P-Automata for Markov Decision Processes

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P-automata provide an automata-theoretic approach to probabilistic verification. Similar to alternating tree automata accepting labelled transition systems, p-automata accept labelled Markov chains (MCs). This paper proposes an extension of p-automata that accept the set of all MCs (modulo bisimulation) obtained from a Markov decision process under its schedulers.

1 Introduction

Model checking of μ-calculus [9] formulas on a Kripke structure (or labelled translation system) is a well studied method for verifying the correctness of discrete state systems [6]. The problem entails whether every execution (infinite tree) of a Kripke structure satisfies a given μ-calculus formula. The satisfiability problem for μ-calculus, on the other hand, is to decide whether there exists an infinite tree which satisfies a given μ-calculus formula. Both these problems are algorithmically feasible, and the key method is the translation to alternating tree automata [13].

The notion of p-automata was introduced in [8] to provide a similar automata-theoretical foundation for the verification of probabilistic systems as alternating tree automata provide for Kripke structures. As alternating tree automata describe a complete framework for abstraction with respect to branching-time logic like, μ-calculus, CTL and CTL* [13], p-automata similarly give a unifying framework for different probabilistic logics.

Every p-automaton defines a set of labeled Markov chains, that is, a p-automaton reads an entire Markov chain as input and it either accepts the Markov chain or rejects it. Analogous to alternating tree automata where acceptance of a Kripke structure is decided by solving 2-player games [13], the acceptance of a labelled Markov chain by a p-automaton is decided by solving *stochastic* 2-player games. In this paper we revisit p-automata defined by [8] and extend it with a new construct for representing Markov decision processes. We view a Markov decision process (MDP) as a set of Markov chains defined by different schedulers and use the extended p-automata to represent this set. Modeling MDPs as p-automata allows us to define a automata theoretical framework for abstraction of MDPs.

The main contribution of this paper is as follows: We extend the p-automata with a construct that captures the non-determinism in the choice of probability distribution. This allows us to model Markov decision processes as p-automata. We show that the extended p-automata are closed under bisimulation, union and intersection, (though, in contrast to [8], the language is no longer closed under negation). We show that the language of the p-automaton obtained from an MDP accepts exactly those Markov chains that are bisimilar to the Markov chains induced by the schedulers of the MDP. In the rest of the paper, when referring to p-automata we will assume the extended p-automata (as defined in Definition 7), unless the contrary is stated explicitly.

The paper is organised as follows. In Section 2, we mention some important definitions and preliminaries. In section 3 and 4, we introduce the p-automata and define the acceptance game. In Section 5, we describe the embedding of an MDP as a p-automaton and conclude in Section 6. Details of some of the proofs are present in the appendix.

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2 Preliminaries and definitions

Let X^Y be the set of functions from the set Y to the set X. For $\varphi \in X^Y$ let $\mathsf{img}(\varphi) \subseteq X$ be the image and $\mathsf{dom}(\varphi) = Y$ be the domain of φ. The set of probability distributions over set X is denoted by $\mathscr{D}_X$ where $\boldsymbol{d} \in \mathscr{D}_X$ iff $\boldsymbol{d} \in \mathbb{R}_+^X$ and $\boldsymbol{d}^T \cdot \boldsymbol{1} = 1$ ($\mathbb{R}_+$ is the set of non-negative reals). For $\mu \in \mathscr{D}_X$, let $\mathsf{supp}(\mu) = \{x \in X \mid \mu(x) > 0\}$ be the support of distribution μ.

Definition 1. *A Markov chain (MC) M is a quintuple (S, P, AP, L, s_{in}) where S is a (countable) set of states, $P(s) \in \mathscr{D}_S$ for all $s \in S$, AP is a set of atomic propositions, $L : S \to 2^{AP}$ is a labeling function, and $s_{in} \in S$ is the initial state (Figure 1).*

An *infinite path* σ through MC M is a sequence of states $\sigma = \{\sigma_i\}_{i \geq 0}$, where for all $i \geq 0$, $P(\sigma_i, \sigma_{i+1}) > 0$. Let $\mathsf{path}(s)$ denote the set of (finite or infinite) paths starting from state s. For a path σ, let $\sigma{\downarrow}$ denote the last state of σ if this exists (i.e., if σ is finite) and $|\sigma|$ denote the length of σ. Let $\mathsf{succ}(s) = \{t \mid P(s,t) > 0\}$ be the successors of state s. A probability measure on sets of infinite paths is obtained in a standard way. Let $(\Omega_s, \mathscr{F}, \Pr)$ be the Borel σ-algebra where Ω_s is the set of infinite paths from state s, $\mathscr{F}$ is the smallest σ-field on cylinder sets of Ω_s, and Pr is the probability measure on $\mathscr{F}$, for a finite path σ, $\Pr(\sigma) = \prod_{0 < i \leq |\sigma|} P(\sigma_{i-1}, \sigma_i)$ [2].

Definition 2. *A* Markov decision process *(MDP) D is a quintuple $(S, \Delta, AP, L, s_{in})$ where S, AP, L, and s_{in} are as before, and $\Delta : S \to 2^{\mathscr{D}_S}$ such that $\Delta(s)$ is a finite set of distributions. (Figure 2) We assume S and $\Delta(s)$ for each $s \in S$ to be finite (unless the contrary is explicitly specified).*

A finite path of an MDP is a sequence of states $\sigma = \sigma_0 \ldots \sigma_n$ such for each $0 < i \leq n$ $\sigma_i \in \mathsf{supp}(\mu)$ for some $\mu \in \Delta(\sigma_{i-1})$. Let $\mathsf{path}(s)$ be the set of (finite and infinite) paths from the state s. Let $\mathsf{succ}(s) = \{t \mid t \in \bigcup_{\mu \in \Delta(s)} \mathsf{supp}(\mu)\}$ be the set of successors of s. As usual, we use *schedulers* to resolve the possible non-determinism in a state.

Definition 3. *A* scheduler *of MDP $D = (S, \Delta, AP, L, s_{in})$ is a function $\eta : S^+ \to \mathscr{D}_{\mathscr{D}_S}$ with $\eta(\sigma) \in \mathscr{D}_{\Delta(\sigma\downarrow)}$. The scheduler η induces the MC $D_\eta = (S^+, P, AP, L', s_{in})$ with $L'(\sigma) = L(\sigma{\downarrow})$, and $P(\sigma, \sigma{\cdot}t) = \sum_{\mu \in \Delta(\sigma\downarrow)} \eta(\sigma)(\mu) \cdot \mu(t)$.*

These schedulers are history-dependent and randomized. Let $\mathsf{HR}(D)$ denote the set of history-dependent randomized schedulers of MDP D.

Definition 4. *Let MC $M = (S, P, AP, L, s_{in})$. The equivalence relation $\mathscr{R} \subseteq S \times S$ is a* probabilistic bisimulation *[10] iff for every $(s, s') \in \mathscr{R}$ it holds:*

1. *$L(s) = L(s')$, and*
2. *for every $C \in S/\mathscr{R}$, we have $\sum_{t \in C} P(s,t) = \sum_{t' \in C} P(s', t')$.*

Let $\sim$ denote the largest probabilistic bisimulation on S. The MCs M_1 and M_2 are probabilistically bisimilar, denoted $M_1 \sim M_2$, if $s_{in}^1 \sim s_{in}^2$ in the disjoint union of M_1 and M_2.

Definition 5. *A stochastic game G is a tuple $(V, E, V_0, V_1, V_p, P, \Omega)$, where (V, E) is a finite directed graph and (V_0, V_1, V_p) is a partition of V. V_0 is the set of Player 0 configurations, V_1 is the set of Player 1 configurations and V_p is the set of stochastic (or probabilistic) configurations. P is a probability transition function $P : V_p \to \mathscr{D}_V$ and $\Omega \subseteq V$ is a set of accepting configurations. A path (also called a play) in the graph (V, E) is winning for Player 0 if it is finite and ends in Player 1 configuration, or it is infinite and ends in a suffix of configurations in Ω. Otherwise, that play is winning for Player 1.*

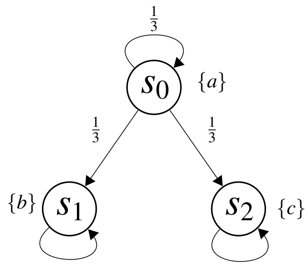

Figure 1: A Markov chain M, with $S = \{s_0, s_1, s_2\}$ $P(s_0, s_0) = P(s_0, s_1) = P(s_0, s_2) = \frac{1}{3}$, $P(s_1, s_1) = P(s_2, s_2) = 1$.

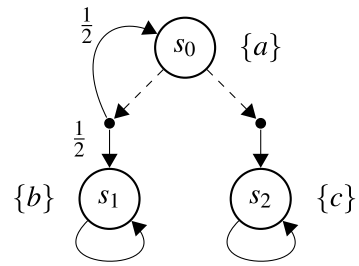

Figure 2: A Markov decision process D. $\Delta(s_0) = \{\mu_1, \mu_2\}$, where $\mu_1(s_0) = \frac{1}{2}, \mu_1(s_1) = \frac{1}{2}$ and $\mu_2(s_2) = 1$.

A stochastic game is called a *weak stochastic game* iff for all maximal connected components (MSCC) C in (V, E), either $C \subseteq \Omega$ or $C \cap \Omega = \emptyset$. On the other hand, if $V_p = \emptyset$ then it is called a *weak game*. A *strategy* of a Player 0 is a function $\gamma : V^* \times V_0 \to \mathscr{D}_V$, with $\gamma(w{\cdot}u)(v) > 0$ implies $(u, v) \in E$. A play $w = v_0 v_1 \ldots$ is consistent with strategy γ if for every $i \geq 0$, $v_i \in V_0$ implies $\gamma(v_0 \ldots v_i)(v_{i+1}) > 0$. Strategies of Player 1 are defined similarly. Let Υ and Π be the set of all strategies for Player 0 and Player 1, respectively. A player 0 strategy γ is memoryless iff $\gamma(w{\cdot}v) = \gamma(w'{\cdot}v)$, for any $w, w' \in V^*$, and it pure iff $\gamma : V^* \times V_0 \to V$, (similarly definitions applies to strategies of player 1).

A pair of strategies $(\gamma, \pi) \in \Upsilon \times \Pi$ of a game G determines an MC $M^{\gamma,\pi}$ (configurations without an out-going transition are made absorbing) whose paths are plays of G according to γ, π. The measure of the set of winning plays of Player 0 starting from a configuration c in $M^{\gamma,\pi}$ is denoted by $\mathsf{val}_0^{\gamma,\pi}(c)$. We have $\mathsf{val}_1^{\gamma,\pi}(c) = 1 - \mathsf{val}_0^{\gamma,\pi}(c)$. The $\mathsf{val}_0(c) = \sup_{\gamma \in \Upsilon} \inf_{\pi \in \Pi} \mathsf{val}_0^{\gamma,\pi}(c)$ and $\mathsf{val}_1(c) = \sup_{\pi \in \Pi} \inf_{\gamma \in \Upsilon} \mathsf{val}_1^{\gamma,\pi}(c)$. If a strategy achieves these values then it is called *optimal*.

Theorem 1. *[11, 5, 4] Let G be a stochastic game and c be one of its configurations. Then G is determined, that is $\mathsf{val}_0(c)+\mathsf{val}_1(c) = 1$. If G is finite and weak, then optimal strategies for both players exist and they are memoryless and pure. If G is a stochastic weak game, then the problem whether $\mathsf{val}_0(c)$ greater than a given quantity $v \in \mathbb{Q}$ can be decided in NP$\cap$co-NP, and if G is weak game then $\mathsf{val}_0(c) = 1$ can be decided in linear time.*

The theorem extends to cases where some configurations have predefined values in $[0, 1]$.

