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P-CONVEXITY AND B-CONVEXITY IN BANACH SPACES

BY

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ABSTRACT. Two properties of B-convexity are shown to hold for P-convexity: (1) Under certain conditions, the direct sum of two P-convex spaces is P-convex. (2) A Banach space is P-convex if each subspace having a Schauder decomposition into finite dimensional subspaces is P-convex.

0. Introduction. In the previous paper [1] the question of whether all B-convex spaces are reflexive was discussed. The concept of a P-convex space was introduced by C. Kottman [4] as follows:

Definition. For a positive integer n, let $P(n, X)$ be the supremum of all numbers r such that there is a set of n disjoint closed balls of radius r inside $U(X) = \{x: \|x\| \le 1\}$. X is said to be P-convex if $P(n, X) < \frac{1}{2}$ for some n. Kottman showed that all P-convex spaces are both B-convex and reflexive. Therefore the question "Is there a B-convex space that is not P-convex?" is of interest.

Many properties of B-convex spaces are not known for P-convex spaces. In this paper we consider two of these properties and prove partial analogs of them for P-convex spaces: The first property is that direct sums of B-convex spaces are B-convex [2]. The proof of this fact for B-convex spaces rests on the invariance of B-convexity under isomorphism, but it is not known whether P-convexity possesses this invariance. Two partial analogs of the direct sum property are obtained, Theorems 1.3 and 1.5, using Ramsey's theorem of combinatorics. The second property is that a space is B-convex if each subspace having a basis is B-convex [1]. A partial analog of this is proved, Theorem 2.1, using one of the direct sum results.

We will use the following characterization of P-convexity from Remark 1.4 of [4]: Let a set of n elements be called δ separated of order n provided the distance between any two elements of the set is at least δ. Then a space X is P-convex if and only if for some positive integer n and some positive number $\epsilon < 2$ there is no $2 - \epsilon$ separated set of order n in $U(X)$.

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I. Direct sums. The results of this section are based on the following theorem proved in 1930 by Ramsey [6].

Theorem (Ramsey). *Let p, q, and r be integers so that $p, q \geq r > 1$. Then there is a number $n(p, q, r)$ having the following property. Let S be a set having $n(p, q, r)$ or more elements. Let the family of all r-subsets of S (where an r-subset is a set having r elements) be divided into two disjoint families, α and β. Then either*

(1) *there is $A \subset S$, a subset with p elements, so that any r subset of A is in α, or*

(2) *there is $B \subset S$, a subset of q elements, so that any r subset of B is in β.*

We use this theorem to prove the following lemma.

1.1 **Lemma.** *Let A and B be sets, P_A a property which a 2-subset of points (a_i, a_j) in A may have, and P_B a property on 2-subsets of points (b_i, b_j) in B. Suppose there is an integer N_A so that if $a_i, \cdots, a_n$, $n \geq N_A$, are distinct points of A, then there is i, j so that (a_1, a_j) has P_A, and there is N_B with the corresponding property for B. Then there is an integer N_{AB} so that if $n \geq N_{AB}$, $a_1, \cdots, a_n$ distinct points of A, $b_1, \cdots, b_n$ distinct points of B, then there is i, j so that both (a_i, a_j) has P_A and (b_i, b_j) has P_B.*

Proof. Let $N_0 = \max(N_A, N_B)$ and let $N_{AB} = n(N_0, N_0, 2)$ from Ramsey's theorem. For $n \geq N_{AB}$ let $S = \{1, \cdots, n\}$. Given $\{a_i\}_{i=1}^n$, let

$\alpha = \{(i, j): (a_i, a_j)$ does not have $P_A\}$,

$\beta = \{(i, j): (b_i, b_j)$ does not have P_B; $(i, j) \notin \alpha\}$.

Now suppose there is no i, j as asserted in the lemma. Then $\alpha \cup \beta$ is the set of all pairs of elements of S. Also $\alpha \cap \beta = \varnothing$, so Ramsey's theorem applies. If Conclusion 1 holds, there is $\{i_n\}_{n=1}^{N_0} \subset S$ so that each $(i_n, i_m) \in \alpha$. Thus $\{a_{i_n}\}_{n=1}^{N_0}$ is a set of N_A or more points, no pair of which has P_A, which is a contradiction. Conclusion 2 yields a similar contradiction.

Lemma 1.1 will be incorporated into the following lemma for ordered pairs $(a, b) \in A \times B$.

