

p -ADIC q -HIGHER-ORDER HARDY-TYPE SUMS

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ABSTRACT. The goal of this paper is to define p -adic Hardy sums and p -adic q -higher-order Hardy-type sums. By using these sums and p -adic q -higher-order Dedekind sums, we construct p -adic continuous functions for an odd prime. These functions contain p -adic q -analogue of higher-order Hardy-type sums. By using an invariant p -adic q -integral on \mathbb{Z}_p , we give fundamental properties of these sums. We also establish relations between p -adic Hardy sums, Bernoulli functions, trigonometric functions and Lambert series.

1. Introduction, definitions, and notations

Recall that the Dedekind eta-function, which was introduced by Dedekind in 1877, is defined by

$$\eta(z) = e^{\frac{\pi iz}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})$$

for $z \in \mathbb{H}$, upper-half plane. In many applications of elliptic modular functions to Number Theory Dedekind eta-function plays a central role. The classical Dedekind sum appears in the transformation formulae of this function. If h and k are coprime integers with $k > 0$, the classical Dedekind sum is defined by

$$s(h, k) = \sum_{\mu \bmod k} \left(\left(\frac{\mu}{k} \right) \right) \left(\left(\frac{h\mu}{k} \right) \right),$$

where the function $((x))$ is defined by

$$((x)) = \begin{cases} x - [x]_G - \frac{1}{2}, & x \text{ is not an integer} \\ 0, & \text{otherwise,} \end{cases}$$

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where $[x]_G$ is the largest integer $\leq x$ ([1], [17]).

Generalized Dedekind sums $s(h, k : p)$ are defined by [1]

$$(1.1) \quad s(h, k : p) = \sum_{a \bmod k} \frac{a}{k} \overline{B}_p\left(\frac{ha}{k}\right),$$

where h and k are coprime positive integers and $\overline{B}_p(x)$ is the p -th Bernoulli function, which is defined by

$$(1.2) \quad \overline{B}_p(x) = B_p(x - [x]_G) = -p! (2\pi i)^{-p} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} m^{-p} e^{2\pi i m x},$$

where $B_p(x)$ is the usual p -th Bernoulli polynomial.

Observe that when $p = 1$, the sum $s(h, k : 1)$ is known as the classical Dedekind sums, $s(h, k)$. Also the sums $s(h, k : p)$ are related to the Lambert series, $G_p(x)$, which are defined as follows:

$$G_p(x) = \sum_{n=1}^{\infty} n^{-p} \frac{x^n}{1 - x^n},$$

where $p \geq 1$.

Using a Mellin transform technique developed by Rademacher, Apostol [1] obtained transformation formulae relating $G_p(e^{2\pi i z})$ to $G_p(e^{2\pi i Az})$ for odd p , where $Az = \frac{az+b}{cz+d}$ is a modular substitution. The sum $s(h, k : p)$ appears in these transformation formulae. The sum $s(h, k : p)$ is expressible as infinite series related to certain Lambert series. A representation of $s(h, k : p)$ as infinite series was given by Apostol [1]:

THEOREM 1. ([1]) *Let $(h, k) = 1$. For odd $p \geq 1$, we have*

$$(1.3) \quad s(h, k : p) = \frac{p!}{(2\pi i)^p} \sum_{\substack{n=1 \\ n \neq 0(k)}}^{\infty} n^{-p} \left(\frac{e^{\frac{2\pi i n h}{k}}}{1 - e^{\frac{2\pi i n h}{k}}} - \frac{e^{-\frac{2\pi i n h}{k}}}{1 - e^{-\frac{2\pi i n h}{k}}} \right).$$

Throughout this paper \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{N} will respectively denote the ring of p -adic rational integers, the field of p -adic rational numbers and the set of natural numbers. Let v_p be the normalized exponential valuation of \mathbb{Q}_p with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q -extension, q is variously considered as an indeterminate. Let \mathbb{C}_p be denoted by the completion of algebraic closure of \mathbb{Q}_p . If $q \in \mathbb{C}_p$, we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$ (for detail see [6], [8]-[12]).

Rosen and Snyder [16] defined p -adic Dedekind sums. They commenced this sums by interpolating certain m -th Bernoulli functions. Let

p be an odd prime. Let a and N be positive integers such that $(p, a) = 1$ and $p|N$. We wish to extend $N^{m-1}B_m(\frac{a}{N})$. The function $F(s; a, N)$ is defined by

$$(1.4) \quad F(s; a, N) = w^{-1}(a) \frac{\langle a \rangle^s}{N} \sum_{j=0}^{\infty} \binom{s}{j} \left(\frac{N}{a}\right)^j B_j,$$

where $s \in \mathbb{Z}_p$, $\langle a \rangle$ denotes the principal unit associated with a , i.e., $\langle a \rangle$ is the unique unit given via the decomposition $a = \langle a \rangle w(a)$ where $w(a) = \lim_{n \rightarrow \infty} a^{p^n}$. Since $\left| \binom{s}{j} \right|_p \leq 1$, $\left| \frac{N}{a} \right|_p < 1$ and $|B_j|_p \leq p$ for all $j \geq 0$ (see [16]), we know that $\sum_{j=0}^{\infty} \binom{s}{j} \left(\frac{N}{a}\right)^j B_j$ converges to a continuous p -adic function of s on \mathbb{Z}_p . Let $s = m$, where m is a positive integer. We set

$$F(m; a, N) = w^{-m-1} N^{m-1} B_m \left(\frac{a}{N} \right).$$

In particular, if $m + 1 \equiv 0 \pmod{p-1}$, then

$$(1.5) \quad F(m; a, N) = N^{m-1} B_m \left(\frac{a}{N} \right).$$

Therefore $F(s; a, N)$ is a continuous p -adic extension of $N^{m-1} B_m \left(\frac{a}{N} \right)$ ([16]).

