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p-Harmonic Obstacle Problems (*).

PART I: Partial Regularity Theory.

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Summary. – *We develop an interior partial regularity theory for vector valued Sobolev functions which locally minimize degenerate variational integrals under the additional side condition that all comparison maps take their values in the closure of a smooth region of the target space. Our results apply to the case of p-energy minimizing mappings $X \to Y$ between Riemannian manifolds including target manifolds Y with non-void boundary.*

0. – Introduction.

In this paper we investigate the partial regularity properties of vector valued functions $u: \Omega \to \mathbf{R}^N$ defined on some n-dimensional region Ω which locally minimize variational integrals of the form

$$E_p(u, \Omega) := \int_\Omega |Du|^p \tag{0.1}$$

under a smooth side condition in the image space. Here $p \in [2, n]$ is a fixed real number and the side condition is formulated as $\mathrm{Im}\,(u) \subset \bar{M}$, where we consider the following three different cases:

a) M is a smooth bounded open subset of Euclidean space $\mathbf{R}^N$ *or*

b) a smooth bounded subdomain of a k-dimensional submanifold Y of $\mathbf{R}^N$ such that $\bar{M} \subset \mathrm{Int}\,(Y)$ *or*

c) a compact submanifold of $\mathbf{R}^N$.

In *a*) and *b*) we are confronted with an *obstacle problem*, *c*) is the extension of the harmonic mapping problem studied by SCHOEN-UHLENBECK in [S, U1, 2] to the p-harmonic case which we included since the partial regularity theory in the unconstrained Riemannian case *c*) follows from our results concerning obstacle problems by simplification of the arguments.

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Introducing the Sobolev space

$$H^{1,p}(\Omega, \bar{M}) := \{v \in H^{1,p}(\Omega, \mathbf{R}^N) : v(x) \in \bar{M} \text{ a.e. on } \Omega\}$$

our main results (see A, B, C, D in section 1) can be summarized as follows:

If $u \in H^{1,p}(\Omega, \bar{M})$ *has the property* $E_p(u, \Omega) \leqslant E_p(v, \Omega)$ *for all* $v \in H^{1,p}(\Omega, \bar{M})$ *such that* spt $(u - v) \subset\subset \Omega$, *then there is a closed subset* Σ *of* Ω *such that* $u \in C^1(\Omega \setminus \Sigma)$ *and*

$$(0.2) \qquad \begin{cases} \mathbf{H} - \dim(\Sigma) \leqslant n - [p] - 1 & \text{for } n > p + 1\,, \\ \Sigma \text{ is discrete} & \text{for } n - 1 \leqslant p < n\,, \\ \Sigma = \phi & \text{for } p = n\,. \end{cases}$$

(The conformally invariant case $p = n$ is already treated in [Go].)

In a series of papers [D], [D, F1, 2], [F1-F6] (compare also [F, W] and [W]) we proved (0.2) for quadratic obstacle problems (i.e. $p = 2$) but none of the methods used there extend to exponents $p > 2$: For example in case a) a minimizer $u \in H^{1,p}(\Omega, \bar{M})$ is a weak solution of a system of the type (see Theorem 2.1)

$$L(u) := -D(|Du|^{p-2} Du) = R(\cdot, u, Du)$$

with a right-hand-side R supported on the coincidence set $[u \in \partial M]$ and growing of order p in Du. Obviously the differential operator L is linear only for $p = 2$ and our previous proofs used

$L(u^i) \to L(u)$ in the sense of distributions if

$u^i \to u$ weakly in $H^{1,p}$ *or*

$L(u^\varrho) = (L(u))^\varrho$ for mollifications.

Since both statements fail to hold for $p > 2$ we had to develop completely new arguments.

