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p-Cross-section Bodies

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Abstract. If K is a convex body in $\mathbb{E}^n$, its cross-section body CK has a radial function in any direction $u \in S^{n-1}$ equal to the maximal volume of hyperplane sections of K orthogonal to u. A generalization called the p-cross-section body C_pK of K, where $p > -1$, is introduced. The radial function of C_pK in any direction $u \in S^{n-1}$ is the pth mean of the volumes of hyperplane sections of K orthogonal to u through points in K. It is shown that C_1K is convex but C_pK is generally not convex when $p > 1$. An inclusion of the form $a_{n,q}C_qK \subseteq a_{n,p}C_pK$, where $-1 < p < q$ and the constant $a_{n,p}$ is the best possible, is established. This is applied to disprove a conjecture of Makai and Martini.

1. Introduction

If K is a convex body in $\mathbb{E}^n$, its cross-section body CK has a radial function in any direction $u \in S^{n-1}$ equal to the maximal volume of hyperplane sections of K orthogonal to u. This body, introduced by Martini [20], is just the intersection body IK of K when K is centered (i.e., symmetric about the origin), and coincides with the projection body ΠK of K in the planar case. (See Section 2 for definitions.) Projection bodies originated in the work of Minkowski, and have applications in the local theory of Banach spaces, stochastic geometry, mathematical economics, and other areas. Intersection bodies were defined more recently by E. Lutwak, and are a crucial concept in the solution of the Busemann-Petty problem. See [7] for an overview and references. Thus the cross-section body has an intrinsic interest as a sort of hybrid of the projection and intersection body. Cross-section bodies also enjoy a fascinating connection with Fermi surfaces of metals (see [7], p. 308), but they are still somewhat mysterious, despite a recent flurry of activity; see, for example, [4], [6], [15], [16], and [22].

The purpose of this paper is to continue the investigation of the cross-section body and to introduce a generalization called the p-cross-section body C_pK of

K, where $p > -1$. The radial function of C_pK in any direction $u \in S^{n-1}$ is the pth mean of the volumes of hyperplane sections of K orthogonal to u through points in K. These are natural objects in affine geometry. In a sense this paper is a continuation of [8], which introduced the pth radial mean body R_pK of K, where $p > -1$. Indeed, the radial function of a suitable dilatate of R_pK in any direction $u \in S^{n-1}$ is the pth mean of the lengths of linear sections of K parallel to u through points in K. The introduction to [8] gives a wider perspective on the pth radial mean bodies.

These families of bodies defined in terms of pth means have a strong unifying effect, linking objects whose definitions make them seem quite unrelated. It was shown in [8] that the bodies R_pK approach the difference body of K as $p \to \infty$ and approach a dilatate of the polar projection body of K as $p \to -1$. Here we see that the bodies C_pK approach the cross-section body of K as $p \to \infty$ and approach a dilatate of the polar difference body of K as $p \to -1$.

If $u \in S^{n-1}$, the function that gives the volumes of hyperplane sections of a convex body K orthogonal to u is sometimes called the *$(n-1)$-dimensional X-ray of K orthogonal to u* in view of its relation to the ordinary (i.e., 1-dimensional) X-ray in tomography. Its connections and applications in tomography, spline theory, and mathematical physics (via the relative of the Steiner symmetral known as the Schwarz symmetral) are explained in [7, Notes 2.1 and 2.3]. Its derivatives at the origin play a fundamental role in the solution of the Busemann-Petty problem mentioned above; in this connection, it has also been called the *parallel section function*. In the case of a metallic object, this function can in principle be measured by an electromagnetic measurement known as the ramp response.

The main results are as follows. In Section 5 we investigate the convexity of p-cross-section bodies. The motivation for this originates in Busemann's theorem, an outcome of Busemann's theory of area in Finsler spaces, which implies that when K is centered, $IK = CK$ is convex. This is an extremely important result in both geometric tomography and Minkowski geometry (see, for example [7] and [25]). In Theorem 5.2 we show that if K is a convex body in $\mathbb{E}^n$, then

$$C_1K = I(R_{n-1}K),$$

and conclude that C_1K is convex. (This formula arises from a connection with the polar pth centroid bodies that appear in a centro-affine inequality obtained by Lutwak and Zhang [14] which generalizes the well-known Blaschke-Santaló inequality for convex bodies symmetric about the origin.) We then use an idea of Brehm [4] together with a result of Cohn [5] on log-concave functions to find for each $n \geq 4$ a computable number p_n such that C_pK is not convex when K is an n-dimensional simplex and $p > p_n$. From this we show in Corollary 5.8 that p-cross-section bodies are generally not convex when $p > 1$. Cohn's result is useful again in Section 6, where we obtain the best-possible inclusion

$$a_{n,q}C_qK \subseteq a_{n,p}C_pK,$$

where $-1 < p < q$ and

$$a_{n,p} = \left(\frac{np+n-p}{n}\right)^{1/p}$$

for nonzero p. (This is the counterpart of a similar inclusion between R_qK and R_pK in [8] that implies two powerful affine isoperimetric inequalities, the Rogers-Shephard inequality and the Zhang projection inequality.) In Corollary 6.4 we deduce that if K has its centroid at the origin, then

$$e^{-1+1/n}IK \subset C_pK,$$

for $p > 0$, a pleasing complement to the inclusion $CK \subset e^{1-1/n}IK$ proved in [15]. In Corollary 6.6 we show that there is an ellipsoid E such that

$$E \subset CK \subset \sqrt{12}E$$

and use this fact to disprove Conjecture 7.12 of [15]. Finally, we note that the simple inclusion

$$\Pi K \subset nCK$$

represents a substantial improvement on Theorem 7.1 of [15].

