(c)| > p^R$, so if we can show that $|f^n(c) - g^n(\gamma)| \leq p^R$, then it must be the case that $|g^n(\gamma)| = |f^n(c)|$. We show this by splitting the absolute value into two:

$$|f^n(c) - g^n(\gamma)| = |f^n(c) - f^n(\gamma) + f^n(\gamma) - g^n(\gamma)| \leq \max\{|f^n(c) - f^n(\gamma)|, |f^n(\gamma) - g^n(\gamma)|\}.$$

It remains to show that we can choose ϵ to satisfy the three stated conditions. All of these facts stem from the continuity of polynomials. For the first condition, we need

$$|A_i(c, c_2, \ldots, c_{d-1}) - A_i(\gamma, \gamma_2, \ldots \gamma_{d-1})| < |A_i(c, c_2, \ldots, c_{d-1})|$$

for $1 \leq i \leq d-1$. Since A_i is a continuous function for all i, we can choose ϵ so that these inequalities are satisfied. The second condition follows for the same reason — f^n is a continuous function, so we can choose ϵ so that if $|\delta_1| < \epsilon$, $|f^n(c) - f^n(c+\delta_1)| \leq p^R$.

For the final condition, we write f^n and g^n as follows:

$$f^n(z) = \sum_{i=1}^{d^n} B_i(c, c_2, \dots, c_{d-1}) z^i, \quad g^n(z) = \sum_{i=1}^{d^n} B_i(\gamma, \gamma_2, \dots, \gamma_{d-1}) z^i.$$

Since the B_i are symmetric polynomials, they are continuous, and we can choose ϵ so that

$$|B_i(c, c_2, \dots c_{d-1}) - B_i(\gamma, \gamma_2, \dots, \gamma_{d-1})| \leq \frac{p^R}{|\gamma|^i}$$

for all i. This guarantees that

$$\max\{\left|B_i(c, c_2, \dots c_{d-1})\gamma^i - B_i(\gamma, \gamma_2, \dots, \gamma_{d-1})\gamma^i\right|\} \leq p^R$$

and therefore, the third condition is satisfied. Thus, since its complement is open, we conclude that $\mathcal{M}_{d,p}$ is a closed subset of $\mathbb{C}_p^{d-1}$. $\square$

CHAPTER 3

The Critical Radius of $\mathcal{M}_{d,p}$

We have seen that, when $p \geq d$, the set $\mathcal{M}_{d,p}$ is easy to describe. A polynomial $f \in \mathcal{P}_{d,p}$ is in $\mathcal{M}_{d,p}$ if and only if all of its critical points have absolute value less than or equal to 1. The proof does not hold when $p < d$, which leads us to ask what $\mathcal{M}_{d,p}$ looks like in this situation. In this section, we explore whether it is possible to have PCB maps with critical points outside the unit disk, and if so, we seek to determine a sharp upper bound for the p-adic absolute value of such critical points.

We can think of $\mathcal{M}_{d,p}$ as a subset of $\mathbb{C}_p^{d-1}$ if we associate $f \in \mathcal{P}_{d,p}$ with the point $(c_1, c_2, \ldots c_{d-1})$, where the c_i are the roots of f', counted with multiplicity. (Since the set of critical points of f is an unordered set, each polynomial in $\mathcal{P}_{d,p}$ has up to $(d-1)!$ points in $\mathbb{C}_p^{d-1}$ associated to it.) By Proposition 2.9, the set $\mathcal{M}_{d,p}$ contains a polydisk $\bar{D}(0, p^s)^{d-1}$. In the previous chapter, we have seen that if $p > d$ or if $d = p^k$, this polydisk has radius 1, and is equal to $\mathcal{M}_{d,p}$. When $p < d$ and $d \neq p^k$, the polydisk described in Proposition 2.9 is contained in $\mathcal{M}_{d,p}$ but is not necessarily equal to all of $\mathcal{M}_{d,p}$.

We define the following quantity, which measures the critical radius of the p-adic Mandelbrot set in $\mathcal{P}_{d,p}$:

$$r(d,p) = \sup_{f \in \mathcal{M}_{d,p}} \max_{\substack{c \in \mathbb{C}_p \\ f'(c)=0}} \{-v_p(c)\}. \tag{3.0.6}$$

This is a (base p) logarithmic measure of the maximum possible absolute value for a critical point of a polynomial $f \in \mathcal{M}_{d,p}$. For small primes, in particular $p < d$, the set $\mathcal{M}_{d,p}$ may be complicated and have a fractal-like boundary. We use $r(d,p)$ as a way to measure the extent to which $\mathcal{M}_{d,p}$ extends beyond the polydisk described in Proposition 2.9. Just as the critical values for quadratic polynomials in the classical Mandelbrot set over $\mathbb{C}$ are contained in a disk of radius 2 [**2**, Theorem 9.10.1], the critical points for polynomials in $\mathcal{M}_{d,p}$ are contained in a disk of radius $p^{r(d,p)}$. Knowing $r(d,p)$ can be useful in searching for post-critically finite polynomials over a given number field, as is done for cubic polynomials over $\mathbb{Q}$ in [**13**], as it gives one a reasonably-sized search space. For $p > d$ or $d = p^k$, we have already seen in Theorem 2.8 and Proposition 2.6 that $r(d,p) = 0$. In the sections that follow, we determine $r(d,p)$ for other combinations of d and p.

In the following table, we summarize the results of this chapter:

	$r(d,p)$	Conditions on d and p	Notes
1	$r(d,p) = 0$	$p > d$	
2	$r(d,p) = 0$	$d = p^k$	$k \in \mathbb{Z},\ k > 0$
3	$r(d,p) = 0$	$d = 2p$	
4	$r(d,p) = 0$	$d = 3p, p \neq 2$	
5	$r(d,p) = \dfrac{p}{d-1}$	$\frac{1}{2}d < p < d$	
6	$r(d,p) \geq \dfrac{a(k-\ell)p^k}{d-1}$	$d = ap^k + b$	k maximal such that $p^k \leq d$, ℓ maximal such that $p^\ell \mid d$, $1 \leq a < p, 0 \leq b < p^k$

Line 1 was established in Theorem 2.8 and line 2 was shown to be true in Proposition 2.6. We will first prove line 6 in Section 1, followed by line 5 in Section 2 and lines 3 and 4 in Section 3.

1. Finding lower bounds for $r(d,p)$

In this section, we provide an example that gives the best known lower bounds for $r(d,p)$ when $p < d$. It gives a positive lower bound for $r(d,p)$ for all $p < d$ except for those for which d has the form $d = ap^k$, where $a < p$.

Proposition 3.1. *Suppose that $p < d$. Let k be the largest integer such that $p^k < d$ and let ℓ be the largest integer such that $p^\ell | d$. Write $d = ap^k + b$, where*

$1 \leq a < p$ *and* $0 \leq b < p^k$. *Then,*

$$r(d,p) \geq \frac{a(k-\ell)p^k}{d-1}.$$

In particular, if $d \neq ap^k$, *then* $r(d,p) > 0$.

PROOF. We first treat the $b = 0$ case. Let $d = ap^k$. Then since $k = \ell$ in this case, we must show that $r(d,p) \geq 0$. Consider the polynomial $f(z) = z^{d-1}(z-d)$. Then, the critical points of f are 0, which is fixed, and $d-1$. Since all of the coefficients of f are p-integral, the polynomial f will map the unit disk to itself. Therefore, since $d-1$ is a p-adic unit, the orbit of $d-1$ is bounded, and we have provided an example of a PCB polynomial with a critical point of absolute value $p^0 = 1$. Thus, $r(d,p) \geq 0$.

Now suppose $b \neq 0$. Let $\alpha \in \mathbb{C}_p$ satisfy the following equation:

$$\alpha^{d-1} = \frac{d^d}{(-ap^k)^{ap^k} b^b}. \tag{3.1.7}$$

Then we claim that the lower bound given in Proposition 3.1 is realized by the following map:

$$f(z) = z^b (z-\alpha)^{ap^k}. \tag{3.1.8}$$

This map has either two or three critical points: α, $\frac{b}{d}\alpha$, and possibly 0. (Zero is a critical point if $b \neq 1$.) Since we chose α to satisfy (3.1.7), it follows immediately that

$$f\left(\frac{b}{d}\alpha\right) = \alpha, f(\alpha) = 0, \text{ and } f(0) = 0.$$

Thus, f is post-critically finite, and therefore PCB, with

$$-v(\alpha) = \frac{a(k-\ell)p^k}{d-1}.$$

By definition, we have $r(d,p) \geq -v(c)$ for any critical point c for a map $f \in \mathcal{M}_{d,p}$, which gives the desired lower bound for $r(d,p)$. □

Since ℓ is necessarily less than or equal to k, we see that $r(d,p) \geq 0$ in all cases, and $r(d,p)$ is strictly greater than zero when $b \neq 0$. Proposition 3.1 does not give a positive lower bound for $r(d,p)$ when $d = ap^k$, where $1 \leq a < p$. We explore this situation further in Section 3.

Note that if f has just two distinct roots, as in (3.1.8), the example given in Proposition 3.1 maximizes $-v(\alpha)$:

PROPOSITION 3.2. *Let m and n be positive integers such that $m+n=d$. Let β be any value in $\mathbb{C}_p$ such that the polynomial*

$$g(z) = z^m(z-\beta)^n \text{ is in } \mathcal{M}_{d,p}.$$

Let $a, b, k,$ and ℓ be defined in terms of d as in Proposition 3.1. Then,

$$-v(\beta) \leq \frac{a(k-\ell)p^k}{d-1}.$$

PROOF. Suppose that $-v(\beta) > 0$. Since $f \in \mathcal{M}_{d,p}$, it is necessary that the critical point $\frac{m}{d}\beta$ have absolute value less than or equal to that of β. Thus, $v(m) \geq v(d)$. Since the other critical points, β and (potentially) 0, have finite orbits, it suffices to examine the orbit of $\frac{m}{d}\beta$. Looking at its first iterate, we see that

$$f\left(\frac{m}{d}\beta\right) = \frac{m^m(-n)^n}{d^d}\beta^d.$$

It is necessary that $-v(f(\frac{m}{d}\beta)) \leq -v(\beta)$, which implies that

$$|\beta^{d-1}| \leq \left|\frac{d^d}{m^m n^n}\right|.$$

This implies that

$$-v(\beta) \leq \frac{mv(m) + nv(n) - dv(d)}{d-1}. \tag{3.1.9}$$

First, we treat the $b=0$ case. In this case, $d = ap^k$, so $v(d) \geq \max\{v(m), v(n)\}$, and thus $-v(\beta) \leq 0$.