The first approach to p-harmonic obstacle problems is due to N. Fusco and the author [F, F] but restricted to a very special case: Assuming that M is diffeomorphic to a ball we could show (using standard arguments from [G]) that local minimizers satisfy Caccioppoli's inequality which gives $Du \in L^q_{\mathrm{loc}}$ for some $q > p$. Then a proper extension of the ideas in [F, H1, 2] and [G, M] proves (0.2) for this simple situation. Moreover we obtained (0.2) for a larger class of functionals than the p-energy introduced in (0.1). But for general nonconvex sets M it is impossible to get Caccioppoli's inequality along the known lines. The heart of our new partial regularity proof is an extension theorem

$$H^{1,p}(S^{n-1}, \bar{M}) \to H^{1,p}(B^n, \bar{M})$$

for maps defined on the sphere S^{n-1} with values in $\overline{M}$. In Theorem E below we show that if the p-energy and the mean oscillation of a function $v \in H^{1,p}(S^{n-1}, \overline{M})$ are small enough then v can be extended to a function $\bar{v} \in H^{1,p}(B^n, \overline{M})$ satisfying proper decay estimates. Since this construction is rather complicated it will be given in a separate paper (part II). With the help of Theorem E we construct suitable comparison functions to get a substitute for Caccioppoli's inequality ($\rightarrow$ hybrid inequality in Theorem 2.3):

If $u \in H^{1,p}(\Omega, \overline{M})$ *is locally minimizing and if for some ball* $r^p \fint_{B_r(x)} |Du|^p \leq \varepsilon$ (= a small absolute constant), *then*

$$\fint_{B_{r/2}(x)} |Du|^p \leq \frac{1}{2} \fint_{B_r(x)} |Du|^p + Cr^{-p} \fint_{B_r(x)} |u - (u)_r|^p .$$

Combining this result with the monotonicity formula for the scaled p-energy ($\rightarrow$ Theorem 2.4) we can prove a *partial higher integrability theorem* saying that $Du \in L^q(B_{r/2}(x))$ for some $q > p$ if the scaled p-energy on the ball $B_r(x)$ is small enough.

In section 3 of the paper we show how to combine this information with the Euler system to get the following partial regularity criterion:

$$r^p \fint_{B_r(x)} |Du|^p \leq \varepsilon \Rightarrow u \text{ is continuous near } x . \tag{0.3}$$

Here and before ε denotes a small positive constant depending only on dimensions and the geometry of M. From (0.3) we immediately deduce ($\rightarrow$ Theorem A)

$$\boldsymbol{H}^{n-p}(\Sigma) = 0$$

for the singular set Σ of the minimizer u.

In order to get the better estimate (0.2) for Σ we replace (0.3) by a smallness condition on the mean oscillation:

$$\fint_{B_r(x)} |u - (u)_r|^p \leq \varepsilon \Rightarrow x \in \Omega \setminus \Sigma . \tag{0.4}$$

For unconstrained problems (or obstacle problems with convex set M) (0.4) trivially follows from (0.3) using Caccioppoli's inequality, here we have to use Theorem E again. Clearly (0.4) is stable under weak convergence and we deduce the following compactness property of sequences $(u_i) \subset H^{1,p}(\Omega, \overline{M})$ of local E_p-minimizers:

If $u_i \rightarrow u_0$ *weakly in* $H^{1,p}(\Omega, \boldsymbol{R}^N)$, *then* u_0 *is regular up to a closed set* Σ_0 *with* $\boldsymbol{H}^{n-p}(\Sigma_0) = 0$.

This enables us to carry out the standard dimension reduction (see [Fe], [S], [S, U1]) by the way proving (0.2); the details are given in section 4.

A final result deals with the case $n-1 \leqslant p < n$: following ideas of Giusti [Gi] we show in Theorem D below that near a singular point x_0 a minimizer locally behaves like $(x-x_0)/|x-x_0|$.

We wish to remark that our theorems may be useful for the study of p-harmonic problems as they occur for example in [B, C, L] and [Wh]; moreover they may apply to constrained problems in nonlinear elasticity. Some applications (homotopy of maps in the Sobolev class $H^{1,p}(\Omega, \bar{M})$, free boundary value problems) are contained in part II, section 3, (compare also [F6]) and in part III we combine our previous results with a boundary regularity theorem to prove the existence of «small» p-harmonic maps between Riemannian manifolds.

1. – Notations and results.