This function interpolates $k^m s(h, k : m)$ where $p|k$ and $(h\mu, p) = 1$ for each $\mu = 1, 2, \dots, k-1$ ([16]):

$$(1.6) \quad k^m s(h, k : m) = \sum_{\mu=1}^{k-1} \mu F(m; (h\mu)_k, k),$$

for all $m + 1 \equiv 0 \pmod{p-1}$, where $(\alpha)_n$ denotes the integer x such that $0 \leq x < n$ and $x \equiv \alpha \pmod{n}$, (for detail see [16]).

p -adic Dedekind sums in the Rosen and Snyder [16] sense are given as follows:

DEFINITION 1. ([16]) Let p be an odd prime. Let h and k relatively prime positive integers such that $p \nmid k$. Let $u \in \mathbb{Z}_p$. The p -adic Dedekind sum is defined to be

$$(1.7) \quad s(u; h, k : p) = \sum_{\mu=1}^{k-1} \mu F(u; h\mu, k).$$

If $p \equiv 1 \pmod{k}$ (p odd, as always), then we obtain immediately that

$$(1.8) \quad s(m; h, k : p) = (1 - p^{m-1}) k^m s(h, k : m),$$

for all m such that $m + 1 \equiv 0 \pmod{p - 1}$.

In [14], Kudo studied on p -adic Dedekind sums different from Rosen and Snyder [16]. By using p -adic measure and p -adic continuous function which interpolates of Euler numbers, Kudo [14] defined p -adic Dedekind sums.

In this paper, we use the notation

$$[x] = [x : q] = \frac{1 - q^x}{1 - q}.$$

Hence,

$$\lim_{q \rightarrow 1} [x] = x$$

for any x with $|x| \leq 1$ in the present p -adic case.

In [9]-[12], Kim defined the q -Volkenborn integration. By using q -Volkenborn integration, he evaluated q -Bernoulli polynomials and numbers. By using non-Archimedean q -integration, he introduced multiple Changhee q -Bernoulli polynomials which form a q -extension of Barnes' multiple Bernoulli polynomials. He also construct the Changhee q -zeta functions which give q -analogs of Barnes' multiple zeta function. He gave the relationships between the Changhee q -zeta function and Daehee q -zeta function. He constructed a new measure. In [6], Kim gave definition of a q -analogue of p -adic Haar distribution as follows: For any positive integer N ,

$$\mu_q(a + p^N \mathbb{Z}_p) = \frac{(1 - q)q^a}{1 - q^{p^N}} = \frac{q^a}{[p^N]}$$

for $0 \leq a < p^N - 1$ and this can be extended to a distribution on \mathbb{Z}_p . This distribution yields an invariant p -adic q -integral for each non-negative integer m and the m -th Carlitz's q -Bernoulli number β_m^* can be represented by this p -adic q -integral as follows:

$$\beta_m^* = \int_{\mathbb{Z}_p} [a]^m d\mu_q(a) = \lim_{N \rightarrow \infty} \sum_{a=0}^{p^N-1} \frac{q^a [a]^m}{[p^N]},$$

where the limit is convergent (see [6]-[8]).

In [13], T. Kim and H. S. Kim defined q -Bernoulli number $\beta_m (= \beta_m(q) \in \mathbb{C}_p)$, by making use of this integral, as follows

$$\int_{\mathbb{Z}_p} q^{-a} [a]^m d\mu_q(a) = \beta_m.$$

Note that $\lim_{q \rightarrow 1} \beta_m(q) = B_m$, where B_m is the m -th Bernoulli number. They also defined q -Bernoulli polynomials by

$$(1.9) \quad \beta_m(x : q) = \int_{\mathbb{Z}_p} q^{-t} [x + t]^m d\mu_q(t)$$

for $x \in \mathbb{Z}_p, m \in \mathbb{N}$.

These numbers $\beta_m(x : q)$ can be represented as

$$\beta_m(x : q) = \sum_{k=0}^m \binom{m}{k} \beta_k(q) [x]^{m-k} \quad (\text{see [13]}).$$

In [8], by using an invariant p -adic q -integrals on \mathbb{Z}_p , Kim constructed a p -adic continuous function for an odd prime to contain a p -adic q -analogue of higher-order Dedekind sums $k^m s(h, k : m + 1)$.

THEOREM 2. ([8]) *Assume that h, k are fixed integers with $(h, k) = (p, k) = 1$. Let*

$$s_q(h, k : m; q^l) = \sum_{M=0}^{k-1} \frac{[M]}{[k]} \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{hM}{k} \right\} : q^l \right]^m d\mu_{q^l}(x),$$

where $\{x\}$ denotes the fractional part of x . Then, there exists a continuous function $s_q(s; h, k : p; q^k)$ on \mathbb{Z}_p which satisfies

$$(1.10) \quad s_q(m + 1; h, k : p; q^k) = [k]^m s_q(h, k : m; q^k) - [k]^m [p : q^k]^{m-1} s_q((p^{-1}h)_k, k : m; q^{pk}),$$

for all $m + 1 \equiv 0 \pmod{p - 1}$, where $(p^{-1}a)_N$ denotes the integer x such that $0 \leq x < N$ and $px \equiv a \pmod{N}$.

Here we express Hardy sums explicitly in terms of $s(h, k)$. In stating Hardy sums we will use the notation of Berndt [3] and Sitaramachandrarao [21].