Now suppose $b \neq 0$. The quantity on the right hand side of (3.1.9) is maximized when either $mv(m)$ or $nv(n)$ is as large as possible, which occurs when one of m or n is equal to ap^k. In this case, we get that $-v(\beta) = \frac{a(k-\ell)p^k}{d-1}$, as desired. $\square$

REMARK 3.3. We note that Proposition 3.1 shows that there is no uniform upper bound for $r(d,p)$. For any positive integer k, if $d = p^k + 1$, then Proposition 3.1 shows us that $r(d,p) \geq k$.

2. Determining $r(d,p)$ for $\frac{1}{2}d < p < d$

The following theorem, which gives the exact value of $r(d,p)$ for certain values of $p < d$, is the main result of this section.

THEOREM 3.4. *For $\frac{1}{2}d < p < d$ we have*

$$r(d,p) = \frac{p}{d-1}.$$

To prove Theorem 3.4, we will rely on the following lemma:

LEMMA 3.5. *Let $f \in \mathcal{M}_{d,p}$, and let r and R be as defined by* (2.2.3) *and* (2.2.4). *If $r > 0$ and $\frac{1}{2}d < p < d$, then $R = r$.*

PROOF. First note that if f is post-critically bounded, then $R \geq r$ is necessary. Recall that

$$R = \max_{1 \leq i \leq d-1} \left\{ \frac{-v(a_i)}{d-i} \right\}. \tag{3.2.10}$$

Since $\frac{d}{2} < p < d$, there is exactly one coefficient a_i with $p|i$, namely, a_p. Thus,

$$|a_i| = |\sigma_{d-i}| \text{ for } i \neq p, \text{ and } |a_p| = p \cdot |\sigma_{d-p}|.$$

The only way that R could be strictly greater than r is if $-v(a_p)/(d-p)$ is uniquely maximal in the formula (3.2.10) for R, with

$$(d-p)r - 1 < -v(\sigma_{d-p}) \leq (d-p)r.$$

In this case, we see that $R = -v(a_p)/(d-p)$ could be as large as $r + \frac{1}{d-p}$. But if this is true, then $f(c_1)$ is dominated by a single term, namely $a_p c_1^p$ with

$$-v(f(c_1)) = -v(a_p c_1^p) = pr + (d-p)R > R,$$

contradicting the fact that f is PCB. Thus $R = r$. $\square$

Now, we are ready to prove Theorem 3.4.

Proof. Suppose that $\frac{1}{2}d < p < d$. Note that Proposition 3.1 shows that $r(d,p) \geq \frac{p}{d-1}$. It remains to show that $\frac{p}{d-1}$ is also an upper bound for $r(d,p)$. Suppose that there is a polynomial $f \in \mathcal{M}_{d,p}$ with a critical point c_1 such that $-v(c_1) > 0$. Recall that we order the critical points of f so that $|c_i| \geq |c_{i+1}|$ for all i. Let $r = -v(c_1)$. Our goal is to show that $r \leq \frac{p}{d-1}$. Lemma 3.5 implies that the critical orbits for f are all contained in $\bar{D}(0, p^r)$. Let m denote the number of critical points with absolute value p^r (with multiplicity), i.e.,

$$m = \max\{i : -v(c_i) = r\}.$$

We break the proof into two cases. The first case we consider is $m < p$.

We will refer to $\{c_1, c_2, \ldots c_m\}$ as the *large critical points*. Each large critical point must lie in one of the disks in the following set, where $f(z_i) = 0$ and $s_i \leq 0$:

$$f^{-1}(\bar{D}(0,p^r)) = \bigcup_{i=1}^{d} \bar{D}(z_i, p^{s_i}).$$

Proposition 3.6. *Let $f \in \mathcal{M}_{d,p}$ and define R as in* (2.2.3)*. Then $f^{-1}(\bar{D}(0,p^R))$ is a union of up to d smaller disks $\bar{D}(z_i, p^{s_i})$, where the z_i are the roots of f and $s_i \leq 0$.*

Proof. We must show that if $f(\bar{D}(z_i, p^{s_i})) = \bar{D}(0, p^R)$, then $s_i \leq 0$. Suppose that $a \in f^{-1}(\bar{D}(0,p^R))$. We will show that there is a root z_0 of f such that $|z_0 - a| \leq 1$. If $|a| \leq 1$, we are done, as 0 is a root of f. So we may proceed assuming that $|a| > 1$. Writing $f(z) = \prod_{i=1}^{d}(z - z_i)$, we see that

$$|f(a)| = \prod_{i=1}^{d} |a - z_i|.$$

At least one factor in this product has absolute value equal to p^R — if $|a| = p^R$, then $|a - 0|$ is such a factor, and if $|a| < p^R$, then there is a root z_1 of f such that $|z_1| = p^R$, and therefore $|a - z_1|$ is such a factor. Since $f(a) \in \bar{D}(0, p^R)$, we have that $|f(a)| \leq p^R$. Without loss of generality, suppose that $|a - z_1| = p^R$. Then,

$$\prod_{i=2}^{d} |a - z_i| \leq 1,$$

which implies that $|a - z_i| \leq 1$ for at least one i. $\square$

Proposition 3.6 allows us to assume that for every critical point c_i for f, there is a root z_i such that $|z_i - c_i| \leq 1$.

Define

$$k = d - p, \quad \text{so} \quad 1 \leq k \leq p - 1.$$

By Lemma 2.7, we must have more than m roots z_i such that $-v(z_i) = r$. Since the Newton polygons for f and f' can only differ at one place (namely, at the pth place), this is only possible if there are exactly k roots of f (and at most $k-1$ critical points) with absolute value p^r. This implies that $-v(a_p) = kr$. Let c_{m+1} be the largest critical point such that $-v(c_{m+1}) < r$ and let $t = -v(c_{m+1})$. Since $a_p = \frac{d}{p}\sigma_k$, we must have $-v(\sigma_k) = kr - 1$, which implies that $t \geq r - 1$. Looking at $f(c_{m+1})$, the sole largest term is $a_p c_{m+1}^p$, which implies that

$$-v(f(c_{m+1})) = kr + pt \geq dr - p.$$

If f is PCB, then $-v(f(c_{m+1})) \leq r$, which gives the inequality $dr - p \leq r$, and the desired bound follows.

Now suppose that the number of large critical points is $m \geq p$. Then, by analysis of the Newton polygons for f and f', either f has a root z_1 with $-v(z_1) > r$, or f has exactly m roots of absolute value p^r. The first possibility does not occur, because if $-v(z_1) > r$, then z_1 must be in the basin of infinity, by Lemma 3.5. This is a contradiction, since z_1 is preperiodic, as 0 is a fixed point for f. So, the largest root z_1 of f satisfies $-v(z_1) = r$ and the number of large critical points is equal to the number of roots of absolute value p^r. By Lemma 2.7, the only way for f to be PCB is if there is a disk $\bar{D}(c_1, p^s)$ mapping p-to-1 onto $\bar{D}(0, p^r)$ containing at least p of the large critical points, where $s \leq 0$ by Proposition 3.6. To proceed with the proof of Theorem 3.4, will again divide into two cases.

First, suppose that $-v(c_i - c_j) \leq 0$ for all critical points c_i, c_j. (Note that in this case, $m = d-1$.) Let $c_i = c_1 + \epsilon_i$, where $-v(\epsilon_i) \leq 0$. Then we have

$$f(c_1) = c_1^d - \frac{d}{d-1}\sigma_1 c_1^{d-1} + \cdots + (-1)^{d-1}\frac{d}{1}\sigma_{d-1}c_1.$$

We will use the fact that

$$\sigma_i = \binom{p+k-1}{i} c_1^i + \delta_i, \text{where } -v(\delta_i) < ir \tag{3.2.11}$$

to rewrite our expression for $f(c_1)$ as follows:

$$\begin{aligned} f(c_1) = c_1^d \Bigg(1 - \frac{d}{d-1}\binom{d-1}{1} + \frac{d}{d-2}\binom{d-1}{2} \\ - \cdots + (-1)^{d-1}\frac{d}{1}\binom{d-1}{d-1}\Bigg) + \epsilon. \end{aligned} \tag{3.2.12}$$

In (3.2.12), ϵ is a function of c_1 and the δ_i, as defined in (3.2.11). It remains to check that the coefficient of c_1^d is a p-adic unit and to determine the largest possible absolute value for ϵ. First, we look at the coefficient of c_1^d in (3.2.12). This coefficient can be rewritten as follows:

$$\begin{aligned} 1 - \frac{d}{d-1}\binom{d-1}{1} + \cdots + (-1)^{d-1}\frac{d}{1}\binom{d-1}{d-1} &= \sum_{i=0}^{d-1}(-1)^i\binom{d}{i} \\ &= (1-1)^d - (-1)^d = (-1)^{d+1}. \end{aligned} \tag{3.2.13}$$

Hence the coefficient of c_1^d is a p-adic unit, so the first term in $f(c_1)$ has absolute value p^{dr}. Since we must have $-v(f(c_1)) \leq r$ in order for f to be PCB, it is necessary that $-v(\epsilon) = dr$ as well. The only term that can possibly be that large is the one corresponding to $a_p c_1^p$. Let $\sigma_j(\epsilon_i)$ denote the j^{th} symmetric function on the ϵ_i. Then, the portion of $a_p c_1^p$ contributing to ϵ is:

$$(-1)^k \frac{d}{p}\left(\binom{d-2}{k-1}\sigma_1(\epsilon_i)c_1^{k-1} + \binom{d-3}{k-2}\sigma_2(\epsilon_i)c_1^{k-2} \right.$$

$$\left. + \cdots + \binom{p}{1}\sigma_{k-1}(\epsilon_i)c_1 + \sigma_k(\epsilon_i)\right) c_1^p.$$

Note that since $\binom{d-i-1}{k-i}$ is a multiple of p for all $i < k$, the last term is the only one that can possibly realize the absolute value p^{dr}. Looking at $x = (-1)^k \frac{d}{p}\sigma_k(\epsilon_i)c_1^p$, we see that

$$-v(x) \leq pr + 1.$$

Since $-v(x) = dr$, we have $dr \leq pr + 1$, which implies that $r \leq \frac{1}{k} = \frac{1}{d-p}$. This is less than or equal to $\frac{p}{d-1}$, and we obtain the desired result.

Now we treat the final case, in which there are at least p large critical points and there exist c_i, c_j such that $-v(c_i - c_j) \geq 0$. Without loss of generality, let $c_i = c_1$, where c_1 is in the disk $\bar{D}(c_1, p^s)$ which contains at least p critical points. Write $f(z) - z$ as $\prod(z - \alpha_i)$, where the α_i are the fixed points of f. Then substituting $z = c_1$ gives the equation

$$f(c_1) - c_1 = \prod_{i=1}^{d}(c_1 - \alpha_i).$$

Since the left hand side of this equation has absolute value at most p^r, the same must be true of the right hand side. Since 0 is a fixed point, we can let $\alpha_d = 0$. Then, since $-v(c_1 - 0) = r$, we are left with

$$-v\left(\prod_{i=1}^{d-1}(c_1 - \alpha_i)\right) \leq 0.$$

This implies that there is some α_i satisfying $-v(c_1 - \alpha_i) \leq 0$. Call this fixed point α.

