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p-Buchsbaum rank 2 bundles on the projective space

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Abstract. *It has been proved by various authors that a normalized, 1-Buchsbaum rank 2 vector bundle on $\mathbb{P}^3$ is a nullcorrelation bundle, while a normalized, 2-Buchsbaum rank 2 vector bundle on $\mathbb{P}^3$ is an instanton bundle of charge 2. We find that the same is not true for 3-Buchsbaum rank 2 vector bundles on $\mathbb{P}^3$, and propose a conjecture regarding the classification of such objects.*

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Introduction

A coherent sheaf E on $\mathbb{P}^3$ is said to be *p-Buchsbaum* if p is the minimal power of the irrelevant ideal which annihilates $H^1_*(E)$. The complete list of p-Buchsbaum rank 2 bundles on $\mathbb{P}^3$ for $p \leq 2$ has been established by several authors, see for example [7, 9, 14, 15, 16]. More precisely, we have the following.

Theorem 1. *Let E be a normalized p-Buchsbaum rank 2 vector bundle on $\mathbb{P}^3$. Then*

- *$p = 0$ if and only if E is direct sum of line bundles;*
- *$p = 1$ if and only if E is a null correlation bundle, i.e. an instanton bundle of charge 1;*
- *$p = 2$ if and only if E is an instanton bundle of charge 2.*

After examining this list, two questions natually arise. First, is every rank 2 instanton bundle of charge k on $\mathbb{P}^3$ k-Buchsbaum? Second, since every bundle is p-Buchsbaum for some sufficiently high p, for which values of p can we find a p-Buchsbaum rank 2 bundle which is not instanton?

The goal of this paper is to provide partial answers to these questions. In particular, we show that every rank 2 instanton bundle of charge 3 is 3-Buchsbaum. However, this is false for instantons of higher charge. On the other hand, we show that the generic instanton of charge 4 or 5 is also 3-Buchsbaum. In addition, we provide an explicit example of a 3-Buchsbaum bundle of rank 2 which is not an instanton, and conjecture that every 3-Buchsbaum rank 2 bundle on $\mathbb{P}^3$ is one of these.

1. Preliminaries

In this section we will fix the notation and recall the basic definitions used throughout this paper.

1.1. Buchsbaum sheaves

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let us denote by $S = \mathbb{K}[x_0, x_1, x_2, x_3]$ the ring of polynomials in four variables, so that $\mathbb{P}^3 := \mathrm{Proj}(S)$, and let $\mathfrak{m} = (x_0, x_1, x_2, x_3)$ denote the irrelevant ideal.

Let V be a $\mathbb{K}$-vector space of dimension $m+1$, with V^* denoting its dual. The projective space $\mathbb{P}(V) = \mathbb{P}^m$ is understood as the set of equivalence classes of m-dimensional subspaces of V, or, equivalently, the equivalence classes of the lines of V^*.

Given a coherent sheaf E on $\mathbb{P}^3$, consider the following graded S-module:

$$H^1_*(E) = \bigoplus_{n\in\mathbb{Z}} H^1(E(n)).$$

Definition 1.1. *A coherent sheaf E on $\mathbb{P}^3$ is said to be p-Buchsbaum if and only if p is the minimal power of the irrelevant ideal which annihilates the S-module $H^1_*(E)$, i.e.*

$$p = \min\left\{t \mid \mathfrak{m}^t H^1_*(E) = 0\right\}.$$

In this work, we will only consider locally free sheaves on $\mathbb{P}^3$.

1.2. Monads and regularity

Recall that a *monad* on a projective variety X of dimension n is a complex of locally free sheaves on X of the form

$$M_\bullet : A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

such that the map α is injective and the map β is surjective. It follows that $E := \ker\beta/\operatorname{Im}\alpha$ is the only nontrivial cohomology of the complex $M_\bullet$. The coherent sheaf E is called the *cohomology* of $M_\bullet$; it is locally free if and only if the map α is injective in every fiber.

The monad $M_\bullet$ is called a *Horrocks monad* if, in addition:

i) A and C are direct sum of invertible sheaves,

ii) $H^1_*(B) = H^{n-1}_*(B) = 0$.

Furthermore, the monad is also called *minimal* if it satisfies

iii) no direct sum of A is isomorphic to a direct sum of B,

iv) no direct sum of C is the image of a line subbundle of B.

Let us recall the following result on minimal Horrocks monads, cf. [12, Theorem 2.3].