3 Weak p-automata

In this section we extend p-automata, as defined in [8] with a new operator $\oplus$.

Definition 6 (Boolean formulas on T)**.** *Let T be any arbitrary set, then $B^+(T)$ is the set of positive boolean formulas generated by the following syntax:*

$$\varphi ::= \; t \mid \mathit{true} \mid \mathit{false} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \tag{1}$$

where $t \in T$.

The closure of $\varphi \in B^+(T)$ is defined as $\mathsf{cl}(\varphi)$, where $\varphi \in \mathsf{cl}(\varphi)$ and if $\varphi_1 \circ \varphi_2 \in \mathsf{cl}(\varphi)$ then $\varphi_1, \varphi_2 \in \mathsf{cl}(\varphi)$, for $\circ \in \{\wedge, \vee\}$. Let Q be any set of states, the following sets are derived from Q:

$$\begin{array}{llll} \|Q\|_> & = \{\|q\|_{\bowtie p} \mid q \in Q, \bowtie \in \{\geq, >\}, p \in [0,1] \cap \mathbb{Q}\} & \|Q\|^* & = \{*(t_1, \ldots, t_n) \mid n \in \mathbb{N}, \forall i, t_i \in \|Q\|_>\} \\ \|Q\|^\vee & = \{\vee(t_1, \ldots, t_n) \mid n \in \mathbb{N}, \forall i, t_i \in \|Q\|_>\} & \|Q\|^\oplus & = \{\oplus(r_1, \ldots, r_n) \mid n \in \mathbb{N}, \forall i, r_i \in \|Q\|^*\} \\ \|Q\| & = \|Q\|^* \cup \|Q\|^\vee \cup \|Q\|^\oplus & & \end{array}$$

We will call the elements of $\|Q\|_>$ guarded states and elements of $\|Q\|^\oplus$ terms. For brevity, we will write $*(t|t \in X)$ for $*(t_1,\dots,t_n)$ where $X = \{t_1,\dots,t_n\}$, (similarly for $\varphi \in \|Q\|^\oplus$ or $\|Q\|^\vee$). For $\varphi = *(\|q_1\|_{\bowtie_1 p_1},\dots,\|q_n\|_{\bowtie_n p_n})$ (or $\vee(\|q_1\|_{\bowtie_1 p_1},\dots,\|q_n\|_{\bowtie_n p_n})$), let the set of guarded states be $\mathsf{gs}(\varphi) = \{q_1,\dots,q_n\}$. If $\varphi = \oplus(r_1,\dots,r_n)$ then the set of terms is $\mathsf{tm}(\varphi) = \{r_1,\dots,r_n\}$. In particular, if $|\mathsf{tm}(\varphi)| = 1$ then $\varphi = \oplus(r)$ is the same as r where $r = *(t_1,\dots,t_n)$. Thus, we consider $\|Q\|^*$ a special case of $\|Q\|^\oplus$.

We will see subsequently that, $\varphi \in \|Q\|^*$ represents the different probabilistic branches, whereas $\varphi \in \|Q\|^\oplus$ represents the non-determinism among the possible probabilistic branching $r \in \mathsf{tm}(\varphi)$.

Definition 7. *A p-automaton A is a tuple $(Q,\Sigma,\delta,\varphi_{in},F)$, where Q is a finite set of states, Σ is a finite alphabet (2^{AP}), $\delta : Q\times\Sigma \to B^+(Q\cup\|Q\|)$ is the transition function, $\varphi_{in} \in B^+(\|Q\|)$ is an initial condition, and $F \subseteq Q$ is an accepting set of states.*

As a convention, p-automata have states, MC have locations, and weak stochastic games have configurations. We will make the following simplification, from hereon we assume that for each $\varphi \in \|Q\|^\oplus$, if a state $q \in \mathsf{gs}(r)$ and $q \in \mathsf{gs}(r')$, where $r,r' \in \mathsf{tm}(\varphi)$ then $r = r'$. A p-automaton $A = (Q,\Sigma,\delta,\varphi_{in},F)$ defines a labeled directed graph $G_A = (Q',E,E_b,E_u)$ (called the *game graph*):

$$
\begin{array}{rcl}
Q' &=& Q\cup cl(\delta(Q,\Sigma))\\
E &=& \{(\varphi_1\wedge\varphi_2,\varphi_i)|\varphi_i \in Q'\setminus Q, 1\le i\le 2\} \,\cup\, \{(q,\delta(q,\sigma)) \mid q\in Q,\sigma\in\Sigma\}\\
 & & \{(\varphi_1\vee\varphi_2,\varphi_i)|\varphi_i \in Q'\setminus Q, 1\le i\le 2\}\\
E_u &=& \{(\varphi\wedge q,q),(q\wedge\varphi,q),(\varphi\vee q,q),(q\vee\varphi,q) \mid \varphi\in Q', q\in Q\}\\
E_b &=& \{(\varphi,q)\ |\varphi\in\|Q\|^\vee, q\in\mathsf{gs}(\varphi)\} \,\cup\, \{(\varphi,q)\ |\varphi\in\|Q\|^\oplus, q\in\mathsf{gs}(\mathsf{tm}(\varphi))\}
\end{array}
$$

where $\delta(Q,\Sigma) = \{\delta(q,\sigma) \mid q\in Q \text{ and } \sigma\in\Sigma\}\cup\{\varphi_{in}\}$.

Example 1. *Let the p-automaton $A = (Q,\Sigma,\delta,\varphi,F)$ be defined as follows: $Q = \{q_1,\dots,q_5\}$, $\Sigma = \{a,b,c\}$, $\varphi = \oplus(*(\|q_1\|_{\ge\frac{1}{2}},\|q_5\|_{\ge\frac{1}{2}}),*(\|q_2\|_{\ge 1}))$, $\delta(q_1,a) = *\|q_3\|_{\ge 1}$, $\delta(q_2,a) = *\|q_4\|_{\ge 1}$, $\delta(q_3,b) = *\|q_3\|_{\ge 1}$, $\delta(q_4,c) = *\|q_4\|_{\ge 1}$, $\delta(q_5,a) = \varphi$ and $F = Q$. The game graph is shown in Figure 3.*

We add markings on the edges to distinguish them. Edges in E_u and E are unmarked and are called *unbounded* and *simple* transitions, respectively. Edge $(\varphi,q) \in E_b$ is called a *bounded* transition and is marked with $\oplus$ if $\varphi \in \|Q\|^\oplus$, else it is marked with $\vee$. Two formulas $\varphi,\varphi' \in Q'$ are related as $\varphi \preceq_A \varphi'$ iff there is a path from φ to φ' in G_A, and let $\preceq_A \cap \preceq_A^{-1}$ be defined as $\equiv_A$. The equivalence class $[\![\varphi]\!]$ of φ with respect to $\equiv_A$ forms a maximal strongly connected component (MSCC) in G_A. An MSCC is *bounded* iff every edge in an MSCC of G_A, is either in $E\cup E_b$, and an MSCC is *unbounded* iff every edge of the MSCC is in $E\cup E_u$.

Definition 8 (uniform weak p-automata)**.** *A p-automaton A is called* uniform *if: 1.) Every MSCC of G_A is either bounded or unbounded. 2.) For every bounded MSCC marked edges are either all marked with $\oplus$ or (exclusively) with $\vee$. 3.) The set of equivalence classes $\{[\![\varphi]\!] \mid \varphi\in Q'\}$ is finite. 4.) For every symbol σ and $\varphi = \oplus(r_1,\dots,r_n)$, either every $q \in r_i$, $\delta(q,\sigma) \in \mathscr{B}^+(\|Q\|)$ or every $q \in r_i$, $\delta(q,\sigma) \in \mathscr{B}^+(Q)$. A (not necessarily uniform) p-automaton A is called* weak *if for all $q \in Q$, either $[\![q]\!]\cap Q \subseteq F$ or $[\![q]\!]\cap F = \emptyset$.*

In the rest of the paper we will only consider uniform weak p-automata.

4 Acceptance games

Let $A = (Q,\Sigma,\delta,\varphi_{in},\Omega)$ be a p-automaton and $M = (S,P,L,\mathsf{AP},s_{in})$ be an MC. The acceptance of M by A depends on the results of a sequence of (stochastic) weak games. Let $\Phi = Q\cup\mathsf{cl}(\delta(Q,\Sigma))$ be the set of formulas appearing in the vertices of the game graph G_A. Consider the partial order

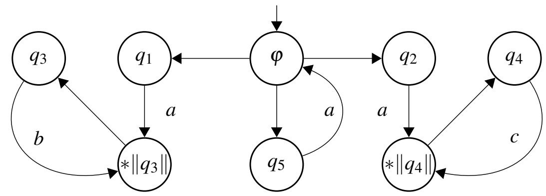

Figure 3: Game graph G_A without unbounded edges.

$$
\begin{array}{ll}
V_0^{T,\varphi} = & \{\langle T,\varphi\rangle\} \cup \{\langle T',\varphi',v\rangle \in R_{T,\varphi}\times \mathsf{Val}_{T,\varphi} \mid \bot \neq \mathsf{val}(T',\varphi') < v\} \cup \\
& \{\langle T',\varphi_1\vee\varphi_2,v\rangle \in R_{T,\varphi}\times \mathsf{Val}_{T,\varphi} \mid \mathsf{val}(T',\varphi_1\vee\varphi_2) = \bot\} \\
V_1^{T,\varphi} = & \{\langle T,\varphi,f\rangle \mid f\in \mathscr{F}^{\oplus}_{T,\varphi}\} \cup \{\langle T',\varphi',v\rangle \in R_{T,\varphi}\times \mathsf{Val}_{T,\varphi} \mid \bot \neq \mathsf{val}(T',\varphi') \geq v\} \cup \\
& \{\langle T',\varphi_1\wedge\varphi_2,v\rangle \in R_{T,\varphi}\times \mathsf{Val}_{T,\varphi} \mid \mathsf{val}(T',\varphi_1\wedge\varphi_2) = \bot\} \\
E^{T,\varphi} = & \{(\langle T,\varphi\rangle,\langle T,\varphi,f\rangle) \mid f\in \mathscr{F}^{\oplus}_{T,\varphi}\} \cup \{(\langle T',\varphi_1\circ\varphi_2,v\rangle,\langle T',\varphi_i,v\rangle) \mid \circ\in\{\wedge,\vee\}, 1\leq i\leq 2, \\
& \qquad (T',\varphi_1\circ\varphi_2,v)\in R_{T,\varphi}\times \mathsf{Val}_{T,\varphi}, \mathsf{val}(T',\varphi_1\circ\varphi_2)=\bot\} \cup \\
& \{(\langle T',\varphi',v\rangle,\langle T',\varphi'\rangle) \mid T'\in \mathsf{succ}(T), \varphi'\in[\varphi], v\in \mathsf{Val}_{T,\varphi}, \mathsf{val}(T,\varphi')=\bot\} \cup \\
& \{(\langle T,\varphi,f\rangle,\langle T',\delta(q,\sigma),f(q,T')\rangle) \mid T'\subseteq \mathsf{succ}(T), q\in I_\varphi, f(q,T')>0,\} \cup \\
& \{(\langle T,\varphi,f\rangle,\langle \{s'\},\delta(q,\sigma),f(q,s')\rangle) \mid s'\in \mathsf{succ}(T), q\in I_\varphi, f(q,T')>0, \delta(q,\sigma)\in \mathscr{B}^+(Q)\}
\end{array}
$$

Table 1: Acceptance game $\mathsf{G}(M,[\varphi])$, Case 1. $\sigma = L(T)$.