In this section we fix our assumptions and give a survey of the main theorems. In the *Riemannian case* let Ω denote a bounded open subset of a n-dimensional Riemannian manifold, $n \geqslant 2$. Y^k is a k-dimensional submanifold of Euclidean space $\boldsymbol{R}^N$ containing a bounded open region M with the following properties:

$$\begin{cases} \partial M \text{ is a nonvoid } C^3 \text{ submanifold of } Y\,, \\ \text{the closure of } M \text{ is compactly contained in } \operatorname{Int}(Y)\,. \end{cases}$$

Since $\bar{M}$ is also a closed subset of $\boldsymbol{R}^N$ the space

$$H^{1,p}(\Omega, \bar{M}) := \{u \in H^{1,p}(\Omega, \boldsymbol{R}^N) \colon u(x) \in \bar{M} \text{ for } \boldsymbol{H}^n\text{-almost } x \in \Omega\}$$

is a weakly closed subclass of $H^{1,p}(\Omega, \boldsymbol{R}^N)$. Here p is a fixed real number in the interval $[2, n]$ and $\boldsymbol{H}^n$ stands for the n-dimensional Hausdorff measure on Ω.

For functions $u \in H^{1,p}(\Omega, \boldsymbol{R}^N)$ we introduce the p-energy (on Ω)

$$E_p(u, \Omega) := \int_\Omega |Du|^p \, d\boldsymbol{H}^n\,, \tag{1.1}$$

and $u \in H^{1,p}(\Omega, \bar{M})$ is locally E_p-minimizing under the side condition $\operatorname{Im}(u) \subset \bar{M}$ iff

$$E_p(u, \Omega) \leqslant E_p(v, \Omega) \tag{1.2}$$

holds for all functions $v \in H^{1,p}(\Omega, \bar{M})$ such that $\operatorname{spt}(u-v) \subset\subset \Omega$. Condition (1.2) states that u has least energy among all functions agreeing with u outside some compact subset of Ω and respecting the obstacle $Y \setminus M$.

In the *Euclidean case* Ω is a bounded subdomain of $\boldsymbol{R}^n$, $M \subset \boldsymbol{R}^N$ denotes a bounded smooth region. Suppose further that for $\alpha, \beta = 1, \ldots, n$, $i, j = 1, \ldots, N$ we are given functions

$$a_{\alpha\beta} = a_{\beta\alpha} \colon \bar{\Omega} \to \boldsymbol{R}\,, \qquad B^{ij} = B^{ji} \colon \bar{\Omega} \times \boldsymbol{R}^N \to \boldsymbol{R}$$

of class C^1 satisfying the ellipticity condition

$$\left.\begin{cases} a_{\alpha\beta}(x)\eta_\alpha\eta_\beta \geqslant \mu_0|\eta|^2 \\ B^{ij}(x, y)\xi^i\xi^j \geqslant \mu_0|\xi|^2 \end{cases}\right\}, \qquad x \in \bar{\Omega}\,, \quad \eta \in \boldsymbol{R}^n\,, \quad y, \xi \in \boldsymbol{R}^N\,, \tag{1.3}$$

for some positive constant μ_0. Here and in the sequel we use standard summation convention: greek (latin) indices repeated twice are summed from 1 to $n(N)$.

The definition of the space $H^{1,p}(\Omega, \bar{M})$ is as above and for functions in this space we introduce the splitting functional

$$F_p(u, \Omega) := \int_\Omega \big(a_{\alpha\beta} B^{ij}(\cdot, u) D_\alpha u^i D_\beta u^j\big)^{p/2}\, dx\,. \tag{1.4}$$

The notation of a F_p-minimizing function in the class $H^{1,p}(\Omega, \bar{M})$ is completely analogous to (1.2).

Finally we define the regular and singular set of $u \in H^{1,p}(\Omega, \boldsymbol{R}^N)$

$$\operatorname{Reg}(u) := \{x \in \Omega \colon u \text{ is continuous in a neighborhood of } x\}\,,$$
$$\operatorname{Sing}(u) := \Omega \setminus \operatorname{Reg}(u)\,.$$

Obviously the regular set is open and the following theorems give informations on the size of Sing (u) if u is locally minimizing in $H^{1,p}(\Omega, \bar{M})$.

THEOREM A (*first estimate of the singular set*). – Suppose that $u \in H^{1,p}(\Omega, \bar{M})$ is locally E_p-or F_p-minimizing. Then

$$\operatorname{Sing}(u) = \Big\{x \in \Omega \colon \liminf_{r \to 0} r^{p-n} \int_{B_r(x)} |Du|^p > 0\Big\}\,,$$

especially $\boldsymbol{H}^{n-p}(\operatorname{Sing}(u)) = 0$ and $\operatorname{Sing}(u) = \emptyset$ for $p = n$.