THEOREM 1.2. *Let X be an arithmetically Cohen–Macaulay variety of dimension $n \geq 3$, and let E be a locally free sheaf on X. Then there is a 1-1 correspondence between collections*

$$\{n_1, \dots, n_r, m_1, \dots, m_s\} \text{ with } n_i \in H^1(E^\vee \otimes \omega_X(k_i)) \text{ and } m_j \in H^1(E(-l_j))$$

for integers k_i's and l_j's, and equivalence classes of Horrocks monads of the form

$$M_\bullet : \bigoplus_{i=1}^{r} \omega_X(k_i) \xrightarrow{\alpha} F \xrightarrow{\beta} \bigoplus_{j=1}^{s} \mathcal{O}_X(l_j),$$

whose cohomology is isomorphic to E.

Moreover, the correspondence is such that $M_\bullet$ is minimal if and only if the elements m_j generate $H^1_*(E)$ and the elements n_i generate $H^1_*(E^\vee \otimes \omega_X)$ as modules.

Recall that a coherent sheaf E on $\mathbb{P}^n$ is said to be m-regular in the sense of Castelnuovo–Mumford if $H^i(\mathbb{P}^n, E(m-i)) = 0$ for $i > 0$. Costa and Miró-Roig studied in [3] the Castelnuovo–Mumford regularity of the cohomology of a certain class of monads which include monads of the following form:

$$\mathcal{O}_{\mathbb{P}^3}(-l)^{\oplus k} \xrightarrow{\alpha} \bigoplus_{j=1}^{2+2k} \mathcal{O}_{\mathbb{P}^3}(b_j) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(d)^{\oplus k}, \tag{1}$$

where $l, k, c \geq 1$ and $-l < b_1 \leq \cdots \leq b_{2+2k} < d$. Specializing [3, Theorem 3.2] to monads of the form (1), one obtains the following result.

PROPOSITION 1.3. *If E is the cohomology of a monad of the form (1), then E is m-regular for any integer m such that*

$$m \geq \max\{(k+2)d - (b_1 + \cdots + b_{k+3}) - 2, l\}.$$

1.3. Cohomology of generic instanton bundles

Recall that a bundle E of rank 2 on $\mathbb{P}^3$ is called an *instanton bundle* if it is isomorphic to the cohomology of a monad of the following form:

$$\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus k} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2+2k} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus k} \quad (2)$$

The integer k is called the *charge* of E; notice that $c_1(E) = 0$ and $c_2(E) = k$. Note also that nullcorrelation bundles are precisely instanton bundles of charge 1.

Alternatively, an instanton bundle can also be defined as a bundle E on $\mathbb{P}^3$ with $c_1(E) = 0$ and satisfying the following cohomological conditions:

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

The Hilbert polynomial of an instanton bundle is given by

$$\begin{aligned} P_E(t) &= 2(k+1)\chi(\mathcal{O}_{\mathbb{P}^3}(t)) - k\chi(\mathcal{O}_{\mathbb{P}^3}(t-1)) - k\chi(\mathcal{O}_{\mathbb{P}^3}(t+1)) \quad (3) \\ &= \frac{1}{3}(t+2)((t+3)(t+1) - 3k) \\ &= \frac{1}{3}(t+2)(t+2+\sqrt{3k+1})(t+2-\sqrt{3k+1}). \end{aligned}$$

Note also that $P_E(t) = h^0(E(t)) - h^1(E(t))$ for $t \geq -2$.

On another direction, recall that a coherent sheaf F on $\mathbb{P}^3$ is said to have *natural cohomology* if for each $t \in \mathbb{Z}$, at most one of the cohomology groups $H^p(F(t))$, where $p = 0, \ldots, 3$, is nonzero; every torsion free coherent sheaf with natural cohomology is in fact locally free [10, Lemma 1.1]. In addition, every rank 2 locally free sheaf with $c_1 = 0$, $c_2 > 0$ and natural cohomology is an instanton bundle [10, p. 365].

Hartshorne and Hirschowitz have shown in [10] that the generic instanton bundle has natural cohomology. More precisely, let $\mathcal{I}(k)$ denote the moduli space of rank 2 locally free instanton sheaves of charge k; this is known to be an affine [4], irreducible [18, 19], nonsingular variety of dimension $8k-3$ [13]. Let $\mathcal{N}(k)$ denote the subset of $\mathcal{I}(k)$ consisting of instanton bundles with natural cohomology; it is easy to see that $\mathcal{N}(k)$ is open within $\mathcal{I}(k)$, and [10, Theorem 0.1 (a)] tells us that it is nonempty.

More recently, Eisenbud and Schreyer have introduced the notion of *supernatural bundles*, see [6, p. 862]: a locally free sheaf on $\mathbb{P}^3$ is called supernatural if it has natural cohomology and its Hilbert polynomial has distinct integral roots. Therefore we see that there exists a rank 2 supernatural bundle with $c_1 = 0$ and $c_2 = k > 0$ if and only if $3k+1$ is a perfect square; the first three possible values for k are $k = 1, 5, 8$.