$(\Phi\backslash \equiv_A, \leq_A)$ where $[\varphi] \leq_A [\varphi]'$ iff $\varphi \preceq_A \varphi'$. Let $T \subseteq S$ non-empty set of locations, where for all $s,s' \in T$, $L(s) = L(s') = \sigma$. We assign $L(T)$ to σ. For a formula $\varphi \in \Phi$, $\mathsf{val}(T,\varphi)$ is calculated for each MSCC $[\varphi]$ inductively according to the partial order $\leq_A$. $\mathsf{val}(T,\varphi)$ is the value $\mathsf{val}_0(T,\varphi)$ of Player 0 in the game $\mathsf{G}(M,[\varphi]) = (V,E,V_0,V_1,V_p,P,\Omega)$ (defined below). When calculating $\mathsf{val}(T,\varphi)$, the value of $\mathsf{val}(T',\varphi')$ is pre-calculated for every $\varphi' \in [\varphi']$, such that $[\varphi] \leq_A [\varphi']$. Initially, we set $\mathsf{val}(T,\varphi) = \bot$. Depending on the MSCC we have the following cases:

Case 1. Let $[\varphi]$ be a non-trivial *bounded* MSCC where marked edges have marking $\oplus$. For $\varphi = \oplus(r_1,\ldots,r_n)$, let $I_\varphi = \{q \mid q \in \mathsf{gs}(r), r \in \mathsf{tm}(\varphi)\}$, and $p_{i,q}$ be the probability bound on the state q in the term r_i, i.e., $r_i = *(\|q\|_{\geq p_{i,q}} \mid q \in \mathsf{gs}(r_i))$. Consider any non empty subset of states of the Markov chain, $T \subseteq S$, such that for any $s,s' \in T, L(s) = L(s')$. Let the label of every state of T be σ. We define the set $R_{T,\varphi}$, which is the set of successor configurations of $\langle T,\varphi\rangle$, and $\mathsf{Val}_{s,\varphi}$, which is the set of possible values of $\mathsf{val}(T,\varphi)$. We need to enforce that the value of every state of $\mathsf{val}(T,\varphi)$ is well defined. Thus, if $\mathsf{val}(T',\varphi) \neq \mathsf{val}(T'',\varphi)$, then for all sets $T \supseteq T' \cup T''$, $\mathsf{val}(T,\varphi) = 0$.

$$
\begin{aligned}
R_{T,\varphi} &= \bigcup_{q\in I_\varphi} \{(T',\varphi') \mid T' \in \mathsf{succ}(T) \text{ and } \varphi' \in \mathsf{cl}(\delta(q,L(T)))\} \\
\mathsf{Val}_{T,\varphi} &= \{0,1\} \cup \{\mathsf{val}(T',\varphi') \mid \langle T',\varphi'\rangle \in R_{T,\varphi}, \mathsf{val}(T',\varphi') \neq \bot\}
\end{aligned}
\tag{2}
$$

Observe, $R_{T,\varphi}$ is finite and hence $\mathsf{Val}_{T,\varphi} \subseteq \mathbb{Q}$ is also finite. Let $\mathscr{F}^{\oplus}_{T,\varphi}$ be a set of functions $I_\varphi \times S \to \mathsf{Val}_{T,\varphi}$ where $f \in \mathscr{F}^{\oplus}_{T,\varphi}$ iff there exists a $\boldsymbol{d} \in \mathscr{D}_{\mathsf{tm}(\varphi)}$ and $\{a_{q,s'}\}_{q\in I_\varphi, s'\in S} \in R^{I_\varphi \times S}$ such that:

$$
\forall q, \forall s \in T \in I_\varphi : \sum_{s'\in succ(s)} a_{q,s'} f(q,s') P(s,s') \geq p_{i,q} \boldsymbol{d}_{r_i}, \text{ and } \forall s' \in \mathsf{succ}(s) : \sum_{q\in I_\varphi} a_{q,s'} = 1 \tag{3}
$$

$\boldsymbol{d}$ and $\{a_{q,s'}\}$ are called *witness* of the function f. Note that, the set $\mathscr{F}^{\oplus}_{s,\varphi}$ is finite, because both the domain and the range are finite sets (but can be exponential in size). The game $\mathsf{G}(M,[\varphi]) = (V,V_0,V_1,V_p,E,P,\Omega)$

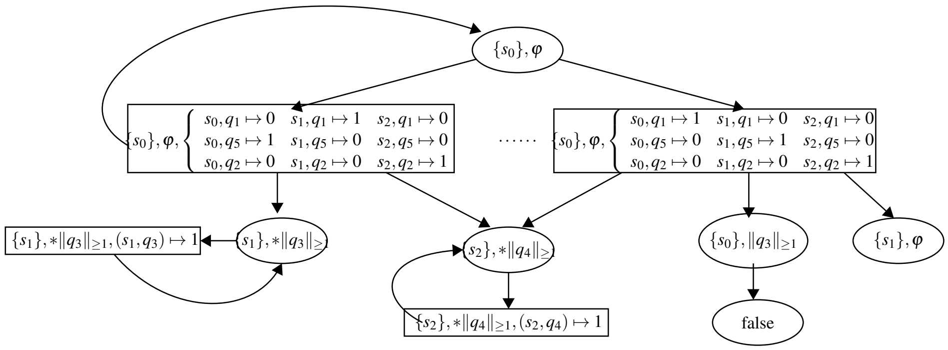

Figure 4: 2-player game (case 1.) generated by p-automaton A in Example 1 and MC M in Figure 1. The oval states are Player 0 states and the rectangle states are Player 1 states. State $\langle \{s_1\}, \varphi \rangle$ belongs to another game and $\mathsf{val}(\{s_1\}, \varphi)$ has been pre-computed.

is defined as follows:

$$V_0 \;=\; \bigcup_{T,\varphi} V_0^{T,\varphi} \quad V_1 = \bigcup_{T,\varphi} V_1^{T,\varphi} \quad V_p = \emptyset \quad E \;=\; \bigcup_{T,\varphi} E^{T,\varphi} \qquad \Omega \;=\; \emptyset \text{ or } V$$

where $V_0^{T,\varphi}, V_1^{T,\varphi}$, and $E^{T,\varphi}$ are defined in Table 1, and $\Omega = V$ if for some $q \in [t]$, $q \in F$ else $\Omega = \emptyset$. Starting from the configuration $\langle T, \varphi \rangle$, the game progresses as follows: At $\langle T, \varphi \rangle$, Player 0 selects a function $f \in \mathscr{F}^{\oplus}_{T,\varphi}$ (i.e., there exist witnesses $\{a_{q,s'}\}$ and $\boldsymbol{d}$). Player 1 can select any subset $T' \subseteq \mathsf{succ}(T)$, such that for every state $s' \in T'$ there is a $q \in I_\varphi$, such that $f(s', q) = 1$ and $\delta(q, a) \in \mathscr{B}^+(\|Q\|)$. Or, it can select $T' = \{s'\}$, where $f(s', q) > 0$ and $\delta(q, \sigma)' \in \mathscr{B}^+(Q)$. Thus, Player 1 can move to $\langle T', \delta(q, \sigma), v \rangle$, where $v = f(s', q)$ for $s' \in T$. A winning play of the game (see Figure 4) for Player 0 is determined by the following rules:

a. A finite play reaches a configuration $\langle T', \varphi', v \rangle$ such that $\mathsf{val}(s', \varphi') \neq \bot$, that is the value of the configuration $\langle s', \varphi' \rangle$, was already determined. Player 0 wins if $v \leq \mathsf{val}(T', \varphi')$ else player 1 wins. Observe again that configuration $\langle T', \varphi', v \rangle$ is a player 1 configuration if $\bot \neq v \leq \mathsf{val}(T', \varphi)$ and a player 0 configuration if $\bot \neq v > \mathsf{val}(T', \varphi')$.

b. If at $\langle T', \varphi', v \rangle$, $\mathsf{val}(T', \varphi') = \bot$ then the play continues with $\langle T', \varphi' \rangle$. An infinite play is winning if it satisfies the weak acceptance condition Ω. That is, if the play stays in V then player 0 wins if and $V \subseteq \Omega$ else player 1 wins.

Case 2. Let $[\varphi]$ be a nontrivial MSCC such that every transition in the graph G_A belonging to $[\varphi]$ are not in E_u and not marked $\oplus$. Details are present in the appendix.

Case 3. Let $[\varphi]$ be a nontrivial MSCC such that all the transitions in $[\varphi]$ of G_A are in $E_u \cup E$. This gives rise to a weak stochastic game.

$$V = \{\langle s, \varphi' \rangle \mid s \in S \text{ and } \varphi' \in [\varphi]\} \quad V_0 = \{\langle s, \varphi_1 \vee \varphi_2 \rangle \in V\} \qquad V_p = (S \times Q) \cap V$$
$$V_1 = \{\langle s, \varphi_1 \wedge \varphi_2 \rangle \in V\} \qquad P(\langle s, q \rangle, \langle s', \delta(q, L(s)) \rangle) = P(s, s') \quad \Omega = \emptyset \text{ or } V$$

where Ω is V if some q in $[\varphi]$ is in F else $\Omega = \emptyset$.

$$E = \{(\langle s, \varphi_1 \wedge \varphi_2\rangle, \langle s, \varphi_i\rangle) \in V \times V \mid 1 \leq i \leq 2\} \cup \{(\langle s, \varphi_1 \vee \varphi_2\rangle, \langle s, \varphi_i\rangle) \in V \times V \mid 1 \leq i \leq 2\}$$
$$\cup \{(\langle s, q\rangle, \langle s', \delta(q, L(s))\rangle) \in V \times V \mid P(s, s') > 0\}$$

By Theorem 1. a value $\mathsf{val}_0(s, \varphi)$ of any configuration $\langle s, \varphi\rangle \in V$ exists. We set $\mathsf{val}(\{s\}, \varphi)$ to this value.

Case 4. Let $[\varphi]$ be a trivial MSCC. It is handled as one of the above cases. The value of the configurations $\mathsf{val}(s, \varphi)$ is obtained from the $\mathsf{val}(s', \varphi')$ which have already been calculated in $\mathsf{G}(M, [\varphi'])$.

M is accepted by A, iff $\mathsf{val}(\{s_{in}\}, \varphi_{in}) = 1$*. The language of A,* $\mathscr{L}(A) = \{M : A \text{ accepts } M\}$*.*

The p-automata defined here has two notable difference than p-automata in [8]. First is the syntactic difference due to the presence of formula $\oplus(\varphi_1, \ldots, \varphi_n)$. Second is the semantic difference were we deal with sets of states of the Markov chains for a bounded MSCC (case 1.). This is crucial for proving correctness of Theorem 3. For unbounded MSCC the description of the acceptance game is same as the original definition.

The number of configurations of the weak game $\mathsf{G}(M, [\varphi])$ is exponential in the size of $[\varphi]$ and Markov chain, when $[\varphi]$ is bounded (case 1.). It is exponential in the size of automaton due to the different function $f \in \mathscr{F}^{\oplus}_{s,\varphi}$. Since, weak games can be solved in polynomial time in the size of the game (and the weak stochastic game can be solved in NP∩co-NP), the problem whether a finite Markov chain is accepted by a p-automaton can be decided in exponential time.

Next we show that the language of a extended p-automaton is closed under probabilistic bi-simulation.