2. Definitions and Preliminaries

As usual, S^{n-1} denotes the unit sphere, B the unit ball, and o the origin in Euclidean n-space $\mathbb{E}^n$. By a *direction*, we mean a unit vector, that is, an element of S^{n-1}. If u is a direction, we denote by $u^\perp$ the $(n-1)$-dimensional subspace orthogonal to u and by l_u the line through the origin parallel to u. Throughout the paper the symbol $\subset$ denotes strict inclusion.

We write V_k for k-dimensional Lebesgue measure in $\mathbb{E}^n$, where $1 \le k \le n$, and where we identify V_k with k-dimensional Hausdorff measure. We also generally write V instead of V_n. We let $\kappa_n = V(B)$ and $\omega_n = V_{n-1}(S^{n-1})$. The notation dz will always mean $dV_k(z)$ for the appropriate k with $1 \le k \le n$.

We say that a set is *centered* if it is centrally symmetric, with center at the origin.

A *convex body* is a compact convex set with nonempty interior. If K is a convex body, we write h_K for its *support function.* (The excellent treatise of Schneider [24] explains such terms in detail.) The *projection body* of K is the centered convex body ΠK defined by

$$h_{\Pi K}(u) = V_{n-1}(K|u^\perp),$$

for each $u \in S^{n-1}$, where $K|u^\perp$ is the orthogonal projection of K on $u^\perp$. (The support function of the projection body is also called the *brightness function* of K.) We denote the *polar body* of K by K^*, and call $\Pi^* K$, the polar body of ΠK, the *polar projection body* of K. The *difference body* DK of K is defined by

$$DK = K + (-K).$$

The support function of DK is the *width function* w_K of K. The *polar difference body* D^*K is the polar body of the difference body of K.

A set L is *star-shaped* with respect to the point x if every line through x which meets L does so in a (possibly degenerate) closed line segment. If L is a compact set that is star-shaped with respect to x, its *radial function* $\rho_L(x,\cdot)$ *with respect to* x is defined, for all $u \in S^{n-1}$ such that the line through x parallel to u intersects L, by

$$\rho_L(x,u) = \max\{c : x + cu \in L\}.$$

The radial function of L with respect to x can be extended to $\mathbb{E}^n \backslash \{x\}$ by

$$\rho_L(x,z) = \frac{1}{r}\rho_L(x,u),$$

where $z = x + ru$, $r > 0$, $u \in S^{n-1}$. We call this the *extended radial function of* L *with respect to* x. When x is the origin, we also denote $\rho_L(o,u)$ by $\rho_L(u)$ and refer to it simply as the radial function of L. By a *star body* we mean a compact set L whose radial function is defined and continuous. Note that this implies that $o \in L$.

Let K be a convex body in $\mathbb{E}^n$. It is not difficult to verify that

$$\rho_{DK}(u) = \max_{x\in K} \rho_K(x,u) = \max_{y\in u^\perp} V_1(K \cap (l_u + y)),$$

for $u \in S^{n-1}$.

The *intersection body* of a star body L is the centered body IL defined by

$$\rho_{IL}(u) = V_{n-1}(L \cap u^\perp) = \frac{1}{n-1} R(\rho_L^{n-1})(u),$$

for each $u \in S^{n-1}$. Here R denotes the *spherical Radon transform*, defined by

$$Rf(u) = \int_{S^{n-1}\cap u^\perp} f(v)\, dv,$$

for $f \in C(S^{n-1})$. If K is a centered convex body, then IK is convex, by Busemann's theorem (see, for example, [23], Theorem 3.9 or [7], Theorem 8.1.10).

The *cross-section body* of a convex body K, introduced by Martini [20] (see also [7], Chapter 8), is the centered body CK defined by

$$\rho_{CK}(u) = \max_{t \in \mathbb{R}} V_{n-1}(K \cap (u^\perp + tu)),$$

for each $u \in S^{n-1}$.

Part (i) of the following result was proved by Martini [19] (the right-hand inclusion was noted earlier by Petty; see [7], p. 308), while part (ii) was established by Makai and Martini [15], Theorem 3.1.

Proposition 2.1. *Let K be a convex body in $\mathbb{E}^n$ containing the origin. Then*

$$\text{(i)} \qquad IK \subseteq CK \subseteq \Pi K.$$

If K has its centroid at the origin, then

$$\text{(ii)} \qquad CK \subseteq \left(\frac{n+1}{n}\right)^{n-1} IK.$$

The inclusions in the previous proposition are the best possible, in the following sense. Clearly, $CK = IK$ if K is centered. (It is stated in [17] that $CK = IK$ if and only if K is centered; this depends on results in [18].) Martini [19] gives a necessary and sufficient condition for $\rho_{CK}(u) = \rho_{\Pi K}(u)$ to hold for a given $u \in S^{n-1}$ when $n \geq 3$; this condition is satisfied, in particular, if K is a cylinder with axis in direction u. He concludes that when $n \geq 3$, $CK = \Pi K$ if and only if K is an ellipsoid. In [15] (see also [7], Theorem 8.3.5), the authors note that $CK = \Pi K$ when $n = 2$ and prove that the constant in Proposition 2.1(ii) cannot be reduced if and only if K is a cone.

A function f with convex support in $\mathbb{E}^n$ is called *log concave* if $\log f$ is concave, that is, if

$$f((1-\alpha)x + \alpha y) \geq f(x)^{1-\alpha} f(y)^\alpha,$$

whenever $0 < \alpha < 1$ and x, y are in the support of f.