Let h and k are integers with $k > 0$, the Hardy sums are defined as follows:

$$\begin{aligned} S(h, k) &= \sum_{j=1}^{k-1} (-1)^{j+1+\lceil \frac{hj}{k} \rceil}_G, \quad s_1(h, k) = \sum_{j=1}^k (-1)^{\lceil \frac{hj}{k} \rceil}_G \left(\left(\frac{j}{k} \right) \right), \\ s_2(h, k) &= \sum_{j=1}^k (-1)^j \left(\left(\frac{hj}{k} \right) \right) \left(\left(\frac{j}{k} \right) \right), \quad s_3(h, k) = \sum_{j=1}^k (-1)^j \left(\left(\frac{hj}{k} \right) \right), \\ s_4(h, k) &= \sum_{j=1}^{k-1} (-1)^{\lceil \frac{hj}{k} \rceil}_G, \quad s_5(h, k) = \sum_{j=1}^k (-1)^{j+\lceil \frac{hj}{k} \rceil}_G \left(\left(\frac{j}{k} \right) \right). \end{aligned}$$

The relations between Hardy sums and Dedekind sums are given as follows:

THEOREM 3. ([21]) *Let $(h, k) = 1$. Then if $h + k$ is odd,*

$$S(h, k) = 8s(h, 2k) + 8s(2h, k) - 20s(h, k);$$

if h is even,

$$s_1(h, k) = 2s(h, k) - 4s(h, 2k);$$

if k is even,

$$(1.11) \quad s_2(h, k) = -s(h, k) + 2s(2h, k);$$

if k is odd,

$$s_3(h, k) = 2s(h, k) - 4s(2h, k);$$

if h is odd,

$$s_4(h, k) = -4s(h, k) + 8s(h, 2k);$$

if $h + k$ is even,

$$s_5(h, k) = -10s(h, k) + 4s(2h, k) + 4s(h, 2k)$$

and each one of $S(h, k)$ ($h + k$ even), $s_1(h, k)$, (h odd), $s_2(h, k)$ (k odd), $s_3(h, k)$ (k even), $s_4(h, k)$ (h even) and $s_5(h, k)$ ($h + k$ odd) is zero.

The proof of this theorem was given by Sitaramachandrarao [21]. Sitaramachandrarao [21], Berndt [3] and Dieter [5] and the author ([17]-[20]) gave fundamental properties of the Hardy Sums.

A brief summary of the paper follows:

In Section 2, we define p -adic Hardy sums. We give relations between these sums, Bernoulli polynomials and trigonometric functions. In Section 3, we shall establish new relations connection between the sums $s(h, k : p)$ and the Lambert series $G_p(e^{2\pi ih/k})$. In Section 4, we define p -adic q -higher-order Hardy-type sums. By using these sums and the sum $s_q(h, k : m; q^l)$, we construct a p -adic continuous function for an odd prime. This functions contain p -adic q -analogue of higher-order Hardy-type sums. By using an invariant p -adic q -integral on \mathbb{Z}_p , we give fundamental properties of q -analogue of p -adic Hardy type sums. We give relations between p -adic q -analogue of higher-order Hardy-type sums and p -adic q -analogue of higher-order Dedekind sums as well.

2. p -adic Hardy sums

Our aim in this section is to define p -adic Hardy sums. By using (1.7) and Theorem 3, we define p -adic Hardy sums. By applying (1.5) we prove the following theorems. p -adic Hardy sums are defined as follows [20]:

DEFINITION 2. Let p be an odd prime. Let h and k be relatively prime positive integers such that $p \nmid k$. Let $s \in \mathbb{Z}_p$. Let $h + k$ be odd,

$$S(s, h, k : p) = 4 \sum_{\mu(\bmod k)} \mu(2F(s; h\mu, 2k) + 2F(s; 2h\mu, k) - 5F(s; h\mu, k)),$$

let h be even,

$$s_1(s, h, k : p) = 2 \sum_{\mu(\bmod k)} \mu(F(s; h\mu, k) - 2F(s; h\mu, 2k)),$$

let k be even,

$$(2.1) \quad s_2(s, h, k : p) = - \sum_{\mu(\bmod k)} \mu(F(s; h\mu, k) - 2F(s; 2h\mu, k)),$$

let k be odd,

$$s_3(s, h, k : p) = 2 \sum_{\mu(\bmod k)} \mu(F(s; h\mu, k) - 2F(s; 2h\mu, k)),$$

let h be odd,

$$s_4(s, h, k : p) = 4 \sum_{\mu(\bmod k)} \mu(-F(s; h\mu, k) + 2F(s; h\mu, 2k)),$$

let $h + k$ be even,

$$\begin{aligned} & s_5(s, h, k : p) \\ &= 2 \sum_{\mu(\bmod k)} \mu(-5F(s; h\mu, k) + 2F(s; 2h\mu, k) + 2F(s; h\mu, 2k)). \end{aligned}$$

THEOREM 4. Let p be an odd prime. Let h and k be relatively prime positive integers such that $p \nmid k$. Let $s \in \mathbb{Z}$. If k is even,

$$\begin{aligned} (2.2) \quad & s_2(s, h, k : p) \\ &= -k^{s-1} \sum_{c=0}^s \binom{s}{c} B_c \left(\frac{h}{k}\right)^{s-c} \left(\frac{1 - 2^{s-c+1}}{s - c + 2}\right) (B_{s-c+2}(k) - B_{s-c+2}); \end{aligned}$$

if k is odd,

$$(2.3) \quad \begin{aligned} & s_3(s, h, k : p) \\ &= 2k^{s-1} \sum_{c=0}^s \binom{s}{c} B_c \left(\frac{h}{k} \right)^{s-c} \left(\frac{1 - 2^{s-c+1}}{s - c + 2} \right) (B_{s-c+2}(k) - B_{s-c+2}). \end{aligned}$$

Proof. We need the following well-known relation:

$$(2.4) \quad \sum_{v=1}^{n-1} v^p = \frac{B_{p+1}(n) - B_{p+1}}{p+1}$$

By using (1.5) and (2.4) in (2.1) and after straightforward calculations, we arrive at the desired result. For detail proof see [20]. Thus we choose to omit the details involved. \square