Proposition 1. *For a p-automaton A and MCs* M_1 *and* M_2 *with* $M_1 \sim M_2$*,* $M_1 \in \mathscr{L}(A)$ *iff* $M_2 \in \mathscr{L}(A)$*.*

Proof. Let $M_1 = (S_1, P, \mathsf{AP}, L, s_{1,in})$ and $M_2 = (S_2, P, \mathsf{AP}, L, s_{2,in})$, with S_1 disjoint from S_2. Let $A = (Q, \Sigma, \delta, \varphi_{in}, \Omega)$, G_1 and G_2 be the acceptance game for MCs M_1 and M_2, respectively. We show that for each configurations $\langle T_1, \varphi\rangle$ and $\langle T_2, \varphi\rangle$ in G_1 and G_2, respectively, if for every $s_1 \in T_1$ there exists a $s_2 \in T_2$ such that $s_1 \sim s_2$ and vice-versa, then $\mathsf{val}(T_1, \varphi) = \mathsf{val}(T_2, \varphi)$. Towards this end, we will construct a winning strategy for player 0 in G_2 from the game G_1 and vice-versa. The details are present in the appendix.

□

Theorem 2. *The language of p-automata is closed under union, intersection, and bisimulation.*

Proof. Closure under union and intersection follows from the presence of $\vee$ and $\wedge$, respectively in the syntax. Closure under bisimulation follows from Proposition 1. □

5 Embedding MDP

In this section we will embed an MDP into an p-automaton. Let $D = (S, \Delta, \mathsf{AP}, L, s_{in})$ be an MDP.

Definition 9 (p-automata for an MDP)**.** *The p-automaton* $A_D = (Q, \Sigma, \delta, \varphi_{in}, \Omega)$ *is defined as follows:* [1]

$$Q = S \times S \quad ; \quad \Omega = Q \quad ; \quad \delta((s, s'), L(s)) = \varphi_{s'} \text{ and } \delta((s, s'), \sigma) = \mathit{false} \quad \text{if } \sigma \neq L(s)$$
$$\varphi_{in} = \oplus(r_i \mid \mu_i \in \Delta(s_{in}), r_i = *(\|(s_{in}, \mu_i, s')\|_{\geq \mu_i(s')} \mid \mu_i(s') > 0))$$
$$\varphi_s = \oplus(r_i \mid \mu_i \in \Delta(s) \text{ and } r_i = *(\|(s, \mu_i, s')\|_{\geq \mu_i(s')} \mid \mu_i(s') > 0))$$

[1] It could be the case that there is some state $q \in Q$ which a guarded state of more than one term of a formula $\varphi \in \|Q\|^{\oplus}$. This can be resolved by renaming and introducing new states.

Example 2. *The MDP in the Figure 2 is embedded in the automaton A defined in the Example 1 and the MC of Figure 1 is accepted by A.*

Theorem 3. *Let D be an MDP and A_D be its p-automaton. a.) For every scheduler η, $D_\eta \in \mathscr{L}(A_D)$ and b.) for every MC $M \in \mathscr{L}(A_D)$ there exists a $\eta \in \mathsf{HR}(D)$ such that $M \sim D_\eta$.*

Proof. We first show that if for any η, $D_\eta \in L(A_D)$, and then we show that if a Markov chain $M \in L(A_D)$, then there exists a scheduler η such that $M \sim D_\eta$.

- Let the MDP D be $(S,\Delta,\Sigma,L,s_{in})$. We will first show that for any scheduler $\eta \in \mathsf{HR}(D)$, $D_\eta = (S^+,\Sigma,P',L,s_0)$ is in $\mathscr{L}(A_D)$. We need to show that $\mathsf{val}(\{s_0\},\varphi_{s_0}) = 1$. We first prove that for any state $w \in S^+$ of D_η, with $w{\downarrow} = s$ the value $\mathsf{val}(\{w\},\varphi_s) = \mathsf{val}(\{w{\cdot}u\},\varphi_u)$ whenever $P'(w,w{\cdot}u) > 0$. Player 0 at the configuration $\langle\{w\},\varphi_s\rangle$ chooses function $f \in \mathscr{F}^{\oplus}_{\{w\},\varphi_s}$, such that the witness are as follows: $\boldsymbol{d} = \eta(w)$, $a_{q,w\cdot u} = 1$ and $f(q,w{\cdot}u) = 1$, where $q = (s,\mu,u)$. Observe, that there exists exactly one state $w{\cdot}u$, such that $f(w{\cdot}u,q) = 1$. Thus player can only move to configurations of the type $\langle\{w{\cdot}u\},\varphi_u\rangle$. Thus, $\mathsf{val}(\{w\},\varphi_s) = \mathsf{val}(\{w{\cdot}u\},\varphi_u)$. In an MSCC where non of the values are known, $\mathsf{val}(\{w\},\varphi_s) = 1$, because the every infinite path is winning. This shows, $\mathsf{val}(\{s_0\},\varphi_{s_0}) = 1$.
- Suppose a finite path $\langle T_0,\varphi_{s_0}\rangle,\ldots,\langle T_n,\varphi_{s_n}\rangle$ is winning for Player 1. That is at $\langle T_n,\varphi_{s_n}\rangle$ it is not the case that Player 0 can find a distribution $\boldsymbol{d}$ such that,

  $$\forall r_i \in \mathsf{tm}(\varphi_{s_n})\ \forall q \in \mathsf{gs}(r_i)\ \forall s \in T_n : \sum_{s' \in \mathsf{succ}(s)} a_{q,s'} f(q,s') = p_{i,q}\boldsymbol{d}_i$$

  and for each $q \in \mathsf{gs}(\mathsf{tm}(\varphi_{s_n}))$ and any set $T' \subseteq \mathsf{succ}(T_n)$, where $\forall s' \in T' : f(q,s') = 1$, $\langle T',\varphi_{s_n}\rangle$ is winning for Player 0. Take any other (arbitrary) play $\langle T'_0,\varphi_{s_0}\rangle,\ldots,\langle T'_n,\varphi_{s_n}\rangle$ (with $T_0 = T'_0 = \{t_0\}$). Then $\langle T_0 \cup T'_0,\varphi_{s_0}\rangle,\ldots,\langle T_n \cup T'_n,\varphi_{s_n}\rangle$ is also winning for Player 1. So it is in her best interest to choose T' as large as possible

  Let $M = (T,\Sigma,P,L,t_0)$, and $M \in \mathscr{L}(A_D)$. The value of configuration $\langle\{t_0\},\varphi_{s_0}\rangle$ is 1, and assume Player 1 plays optimally, i.e., she chooses a set as large as possible. We will construct a map $\eta^\star \subseteq (S^+ \times \mathscr{D}_{\mathscr{D}_s})$. For any possible finite run, $\rho_n = \langle T_0,\varphi_{s_0}\rangle,\ldots,\langle T_n,\varphi_{s_n}\rangle$, with $T_0 = \{s_0\}$, $(s_0,\ldots,s_n,\boldsymbol{d}) \in \eta^\star$, where $\boldsymbol{d}$ is the distribution chosen by Player 0 at $\langle T_n,\varphi_{s_n}\rangle$. Since, Player 1 plays optimally, it cannot be the case that two distinct play $\rho_n = \langle T_0,\varphi_{s_0}\rangle,\ldots,\langle T_n,\varphi_{s_n}\rangle$ and $\rho'_n = \langle T'_0,\varphi_{s_0}\rangle,\ldots,\langle T'_n,\varphi_{s_n}\rangle$ exists. Thus, we see that $\eta^\star \in \mathsf{HR}(D)$.

  Now consider an unrolling of M. Thus, states of M are subsets of T^+. It suffices to show a bisimulation relation between, $D_{\eta^\star}$ and the unrolled M. Let $R \subseteq (T^+ \cup S^+) \times (T^+ \cup S^+)$ be the smallest transitive, reflexive and symmetric relation with the following property:

  - $t_0 R s_0$.
  - For each play $\rho_n = \langle T_0,\varphi_{s_0}\rangle,\ldots,\langle T_n,\varphi_{s_n}\rangle\langle T_{n+1},\varphi_s\rangle$, all $t \in T_{n+1}$, tRs.

  We will show that R is a bi-simulation relation.

  - If tRw then $L(t) = L'(w)$. If $L(t) \neq L'(w)$ then $\langle t,\varphi_{w\downarrow}\rangle$ cannot be winning for Player 0.
  - For each $q \in I_{\varphi_{w\downarrow}}$, we know, $\sum_{t' \in \mathsf{succ}(t)} P(t,t') a_{q,t'} f(q,t') = p_{q,i}\boldsymbol{d}_i$. From this we can deduce, $\sum_{t' \in C,(t',w\cdot s') \in R} P(t,t') = \sum_{w\cdot s' \in C} P'(w,w{\cdot}s')$ (see Appendix for details).

  Thus, R is a bi-simulation relation , and $M \sim D_{\eta^\star}$.

□

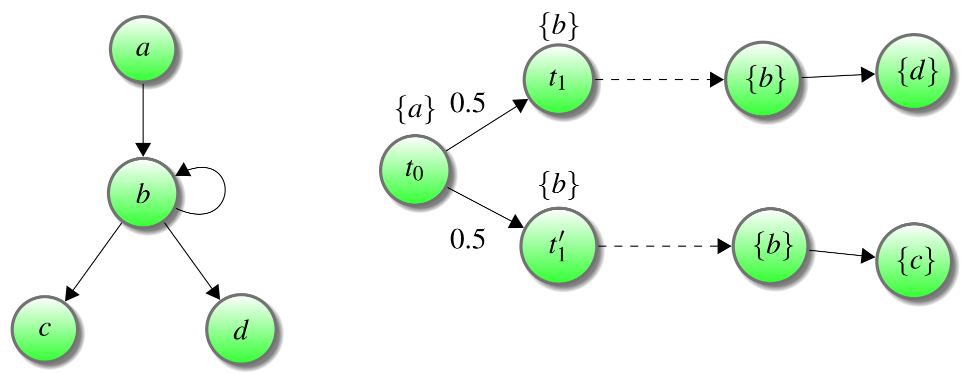

Figure 5: The MDP (left) and a Markov chain (right).

The embedding of MDP relies on the construct $\varphi \in \|Q\|^{\oplus}$. Consider the MDP in Figure 2. At the state s_0 there are two choices of distribution. If we limit the definition of the p-automata to [8] then we have only disjunction (or conjunction) to define the non-determinism at the state s_0 and we cannot accept the MC in Figure 1. We also store the subset of states T that were *induced* by the same $q \in I_\varphi$. Refer to the Figure 5. We need to remember that states t_1 and t'_1, were induced by the same distribution. We end this section by mentioning that any PCTL formula can be embedded as a p-automaton. That is, given any PCTL formula, we can construct a p-automaton such that the set models of the formula is exactly the language of the automaton [8].

6 Conclusion

We have presented an extension of the p-automata [8], and used it to represent the set of MCs which are bisimilar to the MCs induced by the schedulers of an MDP. We have seen that the languages of the p-automata are closed under bi-simulation (union and intersection, trivially). We have addressed the issue of non-determinism of the probability distribution, observed in the concluding remark of [8], and shown that the p-automata are powerful enough to represent various probabilistic systems and logics. Even though the acceptance is still EXPTIME, the number configuration has become also exponential in the size of the Markov chain. Unfortunately, the simulation relation gives only a sound characterize language inclusion. It would be interesting to investigate well behaving fragments for which the simulation relation exactly characterizes language inclusion.

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7 Bibliography

References

[1] Andrea Bianco & Luca de Alfaro (1995): *Model Checking of Probabalistic and Nondeterministic Systems*. In: *Foundations of Software Technology and Theoretical Computer Science, 15th Conference*, pp. 499–513.

[2] Patrick Billingsley: *Probability and Measure*.