THEOREM B (*optimal interior regularity*). – Under the assumptions of A we have

(i) $\boldsymbol{H}$-dim $(\operatorname{Sing}(u)) \leqslant n - [p] - 1$ if $n > p + 1$,

(ii) Sing (u) is discrete for $n - 1 \leqslant p < n$.

$$([p] := \max\{l \in \boldsymbol{N} \colon l \leqslant p\})$$

THEOREM C (*higher regularity*). – Local E_p- or F_p-minimizers $u \in H^{1,p}(\Omega, \bar{M})$ are of class $C^{1,\alpha}$ on the regular set for some $\alpha < 1$; in the quadratic case $p = 2$ we have $u \in H^{2,q}(\mathrm{Reg}\ (u))$ for all finite q.

THEOREM D (*behaviour at isolated singularities*). – Let $p \in [n-1, n)$ and suppose that $u \in H^{1,p}(\Omega, \bar{M})$ is locally E_p- or F_p-minimizing. Then, if $x_0 \in \Omega$ is a singular point we have

$$\limsup_{x \to x_0} |Du(x)||x - x_0| < \infty .$$

COMMENTS. – 1) Our theorems generalize the results obtained in the quadratic case $p = 2$ for which we refer the reader to the papers [D], [D, F1, 2], [F1-5] and [F, W]. But as already mentioned in the introduction none of the methods developed for $p = 2$ extends to larger exponents.

2) For unconstrained quadratic problems A-D can be found in [G, G1, 2] and [S, U1, 2]. A regularity theory for free F_p-minimizers is due to FUSCO-HUTCHINSON [F, H2] and GIAQUINTA-MODICA [GM].

3) In the Riemannian case the regularity properties of free E_p-minimizers (i.e. of local $E_p(\cdot, \Omega)$-minima in the class $H^{1,p}(\Omega, Z)$ for some compact submanifold Z of $\boldsymbol{R}^N$) are not well analyzed. An inspection of our arguments however shows (in the absence of the obstacle all calculations become much easier):

THEOREM. – Let Ω denote an open part of some n-dimensional manifold and suppose that $u \in H^{1,p}(\Omega, Z)$ is locally E_p-minimizing. Then

$$\begin{cases} \boldsymbol{H} - \dim(\mathrm{Sing}\,(u)) \leq n - [p] - 1 \text{ if } n > p + 1 \text{ and} \\ \mathrm{Sing}\,(u) \text{ is discrete for } n - 1 \leq p < n . \end{cases}$$

The compactness of Z can be replaced by the condition that the minimizer takes its values in a bounded subset of Z so that we are in the situation studied by SCHOEN-UHLENBECK [S, U1, 2].

4) In [D, F2] and [F2] we extended the boundary regularity theorems of [J, M] and [S, U2] to quadratic obstacle problems. For general $p > 2$ an analogous result is also true but since the details are rather complicated we shall give a proof of boundary regularity in a separate Part III.

5) In the Euclidean case and for sets M diffeomorphic to the N-dimensional ball we can improve Theorem B by showing

$$\boldsymbol{H} - \dim(\mathrm{Sing}\,(u)) \leq n - [p + \varepsilon] - 1$$

for certain positive ε depending on absolute data. The proof of this fact is based on a Caccioppoli inequality which holds for the restricted class of obstacles. The details can be found in the joint paper [F, F] but the techniques outlined there do not carry over to general sets M.

6) The following simple example shows that singularities are natural in the context of obstacle problems: Assume $p < n$ and let

$$\Omega := B_1^n(0)\,, \qquad M := \{y \in \mathbf{R}^n\colon \tfrac{1}{2} < |y| < 2\}\,, \qquad u_0(x) := x/|x|\,.$$

Since u_0 is in the space $H^{1,p}(\Omega, \bar{M})$ the problem

$$E_p(\cdot, \Omega) \to \min \text{ in } H^{1,p}(\Omega, \bar{M})$$

for boundary values u_0 has at least one solution u with $\operatorname{Sing}(u) \neq \emptyset$ by the No Retraction Theorem. In the special case $n = 3$, $p = 2$ we infer from the theorems in [D, F2] that the singularities of u form a finite subset of $\bar{\Omega}$.