The term *absolute constant* in statements concerning a convex body K in $\mathbb{E}^n$ means a constant independent of n and K.

Suppose that K is a body in $\mathbb{E}^n$ and $\mathcal{L}$ is a family of star bodies in $\mathbb{E}^n$ associated with K. We say that the bodies in $\mathcal{L}$ are *equivalent* if there are nonzero absolute constants c_0 and c_1 such that $c_0 L \subseteq L' \subseteq c_1 L$ whenever $L, L' \in \mathcal{L}$.

3. Bodies defined by pth means

Suppose $p \neq 0$, μ is a finite Borel measure in a set X, and f is a nonnegative μ-integrable function on X. The *pth mean* $M_p f$ of f is

$$M_p f = \left(\frac{1}{\mu(X)} \int_X f(x)^p d\mu(x) \right)^{1/p}.$$

It is easy to show that

$$\lim_{p \to \infty} M_p f = \operatorname{ess} \sup_{x \in X} f(x)$$

and

$$\lim_{p \to 0} M_p f = \exp \left(\frac{1}{\mu(X)} \int_X \log f(x) d\mu(x) \right).$$

The best reference for integral means is still [10], Chapter 6.

Several families of bodies have already been defined using pth means. We mention two of these here.

Let K be a convex body in $\mathbb{E}^n$. The *pth radial mean body* $R_p K$ of K is defined by

$$\rho_{R_p K}(u) = \left(\frac{1}{V(K)} \int_K \rho_K(x, u)^p \, dx \right)^{1/p},$$

for $u \in S^{n-1}$ and nonzero $p > -1$. We have $R_\infty K = DK$ and

$$((p+1)V(K))^{1/p} R_p K \to \Pi^* K,$$

as $p \to -1+$; see [8]. This spectrum of bodies therefore connects the difference body and the polar projection body.

Suppose that C is a compact set in $\mathbb{E}^n$ with $V_n(C) > 0$. The *polar pth centroid body* $\Gamma_p^* C$ of C is defined by

$$\rho_{\Gamma_p^* C}(u) = \left(\frac{1}{V_n(C)} \int_C |u \cdot x|^p \, dx \right)^{-1/p},$$

for $u \in S^{n-1}$ and nonzero $p > -1$. See, for example, [14] (where C is assumed to be a star body and where the definition contains an extra constant factor) and [7], p. 342. We are grateful to Erwin Lutwak and Gaoyong Zhang for permission to include the following unpublished result of theirs.

Proposition 3.1. *Let L be a star body in $\mathbb{E}^n$. Then*

$$\left(\frac{2}{(p+1)V(L)} \right)^{1/p} \Gamma_p^* L \to IL,$$

as $p \to -1+$.

Proof. For $p > -1$, the p-cosine transform $T_p f$ of a function $f \in C(S^{n-1})$ is defined by

$$T_p f(u) = \int_{S^{n-1}} f(v)|u \cdot v|^p \, dv,$$

for $u \in S^{n-1}$. It is known that

$$\lim_{p \to -1+} \frac{p+1}{2} T_p f = Rf,$$

for each $f \in C(S^{n-1})$; see [9] or [11]. Using this fact, a change to spherical polar coordinates, Fubini's theorem, and the continuity of R, we obtain

$$\begin{aligned}
\left(\frac{2}{(p+1)V(L)}\right)^{-1} \rho_{\Gamma_p^* L}(u)^{-p} &= \frac{p+1}{2} \int_L |u \cdot x|^p \, dx \\
&= \frac{p+1}{2(n+p)} \int_{S^{n-1}} \rho_L(v)^{n+p} |u \cdot v|^p \, dv \\
&= \frac{p+1}{2(n+p)} T_p(\rho_L^{n+p})(u) \\
&\to \frac{1}{n-1} R(\rho_L^{n-1})(u) = \rho_{IL}(u)
\end{aligned}$$

as $p \to -1+$. □

When K is a centered convex body, $\Gamma_\infty^* K = K^*$, so the spectrum of polar pth centroid bodies then connects the polar body and the intersection body.

4. The p-cross-section body $C_p K$

Let K be a convex body in $\mathbb{E}^n$. We define the *p-cross-section body* $C_p K$ of K for nonzero $p > -1$ by

$$\begin{aligned}
\rho_{C_p K}(u) &= \left(\frac{1}{V(K)} \int_K V_{n-1}(K \cap (u^\perp + x))^p \, dx\right)^{1/p} \\
&= \left(\frac{1}{V(K)} \int_{\mathbb{R}} V_{n-1}(K \cap (u^\perp + tu))^{p+1} \, dt\right)^{1/p}.
\end{aligned}$$

We can define $C_0 K$ by

$$\rho_{C_0 K}(u) = \exp\left(\frac{1}{V(K)} \int_K \log V_{n-1}(K \cap (u^\perp + x)) \, dx\right),$$

for each $u \in S^{n-1}$. We can also define $C_\infty K$ by

$$\rho_{C_\infty K}(u) = \max_{x \in K} V_{n-1}(K \cap (u^\perp + x)),$$

for each $u \in S^{n-1}$. The bodies $C_p K$ then vary continuously with p.