Next, conjugate f by the affine linear transformation $\phi(z) = z + \alpha$. The new map $f^\phi = \phi^{-1} \circ f \circ \phi$ is of the desired form (monic with $f(0) = 0$) and is PCB because f is PCB. Note that f^ϕ has at least p, but no more than $d-2$, critical points in $\bar{D}(0, p^s)$, where $s \leq 0$. This implies that the number of large critical points for f^ϕ is strictly less than p. We have already dealt with this case, and so we know that all the critical points γ_i for f^ϕ satisfy $-v(\gamma_i) \leq \frac{p}{d-1}$. So, we can conclude that

$$-v(\gamma_i) = -v(c_i - \alpha) \leq \frac{p}{d-1}$$

for all critical points c_i of f. If any c_i satisfies $-v(c_i) \leq \frac{p}{d-1}$, we can conclude that $-v(\alpha) \leq \frac{p}{d-1}$ as well, and we reach the desired conclusion. However, if all critical points have the same absolute value as α, the result does not yet follow.

Suppose that $-v(c_i) = r$ for all i and that there exists c_i such that $c_i \notin \bar{D}(c_1, p^s)$. We have just shown that $-v(c_i - \alpha) \leq \frac{p}{d-1}$, which implies that $-v(c_i - c_j) \leq \frac{p}{d-1}$ for all i, j. Since not all critical points are in $\bar{D}(c_1, p^s)$, there is another disk in the set $V = f^{-1}(\bar{D}(0, p^r))$ containing $n \geq 1$ critical points and (by Lemma 2.7) $n+1$ roots. Thus, since all the large critical points and roots must be contained in V in accordance with Lemma 2.7, the number of roots of absolute value p^r outside $\bar{D}(c_1, p^s)$ must exceed the number of critical points outside $\bar{D}(c_1, p^s)$ by at least one, and therefore, the number of critical points inside $\bar{D}(c_1, p^s)$ must be greater than p, as there are exactly p roots in $\bar{D}(c_1, p^s)$.

Let $c_1, \ldots, c_{p+1} \in \bar{D}(c_1, p^s)$ and write all critical points as before with $c_i = c_1 + \epsilon_i$. Here, $\epsilon_i \leq s \leq 0$ for $2 \leq i \leq p+1$, and $\epsilon_j \leq \frac{p}{d-1}$ for $j > p+1$. Looking at equation (3.2.12), we examine the size of ϵ. Once again, in order to have the necessary cancellation with the leading term, it must be true that $-v(\epsilon) = dr$, which can only be achieved if $-v(\sigma_k(\epsilon_i)) = kr - 1$. But,

$$-v(\sigma_k(\epsilon_i)) \leq (k-2)\frac{p}{d-1}.$$

This gives the following inequality:

$$kr - 1 \leq (k-2)\frac{p}{p+k-1}.$$

This reduces to:

$$r \leq \frac{kp - 2p + d - 1}{k(d-1)}.$$

If we can show that

$$\frac{kp - 2p + d - 1}{k} \leq p,$$

then we are done, and this would give us the desired upper bound. This is true, because

$$\frac{kp - 2p + d - 1}{k} = p + \frac{d - 1 - 2p}{k},$$

and $d - 1 - 2p \leq 0$. Therefore, $r \leq \frac{p}{d-1}$, and the proof is complete. $\square$

3. Determining $r(d, p)$ when $p|d$

Recall that Proposition 3.1 shows that $r(d, p) > 0$ for most values of d and p with $p < d$. In this section, we begin to examine the (d, p) combinations for which Proposition 3.1 does not apply, namely, when $d = ap^k$, with $a < p$ and $k \geq 1$.

Proposition 3.7. *Let $f \in \mathcal{P}_{d,p}$ and suppose $d = 2p$. Then f is PCB if and only if $|c_i| \leq 1$ for all critical points c_i of f. In particular, $r(2p, p) = 0$ and $\mathcal{M}_{d,p}$ is equal to the unit polydisk in $\mathbb{C}_p^{d-1}$.*

Proof. Proposition 2.6 proves this statement if $p = 2$, so we proceed assuming $p \neq 2$. One direction is straightforward. If all the critical points are in the unit disk, then all the coefficients of f are p-integral, and f is PCB.

Now let f be PCB and suppose for contradiction that f has a critical point outside the unit disk, with $-v(c_1) = r > 0$. By comparing the rightmost segments of the Newton polygons for f and f', since the rightmost vertex for f' is one unit above the rightmost vertex for f, we get that, unless the rightmost segment of the Newton polygon for f has horizontal length p, the largest critical point for f is larger than its largest root. Since we can write $f(z) = \prod_{i=1}^{d}(z - z_i)$, where the z_i are the roots of f counted with multiplicity, in this situation $|c_1 - z_i| = |c_1|$ for all i, and therefore $|f(c_1)| = |c_1|^d$. More generally, $|f^n(c_1)| = |c_1|^{d^n}$, and thus f cannot be PCB. If the rightmost segment of the Newton polygon for f has horizontal length equal to p, it is possible for the largest root of f to have the same absolute value as the largest critical point. In this situation, there are exactly p roots z_i with $-v(z_i) = r$, and there are at least p critical points c_i with $-v(c_i) = r$. Suppose there are exactly k such critical points, counted with multiplicity, where $p \leq k \leq 2p - 1$. Now we use Lemma 2.7 to show that this is only possible if they are all contained in a disk centered at a root z_1 that maps p-to-1 onto $\bar{D}(0,p^r)$. Recall that if f is PCB, then each critical point lies in a disk $\bar{D}(z_i,p^{s_i})$ mapping via f onto $\bar{D}(0,p^r)$, where $f(z_i)=0$. Proposition 2.5 implies that $s_i \leq \frac{r}{2p}$ for all i. Since $s_i < r$, it is necessary that each of the k large critical points lie in one of these disks centered at a root z_i with $-v(z_i)=r$. Since there are at least p such critical points and only p such roots, Lemma 2.7 implies that this is only possible if there is one disk $\bar{D}(z_1,p^s)$ mapping via f onto $\bar{D}(0,p^r)$ containing all p such roots and all k of the largest critical points.

Writing $c_i = c_1+\epsilon_i$ for $2\leq i\leq k$, we calculate $f(c_1)$:

$$f(c_1)=c_1^{2p}\left(1-\frac{2p}{2p-1}\binom{k}{1}+\frac{2p}{2p-2}\binom{k}{2}-\cdots+2p\binom{k}{2p-1}\right)+\epsilon,$$

where $-v(\epsilon)<2pr$.

We will reach a contradiction if the coefficient of c_1^{2p} is a p-adic unit, because this would imply that $-v(f(c_1))=2pr>r=R$, and thus that f is not PCB. To reduce the coefficient of c_1^{2p} modulo p, we will use the following lemma:

Lemma 3.8. *Let k be a positive integer. Then*

$$\binom{k}{p}\equiv\left\lfloor\frac{k}{p}\right\rfloor \pmod{p}.$$

Proof. Write $k=ap+b$, where $0\leq b\leq p-1$ and a is a positive integer. Then $\left\lfloor\frac{k}{p}\right\rfloor = a$. Now, consider

$$\binom{k}{p}=\frac{k(k-1)\dots(k-(p-1))}{p!}.$$

Note that, since the terms in the numerator range from $(a-1)p+(b+1)$ to $ap+b$, there is exactly one factor from each congruence class modulo p in the numerator, including ap, which is congruent to $0 \pmod{p}$. After factoring a p from the numerator and denominator, $\binom{k}{p}$ can therefore be expressed as follows:

$$\binom{k}{p}=\frac{a((p-1)!+px)}{(p-1)!}, \text{ where } x \text{ is a positive integer.}$$

Thus,

$$\binom{k}{p} \equiv a \pmod{p},$$

as desired. $\square$

We now return to our proof of Proposition 3.7. Lemma 3.8 implies that the coefficient of c_1^{2p} is congruent to

$$1 - \frac{2p}{p}\binom{k}{p} \equiv 1 - 2 \equiv -1 \pmod{p}.$$

Thus, we reach the desired conclusion, that f is PCB if and only if all the critical points lie in the unit disk. $\square$

Proposition 3.7 states that $r(2p, p) = 0$. The same is true if $d = 3p$, with $p \geq 3$:

PROPOSITION 3.9. *Let $f \in \mathcal{P}_{d,p}$ and suppose $d = 3p$ with $p \geq 3$. Then f is PCB if and only if $|c_i| \leq 1$ for all critical points c_i of f.In particular, when $p \neq 2$, $r(3p, p) = 0$ and $\mathcal{M}_{d,p}$ is equal to the unit polydisk in $\mathbb{C}_p^{d-1}$.*

PROOF. This proof is similar to the proof given for Proposition 3.7, but requires a more elaborate argument similar to that given in the proof of Theorem 3.4. Once again, Proposition 2.6 proves the statement for $p = 3$, so we can assume $p \geq 5$. If $|c_i| \leq 1$ for all critical points c_i of f, it is apparent that f must be PCB, because all coefficients of f are p-integral, and therefore $f(\bar{D}(0,1)) \subset \bar{D}(0,1)$.

Now suppose f is PCB, and suppose further, for contradiction, that f has a critical point c_1 such that $-v(c_1) = r > 0$. By the same argument that appears in the proof of Proposition 3.7, examining the rightmost segment of the Newton polygon for f leads one to conclude that the number of roots of absolute value p^r must be exactly p or $2p$, counting with multiplicity.

First, suppose that the number of roots of f with absolute value p^r is p. Continuing to follow the argument from the proof of Proposition 3.7, we see that we must have at least p critical points of absolute value p^r, and they must all be contained in a p-to-1 disk $\bar{D}(z_1, p^s)$ that maps onto $\bar{D}(0, p^r)$, where z_1 is a root of f with $-v(z_1) = r$. Proposition 2.5 implies that $s \leq \frac{r}{3p}$. Suppose there are exactly k critical points of absolute value p^r, where $k \geq p$, and write $c_i = c_1 + \epsilon_i$ for $2 \leq i \leq k$, where $-v(\epsilon_i) \leq s$. We now examine $f(c_1)$.

$$f(c_1) = c_1^{3p}\left(1 - \frac{3p}{3p-1}\binom{k}{1} + \frac{3p}{3p-2}\binom{k}{2} - \cdots + 3p\binom{k}{3p-1}\right)$$
$$+ \epsilon, \text{where } -v(\epsilon) < 3pr.$$

Modulo p, the coefficient of c_1^{3p} is congruent to

$$1 - \frac{3p}{2p}\binom{k}{p} + \frac{3p}{p}\binom{k}{2p}.$$

First, if $p \leq k < 2p$, Lemma 3.8 implies that this expression is congruent to $1 - \frac{3}{2} \equiv -\frac{1}{2}$ (mod p), which is a p-adic unit. If $k \geq 2p$, then Lemma 3.8 implies that this expression is congruent to $1 - 3 + 3 \equiv 1 \pmod{p}$, which is also a p-adic unit. Therefore, $|f(c_1)| = |c_1|^{3p} = p^{3pr} > p^R$, contradicting the fact that f is PCB.