2. Instanton vs Buchsbaum

We start by introducing the following function on the positive integers

$$m(k) = \left\lfloor \sqrt{3k+1} - 2 \right\rfloor ,$$

where $\lfloor \cdot \rfloor$ denotes the largest positive integer which is smaller than or equal to the argument.

PROPOSITION 2.1. *A rank 2 instanton bundle E is p-Buchsbaum if and only if $h^1(E(p-2)) \neq 0$ and $h^1(E(p-1)) = 0$. In addition, every rank 2 instanton bundle of charge k is p-Buchsbaum for some $m(k)+2 \leq p \leq k$.*

Proof. By Theorem 1.2, we get that $H^1_*(E)$ is generated in $H^1(E(-1))$. Thus if $h^1(E(p-2)) \neq 0$ and $h^1(E(p-1)) = 0$ (and hence $h^1(E(t)) = 0$ for every $t \geq p-1$), then $H^1_*(E)$ must be p-Buchsbaum. Conversely, if E is p-Buchsbaum, then $h^1(E(p-2)) \neq 0$ (otherwise, $H^1_*(E)$ would be annihilated by the $(p-1)$-th power of the irrelevant ideal) and $h^1(E(p-1)) = 0$.

By Proposition 1.3, we have that E is k-regular (cf. also [3, Corollary 3.3]). Hence $H^1(E(k-1)) = 0$, and it follows that every rank 2 instanton bundle is at most k-Buchsbaum.

Finally, note from (3) that for $-1 \leq t \leq m(k)$ we have $\chi(E(t)) < 0$. Since $h^3(E(t)) = 0$ in this range, it follows that $h^1(E(t)) \neq 0$ for $t = m(k)$. Thus every rank 2 instanton bundle is at least $(m(k)+2)$-Buchsbaum. □

Since $m(3)+2 = 3$, the first immediate consequence of the previous Proposition is given by the following Corollary.

COROLLARY 2.2. *Every rank 2 instanton bundle of charge 3 is 3-Buchsbaum.*

However, it is not true that every rank 2 instanton bundle of charge 3 has natural cohomology, as observed in [10, Example 1.6.1]. Indeed, recall that an instanton bundle E is called a *'t Hooft instanton* if $h^0(E(1)) \neq 0$, cf. [1]; more formally, consider the set

$$\mathcal{H}(k) := \{E \in \mathcal{I}(k) \mid h^0(E(1)) \neq 0\} ,$$

which is known to be a locally closed subvariety of $\mathcal{I}(k)$ of dimension $5k+4$, irreducible and rational [1, Theorem 2.5]. On the other hand, let $\mathcal{U}(k) := \mathcal{I}(k) \setminus \mathcal{N}(k)$, the subvariety of "unnatural" instanton bundles.

LEMMA 2.3. *For every $k \geq 3$, we have $\mathcal{H}(k) \subset \mathcal{U}(k)$, while $\mathcal{H}(3) = \mathcal{U}(3)$.*

Proof. If E is a rank 2 instanton bundle of charge $k \geq 3$, then $h^1(E(1)) \neq 0$ (because $\chi(E(-1)) < 0$). Hence if E is a 't Hooft instanton, then it does not have natural cohomology, showing that $\mathcal{H}(k) \subset \mathcal{U}(k)$.

Conversely, let now E be a rank 2 instanton bundle of charge 3 which does not have natural cohomology. We then know that

(i) $h^0(E(t)) = 0$ for $t \leq 0$;

(ii) $h^1(E(t)) = 0$ for $t \neq -1, 0, 1$;

(iii) $h^2(E(t)) = 0$ for $t \neq -5, -4, -3$;

(iv) $h^3(E(t)) = 0$ for $t \geq -4$.

The last two claims are obtained by Serre duality and the fact $E \simeq E^*$. Therefore the only way in which E may fail to have natural cohomology is if $h^0(E(1)) = h^3(E(-5)) \neq 0$. It follows that $\mathcal{U}(3) \subset \mathcal{H}(3)$. □

It would be interesting to determine properties of the $\mathcal{U}(k)$ for $k \geq 4$, particularly its dimension and number of irreducible components. The previous lemma tells us that $\dim \mathcal{U}(k) \geq 5k + 4$.

Another immediate consequence of Proposition 2.1 is the following.

Corollary 2.4. *The generic rank* 2 *instanton bundle of charge k is* $(m(k)+2)$-*Buchsbaum.*

In particular, since $m(4) + 2 = m(5) + 2 = 3$, the generic rank 2 instanton bundle of charges 4 and 5 are 3-Buchsbaum, while instanton bundles of charge $k \geq 6$ are at least 4-Buchsbaum.