1.2 **Lemma.** *Let A, B, P_A, P_B, N_A, N_B, N_{AB} be as in Lemma 1.1 with the additional property that a pair having the same first and second elements of A, (a_i, a_i), always has P_A, and the corresponding property for B. Then if $\{(a_i, b_i)\}_{i=1}^n$ is a set of distinct pairs from $A \times B$ (i.e., any two pairs differ in the first or second entries, or both) and $n \geq N_{AB}N_AN_B$, then there is i, j so that both (a_i, a_j) has P_A and (b_i, b_j) has P_B.*

Proof. Let $a^1, a^2, \cdots, a^{r_A}$ be the distinct values of $\{a_i\}_{i=1}^n$ and write the sets

$$\{(a_i, b_i): a_i = a^1\}, \{(a_i, b_i): a_i = a^2\}, \cdots, \{(a_i, b_i): a_i = a^{r_A}\}.$$

If one of these sets, say the Kth, has N_B or more pairs, then for these pairs $\{b_i: a_i = a^K\}$ are distinct and so there is i, j so that (b_i, b_j) has P_B. By hypothesis $(a_i, a_j) = (a^K, a^K)$ has P_A so the conclusion of the lemma holds. Otherwise each of the sets has less than N_B pairs, so that the total number of pairs in all of the sets is $n < N_B r_A$. Since $N_{AB} N_A N_B \le n$, we have $r_A > N_{AB} N_A$. By choosing one pair from each of the sets, we get a family of pairs $\{(a^n, b_{i(n)})\}_{n=1}^{r_A}$ having distinct first elements, i.e., if $n \neq m$ then $a^n \neq a^m$. Now let $b^1, b^2, \cdots, b^{r_B}$ be the distinct values of $\{b_{i(n)}\}_{n=1}^{r_A}$ and write the sets

$$\{(a^n, b_{i(n)}): b_{i(n)} = b^1\}, \cdots, \{(a^n, b_{i(n)}): b_{i(n)} = b^{r_B}\}.$$

If any of these sets has N_A or more elements, say the Kth, then $\{a^n: b_{i(n)} = b^K\}$ are distinct and there is no j, k so that (a^j, a^k) has P_A and $(b_{i(j)}, b_{i(k)}) = (b^K, b^K)$ has P_B. Since $(a^j, b_{i(j)})$ and $(a^k, b_{i(k)})$ were in the original set of pairs $\{(a_i, b_i)\}_{i=1}^n$ the conclusion of the lemma holds. Otherwise each of the sets has less than N_A pairs, so that the total number of pairs in all the sets is $r_A < r_B N_A$. Since we showed $N_{AB} N_A < r_A$ we have $r_B > N_{AB}$. Take one pair from each of the sets to get $\{(a^{n(j)}, b^j)\}_{j=1}^{r_B}$, a subset of $\{(a_i, b_i)\}_{i=1}^n$, so that if $j \neq k$ then $a^{n(j)} \neq a^{n(k)}$ and $b^j \neq b^k$. Thus $\{a^{n(j)}\}_{j=1}^{r_B}$ and $\{b^j\}_{j=1}^{r_B}$ are distinct points of A and B. Applying Lemma 1.1 to these pairs concludes the proof.

Theorem 1.3. *Let Y and Z be subspaces of X so that $X = Y \oplus Z$. If Y is finite dimensional and Z is P-convex then X is P-convex.*

Proof. Since Z is P-convex there is n_Z, δ so that if $\{z_i\}_{i=1}^n$ are distinct points in $U(Z)$ and $n \ge n_Z$ then there is i, j such that $\|z_i - z_j\| < 2 - \delta$. Since Y is finite dimensional, $U(Y)$ is compact and there is n_Y so that if $\{y_i\}_{i=1}^n$ are distinct points in $U(Y)$ and $n \ge n_Y$ then there is i, j such that $\|y_i - y_j\| < \delta/2$. Let $U(Y) = A$, say (y_i, y_j) has P_A if $\|y_i - y_j\| < \delta/2$, and let $N_A = n_Y$. Let $U(Z) = B$, say (z_i, z_j) has P_B if $\|z_i - z_j\| < 2 - \delta$, and let $N_B = n_Z$. Let $\{y_i + z_i\}_{i=1}^n$ be distinct pairs in $U(X)$, $n \ge N_{AB} N_A N_B$. Then $\{y_i\}_{i=1}^n \subset A$ and $\{z_i\}_{i=1}^n \subset B$.

By Lemma 1.2 there is i, j so that (y_i, y_j) has P_A; i.e., $\|y_i - y_j\| < \delta/2$, and (z_i, z_j) has P_B; i.e., $\|z_i - z_j\| < 2 - \delta$. Thus

$$\|(y_i + z_i) - (y_j + z_j)\| \le \|y_i - y_j\| + \|z_i - z_j\| < 2 - \delta/2.$$

Corollary 1.4. *If X is not P-convex, and Y is a subspace of X of finite codimension, then Y is not P-convex.*

The following theorem can be proved for direct sums of infinite dimensional Banach spaces.