THEOREM 5. Let h and k are integers with $(h, k) = 1$ and let p be an odd prime with $p \geq 1$. If k is even,

$$(2.5) \quad \begin{aligned} s_2(h, k : p) &= \frac{-4ip!}{k(2\pi i)^p} \sum_{\mu=1}^{k-1} \mu \sum_{m=1}^{\infty} m^{-p} \\ &\quad \times \sin\left(\frac{\pi m h \mu}{k}\right) \left(2 \cos\left(\frac{3\pi m h \mu}{k}\right) + \cos\left(\frac{\pi m h \mu}{k}\right) \right), \end{aligned}$$

If k is odd,

$$(2.6) \quad \begin{aligned} s_3(h, k : p) &= \frac{8ip!}{k(2\pi i)^p} \sum_{\mu=1}^{k-1} \mu \sum_{m=1}^{\infty} m^{-p} \\ &\quad \times \sin\left(\frac{\pi m h \mu}{k}\right) \left(2 \cos\left(\frac{3\pi m h \mu}{k}\right) + \cos\left(\frac{\pi m h \mu}{k}\right) \right). \end{aligned}$$

Proof. By using (1.1) in (2.7), we have

$$s_2(h, k : p) = -\frac{1}{k} \sum_{\mu=1}^{k-1} \mu \left(\bar{B}_p\left(\frac{h\mu}{k}\right) - 2\bar{B}_p\left(\frac{2h\mu}{k}\right) \right).$$

Substituting (1.2) into the above and using well-known identity $2i \sin a = e^{ia} - e^{-ia}$ and after some elementary calculations, we obtain the desired result (for detail proof, see [20]). Thus we choose to omit the details involved. \square

REMARK 1. By (1.11), we set

$$(2.7) \quad s_2(h, k : p) = -s(h, k : p) + 2s(2h, k : p),$$

if k is even. Observe that when $p = 1$,

$$s_2(h, k : 1) = -s(h, k : 1) + 2s(2h, k : 1),$$

where the sum $s_2(h, k : 1)$ is known as the sum $s_2(h, k)$ and the sum $s(h, k : 1)$ is known Dedekind sum which is given by (1.1).

LEMMA 1. Let $p \equiv 1 \pmod{k}$ (p odd, as always) and h and k be relatively prime positive integers such that $p \nmid k$. If $h + k$ is odd, then

$$S(m, h, k : p) = (1 - p^{m-1}) k^m 2^{m+3} s(h, 2k : m) + 4(1 - p^{m-1}) k^m (2s(2h, k : m) - 5s(h, k : m));$$

if h is even, then

$$s_1(m, h, k : p) = 2(1 - p^{m-1}) k^m (s(h, k : m) - 2^{m+1} s(h, 2k : m));$$

if k is even, then

$$s_2(m, h, k : p) = -(1 - p^{m-1}) k^m (s(h, k : m) - 2s(2h, k : m));$$

if k is odd, then

$$s_3(m, h, k : p) = 2(1 - p^{m-1}) k^m (s(h, k : m) - 2s(2h, k, m));$$

if h is odd, then

$$s_4(m, h, k : p) = -4(1 - p^{m-1}) k^m (s(h, k : m) - 2^{m+1} s(h, 2k : m));$$

if $h + k$ is even, then

$$s_5(m, h, k : p) = -10(1 - p^{m-1}) k^m s(h, k : m) + 4(1 - p^{m-1}) k^m (s(2h, k : m) + 2^m s(h, 2k : m)),$$

for all m such that $m+1 \equiv 0 \pmod{p-1}$. Each one of $S(m, h, k : p)$ ($h+k$ even), $s_1(m, h, k : p)$, (h odd), $s_2(m, h, k : p)$ (k odd), $s_3(m, h, k : p)$ (k even), $s_4(m, h, k : p)$ (h even) and $s_5(m, h, k : p)$ ($h+k$ odd) is zero.

Proof. By using (1.8), after straightforward calculations, we arrive at the desired result. For detail proof see [20]. Thus we choose to omit the details involved. \square

3. Relations between Hardy sums and Lambert series

The sums $s(h, k : p)$ are related to the Lambert series, $G_p(x)$. The special case $p = 1$ gives $G_1(x) = -\log \prod_{m=1}^{\infty} (1 - x^m)$. Thus, $\log \eta(z)$ is the same as $\frac{\pi iz}{12} - G_1(e^{2\pi iz})$ ([1]-[2], [18]). Generalized Dedekind sums are expressible as infinite series related to certain Lambert series ([18]). In this section we shall establish new relations connection between the sums $s(h, k : p)$ and the Lambert series $G_p(e^{2\pi ih/k})$.

Relation between $s(h, k : p)$ and $G_p(e^{2\pi iz})$ are given as follows.

LEMMA 2. *Let $p \equiv 1 \pmod{k}$ (p odd, as always) and h and k be relatively prime positive integers. Then we have*

$$\begin{aligned} s(h, k : p) &= \frac{p!}{(2\pi i)^p} \sum_{\substack{n=1 \\ n \not\equiv 0(k)}}^{\infty} \sum_{m=1}^{\infty} n^{-p} (e^{\frac{2\pi imnh}{k}} - e^{-\frac{2\pi imnh}{k}}) \\ &= \frac{p!}{(2\pi i)^p} (G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}})). \end{aligned}$$

Proof. By using (1.3) and definition of Lambert series and after straightforward calculations, we arrive at the desired result. \square