In connection with this example we mention the everywhere regularity theorems obtained in [F3, 4], [F, W] and [W] under reasonable restrictions on the geometry of M. In a forthcoming paper we shall give similar results for the p-case.

7) Once having shown (partial) regularity theorems for obstacle problems one should try to describe the topological and analytical properties of the coincidence set (at least for simple geometries) as it is done for $p = 2$ and $N = 1$ in [A, C], [Fr] and [K, S]. The paper [F7] contains some first results in this direction.

2. – Background material.

For obstacle problems with convex set M the minimum property can be transformed in a variational inequality using the fact that $u + t(v - u)$ is admissible for $0 \leqslant t \leqslant 1$ if u is a minimum point and v denotes any function having boundary values u and respecting the obstacle. For general sets M it is not obvious how to linearize the minimum property. This problem was solved for the first time in [F1] and in [D], [D, F2], [F2-4] we improved the technique of linearization. Here we present the final version for p-harmonic obstacle problems.

The second part of this section contains the basic Extension Theorem E which is used for the proof of the hybrid inequality.

Let us start with the Euler system for local minimizers. First we analyze the Euclidean case and introduce some

NOTATIONS. – Since ∂M is smooth the distance function $d(z) := \operatorname{dist}(z, \partial M)$ is regular for $z \in \bar{M}$ near ∂M; by negative reflection we extend d to a smooth function on a tubular neighborhood U of ∂M and define the vector fields

$$\nu(z) := \operatorname{grad} d(z)\,, \qquad v(x, z) := B^{-1}(x, z)(\nu(z))$$

for $x \in \bar{\Omega}$ and $z \in U$, B^{-1} denoting the inverse of $(B^{ij})_{1\leqslant i,j\leqslant N}$: On ∂M ν is just the interior unit normal vector field.

THEOREM 2.1. – Assume that $u \in H^{1,p}(\Omega, \bar{M})$ is locally F_p-minimizing where the functional F_p is defined in (1.4) but now we allow coefficients $a_{\alpha\beta} = a_{\beta\alpha}: \bar{\Omega}\times \mathbf{R}^N \to \mathbf{R}$ of class C^1 satisfying (1.3). Then there exists a Lebesgue measurable density function $\theta: \Omega \to [0, 1]$ such that for all test vectors $\varphi \in C_0^1(\Omega, \mathbf{R}^N)$ the following equation holds:

$$(2.1)\qquad \int_\Omega p a(\cdot, u, Du) A^{ij}_{\alpha\beta}(\cdot, u) D_\alpha u^i D_\beta \varphi^j\, dx + \int_\Omega \frac{p}{2} a(\cdot, u, Du) D_{y^l} A^{ij}_{\alpha\beta}(\cdot, u) D_\alpha u^i D_\beta u^j \varphi^l\, dx = \int_{[u\in\partial M]} \theta \frac{\nu(u)\cdot\varphi}{\nu(u)\cdot v(\cdot, u)} p a(\cdot, u, Du)\cdot \cdot\left\{A^{ij}_{\alpha\beta}(\cdot, u) D_\alpha u^i D_\beta (v^j(\cdot, u)) + \frac{1}{2} D_{y^l} A^{ji}_{\alpha\beta}(\cdot, u) D_\alpha u^i D_\beta u^j v^l(\cdot, u)\right\} dx\,,$$

and the function $\{...\}$ is non negative on $[u \in \partial M]$. Here we abbreviated: $[u \in \partial M] := = \{x \in \Omega: u(x) \in \partial M\}$ and

$$A^{ij}_{\alpha\beta}(x, y) := a_{\alpha\beta}(x, y)\, B^{ij}(x, y)\,, \qquad a(x, y, Q) := (A^{ij}_{\alpha\beta}(x, y) Q^i_\alpha Q^j_\beta)^{p/2-1}.$$

PROOF. – From (1.3) we infer

$$\nu(z)\cdot v(x, z) > 0\,, \qquad x \in \bar{\Omega}\,, \qquad z \in U\,,$$

so that for $\eta \in C_0^1(\Omega)$, $\eta \geqslant 0$, and small positive t the variation

$$u_t := u + t\eta h_\varepsilon(d(u)) v(\cdot, u)$$

is admissible. Here $h_\varepsilon: [0, \infty) \to [0, 1]$ is a fixed smooth function with $h_\varepsilon(s) = 1$ for $0 \leqslant s \leqslant \varepsilon$, $h_\varepsilon(s) = 0$ for $s \geqslant 2\varepsilon$ and $h'_\varepsilon \leqslant 0$.