Now, suppose that the number of roots of f of absolute value p^r is $2p$, with $k \geq 2p$ critical points of absolute value p^r. As in the proof of Theorem 3.4, we split into two subcases, depending on whether all k large critical points are contained in an open disk of radius p^r or not. First suppose that $|c_i - c_j| < p^r$ for all $i, j \in \{1, 2, \ldots, k\}$. We proceed as in the previous case, by writing $c_i = c_1 + \epsilon_i$ for $2 \leq i \leq k$. Then $f(c_1) = Ac_1^{3p} + \epsilon$, where $-v(\epsilon) < 3pr$ and the coefficient A is given as follows:

$$A = 1 - \frac{3p}{3p-1}\binom{k}{1} + \frac{3p}{3p-2}\binom{k}{2} - \cdots + 3p\binom{k}{3p-1}$$
$$\equiv 1 - \frac{3}{2}\binom{k}{p} + 3\binom{k}{2p} \pmod{p}.$$

Using Lemma 3.8, we see that A is a p-adic unit, and therefore $-v(f(c_1)) = 3pr > R$. This contradicts the fact that f is PCB.

Finally, we consider the case where $k \geq 2p$ and there exist two critical points c_i and c_j such that $|c_i - c_j| = p^r$. Using Lemma 2.7, we see that we must either have a p-to-1 disk or a $2p$-to-1 disk containing at least p critical points, and p or $2p$ roots, respectively. Without loss of generality, suppose that $c_1, \ldots c_p$ are in that disk, and that c_k is not. Mimicking an argument that appears in the proof of Theorem 3.4, there exists a fixed point α for f such that $|c_1 - \alpha| \leq 1$. Conjugating f by $\phi(z) = z + \alpha$, we see that $f^\phi = \phi^{-1} \circ f \circ \phi \in \mathcal{M}_{d,p}$, with critical points $\{c_i - \alpha : 1 \leq i \leq 3p - 1\}$. Moreover, the largest critical point of f^ϕ has absolute value p^r, and there are strictly fewer than $2p$ such critical points. This is because we have moved the disk centered at α containing at least p critical points to a disk centered at 0. Thus, there are at least p critical points of absolute value strictly less than p^r, which leaves fewer than $2p$ critical points (but at least one, namely $c_k - \alpha$) of absolute value p^r. We have now reduced to the previous case, and thus we know that all critical points of f^ϕ must lie in the unit disk. This give us our desired contradiction, as $|c_k - \alpha| = p^r$.

All cases have been examined, and the proof is now complete. $\square$

The techiques of the previous two proofs are insufficient to generalize these propositions to the general case where $d = p^k q$, with $q < p$. Nevertheless, these proofs, along with the absence of counterexamples, lead us to make the following conjecture:

CONJECTURE 3.10. *Suppose $d = pq$, where $q < p$, and let $f \in \mathcal{P}_{d,p}$. Then $f \in \mathcal{M}_{d,p}$ if and only if $|c_i| \leq 1$ for all critical points c_i of f. So in particular, $r(pq, p) = 0$.*

CHAPTER 4

A One-Parameter Family of Cubic Polynomials over $\mathbb{C}_2$

We have mentioned that $\mathcal{M}_{d,p}$ can be a complicated and interesting set when $p < d$. In this section, we examine a particular slice of $\mathcal{M}_{3,2}$ to illustrate this assertion. This one-parameter family of cubic polynomials reveals that the boundary of this Mandelbrot set is sometimes complicated and fractal-like.

Consider the following one-parameter family of cubic polynomials, where the parameter $t \in \mathbb{C}_2$:

$$f_t(z) = z^3 - \frac{3}{2}tz^2. \tag{4.0.14}$$

Of the two critical points, one (zero) is a fixed point and the other is t. Note that $t = 1$ corresponds to a post-critically finite map, with the free critical point 1 mapping to the fixed point $-\frac{1}{2}$.

1. The Misiurewicz Point $t = 1$

For the complex Mandelbrot set $\mathcal{M}_{\mathbb{C}}$, a *Misiurewicz point* is a parameter c such that the corresponding polynomial $f_c(z) = z^2+c$ is post-critically finite, with a strictly preperiodic critical orbit. It can be shown that, for Misiurewicz points c, the periodic cycle that the critical orbit falls into is always repelling. Misiurewicz points in the complex Mandelbrot set have a number of interesting properties. They all appear on the boundary of the Mandelbrot set, they are dense in the boundary of $\mathcal{M}_{\mathbb{C}}$, and the boundary of $\mathcal{M}_{\mathbb{C}}$ is self-similar at Misiurewicz points. Moreover, the shape of the Mandelbrot set near a Misiurewicz point c reveals information about the Julia set of f_c. For proofs of these facts, see [**6, 20**]. We now define what it means to be a Misiurewicz point in our p-adic, degree d setting.

Definition 4.1. A polynomial f in the parameter space $\mathcal{P}_{d,p}$ is a *Misiurewicz point* if f is post-critically finite with at least one critical point c such that the orbit of c is strictly preperiodic with a repelling periodic cycle.

For the family defined in (4.0.14), $t = 1$ corresponds to a post-critically finite map. Since $f_t(1) = -\frac{1}{2}$ and $-\frac{1}{2}$ is a repelling fixed point for f_1 (with multiplier $\frac{9}{4}$), f_1 is a Misiurewicz point in $\mathcal{M}_{3,2}$. We now explore the PCB locus of the family f_t near the Misiurewicz point $t = 1$.

Proposition 4.2. *Consider the one-parameter family of cubic polynomials defined in* (4.0.14). *There is a sequence of disks $D_k = \bar{D}(1+2^{2k}, 2^{-(2k+1)})$ converging to $\{1\}$ such that for $t_k \in D_k$ with $k \geq 2$, the corresponding polynomial f_{t_k} is not PCB. There is another sequence of disks $D_m = \bar{D}(1 + 3 \cdot 2^{2m+1}, 2^{-(2m+3)})$, also converging to $\{1\}$, such that for $t_m \in D_m$ with $m \geq 2$, the corresponding polynomial f_{t_m} is PCB.*

Proposition 4.2 shows that $t = 1$ is on the boundary of the p-adic Mandelbrot set for this family of polynomials, in that it is arbitrarily close in the parameter space to parameters corresponding to both PCB and non-PCB maps. For $p > d$, such examples do not exist, as $\mathcal{M}_{d,p}$ is simply the unit polydisk in $\mathcal{P}_{d,p} \simeq \mathbb{C}_p^{d-1}$, which has empty boundary .

Proof of Proposition 4.2. First, consider $t_k \in \bar{D}(1 + 2^{2k}, 2^{-(2k+1)})$. As k approaches infinity, t_k approaches 1 in $\mathbb{C}_2$. We will now show that for $k \geq 2$, the orbit of the critical point t_k under iteration of f_{t_k} is unbounded. Let $t \in \mathbb{C}_2$ and $k \geq 2$ such that $t \equiv 1 + 2^{2k} \pmod{2^{2k+1}}$.

We begin by calculating the first few iterates of t under f_t:

$$f_t(t) = -\frac{1}{2}t^3 \equiv -\frac{1}{2} + 2^{2k-1} \pmod{2^{2k}},$$

$$f_t^2(t) = -\frac{1}{8}t^9 - \frac{3}{8}t^7 \equiv -\frac{1}{2} + 2^{2k-2} \pmod{2^{2k-1}},$$

$$f_t^3(t) = -\frac{1}{512}t^{15}(t^{12} + 9t^{10} + 27t^8 + 27t^6 + 12t^4 + 72t^2 + 108)$$
$$\equiv -\frac{1}{2} + 2^{2k-4} \pmod{2^{2k-3}} \text{ for } k \geq 3.$$

From here, each iterate moves further away from $-\frac{1}{2}$, so that for $2 \leq i \leq k$ we have

$$v\left(f_t^i(t) + \frac{1}{2}\right) = 2k - 2i + 2.$$

Thus, $v(f_t^k(t) + \frac{1}{2}) = 2$ and we can write $f_t^k(t) = -\frac{1}{2} + 4u$, where $|u| = 1$. We now calculate the next two points in the orbit of t:

$$f_t(-\frac{1}{2} + 4u) \equiv -\frac{1}{2} + u \pmod{2}.$$

Let $f_t(-\frac{1}{2} + 4u) = -\frac{1}{2} + u + 2v$, where $|v| \leq 1$. Then, the next iterate will have absolute value 4:

$$f_t(-\frac{1}{2} + u + 2v) \equiv \frac{u}{4} + \frac{v-1}{2} \pmod{1}.$$

Therefore, $|f_t^{k+2}(t)| = 4$. Note that in this case, $R = 1$. So, since $f_t^{k+2}(t) \notin \bar{D}(0, 2^R)$, the orbit of t is unbounded.