3. A 3-Buchsbaum rank 2 bundle with $c_1 = -1$

Theorem 1 tells us, in particular, that the first Chern class of every 1- and 2-Buchsbaum rank 2 bundle on $\mathbb{P}^3$ is zero. In this section, we show that the same is not true for p-Buchsbaum bundles with $p \geq 3$, providing an example of a 3-Buchsbaum rank 2 bundle with $c_1 = -1$.

Indeed, consider the monad

$$\mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \quad , \tag{4}$$

which is the simplest example of a class of monads originally introduced by Ein in [5, eq. 3.1.A]. The existence of such monads can be easily established; consider for instance the following explicit maps

$$\alpha = \begin{pmatrix} -z^2 \\ -w^2 \\ x \\ y \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x \\ y \\ z^2 \\ w^2 \end{pmatrix}$$

where $[x : y : z : w]$ are homogeneous coordinates on $\mathbb{P}^3$.

Let F denote the locally free cohomology of a monad of the form (4); it is a rank 2 bundle with $c_1(F) = -1$ and $c_2(F) = 2$. Ein claims in [5, p. 21], without proof, that F is μ-stable. For the sake of completeness, we include a proof below.

LEMMA 3.1. *Every locally free sheaf F obtained as the cohomology of a monad of the form (4) is μ-stable.*

Proof. First consider the kernel bundle $K := \ker \beta$ defined by the sequence

$$0 \to K \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1).$$

It follows from [2, Theorem 2.7] that K is μ-semistable (but not μ-stable). Therefore, since $\mu(K) = -1$, we must have $h^0(K) = 0$. Now, from the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} K \to F \to 0$$

we have that $h^0(F) = 0$, which implies that F is μ-stable. □

We now show that the bundles considered in this Section are 3-Buchsbaum.

PROPOSITION 3.2. *Every locally free sheaf F obtained as cohomology of a monad of the form (4) is 3-Buchsbaum.*

Proof. By Theorem 1.2, we get that $H^1_*(F)$ is generated in $H^1(F(-1))$. On the other hand, Proposition 1.3 tells us that F is 3-regular, thus $h^1(F(2)) = 0$.

If we also had $h^1(F(1)) = 0$, F would be 2-Buchsbaum, which, by Theorem 1 cannot happen. Therefore F must be 3-Buchsbaum. □

Note also that, since $h^0(F(1)) = h^1(F(1)) = 1$ [11, 2.2], such bundles do not have natural cohomology.

Based on the evidence here presented and also motivated by results due to Roggero and Valabrega in [17], specially Propositions 5 and 6 and Theorem 2 there, we propose the following classification of 3-Buchsbaum rank 2 bundles on $\mathbb{P}^3$.

CONJECTURE 3.3. *Every normalized, 3-Buchsbaum rank 2 bundle on $\mathbb{P}^3$ is either an instanton bundle of charge 3, 4 or 5, if $c_1 = 0$, or the cohomology of a monad of the form (4), if $c_1 = -1$.*

Finally, let us comment on p-Buchsbaum rank 2 bundles on $\mathbb{P}^3$ for $p \geq 4$. An interesting, possible source of examples of such bundles is provided by Ein's *generalized nullcorrelation bundles*, described in [5]. These are bundles obtained as cohomologies of monads of the following two types:

$$\mathcal{O}_{\mathbb{P}^3}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(d) \quad , \qquad (5)$$

and

$$\mathcal{O}_{\mathbb{P}^3}(-d-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-b-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-a-1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(d) \ , \quad (6)$$

where $d > b \geq a \geq 0$. Let us denote the cohomology of such monads by $E_{a,b,d}$ and $F_{a,b,d}$, respectively.

Note that, by Theorem 1.2 and Proposition 1.3, $H^1_*(E_{a,b,d})$ is generated in degree $-d$, and that $E_{a,b,d}$ is $(3d-2)$-regular when $d \geq 1$. Therefore, such bundles are at most $(4d-3)$-Buchsbaum, being precisely $(4d-3)$-Buchsbaum provided $h^1(E_{a,b,d}(3d-4)) \neq 0$.

Similarly, note that $H^1_*(F_{a,b,d})$ is generated in degree $-d$, and that $F_{a,b,d}$ is $3d$-regular. Therefore, such bundles are at most $(4d-1)$ – Buchsbaum, being precisely $(4d-1)$ –Buchsbaum provided $h^1(F_{a,b,d}(3d-2)) \neq 0$.

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Added in proof. Conjecture 3.3 was proposed by the second author in the "Vector Bundles Days II", held in Trieste in occasion of Emilia Mezzetti's birthday. We thank the editor for informing us that the conjecture has been proved in [8].

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