THEOREM 6. *Let $p \equiv 1 \pmod{k}$ (p odd, as always) and h and k be relatively prime positive integers such that $p \nmid k$. If $h + k$ is odd, then*

$$\begin{aligned} S(m, h, k : p) &= \frac{4m!(1 - p^{m-1})k^m}{(2\pi i)^m} (2^{m+1}(G_p(e^{\frac{\pi ih}{k}}) - G_p(e^{-\frac{\pi ih}{k}})) \\ &\quad + 2(G_p(e^{\frac{4\pi ih}{k}}) - G_p(e^{-\frac{4\pi ih}{k}})) - 5(G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}}))); \end{aligned}$$

if h is even,

$$\begin{aligned} s_1(m, h, k : p) &= \frac{2m!(1 - p^{m-1})k^m}{(2\pi i)^m} (-2^{m+1}(G_p(e^{\frac{\pi ih}{k}}) - G_p(e^{-\frac{\pi ih}{k}})) \\ &\quad + (G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}}))); \end{aligned}$$

if k is even, then

$$\begin{aligned} s_2(m, h, k : p) &= -\frac{m!(1 - p^{m-1})k^m}{(2\pi i)^m} (G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}}) \\ &\quad - 2(G_p(e^{\frac{4\pi ih}{k}}) - G_p(e^{-\frac{4\pi ih}{k}}))); \end{aligned}$$

if k is odd, then

$$s_3(m, h, k : p) = \frac{2m!(1 - p^{m-1})k^m}{(2\pi i)^m} (G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}}) - 2(G_p(e^{\frac{4\pi ih}{k}}) - G_p(e^{-\frac{4\pi ih}{k}})));$$

if h is odd, then

$$s_4(m, h, k : p) = -\frac{4m!(1 - p^{m-1})k^m}{(2\pi i)^m} (-2^{m+1}(G_p(e^{\frac{\pi ih}{k}}) - G_p(e^{-\frac{\pi ih}{k}})) + G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}}));$$

if $h + k$ is even, then

$$s_5(m, h, k : p) = -\frac{2m!(1 - p^{m-1})k^m}{(2\pi i)^m} (-2^{m+1}(G_p(e^{\frac{\pi ih}{k}}) - G_p(e^{-\frac{\pi ih}{k}})) + 5(G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}})) - 2(G_p(e^{\frac{4\pi ih}{k}}) - G_p(e^{-\frac{4\pi ih}{k}}))),$$

for all m such that $m+1 \equiv 0 \pmod{p-1}$. Each one of $S(m, h, k : p)$ ($h+k$ even), $s_1(m, h, k : p)$, (h odd), $s_2(m, h, k : p)$ (k odd), $s_3(m, h, k : p)$ (k even), $s_4(m, h, k : p)$ (h even) and $s_5(m, h, k : p)$ ($h+k$ odd) is zero.

Proof. By using Lemma 1 and Lemma 2, after straightforward calculations, we arrive at the desired result. \square

4. Generalized p -adic q -Hardy sums

By using some simple properties of $((x))$, $s(h, k)$, some well-known results due to [4], [15], and [21], and the following basic observation, we define generalized Hardy sums.

$$(-1)^{[x]_G} = 2((x)) - 4\left(\left(\frac{x}{2}\right)\right)$$

if x is not an integer. To give this, observe that if x is not an integer, then

$$\begin{aligned} 2((x)) - 4\left(\left(\frac{x}{2}\right)\right) &= 2(x - [x]_G - \frac{1}{2}) - 4\left(\frac{x}{2} - \left[\frac{x}{2}\right]_G - \frac{1}{2}\right) \\ &\equiv 1 - 2[x]_G + 4\left[\frac{x}{2}\right]_G = 1 - 2[x]_G + 4\left[\frac{[x]_G}{2}\right]_G \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} 1 - 2[x]_G + 4\frac{[x]_G}{2}, & \text{if } [x]_G \text{ is even} \\ 1 - 2[x]_G + 4\left(\frac{[x]_G - 1}{2}\right), & \text{if } [x]_G \text{ is odd} \end{cases} \\
&= (-1)^{[x]_G}.
\end{aligned}$$

By using the above equation, Cenkci, Can and Kurt [4] gave the following definition:

Let h and k be integers with $k > 0$, the Hardy sums are defined as follows

$$\begin{aligned}
S(h, k) &= 4 \sum_{j=1}^{k-1} \left(\left(\frac{(h+k)j}{2k} \right) \right), \\
s_1(h, k) &= \sum_{j=1}^k \left(\left(\frac{j}{k} \right) \right) \left\{ 2 \left(\left(\frac{hj}{k} \right) \right) - 4 \left(\left(\frac{hj}{2k} \right) \right) \right\}, \\
s_2(h, k) &= -4 \sum_{j=1}^{k-1} \left(\left(\frac{hj}{2k} \right) \right), \\
s_5(h, k) &= \sum_{j=1}^k \left(\left(\frac{j}{k} \right) \right) \left\{ 2 \left(\left(\frac{hj}{k} \right) \right) - 4 \left(\left(\frac{(h+k)j}{2k} \right) \right) \right\}.
\end{aligned}$$

By using (1.1), due to Cenkci, Can and Kurt [4], we arrive at the following definition:

DEFINITION 3. Let h and k be integers with $k > 0$, the Hardy sums are defined as follows:

$$\begin{aligned}
S(h, k : m) &= 4 \sum_{j=1}^{k-1} \overline{B}_m \left(\frac{(h+k)j}{2k} \right), \\
s_1(h, k : m) &= \sum_{j=1}^{k-1} \overline{B}_1 \left(\frac{j}{k} \right) \left(2\overline{B}_m \left(\frac{hj}{k} \right) - 4\overline{B}_m \left(\frac{hj}{2k} \right) \right) \\
&= 2s(h, k : m) - 4 \sum_{j=1}^{k-1} \overline{B}_1 \left(\frac{j}{k} \right) \overline{B}_m \left(\frac{hj}{2k} \right), \\
s_2(h, k : m) &= \sum_{j=1}^{k-1} (-1)^j \overline{B}_1 \left(\frac{j}{k} \right) \overline{B}_m \left(\frac{hj}{k} \right),
\end{aligned}$$