[3] T. Brazdil, V. Brozek, V. Forejt & A. Kucera (2006): *Stochastic games with branching-time winning objectives*. In: *Logic in Computer Science, 2006 21st Annual IEEE Symposium on*, pp. 349–358.

[4] Krishnendu Chatterjee, Marcin Jurdziński & Thomas A. Henzinger (2004): *Quantitative Stochastic Parity Games*. In: *Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '04, pp. 121–130.

[5] Anne Condon (1992): *The Complexity of Stochastic Games*. *Information and Computation* 96, pp. 203–224.

[6] E.Allen Emerson, Charanjit S. Jutla & A.Prasad Sistla (2001): *On model checking for the μ-calculus and its fragments*. *Theoretical Computer Science* 258(12), pp. 491 – 522.

[7] Hans Hansson & Bengt Jonsson (1994): *A logic for reasoning about time and reliability*. *Formal Aspects of Computing* 6(5), pp. 512–535.

[8] Michael Huth, Nir Piterman & Daniel Wagner (2012): *p-Automata: New foundations for discrete-time probabilistic verification*. *Performance Evaluation* 69(78), pp. 356 – 378. Selected papers from {QEST} 2010.

[9] Dexter Kozen (1983): *Results on the propositional μ-calculus*. *Theoretical Computer Science* 27(3), pp. 333 – 354. Special Issue Ninth International Colloquium on Automata, Languages and Programming (ICALP) Aarhus, Summer 1982.

[10] Kim G. Larsen & Arne Skou (1991): *Bisimulation through probabilistic testing*. *Information and Computation* 94(1), pp. 1 – 28.

[11] Donald A. Martin (1975): *Borel Determinacy*. *Annals of Mathematics* 102(2), pp. pp. 363–371.

[12] Roberto Segala & Nancy Lynch (1995): *Probabilistic Simulations for Probabilistic Processes*. *Nordic J. of Computing* 2(2), pp. 250–273.

[13] Thomas Wilke (2001): *Alternating tree automata, parity games, and modal μ-calculus*. *Bull. Belg. Math. Soc. Simon Stevin* 8(2), pp. 359–391.

Appendix

Acceptance game: Case 2

Case 2. Let $[t]$ be a nontrivial bounded MSCC with $\vee$ marked edges. This give rise to an stochastic weak game and is same for p-automata with out $\|Q\|^{\oplus}$. Consider $\varphi \in [t] \cap \|Q\|^{\vee}$, that is $\varphi = \vee(\|q_1\|_{\bowtie_1 p_1}, \ldots, \|q_n\|_{\bowtie_n p_n})$. The sets $R_{s,\varphi}, \mathsf{Val}_{s,\varphi}$ are same as (2), $V_0^{s,\varphi}, V_1^{s,\varphi}, E^{s,\varphi}$ is same as in table 1, with the only difference is the set of functions $\mathscr{F}^{\vee}_{s,\varphi} = I_\varphi \times \succ(s) \to \mathsf{Val}_{s,\varphi}$ (rather than $\mathscr{F}^{\oplus}_{s,\varphi}$). A function $f \in \mathscr{F}^{\vee}_{s,\varphi}$ if there exists $a \in \mathbb{R}^{I_\varphi} \times \mathbb{R}^{S}$ such that:

- there is a $q \in I_\varphi$ with $\sum_{s' \in \mathsf{succ}(s)} a_{q,s} f(q,s) P(s,s') \geq p_i$ or,
- there is a $s \in \mathsf{succ}(s)$ with $\sum_{q \in I_\varphi} a_{q,s'} \neq 1$.

The winning condition is same as case 1. In this paper we will not need the terms in $\|Q\|^{\vee}$ and present it here only for completeness.

Proposition 1

For a p-automaton A and MCs M_1 and M_2 with $M_1 \sim M_2$. $M_1 \in \mathscr{L}(A)$ iff $M_2 \in \mathscr{L}(A)$.

Proof. Let $M_1 = (S_1, P, L, s_{1,in})$ and $M_2 = (S_2, P, L, s_{2,in})$, with S_1 disjoint from S_2, hence we use the same function P and L for both MCs with impunity. Let A be $(Q, \Sigma, \delta, \varphi_{in}, \Omega)$, G_1 and G_2 be the acceptance game for MCs M_1 and M_2, respectively. We show that for each configurations $\langle s_1, \varphi\rangle$ and $\langle s_2, \varphi\rangle$ in G_1 and G_2, respectively, if $s_1 \sim s_2$ then $\mathsf{val}(s_1, \varphi) = \mathsf{val}(s_2, \varphi)$, for $[\varphi]$ is a unbounded MSCC. Equivalently, we construct a wining strategy π_2 for Player 0 in G_2 from the winning strategy π_1 of Player 0 in G_1. By symmetry of the argument (presented below), it also follows that we can construct a wining strategy for Player 0 in G_1 from the winning strategy of Player 0 in G_2.

$\mathsf{G}_i(M_i, [\varphi])$ is a stochastic weak game (for $i \in \{1,2\}$). We start from the configurations $c_1 = \langle s_1, \varphi\rangle$ and $c_2 = \langle s_2, \varphi\rangle$ where $s_1 \in S_1$ and $s_2 \in S_2$ and $s_1 \sim s_2$. The claim is, at each step of any play of the games, we move to configurations $\langle s_1', \varphi'\rangle$ and $\langle s_2', \varphi'\rangle$ in G_1 and G_2 (according to strategy π_1 and π_2), respectively, where $s_1' \sim s_2'$.

When φ is of the form $\varphi_1 \wedge \varphi_2$, c_1 and c_2 are Player 1 configurations. If Player 1 chooses (s_2, φ_i) in G_2 then we make Player 1 in G_1 choose (s_1, φ_i) for $i \in \{1,2\}$. When φ is of the form $\varphi_1 \vee \varphi_2$, c_1 and c_2 are Player 0 configuration, Player 0 in G_2 follows the choice of Player 0 in G_1, i.e., if Player 0 chose $\langle s_1, \varphi_i\rangle$ then Player 0 in G_2 chooses $\langle s_2, \varphi_i\rangle$ in G_2 (for $i \in \{1,2\}$). For $\varphi = q \in Q$, the play is resolved by a probabilistic choice. We know that $P(s_1, C_1) = P(s_2, C_2)$ where $C_i \subseteq S_i$ (for $i \in \{1,2\}$) and $C_1 \cup C_2$ is an equivalence class of $\sim$. Thus, for any play that ends in $\langle s_1', \delta(q,\sigma)\rangle$ in G_1, there is a corresponding play in G_2 that ends in $\langle s_2', \delta(q,\sigma)\rangle$, and we have $s_1' \sim s_2'$ where $\sigma = L(s_1) = L(s_2)$. Hence the set of plays that are winning in G_1 have the same probability measure as the set of corresponding play in G_2. Consequently, $\mathsf{val}(s_1, \varphi) = \mathsf{val}(s_2, \varphi)$.

Let $[\varphi]$ be a *bounded* MSCC of G_A where the only marked edges have $\oplus$ as markings. Consider $T_1 \subseteq S_1$ and $T_2 \subseteq S_2$, such that for each $s_1 \in T_1$, there exist $s_2 \in T_2$ such that $s_1 \sim S_2$ and vice-versa. We show that if $\mathsf{val}(T_1, \varphi) = 1$ then $\mathsf{val}(T_2, \varphi) = 1$. Disjunction and conjunctions are handled as before. Let $\varphi \in \|Q\|^{\oplus}$. We have a function $f_1 \in \mathscr{F}^{\oplus}_{s_1,\varphi}$ with witness $\boldsymbol{d}$ and $\{a_{q,s'}\}_{q \in I_\varphi, s' \in \mathsf{succ}(s)}$ for the play in G_1. Define $f_2 : I_\varphi \times \mathsf{succ}(s_2) \to [0,1]$ with $f_2(q, s_2') = f_1(q, s_1')$ for $s_j' \in \mathsf{succ}(s_j)$ $(j \in \{1,2\})$ if $s_1' \sim s_2'$. It remains to show that $f_2 \in \mathscr{F}^{\oplus}_{s_2,\varphi}$. That is, we need to find suitable witnesses $\boldsymbol{d}'$ and $\{a'_{q,s_2'}\}_{q \in I_\varphi, s_2' \in \mathsf{succ}(s_2)}$

for f_2. Let $\boldsymbol{d}' = \boldsymbol{d}$ and choose $\{a'_{q,s'}\}$ such that for each $q \in I_\varphi$ and an equivalence class C:

$$\sum_{s'_1 \in C} a_{q,s'_1} P(s_1, s'_1) \;=\; \sum_{s'_2 \in C} a'_{q,s'_2} P(s_2, s'_2) \tag{4}$$

There could be many possible solution for $\{a'_{q,s'}\}$, we need to find one solution such that $f_2 \in \mathscr{F}^{\oplus}_{s,\varphi}$. For each $q \in I_\varphi$:

$$\begin{aligned}
&\sum_{s'_2 \in \mathsf{succ}(s_2)} a'_{q,s'_2} P(s_2, s'_2) f_2(q, s'_2) && = \sum_{C \in S_2 \cup S_1 \setminus \sim} \left(\sum_{s'_2 \in C} a'_{q,s'_2} P(s_2, s'_2) f_2(q, s'_2) \right) \\
= &\sum_{C \in S_2 \cup S_1 \setminus \sim} \left(\sum_{s'_1 \in C} a_{q,s'_1} P(s_1, s'_1) f_1(q, s'_1) \right) && = \sum_{s'_1 \in \mathsf{succ}(s_1)} a'_{q,s'_1} P(s_1, s'_1) f_1(q, s'_1) = \; p_{i,q} \boldsymbol{d}'(r_i).
\end{aligned}$$

Thus any value of $\{a'_{q,s'_2}\}$ satisfying the constraint (4) also satisfies the first condition of equation (3). Next we show that under the constraint (4) there exist values for $\{a'_{q,s'_2}\}$ such that $\sum_{q \in I_\varphi} a'_{q,s'_2} = 1$ for all $s'_2 \in \mathsf{succ}(s_2)$. For each equivalence class C of $\sim$, starting from equation 4, we can deduce:

$$\begin{aligned}
& \sum_{s'_2 \in C} a'_{q,s'_2} P(s_2, s'_2) && = \sum_{s'_1 \in C} a_{q,s'_1} P(s_1, s'_1) \\
& \text{Summing over } q \in I_\varphi && \\
\mathit{Or}, \quad & \sum_{q \in I_\varphi} \sum_{s'_2 \in C} a'_{q,s'_2} P(s_2, s'_2) && = \sum_{q \in I_\varphi} \sum_{s'_1 \in C} a_{q,s'_1} P(s_1, s'_1) \\
& \text{Changing the order of summation} && \\
\mathit{Or}, \quad & \sum_{s'_2 \in C} \sum_{q \in I_\varphi} a'_{q,s'_2} P(s_2, s'_2) && = \sum_{s'_1 \in C} \sum_{q \in I_\varphi} a_{q,s'_1} P(s_1, s'_1) \\
\mathit{Or}, \quad & \sum_{s'_2 \in C} P(s_2, s'_2) \sum_{q \in I_\varphi} a'_{q,s'_2} && = \sum_{s'_1 \in C} P(s_1, s'_1) \sum_{q \in I_\varphi} a_{q,s'_1} \\
\mathit{Or}, \quad & \sum_{s'_2 \in C} P(s_2, s'_2) \sum_{q \in I_\varphi} a'_{q,s'_2} && = \sum_{s'_1 \in C} P(s_1, s'_1) \\
\mathit{Or}, \quad & \sum_{C \in S_1 \cup S_2 \setminus \sim} \left(\sum_{s'_2 \in C} P(s_2, s'_2) \sum_{q \in I_\varphi} a'_{q,s'_2} \right) && = \sum_{C \in S_1 \cup S_2 \setminus \sim} \left(\sum_{s'_1 \in C} P(s_1, s'_1) \right) \\
\mathit{Or}, \quad & \sum_{s'_2 \in \mathsf{succ}(s_2)} P(s_2, s'_2) \left(\sum_{q \in I_\varphi} a'_{q,s'_2} \right) && = 1
\end{aligned}$$

One such solution of the equation is when $\sum_{q \in I_\varphi} a'_{q,s'_2} = 1$ for every $s'_2 \in \mathsf{succ}(s_2)$.