In view of the above definition,

$$C_\infty K = CK.$$

Moreover, as $p \to -1+$,

$$\rho_{C_p K}(u) \to \frac{V(K)}{w_K(u)} = \frac{V(K)}{h_{DK}(u)} = V(K)\rho_{D^*K}(u),$$

so

$$C_p K \to V(K) D^* K$$

as $p \to -1+$. The p-cross-section bodies therefore form a spectrum connecting the polar difference body and the cross-section body. That these new bodies are natural objects in affine geometry is suggested by the following fact.

Theorem 4.1. *If ϕ is a nonsingular linear transformation and $p > -1$, then*

$$C_p(\phi K) = |\det \phi| \phi^{-t}(C_p K),$$

where ϕ^{-t} is the linear transformation whose matrix is the inverse transpose of that of ϕ.

Proof. Note that

$$\rho_{C_p K}(u) = \left(\frac{1}{V(K)} \int_K V_{n-1}((K-x) \cap u^\perp)^p \, dx \right)^{1/p}$$

$$= \left(\frac{1}{V(K)} \int_K \rho_{I(K-x)}(u)^p \, dx \right)^{1/p}.$$

The theorem is now an easy consequence of the known formulas

$$I(\phi L) = |\det \phi| \phi^{-t}(IL)$$

(see [12] or [7], Theorem 8.1.6) and

$$\rho_{\phi L}(x) = \rho_L(\phi^{-1} x),$$

for $x \in \mathbb{E}^n \backslash \{o\}$ (see [7], p. 20), which hold for any star body L. $\square$

5. Convexity issues

Busemann's theorem shows that when K is centrally symmetric with center x, $CK = I(K-x)$ is convex. Martini [21] asked whether CK is always convex. This was confirmed by Meyer [22] in the case $n=3$, but Brehm [4] showed that when $n \geq 4$, CK is not convex when K is a simplex. Makai and Martini [16] had shown earlier that CK is a parallelepiped when K is a simplex in $\mathbb{E}^3$.

We know that $C_{-1}K$ is convex, since D^*K is convex. We also know that C_pK is an ellipsoid if K is an ellipsoid, by Theorem 4.1. The results of [8] show that C_pK is convex when $n=2$ and $p>0$, and Meyer's result above shows that C_pK is convex when $n=3$ and $p=\infty$.

We shall now prove that C_1K is convex. It is convenient to introduce the following dilatate Z_pK of Γ_p^*K. Let

$$\rho_{Z_pK}(u) = \left(\frac{p+1}{2}\int_K |u\cdot x|^p\,dx\right)^{-1/p},$$

for $u \in S^{n-1}$ and nonzero $p > -1$. By Proposition 3.1, we have $Z_pK \to IK$ as $p \to -1+$, so we can consistently define

$$Z_{-1}K = IK.$$

Lemma 5.1. *Let K be a convex body in $\mathbb{E}^n$ with $o \in \operatorname{int} K$ and let $p \geq -1$ be nonzero. Then*

$$\rho_{Z_p(R_{n+p}K)}(u) = \left(\frac{1}{V(K)}\int_K \rho_{Z_p(K-x)}(u)^{-p}\,dx\right)^{-1/p},$$

for $u \in S^{n-1}$.

Proof. Suppose that $p > -1$ is nonzero. Using spherical polar coordinates, we obtain

$$\begin{aligned}
\rho_{Z_p(R_{n+p}K)}(u)^{-p} &= \frac{p+1}{2}\int_{R_{n+p}K} |u\cdot x|^p\,dx\\
&= \frac{p+1}{2(n+p)}\int_{S^{n-1}} \rho_{R_{n+p}K}(v)^{n+p}|u\cdot v|^p\,dv\\
&= \frac{p+1}{2(n+p)V(K)}\int_{S^{n-1}}\int_K \rho_K(x,v)^{n+p}|u\cdot v|^p\,dx\,dv\\
&= \frac{p+1}{2(n+p)V(K)}\int_K\int_{S^{n-1}} \rho_{K-x}(v)^{n+p}|u\cdot v|^p\,dv\,dx\\
&= \frac{1}{V(K)}\int_K \rho_{Z_p(K-x)}(u)^{-p}\,dx,
\end{aligned}$$

for all $u \in S^{n-1}$. The case $p = -1$ follows by continuity. $\square$

Theorem 5.2. *Let K be a convex body in $\mathbb{E}^n$. Then*

$$C_1K = I(R_{n-1}K).$$

Proof. Using Lemma 5.1 with $p = -1$, we obtain

$$\begin{aligned}
\rho_{C_1K}(u) &= \frac{1}{V(K)} \int_K V_{n-1}(K \cap (u^\perp + x))\, dx \\
&= \frac{1}{V(K)} \int_K \rho_{I(K-x)}(u)\, dx \\
&= \frac{1}{V(K)} \int_K \rho_{Z_{-1}(K-x)}(u)\, dx \\
&= \rho_{Z_{-1}(R_{n-1}K)}(u) = \rho_{I(R_{n-1}K)}(u),
\end{aligned}$$

for all $u \in S^{n-1}$. $\square$

Corollary 5.3. *Let K be a convex body in $\mathbb{E}^n$. Then C_1K is convex.*

Proof. In [8] it was shown that $R_{n-1}K$ is a centered convex body, so C_1K is convex by Theorem 5.2 and Busemann's theorem. $\square$

The previous theorem can be proved by working directly with the definition of C_1K, but the family $\{Z_p(R_{n+p}K) : p \geq -1\}$ seems to be of independent interest as a spectrum linking D^*K and C_1K.