$$\begin{aligned}
 s_3(h, k : m) &= -4 \sum_{j=1}^{k-1} (-1)^j \overline{B}_m \left(\frac{hj}{k} \right), \\
 s_4(h, k : m) &= -4 \sum_{j=1}^{k-1} \overline{B}_m \left(\frac{hj}{k} \right), \\
 s_5(h, k : m) &= \sum_{j=1}^{k-1} \overline{B}_1 \left(\frac{j}{k} \right) \left(2\overline{B}_m \left(\frac{hj}{k} \right) - 4\overline{B}_m \left(\frac{(h+k)j}{2k} \right) \right) \\
 &= 2s(h, k : m) - 4 \sum_{j=1}^{k-1} \overline{B}_1 \left(\frac{j}{k} \right) \overline{B}_m \left(\frac{(h+k)j}{2k} \right),
 \end{aligned}$$

where $s(h, k : m)$ is generalized Dedekind sums.

REMARK 2. Observe that when $0 \leq x < 1$, the Bernoulli function $\overline{B}_m(x)$ reduces to the m -th Bernoulli polynomial(our notation is the same as that in Apostol [1], [2]).

By using Theorem 2, Definition 2, and Definition 3, we construct a p -adic continuous function for an odd prime to contain a p -adic q -analogue of higher-order Hardy sums in this chapter. It is the aim of this chapter to give a q -analogue of p -adic Hardy type sums by using an invariant p -adic q -integral on \mathbb{Z}_p approach the p -adic analogue of higher-order Dedekind sums at $q = 1$ as follows:

THEOREM 7. Assume that h, k are fixed integers with $(h, k) = (p, k) = 1$. Let

$$\begin{aligned}
 S_q(h, k : m; q^l) &= 4 \sum_{j=1}^{k-1} \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{j(h+k)}{2k} \right\} : q^l \right]^m d\mu_{q^l}(x), \\
 s_{1,q}(h, k : m; q^l) &= 2s_q(h, k : m; q^l) \\
 &\quad - 4 \sum_{j=1}^{k-1} \left(\frac{[j]}{[k]} - \frac{1}{2} \right) \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{hj}{2k} \right\} : q^l \right]^m d\mu_{q^l}(x), \\
 s_{2,q}(h, k : m; q^l) &= \sum_{j=1}^{k-1} (-1)^j \left(\frac{[j]}{[k]} - \frac{1}{2} \right) \\
 &\quad \times \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{hj}{k} \right\} : q^l \right]^m d\mu_{q^l}(x),
 \end{aligned}$$

$$\begin{aligned}
s_{3,q}(h, k : m; q^l) &= \sum_{j=1}^{k-1} (-1)^j \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{hj}{k} \right\} : q^l \right]^m d\mu_{q^l}(x), \\
s_{4,q}(h, k : m; q^l) &= -4 \sum_{j=1}^{k-1} \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{hj}{2k} \right\} : q^l \right]^m d\mu_{q^l}(x), \\
s_{5,q}(h, k : m; q^l) &= -4 \sum_{j=1}^{k-1} \left(\frac{[j]}{[k]} - \frac{1}{2} \right) \\
&\quad \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{j(h+k)}{2k} \right\} : q^l \right]^m d\mu_{q^l}(x) \\
&\quad + 2s_q(h, k : m; q^l),
\end{aligned}$$

where $\{x\}$ denotes the fractional part of x . Then, there exist continuous functions $S_q(s, h, k : p; q^k)$, and $s_{y,q}(s, h, k : p; q^k)$, $1 \leq y \leq 4$, on \mathbb{Z}_p which satisfy :

if $h+k$ be odd, then

$$\begin{aligned}
&S_q(m+1, h, k : p; q^k) \\
(4.1) \quad &= 8s_q(m+1; h, 2k : p; q^k) + 8s_q(m+1; 2h, k : p; q^k) \\
&\quad - 20s_q(m+1; h, k : p; q^k);
\end{aligned}$$

if h be even, then

$$\begin{aligned}
(4.2) \quad & \\
s_{1,q}(m+1, h, k : p; q^k) &= 2s_q(m+1; h, k : p; q^k) - 4s_q(m+1; h, 2k : p; q^k);
\end{aligned}$$

if k be even, then

$$\begin{aligned}
(4.3) \quad &s_{2,q}(m+1, h, k : p; q^k) \\
&= -s_q(m+1; h, k : p; q^k) + 2s_q(m+1; 2h, k : p; q^k);
\end{aligned}$$

if k be odd, then

$$\begin{aligned}
(4.4) \quad & \\
s_{3,q}(m+1, h, k : p; q^k) &= 2s_q(m+1; h, k : p; q^k) - 4s_q(m+1; 2h, k : p; q^k);
\end{aligned}$$

if h be odd, then

$$\begin{aligned}
(4.5) \quad &s_{4,q}(m+1, h, k : p; q^k) \\
&= -4s_q(m+1; h, k : p; q^k) + 8s_q(m+1; h, 2k : p; q^k);
\end{aligned}$$

if $h + k$ be even, then

$$(4.6) \quad s_{5,q}(m, h, k : p; q^k) = 4s_q(m + 1; 2h, k : p; q^k) + 4s_q(m + 1; h, 2k : p; q^k) - 10s_q(m + 1; h, k : p; q^k),$$

for all $m + 1 \equiv 0 \pmod{p - 1}$, where $s_q(m + 1; h, k : p; q^k)$ is q -analogue of p -adic Dedekind-type sums.