If now Player 1 in G_2 chooses $\langle T'_2, \delta(q, \sigma), v \rangle$, Player 1 in G_1 is made to choose T_1, such that for each s'_2 in T'_2 there exist $s'_1 \in T_1$, such that $s'_1 \sim s'_2$. This possible because of our choice of f_2.

The case of bounded MSCC with only $\vee$ marked edges is handled similarly. □

Theorem 3

We know, for each $q \in I_{\varphi_{w\downarrow}}$:

$$\sum_{t' \in \mathsf{succ}(t)} P(t, t') a_{q,t'} f(q, t') \;=\; p_{q,i} \boldsymbol{d}_i$$

Or,

$$\sum_{C \in (T \cup S^+) \setminus R} \; \sum_{t' \in C} P(t, t') a_{q,t'} f(q, t') \;=\; p_{q,i} \boldsymbol{d}_i.$$

Let $\rho_n = \langle T_0, \varphi_{s_0}\rangle, \ldots, \langle T_n, \varphi_{s_n}\rangle\langle T_{n+1}, \varphi_s\rangle$, and $w = s_0, \ldots, s_n$. For each $q \in I_{\varphi_{w\downarrow}}$, $q = (w\downarrow, \mu, s')$ $a_{q,t'} > 0$ iff $f(q,t') = 1$ and $(t', w{\cdot}s') \in R$.

$$\sum_{C\in(T^+\cup S^+)\setminus R} \sum_{t'\in C} P(t,t')a_{q,t'}f(q,t') \;=\; \sum_{t'\in C,(t',w\cdot s')\in R} P(t,t')a_{q,t'}f(q,t')$$

Hence,

$$\sum_{t'\in C,(t',w\cdot s')\in R} P(t,t')a_{q,t'}f(q,t') \;=\; p_{q,i}\boldsymbol{d}_i \;=\; P'(w,w{\cdot}s').$$

Summing over all $q \in I_{\varphi_{w\downarrow}}$, i.e. $w{\cdot}s' \in C$,

$$\sum_{w\cdot s'\in C} \sum_{t'\in C,(t',w\cdot s')\in R} P(t,t')a_{q,t'}f(q,t') \;=\; \sum_{w\cdot s'\in C} P'(w,w{\cdot}s').$$

Changing the order of summations,

$$\sum_{t'\in C,(t',w\cdot s')\in R} P(t,t') \sum_{w\cdot s'\in C} a_{q,t'}f(q,t') \;=\; \sum_{w\cdot s'\in C} P'(w,w{\cdot}s').$$

For $q' \notin C$, $a_{q',t} = 0$, hence $\sum_{q\in C} a_{q,t} = \sum_{q\in I_{\varphi_{w\downarrow}}} a_{q,t}$.

$$\sum_{t'\in C,(t',w\cdot s')\in R} P(t,t') \sum_{w\cdot s'\in I_{\varphi_{w\downarrow}}} a_{q,t'}f(q,t') \;=\; \sum_{w\cdot s'\in C} P'(\rho,\rho{\cdot}s').$$

Or,

$$\sum_{t'\in C,(t',w\cdot s')\in R} P(t,t') \;=\; \sum_{w\cdot s'\in C} P'(w,w{\cdot}s').$$

Theorem ??

Let A_1 and A_2 be p-automata. Then:

$$A_1 \leq A_2 \text{ implies } \mathscr{L}(A_1) \subseteq \mathscr{L}(A_2).$$

Proof. Let $M = (S, P, \Sigma, L, s_{in})$ be an arbitrary MC and A_1, A_2 be p-automata $(Q, \Sigma, \delta, \varphi_{in}, F)$, $(U, \Sigma, \delta, \psi_{in}, F)$, respectively. We assume that Q and U are disjoint and hence use the same symbol for the transition relations and final states for the two automata.

We show that, if $\mathsf{val}(s_{in}, \varphi_{in}) = 1$ in the acceptance game of M by A_1 and $\mathsf{val}(\varphi_{in}, \psi_{in}) = 1$ in the simulation game of A_1 by A_2 then $\mathsf{val}(s_{in}, \psi_{in}) = 1$ in the acceptance game of M by A_2. Let the acceptance games of M by A_1 and A_2 be G_1 and G_2, respectively, and the simulation game of A_1 by A_2 be $\mathsf{G}_{\leq}$. Equivalently, we show that the *claim:* $\mathsf{val}(s,\varphi){\cdot}\mathsf{val}(\varphi,\psi) \leq \mathsf{val}(s,\psi)$, is true for any arbitrary $s \in S, \varphi \in Q \cup \mathsf{cl}(\delta(Q,\Sigma))$ and $\psi \in U \cup \mathsf{cl}(\delta(U,\Sigma))$. A triplet of configurations c_1, c_2 and c_3 is said to be *matching*, where c_1, c_2 and c_3 are configurations of the game $\mathsf{G}_1, \mathsf{G}_{\leq}$ and G_2, respectively, if the first component of c_1 is equal to the first component of c_3, the second component of c_1 is equal to the second component of c_2 and the second component of c_2 is equal to the second component of c_3 ($c_1 = \langle s,\varphi\rangle, c_2 = \langle\varphi,\psi\rangle, c_3 = \langle s,\psi\rangle$) .

We proceed by induction on the partial order $\preceq$, and when considering configurations in $([\varphi],[\psi])$, we assume that the claim holds for every configuration in the pair $([\varphi'],[\psi'])$, where $([\varphi],[\psi]) \preceq ([\varphi'],[\psi'])$.

Effectively, we construct a winning strategy for Player 0 in G_2 from the strategies of the Players in G_1 and $\mathsf{G}_{\leq}$. We have the following cases:
Case 1. If $\varphi \in Q$ and $\psi \in \|U\|^{\oplus}$ then $\mathsf{val}(\varphi,\psi) = 0$, and the claim follows trivially.

Case 2. Let $[\varphi]$ and $[\psi]$ be unbounded MSCCs, where $\mathsf{G}_1(M,[\varphi])$ and $\mathsf{G}_2(M,[\psi])$ are weak stochastic game and $\mathsf{G}_{\leq}([\varphi],[\psi])$ is stochastic game. Consider three matching configurations $c_1 = (s,\alpha)$, $c_2 = (\alpha,\beta)$ and $c_3 = (s,\beta)$, such that $\alpha \in [\varphi]$ and $\beta \in [\psi]$.

If $\alpha = \alpha_1 \wedge \alpha_2$ and β is not a conjunction then c_2 is a Player 0 configuration. Suppose Player 0 at c_2 chose $\langle \alpha_i, \beta\rangle$, then Player 1 at c_1 is made to choose (s,α_i). Else if $\beta = \beta_1 \wedge \beta_2$ then c_3 is a Player 1 configuration and if he chose $\langle s, \beta_i\rangle$ then Player 1 at c_2 chooses $\langle \alpha, \beta_i\rangle$. If $\alpha = \alpha_1 \vee \alpha_2$ (c_1 is a Player 0 configuration) and if she chose $\langle s,\alpha_i\rangle$ at c_1 then Player 1 at c_2 chooses $\langle \alpha_i,\beta\rangle$. If $\beta = \beta_1 \vee \beta_2$ and Player 0 chooses $\langle \alpha,\beta_i\rangle$ at c_2 then Player 0 in c_3 chooses (s,β_i). If $\alpha = q$ and $\beta = u$ then c_1 and c_3 are stochastic configurations and c_2 is a Player 1 configuration. Player 1 is made to select the action $\sigma = L(s)$ and reach a configuration $\langle \delta(q,\sigma), \delta(u,\sigma)\rangle$ and next configuration in G_1 and G_2 is $\langle s', \delta(q,\sigma)\rangle$ and $\langle s', \delta(u,\sigma)\rangle$, respectively. Note that these choices of moves always ensures that we move from one matching triplet to the next.

Consider three matching paths in the games G_1, $\mathsf{G}_{\leq}$ and G_2. If the path in $\mathsf{G}_{\leq}$ is infinite then, and the corresponding path in G_1 is winning, then by the winning condition of $\mathsf{G}_{\leq}$, the respective path in G_2 is also winning. If it is finite then the triplet of paths end in configuration $(\langle s'',\alpha'\rangle, \langle \alpha',\beta'\rangle, \langle s'',\beta'\rangle)$, where $\langle \alpha',\beta'\rangle \notin ([\varphi],[\psi])$. Since, $([\varphi],[\psi])$ is a weak game $\mathsf{val}(\alpha',\beta') \geq \mathsf{val}(\alpha,\beta)$. By assumption $\mathsf{val}(s'',\alpha')\cdot\mathsf{val}(\alpha',\beta') \leq \mathsf{val}(s'',\beta')$ or $\mathsf{val}(s'',\alpha')\cdot\mathsf{val}(\alpha,\beta) \leq \mathsf{val}(s'',\beta')$. The inequality holds for every matching paths in all three games thus, $\mathsf{val}(s,\alpha)\cdot\mathsf{val}(\alpha,\beta) \leq \mathsf{val}(s,\beta)$.

Case 3. Suppose $[\varphi]$ and $[\psi]$ are bounded MSCCs, $G_1(M,[\varphi])$, $G_{\leq}([\varphi],[\psi])$ and $G_2(M,[\psi])$ are all weak games. Consider a triplet of configurations $(\langle s,\alpha\rangle, \langle \alpha,\beta\rangle, \langle s,\beta\rangle)$. We assume $\mathsf{val}(s,\alpha) = 1$ and $\mathsf{val}(\alpha,\beta) = 1$, else $\mathsf{val}(s,\alpha)\cdot\mathsf{val}(\alpha,\beta) \leq \mathsf{val}(s,\beta)$ follows immediately.