The minimality of u gives

$$\lim_{t\to 0+} \frac{1}{t}\{F_p(u_t, \Omega) - F_p(u, \Omega)\} \geqslant 0$$

and by the Riesz Representation Theorem we find a Radon measure $\lambda \geqslant 0$ on Ω

such that

$$(2.2)\quad \int_\Omega pa(\cdot,u,Da)A^{ij}_{\alpha\beta}(\cdot,u)D_\alpha u^i D_\beta\big(\eta h_\varepsilon(d(u))v(\cdot,u)^j\big)\,dx + \\ + \int_\Omega \frac{p}{2}\,a(\cdot,u,Du)D_{y^l}A^{ij}_{\alpha\beta}(\cdot,u)D_\alpha u^i D_\beta u^j \eta h_\varepsilon(d(u))v(\cdot,u)^l\,dx = \int_\Omega \eta\,d\lambda\,.$$

Clearly λ is independent of the parameter ε (for $\varepsilon\neq\varepsilon'$ use the variation $u + t\eta[h_\varepsilon(d(u)) - h_{\varepsilon'}(d(u))]v(\cdot,u)$ which is admissible for $|t|\ll 1$). In order to get a bound for λ we fix $\eta\geqslant 0$ and pass to the limit $\varepsilon\to 0$ in (2.2). The first integral on the left-hand-side of (2.2) splits into three parts for which we get:

$$\int_\Omega pa(\cdot,u,Du)A^{ij}_{\alpha\beta}(\cdot,u)D_\alpha u^i D_\beta\eta h_\varepsilon(d(u))v(\cdot,u)^j\,dx \xrightarrow[\varepsilon\to 0]{} \\ \to \int_{[u\in\partial M]} pa(\cdot,u,Du)a_{\alpha\beta}(\cdot,u)B^{ij}(\cdot,u)\big(B^{-1}(\cdot,u)\nu(u)\big)^j D_\alpha u^i D_\beta\eta\,dx = \\ = \int_{[u\in\partial M]} pa(\cdot,u,Du)a_{\alpha\beta}(\cdot,u)\nu^i(u)D_\alpha u^i D_\beta\eta\,dx = 0$$

since $\nu(u)^i D_\alpha u^i = D_\alpha(d(u)) = 0$ on the set $[u\in\partial M]$,

$$\int_\Omega pa(\cdot,u,Du)A^{ij}_{\alpha\beta}(\cdot,u)D_\alpha u^i D_\beta\big(h_\varepsilon(d(u))\big)\eta v(\cdot,u)^j\,dx = \\ = \int_\Omega pa(\cdot,u,Du)h'_\varepsilon(d(u))\,a_{\alpha\beta}(\cdot,u)B^{ij}(\cdot,u)v(\cdot,u)^j D_\alpha u^i D_\beta(d\circ u)\eta = \\ = \int_\Omega pa(\cdot,u,Du)h'_\varepsilon(d(u))\,\eta a_{\alpha\beta}(\cdot,u)D_\alpha u^i\nu^i D_\beta u^j \nu^j\,dx \leqslant 0\,,$$

$$\int_\Omega pa(\cdot,u,Du)A^{ij}_{\alpha\beta}(\cdot,u)D_\alpha u^i\eta h_\varepsilon(d(u))D_\beta(v(\cdot,u)^j)\,dx \xrightarrow[\varepsilon\to 0]{} \\ \to \int_{[u\in\partial M]} pa(\cdot,u,Du)A^{ij}_{\alpha\beta}(\cdot,u)D_\alpha u^i\eta D_\beta(v(\cdot,u)^j)\,dx\,.$$

Combining these results with (2.2) the Radon Nikodym Theorem gives the existence of a density function $\theta\colon \Omega\to[0,1]$ such that λ can be written as

$$(2.3)\quad \lambda = \chi_{[u\in\partial M]}\theta pa(\cdot,u,Du)\{A^{ij}_{\alpha\beta}(\cdot,u)D_\alpha u^i D_\beta(v(\cdot,u)^j) + \\ + \tfrac{1}{2}D_{y^l}A^{ij}_{\alpha\beta}(\cdot,u)D_\alpha u^i D_\beta u^j v(\cdot,u)^l\} \llcorner L^n\,.$$

Especially the expression $\{...\}$ is non negative on $[u\in\partial M]$ and formula (2.3) shows that (2.2) is also valid for $\eta\in \mathring{H}^{1,p}\cap L^\infty(\Omega)$.