It would be interesting to know whether the bodies $Z_p(R_{n+p}K)$ are convex for all $p \geq -1$. This is true for $p \geq 1$, since by Minkowski's integral inequality, Γ_p^*C is then convex for any compact set C. We know that $R_{n+p}K$ is a centered convex body, by [8], Theorem 4.3, but it seems to be unknown whether Γ_p^*K is convex when K is a centered convex body and $-1 < p < 1$.

In order to state our next theorem, we require some technical lemmas. The following result of Cohn [5] (see also the paper of Borell [3] for a generalization) will be useful now and also in the next section.

Proposition 5.4. *Let f be positive and concave on (a, b). Then the function*

$$F(p) = (p+1) \int_a^b f(t)^p\, dt$$

is log concave for $p > 0$. Moreover, $\log F$ is linear in an interval $[p_0, p_1]$ if and only if the decreasing rearrangement of f is of the form $c(t-a)$ for some constant c.

It will be convenient to let

$$a_{n,p} = \left(\frac{np + n - p}{n} \right)^{1/p},$$

for nonzero $p > -1$ and

$$a_{n,0} = e^{(n-1)/n} = \lim_{p \to 0} a_{n,p}.$$

Lemma 5.5. *Let $n > 1$ and suppose that f is positive and concave on (a, b). Let*

$$G(p) = a_{n,p} \left(\frac{\displaystyle\int_a^b f(t)^{(n-1)(p+1)}\, dt}{\displaystyle\int_a^b f(t)^{n-1}\, dt} \right)^{1/p},$$

for $p > -1$. Then $G(q) \le G(p)$ for $-1 < p < q$, with equality if and only if the decreasing rearrangement of f is of the form $c(t-a)$ for some constant c.

Proof. Suppose that $0 < p < q$. Then $0 < p/q < 1$ and

$$(n-1)(p+1) = \left(1 - \frac{p}{q}\right)(n-1) + \frac{p}{q}(n-1)(q+1).$$

Proposition 5.4 implies that

$$F((n-1)(p+1)) \ge F(n-1)^{1-p/q} F((n-1)(q+1))^{p/q}.$$

This is equivalent to

$$(np + n - p) \int_a^b f(t)^{(n-1)(p+1)}\, dt \ge \left(n \int_a^b f(t)^{n-1}\, dt \right)^{1-p/q} \times \left((nq + n - q) \int_a^b f(t)^{(n-1)(q+1)}\, dt \right)^{p/q},$$

or $G(q) \le G(p)$. If $-1 < p < q < 0$, we have $0 < q/p < 1$, and the inequality $G(q) \le G(p)$ again results from interchanging p and q in the above argument. Therefore this inequality holds for $-1 < p < q$ by continuity. The equality conditions follow from those of Proposition 5.4. □

Lemma 5.6. *For $n > 2$ and $p > 0$, let*

$$g(n,p) = 2^p(n-1)^{p+1}(np+n-p)B(np+n-2p-1,p+2).$$

Then $g(n,p)^{1/p}$ is strictly decreasing for $p > 0$ and

$$\lim_{p\to\infty} g(n,p)^{1/p} = 2\left(\frac{n-2}{n-1}\right)^{n-2}.$$

Proof. We have

$$g(n,p)^{1/p} = 2(n-1)\left((n-1)(np+n-p)B(np+n-2p-1,p+2)\right)^{1/p},$$

where $B(\cdot,\cdot)$ denotes the Beta function. Let $f(t) = (t^{n-2}(1-t))^{1/(n-1)}$. Then

$$f''(t) = -\frac{(n-2)(t^{n-2}(1-t))^{1/(n-1)}}{(n-1)^2t^2(1-t)^2},$$

so f is positive and strictly concave on $(0,1)$. Therefore,

$$\begin{aligned} & a_{n,p}\left(\frac{\displaystyle\int_0^1 (t^{n-2}(1-t))^{p+1}\,dt}{\displaystyle\int_0^1 t^{n-2}(1-t)\,dt}\right)^{1/p} \\ &= \left((n-1)(np+n-p)B(np+n-2p-1,p+2)\right)^{1/p} \end{aligned}$$

is strictly decreasing for $p > 0$, by Lemma 5.5.

The limit of $g(n,p)^{1/p}$ as $p\to\infty$ is obtained by a routine application of Stirling's formula. □

Theorem 5.7. *Let K be an n-dimensional simplex in $\mathbb{E}^n$, $n \geq 4$. Then C_pK, $p > 0$ is not convex when $p > p_n$, where p_n is the unique real number for which $g(n,p_n) = 1$.*

Proof. The proof closely follows that of Brehm [4] for the case $p = \infty$, and we shall refer to that paper for some details.

By Theorem 4.1, we may assume that $K = \triangle^n$, where $\triangle^n$ is the regular simplex in $\mathbb{E}^n$ of side length 1 with centroid at o. Let u_1, u_2 be unit vectors in the direction of two vertices of K, and let $u_3 = (u_1+u_2)/\|u_1+u_2\|$. For $i = 1,2$, a hyperplane orthogonal to u_i that intersects $\triangle^n$ does so in a regular $(n-1)$-dimensional simplex of side length s/h_n, $0 \leq s \leq h_n$, where

$$h_n = w_{\triangle^n}(u_1) = \left(\frac{n+1}{2n}\right)^{1/2}.$$

The quantity h_n is the width of $\triangle^n$ in a direction orthogonal to one of its facets. From this (or see [4]) we obtain

$$V_{n-1}(\triangle^{n-1}) = \frac{h_{n-1}}{n-1} V_{n-2}(\triangle^{n-2}) = \frac{1}{n-1}\left(\frac{n}{2(n-1)}\right)^{1/2} V_{n-2}(\triangle^{n-2}).$$