Proof. Proof of this Theorem is similar to that of Theorem 5, which is given by Kim [8]. q -analogue of (1.4) is given by [8]: Let w denote Teichmuller character $(\text{mod } p)$. For $x \in \mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$, we set $\langle x \rangle = \langle x : q \rangle = w^{-1}(x)[x]$. Let p be an odd prime, a and N positive integers such that $(p, a) = 1$ and $p|N$. Set

$$F_q(s, a, N; q^N) = \frac{w^{-1}(a)\langle a \rangle^s}{[N]} \sum_{j=0}^{\infty} \binom{s}{j} q^{aj} \frac{[N]^j}{[a]^j} \beta_j(q^N)$$

for $s \in \mathbb{Z}_p$, and let

$$Y_m(a, N, w^{-1}; q^l) = \frac{w^{-1}(a)}{[N]^{m-1}} \int_{\mathbb{Z}_p} q^{-lx} \left[x + \frac{a}{l} : q^l \right]^m d\mu_{q^l}(x),$$

where $m, l, N \in \mathbb{N}$. It is easy to see in [6] that

$$\sum_{j=0}^{\infty} \binom{s}{j} q^{aj} \frac{[N]^j}{[a]^j} \beta_j(q^N)$$

converges to continuous p -adic function of s on \mathbb{Z}_p . Set $s = m$ in the above, we get

$$F_q(s, a, N; q^N) = Y_m(a, N, w^{-1}; q^{-N}).$$

If $m + 1 \equiv 0 \pmod{p - 1}$, then by using (1.9), we obtain

$$\begin{aligned} F_q(m, a, N; q^N) &= \frac{1}{[N]^{m-1}} \int_{\mathbb{Z}_p} q^{-Nx} \left[x + \frac{a}{N} : q^N \right]^m d\mu_{q^N}(x) \\ &= \frac{\beta_m(\frac{a}{N}, q^N)}{[N]^{m-1}}. \end{aligned}$$

Consequently, $F_q(m, a, N; q^N)$ is a continuous p -adic extension of $[N]^{1-m} \beta_m(\frac{a}{N}, q^N)$.

By (1.6) and (1.7), higher-order Hardy-type sums $S_q(h, k : m; q^l)$ and $s_{y,q}(h, k : m; q^l)$, $1 \leq y \leq 4$, are given by

$$\begin{aligned}
 S_q(h, k : m; q^l) &= 4 \sum_{j=1}^{k-1} \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{j(h+k)}{2k} \right\} : q^l \right]^m d\mu_{q^l}(x), \\
 s_{1,q}(h, k : m; q^l) &= 2s_q(h, k : m; q^l) \\
 &\quad - 4 \sum_{j=1}^{k-1} \left(\frac{[j]}{[k]} - \frac{1}{2} \right) \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{hj}{2k} \right\} : q^l \right]^m d\mu_{q^l}(x) \\
 s_{2,q}(h, k : m; q^l) &= \sum_{j=1}^{k-1} (-1)^j \left(\frac{[j]}{[k]} - \frac{1}{2} \right) \\
 &\quad \cdot \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{hj}{k} \right\} : q^l \right]^m d\mu_{q^l}(x), \\
 s_{3,q}(h, k : m; q^l) &= \sum_{j=1}^{k-1} (-1)^j \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{hj}{k} \right\} : q^l \right]^m d\mu_{q^l}(x), \\
 s_{4,q}(h, k : m; q^l) &= -4 \sum_{j=1}^{k-1} \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{hj}{2k} \right\} : q^l \right]^m d\mu_{q^l}(x), \\
 s_{5,q}(h, k : m; q^l) &= -4 \sum_{j=1}^{k-1} \left(\frac{[j]}{[k]} - \frac{1}{2} \right) \int_{\mathbb{Z}_p} q^{-lx} \\
 &\quad \cdot \left[x + \left\{ \frac{j(h+k)}{2k} \right\} : q^l \right]^m d\mu_{q^l}(x) + 2s_q(h, k : m; q^l),
 \end{aligned}$$

where $s_q(h, k : m; q^l)$ is higher-order Dedekind sums.

If $m+1 \equiv 0 \pmod{p-1}$, we have

$$[k]^m s_q(h, k : m; q^k) = \sum_{\mu=1}^{k-1} [\mu] F_q \left(m; (h\mu)_k, k; q^k \right).$$

If $h+k$ is odd, then

$$[k]^m S_q(h, k : m; q^k)$$

$$= 8 \sum_{\mu=1}^{k-1} [\mu] \left(F_q \left(m; (h\mu)_k, 2k; q^k \right) + F_q \left(m; (2h\mu)_k, k; q^k \right) \right) - 20 \sum_{\mu=1}^{k-1} [\mu] F_q \left(m; (h\mu)_k, k; q^k \right);$$

if h is even, then

$$[k]^m s_{1,q} \left(h, k : m; q^k \right) = \sum_{\mu=1}^{k-1} [\mu] \left(2F_q \left(m; (h\mu)_k, k; q^k \right) - 4F_q \left(m; (h\mu)_k, 2k; q^k \right) \right);$$

if k is even, then

$$[k]^m s_{2,q} \left(h, k : m; q^k \right) = \sum_{\mu=1}^{k-1} [\mu] \left(-F_q \left(m; (h\mu)_k, k; q^k \right) + 2F_q \left(m; (2h\mu)_k, k; q^k \right) \right);$$

if k is odd, then

$$[k]^m s_{3,q} \left(h, k : m; q^k \right) = \sum_{\mu=1}^{k-1} [\mu] \left(2F_q \left(m; (h\mu)_k, k; q^k \right) - 4F_q \left(m; (2h\mu)_k, k; q^k \right) \right);$$

if h is odd, then

$$[k]^m s_{4,q} \left(s, h, k : p; q^k \right) = \sum_{\mu=1}^{k-1} [\mu] \left(-4F_q \left(m; (h\mu)_k, k; q^k \right) + 8F_q \left(m; (h\mu)_k, 2k; q^k \right) \right);$$