Now we turn our attention to the other sequence of disks, and our goal will be to show that f_t is PCB for all t in those disks. Let $t_m \in \bar{D}(1 + 3 \cdot 2^{2m+1}, 2^{-(2m+3)})$. For ease of notation, let $t = t_m$ for some $m \geq 2$. Then $t = 1 + (3 + 4u) \cdot 2^{2m+1}$ for some u with $|u| \leq 1$. Once again, we begin by calculating the first few iterates of t:

$$f_t(t) \equiv -\frac{1}{2} - 3(3 + 4u)2^{2m} \pmod{2^{4m+1}},$$

$$f_t^2(t) \equiv -\frac{1}{2} - 15(3 + 4u)2^{2m-1} \pmod{2^{4m-1}},$$

$$f_t^3(t) \equiv -\frac{1}{2} - 141(3 + 4u)2^{2m-3} \pmod{2^{4m-3}}.$$

In general, for $3 \leq i \leq m + 1$, we have

$$f_t^i(t) \equiv -\frac{1}{2} - c_i(3 + 4u)2^{2m-2i+3} \pmod{2^{2m-2i+7}}, \text{ where } c_i \equiv 1 \pmod{4}.$$

More specifically, $c_i = 9c_{i-1} + 3 \cdot 2^{2i-3}$. This shows that

$$f_t^{m+1}(t) \equiv -\frac{1}{2} - (3 + 4u) \cdot 2 \equiv \frac{3}{2} \pmod{8}.$$

Thus, we can write $f_t^{m+1}(t) = \frac{3}{2} + 8w$, for some w such that $|w| \le 1$. Calculating one more iterate, we see that

$$f_t^{m+2}(t) = f_t\left(\frac{3}{2} + 8w\right) \equiv 0 \pmod 2.$$

This ensures that the orbit of t is bounded, because f_t maps $\bar{D}(0, \frac{1}{2})$ to itself. While the orbit of t is repelled from the fixed point near $-\frac{1}{2}$, it eventually falls either into the basin of attraction of the superattracting fixed point 0, or it falls into the circle $\{z : |z| = \frac{1}{2}\}$, which f_t maps to itself. $\square$

Proposition 4.2 shows that this 2-adic Mandelbrot set, i.e., the set

$$\{t \in \mathbb{C}_2 : f_t \text{ is PCB}\},$$

has a complicated boundary. While this set is difficult to visualize over $\mathbb{C}_2$, we can begin to draw it if we restrict to $\mathbb{Q}_2$. For the remainder of this chapter, let

$$\mathcal{M} = \{t \in \mathbb{Q}_2 : f_t \text{ is PCB}\}.$$

For $|t| > 1$, note that $R = -v(\frac{3}{2}t) = -v(t) + 1$. We now calculate $|f_t(t)|$:

$$|f_t(t)| = \left|-\frac{1}{2}t^3\right| = 2\,|t|^3 > 2\,|t| = 2^R.$$

Therefore, the orbit of t is unbounded for all t outside the unit disk. Next note that f_t maps $\bar{D}(0,1)$ to itself for $|t| \le \frac{1}{2}$. So $t \in \mathcal{M}$ for $|t| \le \frac{1}{2}$. If we are interested in the boundary of $\mathcal{M}$, we therefore only have to consider t for which $|t| = 1$.

We can represent a neighborhood in $\mathbb{Q}_2$ as a binary tree, as every disk in $\mathbb{Q}_2$ is comprised of two disjoint disks. For example, the disk $\bar{D}(1, \frac{1}{2}) = \{t \in \mathbb{Q}_2 : |t| = 1\}$, which will be the root of our tree, is comprised of $\bar{D}(1, \frac{1}{4})$ and $\bar{D}(3, \frac{1}{4})$. Each disk in turn branches into two smaller disks. Traversing down the tree, one "zooms in" on a point in $\mathbb{Q}_2$. See Figure 1 for a depiction of the first few levels of this tree. We color a node black if the entire disk is in $\mathcal{M}$, we color a node white if the entire disk is outside $\mathcal{M}$, and we color a node gray if it contains some points in $\mathcal{M}$ and some points outside it. The number that labels each node denotes the center of the disk the node represents. As one moves down the left side of the tree, one zooms in on

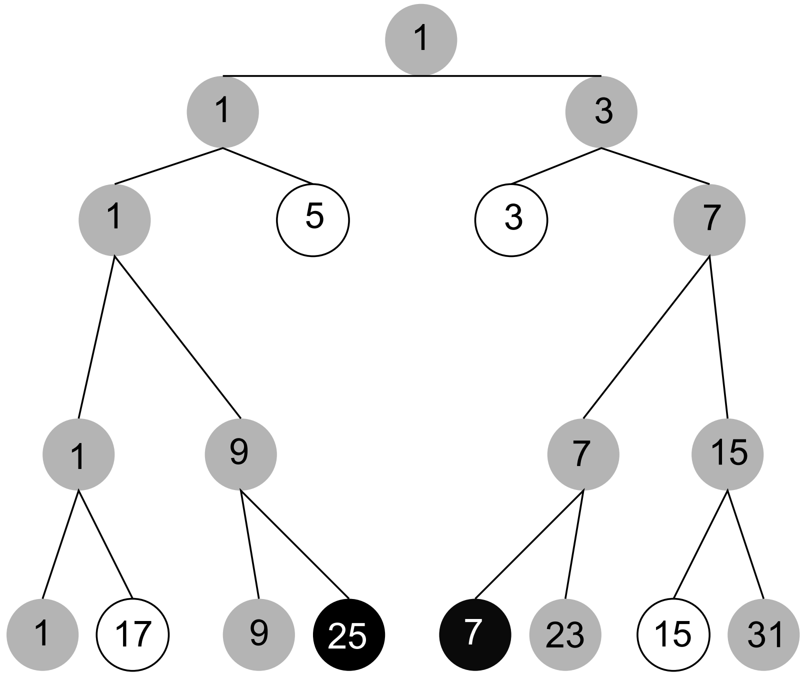

FIGURE 1. Critical orbit behavior for f_t with $|t| = 1$.

the post-critically finite boundary point $t = 1$. This tree is symmetrical, because $f_t^n(t) = -f_{-t}^n(-t)$, and so f_t is PCB if and only if f_{-t} is PCB. In Figure 2, we depict the disk $\bar{D}(1, \frac{1}{8})$ and the tree that emanates from it to give a sense of the complexity of $\mathcal{M}$. You can find the data supporting the coloring in Figure 2 in Appendix A.

Note that as one zooms in on $t = 1$, a self-similar pattern emerges, as illustrated in Figure 3 and as shown in Proposition 4.2. This is reminiscent of the classical Mandelbrot set over $\mathbb{C}$ and its fractal-like boundary. Beginning at $\bar{D}(1, 2^{-(2k+1)})$ for any $k > 1$, we see in Figure 3 that the pattern repeats every time we move two levels down the tree toward 1. The disk $\bar{D}(1+2^{2k+2}, 2^{-(2k+3)})$ corresponds to non-PCB maps, while the disks $\bar{D}(1 + 3 \cdot 2^{2k+1}, 2^{-(2k+3)})$ and $\bar{D}(1 + 5 \cdot 2^{2k+1}, 2^{-(2k+4)})$ correspond to PCB maps. The first two of these three assertions are shown in Proposition 4.2. We now prove the third assertion.

PROPOSITION 4.3. *If $t \in \mathbb{Q}_2$ is of the form $t \equiv 1+5\cdot 2^{2k+1} \pmod{2^{2k+4}}$ for $k \geq 2$, then f_t is PCB.*

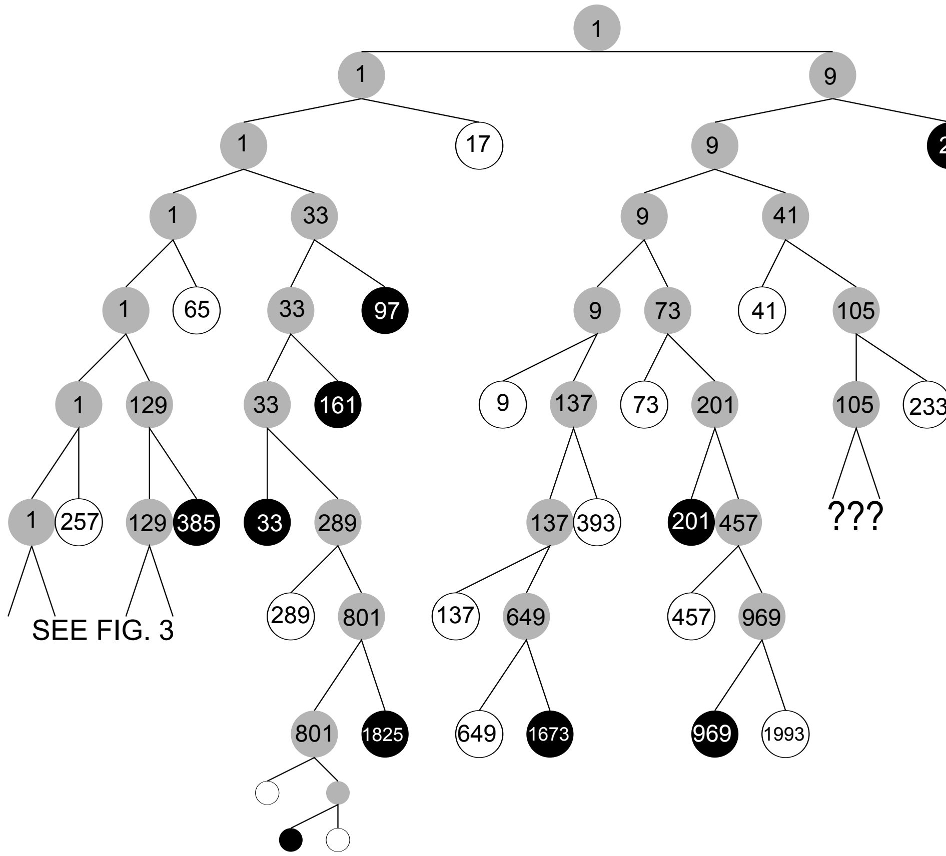

FIGURE 2. Critical orbit behavior for f_t with $t \in \bar{D}(1, \frac{1}{8})$.

PROOF. Let $t = 1 + 5 \cdot 2^{2k+1} + 2^{2k+4}u$, where $u \in \mathbb{Z}_2$. We calculate the first few iterates of t under f_t:

$$f_t(t) = -\frac{1}{2}t^3 \equiv -\frac{1}{2} - 3(5+8u) \cdot 2^{2k} \pmod{2^{4k}},$$

$$f_t^2(t) \equiv -\frac{1}{2} - 15(5+8u) \cdot 2^{2k-1} \pmod{2^{4k-1}},$$

$$f_t^3(t) \equiv -\frac{1}{2} - 141(5+8u) \cdot 2^{2k-3} \pmod{2^{4k-3}}.$$

If $k = 2$, the congruence given for $f_t^3(t)$ shows that $f_t^3(t) \equiv -\frac{5}{2} \pmod{16}$. Now suppose $k \geq 3$. By induction, we see that for $3 \leq n \leq k$,

$$f_t^n(t) \equiv -\frac{1}{2} - c_n(5+8u) \cdot 2^{2k-2n+3} \pmod{2^{2k-2n+7}}, \text{ where } c_n \equiv 5 \pmod{8}.$$

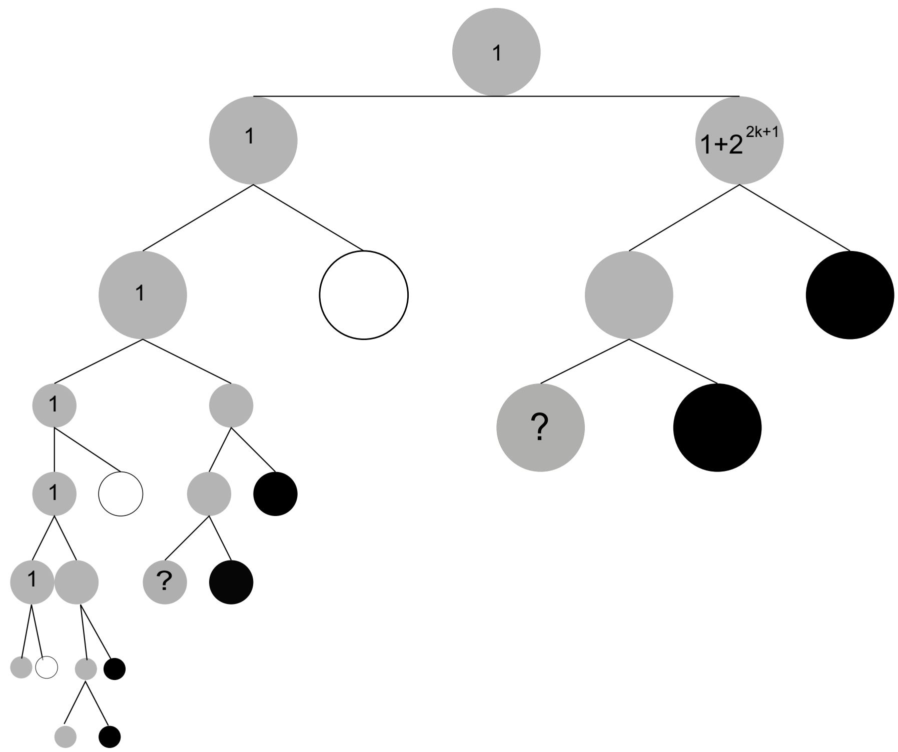

FIGURE 3. Critical orbit behavior for f_t as $t \to 1$. The top node corresponds to the disk $\bar{D}(1, 2^{-(2k+1)})$ for $k \geq 2$.