Using these expressions, we have for $p > 0$ and $i = 1, 2$,

$$\begin{aligned}
\rho_{C_p\triangle^n}(u_i)^p &= \int_0^{h_n} \left(V_{n-1}(\triangle^{n-1}) \left(\frac{s}{h_n}\right)^{n-1} \right)^{p+1} ds \\
&= h_n \int_0^1 \left(V_{n-1}(\triangle^{n-1}) t^{n-1} \right)^{p+1} dt \\
&= \frac{h_n V_{n-1}(\triangle^{n-1})^{p+1}}{np+n-p} \\
&= \left(\frac{n+1}{2n}\right)^{1/2} \frac{1}{(n-1)^{p+1}} \left(\frac{n}{2(n-1)}\right)^{(p+1)/2} \frac{V_{n-2}(\triangle^{n-2})^{p+1}}{np+n-p}.
\end{aligned}$$

A hyperplane orthogonal to u_3 that intersects $\triangle^n$ does so in a cylinder of height $(1 - s/w_n)$ and base a regular $(n-2)$-dimensional simplex of side length s/w_n, $0 \le s \le w_n$, where

$$w_n = w_{\triangle^n}(u_3) = \frac{1}{2}\left(\frac{n+1}{n-1}\right)^{1/2}.$$

Therefore

$$\begin{aligned}
\rho_{C_p\triangle^n}(u_3)^p &= \int_0^{w_n} \left(V_{n-2}(\triangle^{n-2}) \left(\frac{s}{w_n}\right)^{n-2} \left(1 - \frac{s}{w_n}\right) \right)^{p+1} ds \\
&= w_n \int_0^1 \left(V_{n-2}(\triangle^{n-2}) t^{n-2} (1-t) \right)^{p+1} dt \\
&= w_n V_{n-2}(\triangle^{n-2})^{p+1} \int_0^1 t^{np+n-2p-2} (1-t)^{p+1}\, dt \\
&= \frac{1}{2}\left(\frac{n+1}{n-1}\right)^{1/2} V_{n-2}(\triangle^{n-2})^{p+1} B(np+n-2p-1\ , p+2).
\end{aligned}$$

From [4], we have

$$\|u_1 + u_2\| = 2\left(\frac{n-1}{2n}\right)^{1/2},$$

and by [7], Lemma 5.1.4 we know that $C_p\triangle^n$ is not convex if

$$\|u_1+u_2\|\rho_{C_p\triangle^n}(u_3)^{-1} > \rho_{C_p\triangle^n}(u_1)^{-1}+\rho_{C_p\triangle^n}(u_2)^{-1}.$$

Substituting the quantities above into the previous inequality, we conclude that $C_p\triangle^n$ is not convex when $g(n,p)<1$. Since

$$g(n,1)=\frac{2n-2}{2n-3}>1$$

and

$$\lim_{n\to\infty} g(n,p)^{1/p}=2\left(\frac{n-2}{n-1}\right)^{n-2}<1$$

when $n\geq 4$, the result follows from Lemma 5.6. □

The proof of the previous theorem also applies to $n=3$, but its statement is then vacuous, since by Lemma 5.6 we have $g(3,p)>1$ for all $p>0$. Of course, this must be the case by Meyer's result [22] that CK is convex when $n=3$. We find by numerical computation, using Mathematica, the approximate values of p_n listed in the right-hand column of the following table.

n	p_n
4	9.09
5	4.695
6	3.371
7	2.741
8	2.3743
9	2.1347
10	1.966

***Corollary* 5.8.** *For $p>1$, p-cross-section bodies are generally not convex.*

Proof. A straightforward application of Stirling's formula shows that for $p>0$,

$$\lim_{n\to\infty} g(n,p)^{1/p}=\frac{2\Gamma(p+1)^{1/p}}{(p+1)}.$$

It follows from Lemma 5.6 that this limit decreases for $p>0$. Since the limit equals 1 when $p=1$, we have $p_n\to 1$ as $n\to\infty$. Consequently, if $p>1$, there is an n such that $p_n<p$. The result now follows from Theorem 5.7. □

It is possible that C_pK is always convex when $-1 < p \leq p_n$ or at least when $-1 < p \leq 1$. We conjecture that C_pK is convex for all $p > -1$ when K is centered or when $n = 3$.

6. Inclusion results

Jensen's inequality states that if $M_q f$ exists, then

$$M_p f \leq M_q f,$$

for $p \leq q$, with equality if and only if f is constant, as in [10], Sections 6.10 and 6.11. It follows that

$$V(K)\rho_{D^*K}(u) \leq \rho_{C_pK}(u) \leq \rho_{C_qK}(u) \leq \rho_{C_\infty K}(u) = \rho_{CK}(u),$$

when $-1 < p \leq q$. By the equality conditions for Jensen's inequality and those for the Brunn-Minkowski inequality, equality holds if and only if K is the Minkowski sum of an $(n-1)$-dimensional convex body contained in a hyperplane orthogonal to u and a line segment; in short, a (not necessarily right) cylinder with base orthogonal to u. From this we can obtain

$$V(K)D^*K \subset C_pK \subset C_qK \subset CK.$$

The inclusions with possible equality follow at once, but the argument of Martini [20], Theorem 3 (see also [7], p. 345 for other references), used to derive the outer strict inclusion

$$V(K)D^*K \subset CK,$$

applies equally well to the other inclusions in view of the equality conditions for the radial functions given above. Indeed, Martini's proof shows that equality of the radial functions cannot hold for more than n directions, and holds for precisely n linearly independent directions if and only if K is a parallelotope.