Consider now a vector field $T\colon \mathbf{R}^N \to \mathbf{R}^N$ supported in a small ball centered at ∂M with the property $T\cdot\nu = 0$. If $\Phi(s, z)$ denotes the flow of T then $v_t := \Phi\big(t\eta h_\varepsilon(d(u)), u\big)$ is admissible for $\eta \in C_0^1(\Omega)$ and $|t|$ small. From the minimality of u we infer

$$(2.4)\qquad \int_\Omega pa(\cdot, u, Du) A_{\alpha\beta}^{ij}(\cdot, u) D_\alpha u^i D_\beta\big(\eta h_\varepsilon(d(u))\, T^j(u)\big)\, dx + \\ + \int_\Omega \frac{p}{2}\, a(\cdot, u, Du) D_{y^l} A_{\alpha\beta}^{ij}(\cdot, u) D_\alpha u^i D_\beta u^j \eta h_\varepsilon(d(u))\, T^l(u)\, dx = 0\,.$$

Finally we cover ∂M with small balls $B_k = B_R(y_k)$, $y_k \in \partial M$, $k = 1, \dots, L$, such that $B_{3R}(y_k) \subset\subset U$ and choose a partition of the unity $\{\varphi_k\}$ with the properties

$$\operatorname{spt}(\varphi_k) \subset\subset B_{2R}(y_k)\,, \qquad \sum_{k=1}^{L} \varphi_k = 1 \ \text{ on } \bigcup_{k=1}^{L} B_k \supset \partial M\,.$$

Let $T_{k,1}, \dots, T_{k,N-1}$ denote vector fields such that

$$\operatorname{spt}(T_{k,i}) \subset\subset B_{3R}(y_k)\,, \qquad T_{k,i}\cdot T_{k,j} = \delta_{i,j}\,, \qquad T_{k,i}\cdot\nu = 0 \ \text{ on } B_{2R}(y_k)\,.$$

For $k \in \{1, \dots, L\}$ fixed we write φ, T_i instead of φ_k, $T_{k,i}$. A test vector $\psi \in C_0^1(\Omega, \mathbf{R}^N)$ has the decomposition

$$\varphi(u)\psi = \eta v(\cdot, u) + \sum_{i=1}^{N-1} \eta_i T_i(u)$$

with coefficients

$$(2.5)\qquad \begin{cases} \eta = \varphi(u)\psi\cdot\nu(u)/\big(\nu(u)\cdot v(\cdot, u)\big) \\ \eta_i = \varphi(u) T_i(u)\cdot\psi - \eta v(\cdot, u)\cdot T_i(u) \end{cases}$$

of class $H^{1,p} \cap L^\infty$ compactly supported in Ω. Using (2.2) and (2.4) we see

$$(2.6)\qquad \int_\Omega pa(\cdot, u, Du) A_{\alpha\beta}^{ij}(\cdot, u) D_\alpha u^i D_\beta \Big\{\Big[\sum_{i=1}^{N-1} \eta_l T_l(u) + \eta v(\cdot, u)\Big]^j h_\varepsilon(d(u))\Big\}\, dx + \\ + \int_\Omega \frac{p}{2}\, a(\cdot, u, Du) D_{y^h} A_{\alpha\beta}^{ij}(\cdot, u) D_\alpha u^i D_\beta u^j \cdot \\ \cdot \Big[\sum_{l=1}^{N-1} \eta_l T_l(u) + \eta v(\cdot, u)\Big]^h h_\varepsilon(d(u))\, dx = \int_\Omega \eta\, d\lambda\,.$$