Letting $n = k$, we find that $f_t^k(t) \equiv -\frac{17}{2} \pmod{64}$. A quick calculation shows that $f(\bar{D}(-\frac{17}{2}, 2^{-6})) = \bar{D}(-\frac{5}{2}, 2^{-4})$, so for $k \geq 2$, $f_t^{k+1}(t) \in \bar{D}(-\frac{5}{2}, 2^{-4})$. The following lemma completes the proof:

LEMMA 4.4. *If $t \in \mathbb{Z}_2$ with $t \equiv 1 \pmod{32}$, then f_t maps the $\mathbb{Q}_2$ points of the disks $\bar{D}(-\frac{5}{2}, 2^{-4})$ and $\bar{D}(3, 2^{-2})$ to each other.*

PROOF. The proof is a straightforward calculation. Suppose $t = 1 + 32u$ for some u with $|u| \leq 1$. First, let $z \in \bar{D}(-\frac{5}{2}, 2^{-4}) \cap \mathbb{Q}_2$. Then we can write $z = -\frac{5}{2} + 16w$, where $w \in \mathbb{Z}_2$. We calculate $f_t(z)$:

$$f_t(z) = \left(\frac{25}{4} - 80w + 256w^2\right)(-4 + 16w - 48u) \equiv 3 \pmod{4}.$$

We have seen that $f_t(z) \in \bar{D}(3, 2^{-2}) \cap \mathbb{Q}_2$. Let $y = f_t(z) = 3 + 4v$, where $v \in \mathbb{Z}_2$. We now calculate $f_t(y)$:

$$f_t(y) = (9 + 24v + 16v^2)(\frac{3}{2} + 4v - 48u) \equiv \frac{27}{2} + 8v(9 + 3v) \pmod{16}.$$

Note that since $v \in \mathbb{Z}_2$, either $2 \mid v$ or $2 \mid (9 + 3v)$, so $8v(9 + 3v) \equiv 0 \pmod{16}$. We conclude that $f_t(y) \in \bar{D}(-\frac{5}{2}, 2^{-4})$. Thus, f_t maps the $\mathbb{Q}_2$ points of the disks $\bar{D}(-\frac{5}{2}, 2^{-4})$ and $\bar{D}(3, 2^{-2})$ to each other. □

Returning to the proof of Proposition 4.3, we see that after $k+1$ iterates, the orbit of t falls into a 2-cycle of disks in $\mathbb{Q}_2$, namely, $\bar{D}(-\frac{5}{2}, 2^{-4}) \to \bar{D}(3, 2^{-2}) \to \bar{D}(-\frac{5}{2}, 2^{-4})$. Therefore, the orbit of t is bounded, and f_t is PCB. □

With Proposition 4.2 and Proposition 4.3, we have now justified the black and white colorings in Figure 3. It is important to note that the patterns established in Proposition 4.2 hold over $\mathbb{C}_2$, while the pattern established in Proposition 4.3 only holds for parameters in $\mathbb{Q}_2$. Over $\mathbb{C}_2$, if $t \equiv 1 \pmod{32}$, f_t^2 maps the disk $\bar{D}(-\frac{5}{2}, 2^{-4})$ to $\bar{D}(-\frac{5}{2}, 2^{-3})$. There is enough cancellation over $\mathbb{Q}_2$ that $\bar{D}(-\frac{5}{2}, 2^{-4})$ maps to itself via f_t^2, but over $\mathbb{C}_2$, the map f_t^2 is expanding, and boundedness is not guaranteed, or even expected. We have established that $t = 1$ is a boundary point in $\mathcal{M}$ and that the boundary of $\mathcal{M}$ has a self-similar pattern as one zooms in on $t = 1$. This is reminiscent of Misiurewicz points for the complex Mandelbrot set. We now examine another similarity between this Misiurewicz point $t = 1$ in $\mathcal{M}$ and those which can be found in the complex Mandelbrot set $\mathcal{M}_{\mathbb{C}}$.

By a theorem of Tan Lei [**20**], the boundary of the complex Mandelbrot set $\mathcal{M}_{\mathbb{C}}$ near a Misiurewicz point c resembles the Julia set of the polynomial $f(z) = z^2 + c$ near the critical value c. The Julia set of a polynomial defined over $\mathbb{C}_2$ is the boundary of the set of points with bounded orbits. For a proof of this fact, see [**4**, Proposition 4.37]. In the interest of comparing the patterns we have found in $\mathcal{M}$ near the point $t = 1$ to the Julia set of f_1, we now examine which points near the critical value $z = -\frac{1}{2}$ have bounded orbits under iteration of $f_1(z) = z^3 - \frac{3}{2}z^2$. Figure 4 shows the pattern that exists for the disk $\bar{D}(-\frac{1}{2}, \frac{1}{2})$. Much as in Figures 1 through 3, we color

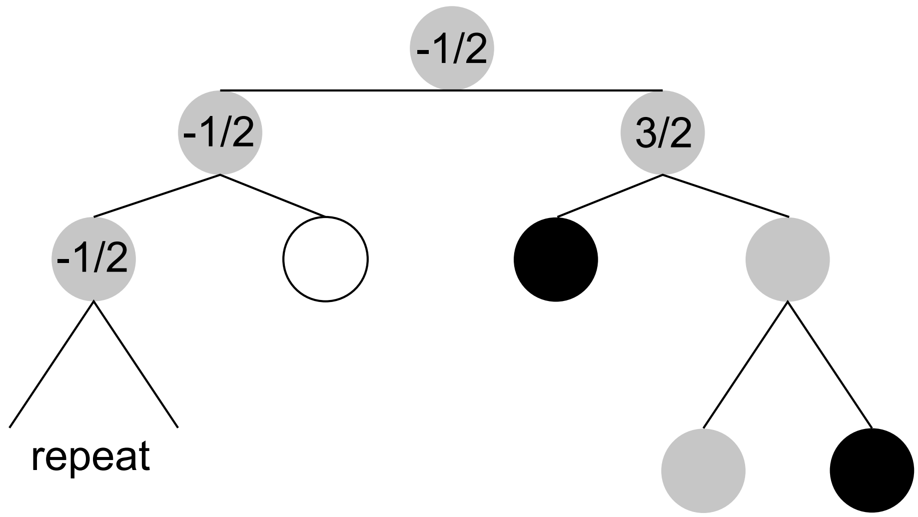

FIGURE 4. Boundedness of orbits under iteration of $f_1(z) = z^3 - \frac{3}{2}z^2$

a node black if every point in the corresponding disk has a bounded orbit, we color a node white if every point in the corresponding disk has an unbounded orbit, and we color a node gray if there are some points in the disk with bounded orbits and others with unbounded orbits. Points in the Julia set will belong to gray disks at every level. In the following proposition, we prove the boundedness pattern that is illustrated in Figure 4.

PROPOSITION 4.5. *Let $f_1(z) = z^3 - \frac{3}{2}z^2$. Then, for any positive integer k, all points contained in the disk $\bar{D}(-\frac{1}{2} + 2^{2k+2}, 2^{-(2k+3)})$ have unbounded orbits under iteration of f. Moreover, all points belonging to the disks $\bar{D}(-\frac{1}{2} + 2^{2k+1}, 2^{-(2k+3)})$ and $\bar{D}(-\frac{1}{2} - 2^{2k+1}, 2^{-(2k+4)}) \cap \mathbb{Q}_2$ have bounded orbits.*

PROOF. We begin with $z \in \bar{D}(-\frac{1}{2} + 2^{2k+2}, 2^{-(2k+3)})$. By induction, we see that for $1 \leq n \leq k+1$, $f_1^n(z)$ satisfies

$$f_1^n(z) \equiv -\frac{1}{2} + 2^{2k+2-2n} \pmod{2^{2k+3-2n}}.$$

Then, we can write $f_1^{k+1}(z) = \frac{1}{2} + 2w$, where $|w| \leq 1$. Calculating one more iterate, we see that

$$|f_1^{k+2}(z)| = \left| f_1\left(\frac{1}{2} + 2w\right)\right| = \left|\left(\frac{1}{4} + 2w + 4w^2\right)(-1 + 2w)\right| = 4.$$

Since $R = 1$, and therefore all points of absolute value greater than 2^1 have unbounded orbits, we see that the orbit of z is unbounded.

Now we turn our attention to the two bounded cases. First suppose that $z \in \bar{D}(-\frac{1}{2} + 2^{2k+1}, 2^{-(2k+3)})$. Again, we proceed by induction to see that, for $1 \leq n \leq k$, $f_1^n(z)$ satisfies

$$f_1^n(z) = -\frac{1}{2} + 2^{2k+1-2n} \pmod{2^{2k+3-2n}}.$$

Then, we can write $f_1^k(z) = \frac{3}{2} + 8w$, where $|w| \leq 1$. Calculating one more iterate, we see that

$$|f_1^n(z)| = \left| f_1\left(\frac{3}{2} + 8w\right)\right| = \left| \left(\frac{9}{4} + 24w + 64w^2\right)(8w)\right| \leq \frac{1}{2}.$$

Since f_1 maps $\bar{D}(0, \frac{1}{2})$ to itself, we see that the orbit of z is bounded.