The constant $a_{n,p}$ in the next theorem is that defined in the previous section.

Theorem 6.1. *Let K be a convex body in $\mathbb{E}^n$ and let $u \in S^{n-1}$. If $-1 < p < q$, then*

$$\rho_{CK}(u) \leq a_{n,q}\rho_{C_qK}(u) \leq a_{n,p}\rho_{C_pK}(u) \leq nV(K)\rho_{D^*K}(u).$$

In each inequality, equality holds if and only if K is the convex hull of an $(n-1)$-dimensional convex body contained in a hyperplane orthogonal to u and a line segment; in short, a (not necessarily right) cone or double cone with base orthogonal to u.

Proof. Suppose that

$$f(t) = V_{n-1}(K \cap (u^\perp + tu))^{1/(n-1)}$$

has support $[a, b]$. Then

$$\begin{aligned} a_{n,p}\rho_{C_pK}(u) &= a_{n,p}\left(\frac{1}{V(K)}\int_a^b V_{n-1}(K\cap(u^\perp+tu))^{p+1}\,dt\right)^{1/p} \\ &= a_{n,p}\left(\frac{\displaystyle\int_a^b f(t)^{(n-1)(p+1)}\,dt}{\displaystyle\int_a^b f(t)^{n-1}\,dt}\right)^{1/p}. \end{aligned}$$

Since f is positive on (a, b) and concave by the Brunn-Minkowski inequality, the middle inequality in the statement of the theorem follows from Lemma 5.5. The left- and right-hand inequalities are just the limiting cases of the middle inequality as $p \to -1+$ and $q \to \infty$.

The equality conditions follow immediately from those of Lemma 5.5 and those of the Brunn-Minkowski inequality. □

***Corollary* 6.2.** *Let K be a convex body in $\mathbb{E}^n$. If $-1 < p < q$, then*

$$CK \subseteq a_{n,q}C_qK \subseteq a_{n,p}C_pK \subseteq nV(K)D^*K.$$

In each inclusion, equality holds if and only if $n = 2$ and K is a triangle.

Proof. The inclusions and equality condition for $n = 2$ follow directly from the previous theorem. Martini [20], Theorem 5 proved the outer inclusion

$$CK \subseteq nV(K)D^*K,$$

showing that if equality of the radial functions holds for a set of directions containing $n+1$ directions in general position, $n \geq 3$, then it holds for precisely $n+1$ directions in general position, and this occurs if and only if K is a simplex. In particular, the inclusions are strict when $n \geq 3$. The proof uses only the equality conditions of Theorem 6.1, so it applies also to the other inclusions in the statement of the corollary. □

***Corollary* 6.3.** *Let K be a convex body in $\mathbb{E}^n$. Then for $p > 0$,*

$$e^{-1+1/n}CK \subset C_pK.$$

Proof. Since $a_{n,p}$ decreases for $p > 0$, Corollary 6.2 implies that $CK \subset a_{n,0}C_pK$. □

The previous corollary and Proposition 2.1(ii) yield the following result.

***Corollary* 6.4.** *Let K be a convex body in $\mathbb{E}^n$ with centroid at the origin. Then for $p > 0$,*

$$e^{-1}IK \subset e^{-1+1/n}IK \subset C_pK \subset e^{1-1/n}IK \subset eIK.$$

The previous two corollaries show that for $p > 0$, all the bodies C_pK are equivalent, and when K has its centroid at the origin, these bodies are also equivalent to IK.

***Lemma* 6.5.** *Let K be a convex body in $\mathbb{E}^n$ with $o \in \operatorname{int} K$. Then*

$$\frac{V(K)}{\sqrt{12}}\Gamma_2^*K \subseteq CK.$$

Proof. Fix $u \in S^{n-1}$, and suppose that

$$g(t) = V_{n-1}(K \cap (u^\perp + tu))$$

has support $[-a, b]$. Then

$$\rho_{\Gamma_2^*K}(u) = \left(\frac{1}{V(K)}\int_{-a}^{b} t^2 g(t)\,dt\right)^{-1/2}$$

and $\rho_{CK}(u) = \max g(t) = M$, say.

Suppose that $\int_0^b g(t)\,dt = m_+$, let G be the function such that $G(t) = M$ for $0 \le t \le m_+/M$ and $G(t) = 0$ otherwise, and let $h = G - g$. Then $h(t) \ge 0$ for $0 \le t \le m_+/M$ and $h(t) \le 0$ for $m_+/M < t \le b$. Since $\int_0^b h(t)\,dt = 0$, it follows that

$$\int_0^s h(t)\,dt \ge 0$$

for all $s \in [0, b]$. By Hardy's lemma (see [10], Theorem 399), we have

$$\int_0^b j(t)h(t)\,dt \ge 0,$$

for any nonnegative, decreasing, continuous function j on $[0, b]$. If we take $j(t) = b^2 - t^2$, $0 \le t \le b$, we get

$$\int_0^b t^2 G(t)\,dt \le \int_0^b t^2 g(t)\,dt.$$

This yields

$$\frac{m_+^3}{3M^2} \le \int_0^b t^2 g(t)\,dt.$$

If $\int_{-a}^{0} g(t)\,dt = m_-$, the same argument gives

$$\frac{m_-^3}{3M^2} \le \int_{-a}^{0} t^2 g(t)\,dt.$$

Therefore

$$\int_{-a}^{b} t^2 g(t)\,dt \ge \frac{m_-^3 + m_+^3}{3M^2} \ge \frac{(m_- + m_+)^3}{12M^2} = \frac{V(K)^3}{12M^2}.$$