The cases of conjunctions and disjunctions are handled as in *case 2*. The interesting case is when $\alpha \in \|Q\|^{\oplus}$ and $\beta \in \|U\|^{\oplus}$, where $\alpha = \oplus(r_1,\dots,r_n)$ and $\beta = \oplus(t_1,\dots,t_m)$. It follows that $\langle s,\alpha\rangle$ and $\langle s,\beta\rangle$ are Player 0 configurations and $\langle \alpha,\beta\rangle$ is a Player 1 configuration. Suppose Player 0 at $\langle s,\alpha\rangle$ selects a function f with witness $\boldsymbol{d}$ and $\{a_{q,s'}\}$, where $s' \in \mathsf{succ}(s)$, $q \in I_\alpha$.

$$\forall r_i \in \mathsf{tm}(\alpha), q \in \mathsf{gs}(r_i) \sum_{s' \in \mathsf{succ}(s)} a_{q,s'} f(s',q) P(s,s') \geq p_{i,q} \boldsymbol{d}_{r_i} \text{ and } \forall s' \in \mathsf{succ}(s) \sum_{q \in I_\alpha} a_{q,s'} = 1$$

We make Player 1 of $G_{\leq}$ choose an action $\sigma = L(s)$ and move to Player 0 configuration $\langle \alpha,\beta,\sigma\rangle$. Configuration $\langle r_i,\beta,\sigma\rangle$ is winning for Player 0, for each $r_i \in \mathsf{tm}(\alpha)$, in the game $\mathsf{G}_{\leq}$. Let $f^i \in \mathscr{F}^{\oplus}_{r_i,\beta}$ be the function with witness $\{a^i_{q,u}\}_{q \in I_{r_i}, u \in I_\beta}$ and $\boldsymbol{c}^i \in \mathscr{D}_{\mathsf{tm}(\beta)}$. Thus for each r_i we have:

$$\forall u \in I_\beta \sum_{q \in I_{r_k}} a_{q,u} f^i(q,u) p_{i,q} \geq \boldsymbol{c}^i_{t_j} \cdot p_{j,u}. \text{ and } \forall q \in I_{r_i} \sum_{u \in I_\beta} a_{q,u} = 1.$$

Player 0 in game G_2 selects a function f'' such that $f''(u,s')$ is then the minimum value in $\mathsf{Val}_{s',\beta}$ that is at least $\max_{q \in \mathsf{gs}(r_i), r_i \in \mathsf{tm}(\alpha)} f(q,s') f^i(q,u)$. The reason for choosing such f'' will soon become clear. The witness of f'' are as follows: $a_{u,s'} = \sum_{r_i \in \mathsf{tm}(\alpha)} \sum_{q \in \mathsf{gs}(r_i)} a_{q,s'} a^i_{q,u}$ and for each $t_j \in \mathsf{tm}(\beta)$, $\boldsymbol{d}'_{t_j} = \sum_{r_i \in \mathsf{tm}(\alpha)} \boldsymbol{d}_{r_i} \cdot \boldsymbol{c}^i_{t_j}$. Intuitively, Player 0 in G_1 gives the distribution $\boldsymbol{d}$ on the guarded terms $r_k \in \mathsf{tm}(\alpha)$ and in the game $\mathsf{G}_{\leq}$ gives the distribution to simulate each r_i by β. This determines the distribution $\boldsymbol{d}'$ for the game G_2.

We will now show that $f'' \in \mathscr{F}^{\oplus}_{s,\beta}$. For each $s' \in \mathsf{succ}(s)$:

$$\sum_{u \in I_\beta} a_{u,s'} = \sum_{r_i \in \mathsf{tm}(\alpha)} \sum_{q \in \mathsf{gs}(r_i)} a_{q,s'} \left(\sum_{u \in I_\beta} a_{q,u} \right) = \sum_{q \in I_\alpha} a_{q,s'} = 1.$$

Consider any $u \in \mathsf{gs}(t_j)$, where $t_j \in \mathsf{tm}(\beta)$:

$$\begin{aligned}
& \sum_{s' \in \mathsf{succ}(s)} a_{u,s'} f''(u,s') P(s,s') \\
= & \sum_{s' \in \mathsf{succ}(s)} \left(\sum_{r_i \in \mathsf{tm}(\alpha)} \sum_{q \in \mathsf{gs}(r_i)} a_{q,s'} a^i_{q,u} \right) f''(u,s') P(s,s') \\
= & \sum_{s' \in \mathsf{succ}(s)} \left(\sum_{r_i \in \mathsf{tm}(\alpha)} \sum_{q \in \mathsf{gs}(r_i)} a_{q,s'} a^i_{q,u} \right) \max_{q \in \mathsf{gs}(r_i), r_i \in \mathsf{tm}(\alpha)} f(u,s') f^i(q,u) P(s,s') \\
\geq & \sum_{s' \in \mathsf{succ}(s)} \sum_{r_i \in \mathsf{tm}(\alpha)} \sum_{q \in \mathsf{gs}(r_i)} a_{q,s'} a^i_{q,u} f(q,s') f^i(q,u) P(s,s') \\
= & \sum_{r_i \in \mathsf{tm}(\alpha)} \sum_{q \in \mathsf{gs}(r_i)} a^i_{q,u} f^i(q,u) \sum_{s' \in \mathsf{succ}(s)} a_{q,s'} f(q,s') P(s,s') \\
\geq & \sum_{r_i \in \mathsf{tm}(\alpha)} \sum_{q \in \mathsf{gs}(r_i)} a^i_{q,u} f^i(q,u) \boldsymbol{d}_{r_i} p_{i,q} \\
= & \sum_{r_i \in \mathsf{tm}(\alpha)} \boldsymbol{d}_{r_i} \sum_{q \in \mathsf{gs}(r_i)} a_{q,u} f'(q,u) p_{i,q} \\
\geq & \sum_{r_i \in \mathsf{tm}(\alpha)} \boldsymbol{d}_{r_i} \boldsymbol{c}^i_{t_j} p_{j,u} \\
= & \boldsymbol{d}'_{t_j} p_{j,u}
\end{aligned}$$

If Player 1 in G_2 chooses $\langle s', \delta(u,\sigma), f''(u,s') \rangle$, we make Player 1 in G_1 choose a state $q \in I_\beta$ (and hence a term r_i such that $q \in \mathsf{gs}(r_i)$) such that $f(q,s') f^i(q,u)$ is maximal and move to $\langle s', \delta(q,\sigma), f(q,s') \rangle$, correspondingly, we make Player 1 in $G_\leq$ to move to $\langle \delta(q,\sigma), \delta(u,\sigma), f^i(q,u) \rangle$.

Consider a triplet of matching paths from G_1, $\mathsf{G}_\leq$ and G_2. Suppose the play continues inside of the MSCC pair of $\mathsf{G}_\leq$, indefinitely. Then the play in G_1 is winning because the play is according to a winning strategy of Player 0 in G_1, for the same reason the play in $\mathsf{G}_\leq$ is winning. Because of the winning condition of $\mathsf{G}_\leq$, the corresponding play in G_2 is also winning.

Suppose the plays in $\mathsf{G}_\leq$ reach a triplet of configurations $(\langle s'', \alpha'', v_1 \rangle, \langle \alpha'', \beta'', v_2 \rangle, \langle s'', \beta'', v_3 \rangle)$, where $([\alpha''], [\beta'']) \neq ([\varphi], [\psi])$. We have $\mathsf{val}(s'', \alpha'') \cdot \mathsf{val}(\alpha'', \beta'') \leq \mathsf{val}(s'', \beta'')$ from the induction hypothesis. We have to show $\mathsf{val}(s'', \beta'') \geq v_3$. Let the triplet of configurations $(\langle s', \alpha' \rangle, \langle \alpha', \beta' \rangle, \langle s', \beta' \rangle)$ be the last configurations such that $\langle \alpha', \beta' \rangle$ is inside the MSCC pair $([\varphi], [\psi])$. Player 1 in G_1 at configuration $\langle s', \alpha' \rangle$ chooses a q such that $f(q,s') f^i(q,u)$ is maximum. As the plays in G_1 and $\mathsf{G}_\leq$ are winning for Player 0, $\mathsf{val}(s'', \alpha'') \geq v_1$ and $\mathsf{val}(\alpha'', \beta'') \geq v_2$. This makes $\mathsf{val}(s'', \beta'') \geq v_1 v_2$. Observe that v_3 is the minimum value in $\mathsf{Val}_{s', \beta'}$ which is at least $\min_{q \in I'_\alpha} f^i(q,u) f(q,s')$ thus $\min_{q \in I_{\alpha'}} f^i(q,u) f(q,s') \leq v_3 \leq x \in \mathsf{Val}_{s', \beta'}$. Since, $\mathsf{Val}_{s', \beta'}$ includes $\mathsf{val}(s'', \beta'')$, therefore, $\mathsf{val}(s'', \beta'') \geq v_3$.

Case 4. Let $[\varphi]$ is bounded and $[\psi]$ is unbounded MSCCs. $\mathsf{G}_1(M, [\varphi])$ is a weak game, whereas $\mathsf{G}_\leq([\varphi], [\psi])$ and $\mathsf{G}_2(M, [\psi])$ are weak stochastic games. Consider a matching triplet of configurations $(\langle s, \varphi \rangle, \langle \varphi, \psi \rangle, \langle s, \psi \rangle)$. The interesting case is when $\mathsf{val}(s, \varphi) = 1$, else the claim follows trivially. If ψ is a conjunction of the form $\psi_1 \wedge \psi_2$, then Player 1 in G_2 chooses the next configuration $\langle s, \psi_i \rangle$, then Player 1 in $G_\leq$ chooses $\langle \varphi, \psi_i \rangle$ in $\mathsf{G}_\leq$. If $\psi = \psi_1 \vee \psi_2$ we need to decide which of the disjunct to choose. If $\varphi = \varphi_1 \wedge \varphi_2$, then $\langle \varphi, \psi \rangle$ is Player 0 configuration in $G_\leq$. If Player 0 chooses $\langle \varphi_i, \psi \rangle$ then Player 1 in G_1

is made to choose $\langle s,\varphi_i\rangle$. If $\varphi=\varphi_1\vee\varphi_2$, and Player 0 in G_1 moves to configuration $\langle s,\varphi_i\rangle$, then Player 1 in $G_\leq$ moves to $\langle\varphi_i,\psi\rangle$. If $\varphi\in\|Q\|^\oplus$ then $\langle\varphi,\psi\rangle$ is a Player 0 configuration. If she chooses (φ,ψ_i) then Player 0 in G_2 chooses (s,ψ_i).

The remaining case is $\gamma\in\|Q\|^\oplus$ and $\varepsilon\in U$, where $\gamma=\oplus(r_1,\ldots,r_n)$ and $\varepsilon=u$. $\langle s,\gamma\rangle$ in G_1 is Player 0 configuration and she chooses a function $f\in\mathscr{F}^\oplus_{s,\varphi}$ with witness $\boldsymbol{d}\in\mathscr{D}_{\mathsf{tm}(\varphi)}$ and $\{a_{q,s'}\}_{q\in I_\varphi,s'\in\mathsf{succ}(s)}$ and moves to the configuration $\langle s,\varphi,f\rangle$. Player 1 at configuration $\langle\gamma,u\rangle$ in $\mathsf{G}_\leq$ chooses the action $\sigma=L(s)$ and then chooses a configurations $\langle r_k,u\rangle$ such that $\mathsf{val}(\gamma,u)$ is minimum (i.e., he plays his best possible move).

$\langle s,\varphi,f\rangle$ in the game G_1 is a Player 1 configuration and there could be more than one way for the game to evolve to the next matching triplet configurations. For example, if $f(q,s')>0$ and $f(q',s')>0$, $q,q'\in I_\varphi$, then it possible to have the next matching triplets as $\langle s,\delta(q,\sigma),f(q,s')\rangle$, $\langle\delta(q,\sigma),\delta(u,\sigma)\rangle$, and $\langle s',\delta(u,\sigma)\rangle$ or $\langle s,\delta(q',\sigma),f(q',s')\rangle$, $\langle\delta(q',\sigma),\delta(u,\sigma)\rangle$, and $\langle s',\delta(u,\sigma)\rangle$. We prove that the claim holds for any of matching triplets arising from different choice of $q\in I_\varphi$, i.e., it holds in the worst case. Equivalently, we show that the claim holds for a *choice of* q for which the value $\mathsf{val}(s,\alpha)\cdot\mathsf{val}(\alpha,\beta)$ is maximum. Consider the triplet of matching plays (where the configurations are step wise matching) from matching configurations $\langle s,\alpha\rangle$, $\langle\alpha,\beta\rangle$ and $\langle s,\beta\rangle$. We have the following cases:

4.a. The triplet of configurations $\langle s,\alpha\rangle$, $\langle\alpha,\beta\rangle$ and $\langle s,\beta\rangle$ where $\langle\alpha,\beta\rangle$ is not in the pair of equivalence classes $([\varphi],[\psi])$. The claim follows from induction hypothesis $\mathsf{val}(s,\alpha)\cdot\mathsf{val}(\alpha,\beta)\leq\mathsf{val}(s,\beta)$.