Finally, we consider $z \in \bar{D}(-\frac{1}{2} - 2^{2k+1}, 2^{-(2k+4)}) \cap \mathbb{Q}_2$. Note that we are restricting to $\mathbb{Q}_2$ in this case, as we did in Proposition 4.3. First, we see by induction that for $1 \leq n \leq k$, $f_1^n(z)$ satisfies

$$f_1^n(z) = -\frac{1}{2} - 2^{2k+1-2n} \pmod{2^{2k+4-2n}}.$$

So, $f_1^k(z) \in \bar{D}(-\frac{5}{2}, 2^{-4})$. Now, we apply Lemma 4.4 to complete the proof. □

Combining the results in Propositions 4.2, 4.3, and 4.5, we see that the Julia set near the critical value $-\frac{1}{2}$ exhibits the same self-similar pattern as the Mandelbrot set $\mathcal{M}$ near the corresponding Misiurewicz point $t = 1$. This example shows that there can be a relationship between Mandelbrot sets and Julia sets in the p-adic setting, as there is in the complex quadratic case. This example is sufficiently generic that we expect to see these similarities more generally for Misiurewicz points in parameter spaces defined over p-adic fields, but this question is beyond the scope of this thesis.

2. An exploration of the intricate segments of the boundary

We have seen that in a disk $D = \bar{D}(1, 2^{-(2m+1)}) \cap \mathbb{Q}_2$ with $m \geq 2$, at least one half of the parameters $t \in D$ belong to $\mathcal{M}$ and at least one third of parameters $t \in D$ correspond to non-PCB polynomials. These facts follow from summing the geometric series that result from the patterns in Figure 3. The remaining one sixth of D consists of disks of the form $\bar{D}(1+2^{2k+1}, 2^{-(2k+4)})$, where $k \geq m$. These are the disks labeled with question marks in Figure 3. In this section, we explore these regions, where the boundary of $\mathcal{M}$ appears to be most intricate.

We provide a large collection of data in Table 3 in Appendix A, listing whether or not f_t is PCB for many t values that can be found in these disks for k values ranging from 2 to 11. For each k in this range, we say whether or not f_t is PCB for $t = 1 + 2^{2k+1} + 2^{2k+4}i$, where i ranges from 0 to 31. This gives us a parameter t from each of the 32 disks of radius $2^{-(2k+9)}$ centered at $\mathbb{Q}_2$-points and emanating from the disk $\bar{D}(1+2^{2k+1}, 2^{-(2k+4)})$. In Table 1, we list the results for each of the ten different k values side-by-side, to show that there is no apparent pattern from level to level in these regions. Each column corresponds to the k value listed at the top. Each row corresponds to the i value listed on the left. Note that we label all parameters that correspond to non-PCB polynomials with a 0, and we label all parameters that correspond to PCB polynomials with a 1.

If there were a pattern of PCB behavior in these disks, we would see entire rows of all 0s or all 1s. The fact that we do not see this pattern indicates that the self similarity of the boundary of $\mathcal{M}$ near $t = 1$ fails on these disks. This is to be expected, as something similar is true with the boundary of the complex Mandelbrot set near Misiurewicz points. The boundary of $\mathcal{M}_{\mathbb{C}}$ near a Misiurewicz point is quasi-self-similar, but not exactly self-similar. The same appears to be true in this case.

In Table 3, which can be found in Appendix A, there is an n value listed with each t value. If f_t is not PCB, n is minimal such that $|f_t^n(t)| > 2$, guaranteeing that f_t is not PCB. If f_t is PCB, n is minimal such that $f_t^n(t) \in \bar{D}(0, \frac{1}{2}) \cup \bar{D}(3, \frac{1}{4})$, guaranteeing that f_t is PCB. As the radius of a disk increases by a factor of at most 4 with each iterate of f_t, we lose at most 2 degrees of precision every time we iterate f_t. Thus, the n value n_0 associated with a t value t_0 tells us that if $t \in \bar{D}(t_0, 2^{-2n_0-2})$, then $f_t^{n_0}(t) \in \bar{D}(f_{t_0}^{n_0}(t_0), 2^{-2})$, which implies that $t \in \mathcal{M}$ if and only if $t_0 \in \mathcal{M}$. For example, the first line of Table 3 tells us that when $t = 33$, $n = 6$. This means that it takes 6 iterates for the orbit of t to enter $\bar{D}(0, \frac{1}{2}) \cup \bar{D}(3, \frac{1}{4})$. We can also deduce that if $t \equiv 33 \pmod{2^{14}}$, then f_t is also PCB. These n values give a conservative estimate — we see from Figure 2 that in fact, it is only necessary that $t \equiv 33 \pmod{2^9}$ for f_t to be PCB, but this requires a more careful calculation.

The data collected in Table 3 and summarized in Table 1 show that the boundary of $\mathcal{M}$ is quite intricate for $t \in \bar{D}(1+2^{2k+1}, 2^{-(2k+4)})$ and exhibits no apparent pattern from level to level, quite unlike what we see in the other regions of $\mathbb{Z}_2$.

Table 1. PCB behavior for $t = 1 + 2^{2k+1} + 2^{2k+4}i$

i	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
0	1	0	1	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1	1	1
2	1	1	0	1	1	0	0	1	1	1
3	0	0	0	1	1	1	0	0	0	1
4	1	1	0	1	1	1	1	1	1	1
5	0	1	1	0	0	1	1	1	0	1
6	1	1	1	0	1	0	0	1	1	1
7	1	1	1	0	0	1	1	1	0	0
8	1	0	1	1	1	1	1	1	1	0
9	0	1	1	1	1	1	0	1	1	1
10	1	1	1	0	1	0	1	0	1	0
11	1	0	0	1	1	1	0	1	0	1
12	1	1	1	0	1	0	1	0	0	1
13	0	1	1	1	1	1	1	0	1	0
14	1	1	1	0	0	1	1	1	1	1
15	1	1	1	1	1	1	0	1	1	0
16	1	0	0	1	0	1	0	1	1	1
17	0	0	1	1	1	1	1	0	0	0
18	1	1	1	1	1	1	1	1	1	1
19	0	1	0	1	0	1	1	1	1	1
20	1	1	1	0	1	1	1	1	0	1
21	0	1	0	0	0	0	1	1	1	1
22	1	1	1	1	1	0	0	0	1	0
23	1	1	1	1	1	1	1	1	0	1
24	1	0	0	1	1	1	1	1	1	1
25	0	1	0	1	0	0	1	1	1	1
26	1	1	1	1	0	0	1	1	1	0
27	0	1	1	0	1	1	1	1	1	1
28	1	1	1	0	1	1	0	1	0	1
29	0	1	1	1	0	1	1	1	1	1
30	1	1	0	0	1	1	0	1	0	1
31	1	0	1	1	1	1	1	0	1	1

APPENDIX A

Data for cubic family in Chapter 4

1. Data supporting Figure 2

All calculations were done in SAGE [**19**]. Table 2 lists the disks in $\mathbb{Q}_2$ that are depicted in Figure 2. The first column specifies the disk. The next two columns list a positive integer n and region containing $f_t^n(t)$ for all t in the specified disk. The final two columns list whether or not f_t is PCB with a reference if necessary. If f_t is not PCB, we will label it *post-critically unbounded*, or PCU. We remind the reader that if $f_t^n(t) \in \mathbb{Q}_2 \setminus \bar{D}(0,2)$ for any n, then f_t is not PCB. If $f_t^n(t) \in \bar{D}(0,\frac{1}{2})$ for any n, then f_t is PCB. For the disks listed in the first three rows, we assume $k \geq 2$.

Table 2: Data supporting Figure 2

Disk	n	Location of $f_t^n(t)$	PCB/PCU	Reference
$\bar{D}(1+2^{2k}, 2^{-(2k+1)})$	$k+2$	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	Proposition 4.2
$\bar{D}(1+3\cdot 2^{2k+1}, 2^{-(2k+3)})$	$k+2$	$\bar{D}(0,\frac{1}{2})$	PCB	Proposition 4.2
$\bar{D}(1+5\cdot 2^{2k+1}, 2^{-(2k+4)})$	$k+1$	$\bar{D}(-\frac{5}{2}, 2^{-4})$	PCB	Proposition 4.3
$\bar{D}(25, 2^{-5})$	3	$\bar{D}(0,\frac{1}{2})$	PCB	
$\bar{D}(41, 2^{-7})$	7	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(9, 2^{-8})$	9	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(73, 2^{-8})$	9	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(233, 2^{-8})$	12	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(33, 2^{-9})$	6	$\bar{D}(-\frac{5}{2}, 2^{-4})$	PCB	Proposition 4.3
$\bar{D}(393, 2^{-9})$	12	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(201, 2^{-9})$	4	$\bar{D}(\frac{35}{2}, 2^{-6})$	PCB	3-cycle A
$\bar{D}(289, 2^{-10})$	9	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	

Table 2: Data supporting Figure 2

Disk	n	Location of $f_t^n(t)$	PCB/PCU	Reference
$\bar{D}(137, 2^{-10})$	14	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(457, 2^{-10})$	14	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(1825, 2^{-11})$	9	$\bar{D}(-\frac{5}{2}, 2^{-4})$	PCB	Proposition 4.3
$\bar{D}(649, 2^{-11})$	19	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(1673, 2^{-11})$	5	$\bar{D}(9, 2^{-6})$	PCB	5-cycle $B1$
$\bar{D}(969, 2^{-11})$	5	$\bar{D}(9, 2^{-6})$	PCB	5-cycle $B2$
$\bar{D}(1993, 2^{-11})$	19	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(801, 2^{-12})$	13	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	
$\bar{D}(2849, 2^{-13})$	5	$\bar{D}(\frac{4767}{2}, 2^{-13})$	PCB	4-cycle C
$\bar{D}(6945, 2^{-13})$	17	$\mathbb{Q}_2 \setminus \bar{D}(0,2)$	PCU	

Cycles Referenced in Table 2

3-cycle A:

$$\bar{D}\left(\frac{35}{2}, 2^{-6}\right) \to \bar{D}(1, 2^{-4}) \to \bar{D}\left(-\frac{89}{2}, 2^{-8}\right) \to \bar{D}\left(\frac{35}{2}, 2^{-6}\right)$$

5-cycle $B1$:

$$\bar{D}(9, 2^{-6}) \to \bar{D}\left(\frac{423}{2}, 2^{-11}\right) \to \bar{D}\left(\frac{83}{2}, 2^{-9}\right) \to \bar{D}(-17, 2^{-7}) \to \bar{D}\left(-\frac{93}{2}, 2^{-8}\right) \to \bar{D}(9, 2^{-6})$$

5-cycle $B2$:

$$\bar{D}(9, 2^{-6}) \to \bar{D}\left(\frac{-537}{2}, 2^{-11}\right) \to \bar{D}\left(\frac{499}{2}, 2^{-9}\right) \to \bar{D}(75, 2^{-7}) \to \bar{D}\left(-\frac{157}{2}2^{-8}\right) \to \bar{D}(9, 2^{-6})$$

