BULL. AUSTRAL. MATH. SOC. 20F28
VOL. 29 (1984), 35-37.

P-GROUPS WITH NON-ABELIAN AUTOMORPHISM GROUPS AND ALL AUTOMORPHISMS CENTRAL

J. J. MALONE

For each odd prime p this paper gives an example of a non-abelian p-group with a non-abelian automorphism group in which each automorphism is central.

In [1] M.J. Curran gave an example of a non-abelian 2-group with a non-abelian automorphism group in which every automorphism is central, and also provided some background material indicating the significance of this example. Here, the conditions needed in order to construct such examples are examined and the insights gained used to construct the groups described in the title of this paper.

Several observations are in order. First, if the central automorphisms of a group G fix $Z(G)$ elementwise then any two central automorphisms commute. This follows readily upon computing $\alpha\beta(g)$ and $\beta\alpha(g)$ for $g \in G$ and $\alpha, \beta \in \text{Aut}_C G$. Secondly, if the inner automorphisms of G are central, it is easy to see that each commutator is in $Z(G)$. Incidentally, this says that G, if not abelian, is nilpotent of class 2. Thirdly, recall that a central automorphism of G must fix G' elementwise. The net result of these observations is that, in order for two central automorphisms to be non-commuting, one of them must map an element which is in $Z(G)$ but not in G' away from itself. The automorphism ψ given by Curran is such a central automorphism.

Received 16 September 1983.

In order to construct the required examples for odd p , the groups given in [2] by R. Faudree will be utilized. (Faudree's 2-group will not be used because, as noted in [4], it does not have the properties claimed for it.) Hence, for each odd prime p , we consider the group

$$F = \langle a_1, a_2, a_3, a_4 \mid (a_i, a_j, a_k) = 1 \text{ and } a_i^{p^2} = 1 \text{ for } 1 \le i, j, k \le 4 ; (a_1,a_2) = a_1^p ; (a_1,a_3) = a_3^p ; (a_1,a_4) = a_4^p ; (a_2,a_3) = a_2^p ; (a_2,a_4) = 1 ; (a_3,a_4) = a_3^p \rangle .$$

Lemma 7 of [2] establishes that each automorphism of F is central while Lemma 2 says that $F' = Z(F) = F = U_p(F)$ where $U_p(F)$ is the set of elements of G whose orders divide p. Since $F' = Z(F)$, the observations made above tell us that Aut F is abelian.

The idea now is to take the direct product of F and another group in such a way that the new group still has all automorphisms central but does have an automorphism which does not fix some element of the centre. This is accomplished by taking $G = F \times B$ where we set $B = \langle b \mid b^p = 1 \rangle$. Then, $b \in Z(G)$ but $b \notin G'$. Note that $Z(G) = Z(F) \times B = U_p(G)$.

PROPOSITION. *The non-abelian group G has a non-abelian automorphism group in which each automorphism is central.*

Proof. Let $\alpha \in \text{Aut } G$, let ν be the natural homomorphism of G onto G/B , and let ρ be the isomorphism which identifies G/B with F . Furthermore, let $a \in F \subset G$ and note that $\alpha(a)$ has the form a^*b^m with $a^* \in F$. Consider $\rho\nu\alpha$ restricted to F :

$$\rho\nu\alpha(a) = \rho\nu(a^*b^m) = \rho(a^*B) = a^* .$$

Since the kernel of this restriction is trivial, the restriction is an automorphism of F . Then, since all automorphisms of F are central, $a^* = az$ where $z \in Z(F)$. Hence, $\alpha(a) = azb$ with $zb \in Z(G)$. We also have that $\alpha(b) = d = b(b^{-1}d)$ where d is some element of order p and so is in $Z(G)$. It now follows that α is central.

Finally, we exhibit two automorphisms of G which do not commute. Let θ be the automorphism of G which maps a_1 to a_1b and fixes the remaining generators and let ψ be the automorphism which maps b

to ba_4^p and fixes the remaining generators. By checking that the images of the generators of G satisfy the pattern of the defining relations, it is readily seen that, in fact, θ and ψ are automorphisms. Also, ψ is the automorphism we are seeking; there is an element of $Z(G)$, namely b, which it does not fix. Since $\psi\theta(a_1) = a_1ba_4$ and $\theta\psi(a_1) = a_1b$, Aut G is non-abelian. Thus we have the result.

Jonah and Konvisser [3] have given another family of p-groups having all automorphisms central. It appears that the technique applied above could also be used with most (see [5] for comments) of these groups to produce more examples of non-abelian groups of central automorphisms.

The examples presented here and in [1] have made use of groups with direct factors. It would be of interest to know if there is a group G with no direct factors for which Aut $G = \text{Aut}_c G$ and Aut G is non-abelian.

References

[1] M.J. Curran, "A non-abelian automorphism group with all automorphisms central", *Bull. Austral. Math. Soc.* 26 (1982), 393-397.

[2] R. Faudree, "Groups in which each element commutes with its endomorphic images", *Proc. Amer. Math. Soc.* 27 (1971), 236-240.

[3] P. Jonah and M. Konvisser, "Some non-abelian p-groups with abelian automorphism groups", *Arch. Math. (Basel)* 26 (1975), 131-133.

[4] J.J. Malone, "More on groups in which each element commutes with its endomorphic images", *Proc. Amer. Math. Soc.* 65 (1977), 209-214.

[5] J.J. Malone, "A non-abelian 2-group whose endomorphisms generate a ring, and other examples of E-groups", *Proc. Edinburgh Math. Soc.* 23 (1980), 57-59.

Department of Mathematical Sciences
Worcester Polytechnic Institute
Worcester
Massachusetts, 01609
U. S. A.