if $h + k$ is even, then

$$[k]^m s_{5,q} \left(s, h, k : p; q^k \right) = \sum_{\mu=1}^{k-1} [\mu] \left(4F_q \left(m; (2h\mu)_k, k; q^k \right) + 4F_q \left(m; (h\mu)_k, 2k; q^k \right) \right) - 10 \sum_{\mu=1}^{k-1} [\mu] F_q \left(m; (h\mu)_k, k; q^k \right),$$

where $(y)_k$ denotes the integer x such that $0 \leq x < n$ and $x \equiv \alpha \pmod{k}$. It is easy to check in [8] that for $s \in \mathbb{Z}_p$, p -adic q -Hardy-type sums defined as follows: if $h + k$ is odd,

$$\begin{aligned}
& S_q(s, h, k : p; q^k) \\
&= \sum_{\mu=1}^{k-1} [\mu] (8F_q(s; h\mu, 2k; q^k) + 8F_q(s; 2h\mu, k; q^k) \\
&\quad - 20F_q(s; h\mu, k; q^k)), \\
& s_{1,q}(s, h, k : p; q^k) \\
&= \sum_{\mu=1}^{k-1} [\mu] (2F_q(s; h\mu, k; q^k) - 4F_q(s; h\mu, 2k; q^k)), \text{ if } h \text{ is even,} \\
& s_{2,q}(s, h, k : p; q^k) \\
&= \sum_{\mu=1}^{k-1} [\mu] (-F_q(s; h\mu, k; q^k) + 2F_q(s; 2h\mu, k; q^k)), \text{ if } k \text{ is even,} \\
& s_{3,q}(s, h, k : p; q^k) \\
&= \sum_{\mu=1}^{k-1} [\mu] (2F_q(s; h\mu, k; q^k) - 4F_q(s; 2h\mu, k; q^k)), \text{ if } k \text{ is odd,} \\
& s_{4,q}(s, h, k : p; q^k) \\
&= \sum_{\mu=1}^{k-1} [\mu] (-4F_q(s; h\mu, k; q^k) + 8F_q(s; h\mu, 2k; q^k)), \text{ if } h \text{ is odd,}
\end{aligned}$$

and if $h + k$ is even

$$\begin{aligned}
s_{5,q}(s, h, k : p; q^k) &= \sum_{\mu=1}^{k-1} [\mu] (4F_q(s; h\mu, 2k; q^k) \\
&\quad + 4F_q(s; 2h\mu, k; q^k) - 10F_q(s; h\mu, k; q^k)).
\end{aligned}$$

Thus, there exist continuous functions $S_q(s, h, k : p; q^k)$, and $s_{y,q}(s, h, k : p; q^k)$, $1 \leq y \leq 4$, on \mathbb{Z}_p which satisfy (1.10) and (4.1)-(4.6) for $m + 1 \equiv 0 \pmod{p - 1}$. So we arrive at the desired result. \square

REMARK 3. In [8], Kim defined the sum $s_q(h, k : m; q^l)$ as follows

$$s_q(h, k : m; q^l) = \sum_{j=0}^{k-1} \frac{[j]}{[k]} \beta_m\left(\frac{hj}{k}, q^l\right).$$

Observe that when $q \rightarrow 1$, the sum $s_1(s, h, k : p; 1)$ reduces to p -adic analogue of higher-order Dedekind sums $k^m s_{m+1}(h, k)$. Here we note that

$$\begin{aligned} S_q(h, k : m; q^l) &= 4 \sum_{j=1}^{k-1} \beta_m\left(\frac{j(h+k)}{2k}, q^l\right), \\ s_{1,q}(h, k : m; q^l) &= 2 \sum_{j=0}^{k-1} \frac{[j]}{[k]} \beta_m\left(\frac{hj}{k}, q^l\right) - 4 \sum_{j=1}^{k-1} \left(\frac{[j]}{[k]} - \frac{1}{2}\right) \beta_m\left(\frac{jh}{2k}, q^l\right), \\ &= 2s_q(h, k : m; q^l) - 4 \sum_{j=1}^{k-1} \left(\frac{[j]}{[k]} - \frac{1}{2}\right) \beta_m\left(\frac{jh}{2k}, q^l\right), \\ s_{2,q}(h, k : m; q^l) &= \sum_{j=0}^{k-1} (-1)^j \left(\frac{[j]}{[k]} - \frac{1}{2}\right) \beta_m\left(\frac{hj}{k}, q^l\right), \\ s_{3,q}(h, k : m; q^l) &= \sum_{j=0}^{k-1} (-1)^j \beta_m\left(\frac{hj}{k}, q^l\right), \\ s_{4,q}(h, k : m; q^l) &= -4 \sum_{j=1}^{k-1} \beta_m\left(\frac{jh}{2k}, q^l\right), \\ s_{5,q}(h, k : m; q^l) &= 2 \sum_{j=0}^{k-1} \frac{[j]}{[k]} \beta_m\left(\frac{hj}{k}, q^l\right) \\ &\quad - 4 \sum_{j=1}^{k-1} \left(\frac{[j]}{[k]} - \frac{1}{2}\right) \beta_m\left(\frac{j(h+k)}{2k}, q^l\right) \\ &= 2s_q(h, k : m; q^l) \\ &\quad - 4 \sum_{j=1}^{k-1} \left(\frac{[j]}{[k]} - \frac{1}{2}\right) \beta_m\left(\frac{j(h+k)}{2k}, q^l\right). \end{aligned}$$

It is easy to see that if $q \rightarrow 1$, then $S_1(s, h, k : p; 1)$ and $s_{y,1}(s, h, k : p; 1)$, $1 \leq y \leq 4$ are the p -adic analogue of higher-order Hardy-type sums $k^m S_{m+1}(h, k)$ and $k^m s_{y,m+1}(h, k)$, $1 \leq y \leq 4$, respectively.

Finally we conclude this paper by raising the following question:

Find the reciprocity law for p -adic q -higher-order Hardy-type sums?

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