By the previous paragraph, this is equivalent to

$$\frac{V(K)}{\sqrt{12}}\rho_{\Gamma_2^* K}(u) \le \rho_{CK}(u),$$

which proves the lemma. □

Fradelizi [6] independently proved the previous lemma under the assumption that K has its centroid at the origin. In fact the latter assumption is easily seen to be unnecessary, and [6], Theorem 9 also provides best-possible constants $c_{0,p}$ and $c_{1,p,n}$ such that

$$c_{0,p}\Gamma_p^* K \subset CK \subset c_{1,p,n}\Gamma_p^* K,$$

for K with centroid at the origin and $p \ge 1$. (That the inclusions are strict follows from the conditions for equality of the radial functions, given in [6], which allow the arguments of Martini [20], Theorems 3 and 5, to be applied as we did above.) The constant $c_{0,2} = V(K)/\sqrt{12}$, so Lemma 6.5 is the best possible, and $c_{1,2,n}$ can be evaluated from the formula in [6] to yield

$$CK \subset \left(\frac{n^3}{(n+2)(n+1)^2}\right)^{1/2} V(K)\Gamma_2^* K \subset V(K)\Gamma_2^* K,$$

for K with centroid at the origin.

The body $\Gamma_2^* K$ is always a centered ellipsoid. (See, for example, [13]; the proof does not require the general assumption in that paper that the body contains the origin.) When K does not contain the origin in its interior, the inclusion $CK \subset V(K)\Gamma_2^* K$ and the one in Lemma 6.5 still hold if we replace $\Gamma_2^* K$ by $\Gamma_2^*(K - x)$, where x is the centroid of K. From these facts we obtain the following corollary.

Corollary 6.6. *Let K be a convex body in $\mathbb{E}^n$. There is an ellipsoid E such that*

$$E \subset CK \subset \sqrt{12}E.$$

Makai and Martini [15], Conjecture 7.2, second part, conjectured that if K is centrally symmetric, there is an absolute constant c such that $\Pi K \subset cCK$, where c is the appropriate constant for the cross-polytope. This is false, however. Indeed, Proposition 2.1 and Corollary 6.6 would then imply that

$$E \subset \Pi K \subset c\sqrt{12}E.$$

But every centered n-dimensional zonoid is a projection body (see [7], Theorem 4.1.11), so this in turn would imply that the volume ratio of zonoids are bounded by an absolute constant, contradicting the fact that they can be of order as large as $\sqrt{n}$. In fact, this conjecture is false even for a centered cube. To see this, note that by a result of Ball [1], [2], the maximal central section of a centered unit cube K has volume $\sqrt{2}$. Since $\Pi K = 2K$, this implies that if u is parallel to a diagonal of K, we have $\rho_{\Pi K}(u) = \sqrt{n}$, while $\rho_{CK}(u) \leq \sqrt{2}$, so $\rho_{\Pi K}(u) \geq \sqrt{n/2}\rho_{CK}(u)$.

We also note the following simple result that substantially improves on [15], Theorem 7.1.

***Theorem* 6.7.** *Let K be a convex body in $\mathbb{E}^n$. Then*

$$\Pi K \subset nCK.$$

Proof. From the known inclusion $DK \subseteq nV(K)\Pi^* K$ of A. M. Macbeath, in which equality holds if and only if K is a simplex (see, for example, [8] or [7], p. 345), it follows that

$$\Pi K \subseteq nV(K)D^* K.$$

Combining this with the inclusion $V(K)D^*K \subset CK$ noted at the beginning of this section, we immediately obtain the desired inclusion. □

7. A variant of C_pK

Suppose that K is a convex body in $\mathbb{E}^n$. With notation introduced in Section 2, we define a variant E_pK of the p-cross-section body C_pK by

$$\rho_{E_pK}(u) = \left(\frac{1}{w_K(u)}\int_{\mathbb{R}} V_{n-1}(K \cap (u^\perp + tu))^p \, dt\right)^{1/p},$$

for each $u \in S^{n-1}$ and $p \geq 1$. The expression on the right is a pth mean, so by the argument applied to C_pK at the beginning of Section 6, we have

$$V(K)D^*K = E_1K \subset E_pK \subset E_qK \subset E_\infty K = CK,$$

when $1 < p < q$. It can also be shown that

$$CK \subseteq b_{n,q}E_qK \subseteq b_{n,p}E_pK \subseteq nV(K)D^*K,$$

where

$$b_{n,p} = (np - p + 1)^{1/p}$$

and $1 < p < q$, with equality in each inclusion if and only if $n = 2$ and K is a triangle. (Instead of Proposition 5.4, a suitable version of [8], Lemma 5.3 can be applied.)

For $p > 0$, the equation

$$\rho_{E_{p+1}K}^{p+1} = (p+1)V(K)\rho_{D^*K}\rho_{C_pK}^{p}$$

relates two of the classes of bodies we have introduced.

It is, of course, possible to extend the definition of E_pK to $p > 0$. However, E_pK is, in general, a nonconvex star body when $0 < p < 1$, as can be directly verified when K is a centered square, for example. The above relationships show that E_1K is convex and $E_\infty K$ is generally not convex. Calculations for the case when K is an n-dimensional simplex, similar to those performed in Section 5, can be carried out, and leave open the possibility that E_pK is convex for all convex bodies K in $\mathbb{E}^n$ when $1 \le p \le 5$.

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