4-cycle C:

$$\bar{D}\left(\frac{4767}{2}, 2^{-13}\right) \to \bar{D}\left(\frac{2447}{2}, 2^{-11}\right) \to \bar{D}\left(\frac{683}{2}, 2^{-9}\right)$$
$$\to \bar{D}(33, 2^{-7}) \to \bar{D}\left(\frac{4767}{2}, 2^{-13}\right)$$

2. Data for Disks Labeled with Question Marks in Figure 3

In Chapter 4, we noted that the PCB locus of f_t as defined in (4.0.14) was complicated when $t \in \bar{D}(1 + 2^{2k+1}, 2^{-(2k+4)})$, for $k \geq 2$. In Table 3, we consider some values of t in these disks, and calculate using Sage [**19**] whether or not f_t is PCB. We consider k values from 2 to 11, and for each k value, we include one representative from each of the 32 disks of radius $2^{-(2k+9)}$, centered at $\mathbb{Q}_2$-points, emanating from $\bar{D}(1 + 2^{2k+1}, 2^{-(2k+4)})$. Each set of 32 t values for a given k are grouped together. For example, the first 32 entries correspond to $t = 33 + 2^8 i$, where i ranges from 0 to 31, and the next set of 32 entries correspond to $t = 129 + 2^{10} i$, for $0 \leq i \leq 31$. The second column in Table 3 indicates whether or not f_t is PCB for the stated value of t. We put a 0 in this column if f_t is PCU, and we put a 1 in this column if f_t is PCB. Recall that f_t is PCU if the orbit of t ever exits the disk $\bar{D}(0, 2)$, and f_t is PCB if the orbit of t ever enters the disk $\bar{D}(0, \frac{1}{2})$. Recall further that, by Lemma 4.4, f_t is PCB if the orbit of t ever enters the disk $\bar{D}(3, \frac{1}{4})$. If the orbit of t is not clearly PCB or PCU after 10,000 iterates — namely, if it does not fall into one of the three regions mentioned in the previous two sentences — it is labeled with a 2. This only occurs once in our data set, for $t = 2849$. Handling this case separately, we see that f_t is PCB. See Table 2 for a reference for this fact. Finally, the third column of Table 3, labeled n, indicates how many iterates it takes for the orbit of t to enter one of the 3 key regions mentioned above. In the one case where our code returns a 2, this n value is listed as 9999.

TABLE 3. Data for Disks of the form $\bar{D}(1+2^{2k+1}, 2^{-(2k+4)})$

t	PCB?	n
33	1	6
289	0	8
545	1	6
801	0	12
1057	1	6
1313	0	8
1569	1	6
1825	1	7
2081	1	6
2337	0	8
2593	1	6
2849	2	9999
3105	1	6
3361	0	8
3617	1	6
3873	1	7
4129	1	6
4385	0	8
4641	1	6
4897	0	12
5153	1	6
5409	0	8
5665	1	6
5921	1	7
6177	1	6
6433	0	8
6689	1	6
6945	0	16
7201	1	6
7457	0	8
7713	1	6
7969	1	7

t	PCB?	n
129	0	10
1153	1	12
2177	1	8
3201	0	16
4225	1	9
5249	1	14
6273	1	8
7297	1	11
8321	0	10
9345	1	11
10369	1	8
11393	0	16
12417	1	13
13441	1	14
14465	1	8
15489	1	20
16513	0	10
17537	0	13
18561	1	8
19585	1	23
20609	1	9
21633	1	21
22657	1	8
23681	1	11
24705	0	10
25729	1	11
26753	1	8
27777	1	17
28801	1	14
29825	1	18
30849	1	8
31873	0	13

t	PCB?	n
513	1	13
4609	1	14
8705	0	21
12801	0	12
16897	0	61
20993	1	19
25089	1	17
29185	1	10
33281	1	33
37377	1	23
41473	1	41
45569	0	18
49665	1	22
53761	1	21
57857	1	13
61953	1	10
66049	0	46
70145	1	32
74241	1	16
78337	0	12
82433	1	52
86529	0	16
90625	1	23
94721	1	10
98817	0	32
102913	0	22
107009	1	17
111105	1	11
115201	1	30
119297	1	20
123393	0	15
127489	1	10

Table 3. Data for Disks of the form $\bar{D}(1+2^{2k+1}, 2^{-(2k+4)})$

t	PCB?	n	t	PCB?	n	t	PCB?	n
2049	1	23	8193	1	36	32769	1	23
18433	1	32	73729	1	14	294913	1	41
34817	1	12	139265	1	34	557057	0	247
51201	1	64	204801	1	34	819201	1	59
67585	1	38	270337	1	35	1081345	1	211
83969	0	59	335873	0	87	1343489	1	30
100353	0	29	401409	1	39	1605633	0	68
116737	0	22	466945	0	109	1867777	1	512
133121	1	76	532481	1	37	2129921	1	34
149505	1	19	598017	1	66	2392065	1	235
165889	0	14	663553	1	68	2654209	0	63
182273	1	141	729089	1	18	2916353	1	328
198657	0	40	794625	1	24	3178497	0	32
215041	1	20	860161	1	140	3440641	1	41
231425	0	44	925697	0	45	3702785	1	57
247809	1	19	991233	1	26	3964929	1	188
264193	1	56	1056769	0	167	4227073	1	285
280577	1	19	1122305	1	140	4489217	1	282
296961	1	12	1187841	1	20	4751361	1	112
313345	1	69	1253377	0	92	5013505	1	54
329729	0	64	1318913	1	69	5275649	1	343
346113	0	17	1384449	0	42	5537793	0	114
362497	1	27	1449985	1	100	5799937	0	58
378881	1	16	1515521	1	85	6062081	1	142
395265	1	39	1581057	1	68	6324225	1	33
411649	1	21	1646593	0	274	6586369	0	237
428033	1	19	1712129	0	202	6848513	0	436
444417	0	20	1777665	1	50	7110657	1	187
460801	0	43	1843201	1	43	7372801	1	284
477185	1	94	1908737	0	34	7634945	1	130
493569	0	39	1974273	1	21	7897089	1	182
509953	1	55	2039809	1	101	8159233	1	86

Table 3. Data for Disks of the form $\bar{D}(1+2^{2k+1}, 2^{-(2k+4)})$

t	PCB?	n
131073	1	170
1179649	1	452
2228225	0	1248
3276801	0	28
4325377	1	620
5373953	1	503
6422529	0	927
7471105	1	123
8519681	1	326
9568257	0	274
10616833	1	29
11665409	0	47
12713985	1	202
13762561	1	969
14811137	1	609
15859713	0	638
16908289	0	168
17956865	1	166
19005441	1	593
20054017	1	70
21102593	1	186
22151169	1	201
23199745	0	904
24248321	1	18
25296897	1	234
26345473	1	761
27394049	1	419
28442625	1	791
29491201	0	282
30539777	1	111
31588353	0	573
32636929	1	137

t	PCB?	n
524289	1	890
4718593	1	756
8912897	1	615
13107201	0	141
17301505	1	949
21495809	1	899
25690113	1	426
29884417	1	85
34078721	1	741
38273025	1	477
42467329	0	760
46661633	1	192
50855937	0	33
55050241	0	123
59244545	1	875
63438849	1	446
67633153	1	886
71827457	0	773
76021761	1	915
80216065	1	293
84410369	1	965
88604673	1	736
92798977	0	153
96993281	1	258
101187585	1	691
105381889	1	570
109576193	1	659
113770497	1	298
117964801	1	683
122159105	1	218
126353409	1	1037
130547713	0	149

t	PCB?	n
2097153	1	233
18874369	1	45
35651585	1	1712
52428801	0	274
69206017	1	422
85983233	0	1646
102760449	1	368
119537665	0	1223
136314881	1	1292
153092097	1	325
169869313	1	204
186646529	0	421
203423745	0	42
220200961	1	1413
236978177	1	2333
253755393	1	107
270532609	1	1929
287309825	0	994
304087041	1	2118
320864257	1	3395
337641473	0	75
354418689	1	1690
371195905	1	63
387973121	0	4164
404750337	1	347
421527553	1	156
438304769	1	162
455081985	1	118
471859201	0	1739
488636417	1	1677
505413633	0	1000
522190849	1	250

TABLE 3. Data for Disks of the form $\bar{D}(1 + 2^{2k+1}, 2^{-(2k+4)})$

t	PCB?	n
8388609	1	3379
75497473	1	446
142606337	1	2148
209715201	1	2839
276824065	1	2989
343932929	1	3088
411041793	1	3842
478150657	0	1758
545259521	0	3707
612368385	1	384
679477249	0	680
746586113	1	3245
813694977	1	712
880803841	0	450
947912705	1	4093
1015021569	0	406
1082130433	1	338
1149239297	0	798
1216348161	1	179
1283457025	1	5407
1350565889	1	379
1417674753	1	681
1484783617	0	5024
1551892481	1	1047
1619001345	1	3239
1686110209	1	3736
1753219073	0	3394
1820327937	1	2554
1887436801	1	323
1954545665	1	2095
2021654529	1	7420
2088763393	1	2500

List of Notation

$\bar{D}(a,s)$	closed disk centered at a of radius r in $\mathbb{C}_p$, page 9
$\mathbb{C}$	complex numbers, page 5
$\mathbb{C}_p$	completion of algebraic closure of $\mathbb{Q}_p$, page 5
$\mathcal{K}_f$	filled Julia set of f, page 9
λ	multiplier of a periodic point, page 6
$\lvert\cdot\rvert_p = \lvert\cdot\rvert$	p-adic absolute value on $\mathbb{C}_p$, page 5
$\mathcal{M}_{d,p}$	p-adic, degree d Mandelbrot set, page 8
$\mathcal{M}$	PCB locus for the cubic family $f_t(z) = z^3 - \frac{3}{2}z^2$, page 32
$\mathcal{M}_{\mathbb{C}}$	complex Mandelbrot set, page 2
$\mathcal{O}_f(z)$	orbit of z under iteration of f, page 5
$\mathcal{P}_{d,p}$	parameter space of $f \in \mathbb{C}_p[z]$, degree d, monic, with $f(0) = 0$, page 7
$\mathbb{Q}_p$	p-adic (rational) numbers, page 5
σ_i	i^{th} symmetric function on the critical points of f, page 9
$\mathbb{Z}_p$	ring of integers in $\mathbb{Q}_p$, page 34
f^n	n^{th} iterate of the function f, page 5
R	smallest number such that $\mathcal{K}_f \subset \bar{D}(0, p^R)$, page 9
r	negation of minimal p-adic valuation of critical points, page 9
$r(d,p)$	critical radius of $\mathcal{M}_{d,p}$, page 14
$v_p(\cdot) = v(\cdot)$	p-adic valuation on $\mathbb{C}_p$, page 9
PCB	post-critically bounded, page 8
PCU	post-critically unbounded, page 42

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