4.b. For every choice of matching of triplets during the evolution of the game, every play from $\langle\alpha,\beta\rangle$ stays in $([\varphi],[\psi])$ and are winning for Player 0 in $\mathsf{G}_\leq$. If the matching play in G_1 starting from $\langle s,\alpha\rangle$ is winning, then the matching play in G_2 from $\langle s,\beta\rangle$ are also winning for Player 0. Suppose this is not the case and there is a play from $\langle s,\beta\rangle$ that is not winning. Consider any corresponding matching play in $\mathsf{G}_\leq$, together they define a matching play in G_1. If the play is not winning in G_2 then the matching play in G_1 is also not winning, which cannot happen as $\mathsf{val}(s,\alpha)=1$ and G_1 is a weak game.

4.c. For every choice of matching of triplets the play stays in $([\varphi],[\psi])$ and are not winning for Player 0 in $\mathsf{G}_\leq$. Then $\mathsf{val}(\alpha,\beta)=0$ and the claim follows trivially.

4.d. The triplet of configurations $\langle s,\alpha\rangle$, $\langle\alpha,\beta\rangle$ and $\langle s,\beta\rangle$ such that for all choices of matching of triplets, not all the plays in $([\varphi],[\psi])$ are winning for Player 0 in $\mathsf{G}_\leq$ but probability of the set of winning plays is greater than zero. Here we explicitly assume that the MC M and automata A_1,A_2 are finite. Every time a configuration $\langle s,\alpha\rangle$ is revisited, the same function $f\in\mathscr{F}^\oplus_{s,\alpha}$ is chosen. Hence, the number of different matching configurations is finite.

We show that the claim, $\mathsf{val}(s,\alpha)P(\alpha,\beta)\leq P(s,\beta)$, holds, where $P(s,\beta)$ and $P(\alpha,\beta)$ is the worst case probability of reaching one of the three types of configurations covered in the previous three cases. Let $P_n(\alpha,\beta)$ and $P_n(s,\beta)$ be the probability of reaching one of the three types of configurations (defined in case 4.a, 4.b and 4.c) in n steps from $\langle\alpha,\beta\rangle$ and $\langle s,\beta\rangle$, respectively by matching paths, when matching triplet are chosen such that, $P_n(\alpha,\beta)\cdot\mathsf{val}(s,\alpha)$ is maximum. We show that $P_n(s,\beta)\geq P_n(\alpha,\beta)\cdot\mathsf{val}(s,\alpha)$ for any n. We proceed by induction on n.

If $\langle s,\alpha\rangle,\langle\alpha,\beta\rangle,\langle s,\beta\rangle$ is one of the three configurations from case 4.a, 4.b and 4.c then $P_0(\alpha,\beta)=\mathsf{val}(\alpha,\beta)$ and $P_0(s,\beta)=\mathsf{val}(s,\beta)$, else zero. Now, consider the triplet $\langle s,\alpha\rangle$, $\langle\alpha,u\rangle$ and $\langle s,u\rangle$, where $\alpha=\oplus(r_1,\ldots,r_n)$ and $P_0(\alpha,u)=0$ and $P_0(s,u)=0$ but for some successor $s'\in\mathsf{succ}(s)$ and $q\in I_\alpha$, $P_0(s',\delta(u,\sigma))>0$ and $P_0(\delta(q,\sigma),\delta(u,\sigma))>0$ $(\sigma=L(s))$. Let $f\in\mathscr{F}^\oplus_{s,\alpha}$ be the function chosen by

Player 0 at $\langle s,\alpha\rangle$ with witnesses $\{a_{q,s'}\}_{q\in I_\alpha, s'\in S}$ and $\boldsymbol{d}$. We have:

$$P_1(s,u) \;=\; \sum_{s'\in\mathsf{succ}(s)} P(s,s')\cdot P_0(s',\delta(u,\sigma)). \tag{5}$$

And,

$$\begin{aligned} P_1(\alpha,u) &= \min_{r_k\in\mathsf{tm}(\alpha)} \sum_{q\in\mathsf{gs}(r_k)} P_0(\delta(q,\sigma),\delta(u,\sigma))p_{k,q} \\ &\le \sum_{r_i\in\mathsf{tm}(\alpha)} \boldsymbol{d}_{r_i} \sum_{q\in\mathsf{gs}(r_i)} P_0(\delta(q,\sigma),\delta(u,\sigma))p_{i,q} \end{aligned} \tag{6}$$

For each $s'\in\mathsf{succ}(s)$ let $q_{s'}$ be the choice of q such that $\mathsf{val}(s',\delta(q_{s'},\sigma))\cdot\mathsf{val}(\delta(q_{s'},\sigma),\delta(u,\sigma))$ is maximum. By construction, $P_0(s',\delta(u,\sigma)) \ge P_0(\delta(q,\sigma),\delta(u,\sigma))\cdot\mathsf{val}(s',\delta(q,\sigma))$, since $\mathsf{val}(s',\delta(u,\sigma)) \ge \mathsf{val}(\delta(q,\sigma),\delta(u,\sigma))\mathsf{val}(s',\delta(q,\sigma))$. From Equation 5.

$$P_1(s,u) \;\ge\; \sum_{s'\in\mathsf{succ}(s)} P(s,s')P_0(\delta(q,\sigma),\delta(u,\sigma))\mathsf{val}(s',\delta(q,\sigma)) \tag{7}$$

Since $f\in\mathscr{F}^{\oplus}_{s,\alpha}$, $\sum_{r_i\in\mathsf{tm}(\alpha)}\sum_{q\in\mathsf{gs}(r_i)} a_{q,s'} = 1$. Therefore:

$$P_1(s,u) \ge \sum_{s'\in\mathsf{succ}(s)} P(s,s')\left(\sum_{r_i\in\mathsf{tm}(\alpha)}\sum_{q\in\mathsf{gs}(r_i)} a_{q,s'}\right)\cdot P_0(\delta(q,\sigma),\delta(u,\sigma))\cdot\mathsf{val}(s',\delta(q,\sigma))$$

Since the configuration $\langle s',\delta(q,\sigma)\rangle$ is winning for Player 0, $\mathsf{val}(s',\delta(q,\sigma)) \ge f(q,s')$.

$$P_1(s,u) \ge \sum_{s'\in\mathsf{succ}(s)} P(s,s')\left(\sum_{r_i\in\mathsf{tm}(\alpha)}\sum_{q\in\mathsf{gs}(r_i)} a_{q,s'}\right)\cdot P_0(\delta(q_{s'},\sigma),\delta(u,\sigma))\cdot f(s',q_{s'})$$

We can distribute $a_{q,s'}$ according to the following:

$$\begin{aligned} P_1(s,u) &\ge \sum_{s'\in\mathsf{succ}(s)}\sum_{r_i\in\mathsf{tm}(\alpha)}\sum_{q\in\mathsf{gs}(r_i)} P_0(\delta(q,\sigma),\delta(u,\sigma))\cdot P(s,s')a_{q,s'}f(s',q) \\ &\ge \sum_{r_i\in\mathsf{tm}(\alpha)}\sum_{q\in\mathsf{gs}(r_i)} P_0(\delta(q,\sigma),\delta(u,\sigma))\cdot \sum_{s'\in\mathsf{succ}(s)} P(s,s')a_{q,s'}f(s',q) \\ &\ge \sum_{r_i\in\mathsf{tm}(\alpha)}\sum_{q\in\mathsf{gs}(r_i)} P_0(\delta(q,\sigma),\delta(u,\sigma))\cdot p_{i,q}\boldsymbol{d}_{r_i} \\ &\ge\; P_1(\alpha,u) \end{aligned}$$

Assume now that the claim holds for all configurations triplets and for n steps. We consider the configuration triplets $\langle s,\alpha\rangle$, $\langle\alpha,u\rangle$ and $\langle s,\beta\rangle$ where $\alpha=\oplus(r_1,\ldots,r_n)$ and $u\in U$. As before, let $f\in\mathscr{F}^{\oplus}_{\alpha,s}$ be the function chosen by Player 0 at the configuration $\langle s,\alpha\rangle$, with witnesses $\{a_{q,s'}\}_{q\in I_\alpha,s'\in S}$ and $\boldsymbol{d}\in\mathscr{D}_{\mathsf{tm}(\alpha)}$. (For configurations with conjunction and disjunction, the matching paths are determined in their respective game by the strategies defined before.) We have:

$$\begin{aligned} P_{n+1}(s,u) &= \sum_{s'\in\mathsf{succ}(s):\exists q:a_{q,s'}>0} P(s,s')\cdot P_n(s',\delta(u,\sigma)) \\ P_{n+1}(\alpha,u) &= \min_{r_i\in\mathsf{tm}(\alpha)}\sum_{q\in\mathsf{gs}(r_i)} p_{i,q}P_n(\delta(q,\sigma),\delta(u,\sigma)) \\ &\le \sum_{r_i\in\mathsf{tm}(\alpha)}\sum_{q\in\mathsf{gs}(r_i)} \boldsymbol{d}_{r_i}p_{i,q}P_n(\delta(q,\sigma),\delta(u,\sigma)) \end{aligned} \tag{8}$$

By induction hypothesis :

$$P_n(s',\delta(u,\sigma)) \geq P_n(\delta(q_{s'},\sigma),\delta(u,\sigma))\cdot \mathsf{val}(s',\delta(q_{s'},\sigma)) \tag{9}$$

Or,

$$P_{n+1}(s,u) \geq \sum_{s'\in \mathsf{succ}(s)} P(s,s')\mathsf{val}(s',\delta(q_{s'},\sigma))P_n(\delta(q_{s'},\sigma),\delta(u,\sigma))$$

Choose $q_{s'}$ for each s' such that $\mathsf{val}(s',\delta(q_{s'},\sigma))P_n(\delta(q_{s'},\sigma),\delta(u,\sigma)$ is maximum.

$$\begin{aligned}
P_{n+1}(s,u) &= \sum_{s'\in \mathsf{succ}(s)} P(s,s')\Big(\sum_{r_i\in \mathsf{tm}(\alpha)}\sum_{q\in \mathsf{gs}(r_i)} a_{q,s'}\Big)\mathsf{val}(s',\delta(q_{s'},\sigma))P_n(\delta(q_{s'},\sigma),\delta(u,\sigma)) \\
&\geq \sum_{s'\in \mathsf{succ}(s)} P(s,s')\Big(\sum_{r_i\in \mathsf{tm}(\alpha)}\sum_{q\in \mathsf{gs}(r_i)} a_{q,s'}\Big)f(q_{s'},s')P_n(\delta(q_{s'},\sigma),\delta(u,\sigma)) \\
&\geq \sum_{r_i\in \mathsf{tm}(\alpha)}\sum_{q\in \mathsf{gs}(r_i)} P_n(\delta(q,\sigma),\delta(u,\sigma)) \sum_{s'\in \mathsf{succ}(s)} P(s,s')a_{q,s'}f(q,s') \\
&\geq \sum_{r_i\in \mathsf{tm}(\alpha)}\sum_{q\in \mathsf{gs}(r_i)} P_n(\delta(q,\sigma),\delta(u,\sigma))p_{i,q}\boldsymbol{d}(r_i) \\
&= P_{n+1}(\alpha,u)
\end{aligned}$$

This concludes the proof. □