Theorem 1.5. *Let $Y \oplus Z$ be the direct sum of two P-convex Banach spaces normed by $\|(y, z)\| = \max(\|y\|, \|z\|)$. Then $Y \oplus Z$ is P-convex.*

Proof. Since Y is P-convex, there is n_Y, ϵ_Y so that if $\{y_i\}_{i=1}^n$ are distinct points in $U(Y)$ and $n \geq n_Y$ we have some i, j so that $\|y_i - y_j\| < 2 - \epsilon_Y$. Similarly there is n_Z, ϵ_Z with this property for points in $U(Z)$. Let $A = U(Y)$. Say (y_i, y_j) has P_A if $\|y_i - y_j\| < 2 - \epsilon$, where $\epsilon = \min(\epsilon_Y, \epsilon_Z)$. Let $N_A = n_Y$. Similarly let $B = U(Z)$ and define P_B and N_B. Let $\{(y_i, z_i)\}_{i=1}^n$ be distinct pairs in $U(Y \oplus Z)$, $n \geq N_{AB} N_A N_B$. Then $\{y_i\}_{i=1}^n \subset A$, $\{z_i\}_{i=1}^n \subset B$. By Lemma 1.2 there is i, j so that (y_i, y_j) has P_A; i.e., $\|y_i - y_j\| < 2 - \epsilon$ and (z_i, z_j) has P_B; i.e., $\|z_i - z_j\| < 2 - \epsilon$ and thus

$$\|(y_i, z_i) - (y_j, z_j)\| = \|(y_i - y_j, z_i - z_j)\| < 2 - \epsilon.$$

2. **Subspaces.** We will use the following

Definition. A sequence $\{M_i\}$ of closed subspaces of a Banach space X is a Schauder decomposition of X if every element u of X has a unique, norm-convergent expansion $u = \sum_{i=1}^{\infty} u_i$, where $u_i \in M_i$ for $i = 1, 2, \cdots$.

Grinblyum [3] has characterized Schauder decompositions as follows.

Theorem. *A sequence $\{M_i\}$ of closed subspaces of X is a Schauder decomposition of X if and only if there is a constant K such that for all integers m, n and all sequences $\{u_i\}$ with $u_i \in M_i$ we have $\|\sum_{i=1}^n u_i\| \leq k\|\sum_{i=1}^{n+m} u_i\|$.*

The following theorem is the P-convex analog to the B-convex subspaces with basis property.

Theorem 2.1. *If X is not P-convex, it contains a subspace having a Schauder decomposition into finite dimensional subspaces which is not P-convex.*

Proof. Let $\{\delta_i\}$ and $\{\epsilon_i\}$ be sequences of positive numbers less than one tending toward zero. Let $\{k_i\}$ be a sequence of integers tending to infinity. A sequence $p(m)$ of integers and a sequence $\{x_i\}$ of vectors will be constructed with the following properties:

Let L denote the closed span $[x_i]$ and let $L_m = [x_i]_{p(m-1)+1}^{p(m+1)}$, then

(1) For each $m = 1, 2, \cdots$ there is a $2 - \epsilon_m$ separated set of order k_m in $U(L_m)$.

(2) For any integers n, q and any $\{u_i\}$, $u_i \in L_i$, we have $\|\sum_{i=1}^n u_i\| \leq (1 + \delta_n)\|\sum_{i=1}^{n+q} u_i\|$.

By property (1) L is not P-convex and by property (2) $\{L_m\}$ is a Schauder decomposition of L.

The construction is by induction on m as in the B-convex Theorem 2.3 of [1]. Let $m = 1$. Since X is not P-convex it contains a $2 - \epsilon_1$ separated set of order k_1. Let L_1 be the span of this set; let $\{x_i\}_{i=1}^{p(1)}$ be a linearly independent set spanning L_1. Choose $\{f_i\}_{i=1}^{q(1)} \subset U(L_1^*)$ by Lemma 2.1 of [1] and extend to X so that if $x \in L_1$,

$$\|x\| \leq (1+\delta_1) \max\{f_i(x): i = 1, \cdots, q(1)\}.$$

Let $\Lambda_1 = \bigcap_{i=1}^{q(1)} f_i^{-1}(0)$. Let P_1 be the projection from $L_1 \oplus \Lambda_1 \longrightarrow L_1$. Then $\|P_1\| \leq 1 + \delta_1$. Since Λ_{m-1} is of finite codimension, it is not P-convex by Corollary 1.4, so that the induction step can be carried out. Property (2) follows from the fact that $\|P_m\| \leq 1 + \delta_m$.

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