Remembering the decomposition (2.5) and introducing the index k again (2.6) reads

$$\int_\Omega pa(\cdot, u, Du)A^{ij}_{\alpha\beta}(\cdot, u)D_\alpha u^i D_\beta\big(\varphi_k(u)h_\varepsilon(d(u))\,\psi^j\big)dx + \\ +\int_\Omega \frac{p}{2}\,a(\cdot, u, Du)D_{y^h}A^{ij}_{\alpha\beta}(\cdot, u)D_\alpha u^i D_\beta u^j\varphi_k(u)h_\varepsilon(d(u))\,\psi^h\,dx = \\ =\int_\Omega \psi\cdot\nu(u)/\big(\nu(u)\cdot v(\cdot, u)\big)\,\varphi_k(u)\,d\lambda\,. \tag{2.7}$$

Obviously we can arrange $\sum_{k=1}^{L}\varphi_k = 1$ on the set where $h_\varepsilon\circ d \neq 0$ so that (2.1) follows from (2.7) by first taking the sum with respect to k and then adding the identity

$$\frac{d}{dt|0}\,F_p(w_t, \Omega) = 0\,, \qquad w_t := u + t[1 - h_\varepsilon(d(u))]\,\psi\,,$$

to the resulting equation. □

We turn to the Riemannian version of Theorem 2.1: Let d_Y denote the Riemannian distance of points y, z in Y and $\varrho(z) := d_Y(z, \partial M)$ for $z \in \overline{M}$ near ∂M. By negative reflection we extend ϱ to a smooth function in a tubular neighborhood U of ∂M in Y. If we let $\nu(z) := \operatorname{grad}_Y \varrho(z)$, then the same calculations as before give

$$\int_\Omega p|du|^{p-2}du^i\cdot d\big(\eta h_\varepsilon(\varrho(u))\,\nu^i(u)\big)\,d\boldsymbol{H}^n = \int_\Omega \eta\,d\lambda$$

for a Radon measure $\lambda \geqslant 0$ on Ω being of the form

$$\lambda = p\theta\chi_{[u\in\partial M]}|du|^{p-2}du^i\cdot d(\nu^i(u)) \llcorner \boldsymbol{H}^n\,,$$

θ denoting a $\boldsymbol{H}^n$-measurable density function $\Omega \to [0, 1]$. Here du^i is the gradient of u^i with respect to the metric on Ω.

Similar arguments as in the Euclidean case imply $N-1$ tangential equations and we arrive at

$$\int_\Omega p|du|^{p-2}du^i\cdot d\varphi^i\,d\boldsymbol{H}^n = \int_\Omega \varphi\cdot\nu(u)\,d\lambda \tag{2.8}$$

for vector fields $\varphi \in \mathring{H}^{1,p}(\Omega, \boldsymbol{R}^N) \cap L^\infty$ along u. If $\Pi\colon (Y)_\varepsilon \to Y$ is the smooth nearest point retraction defined on a suitable tubular neighborhood of Y and if $\psi \in C^1_0(\Omega, \boldsymbol{R}^N)$ is arbitrary, then $\varphi := D\Pi|_u(\psi)$ is a field along u for which (2.8) holds and one easily shows

THEOREM 2.2. – Suppose that we are in the Riemannian case and that u is locally E_p-minimizing in the space $H^{1,p}(\Omega, \bar{M})$. Then there is a $\boldsymbol{H}^n$-measurable density function $\theta\colon \Omega \to [0, 1]$ such that

$$\int_\Omega p|du|^{p-2}\{du^i\cdot d\psi^i + du^i\cdot du^k D^2_{lk}\Pi^i(u)\psi^l\}\, d\boldsymbol{H}^n = \int_{[u\in\partial M]} p\theta\psi\cdot\nu|du|^{p-2}du^i\cdot d(\nu(u)^i)\, d\boldsymbol{H}^n$$

holds for all $\psi \in \mathring{H}^{1,p}(\Omega, \boldsymbol{R}^N) \cap L^\infty$.

REMARK. – The quadratic case of Theorem 2.2 is treated in [D]. □

The regularity theory for free F_p-minima is heavily based on the Caccioppoli inequality (compare [F, H1, 2], [G], [G, G1, 2], [G, M]). But if one considers minimisation problems for mappings taking values in some prescribed non convex set (a manifold or smooth subregion of a manifold) it is not clear if Caccioppoli's inequality continues to hold. The following Theorem E is the basic tool for overcoming this difficulty since it enables us to construct suitable comparison functions which we use to prove a weak form of Caccioppoli's inequality near points with small scaled p-energy. Since the proof of Theorem E is rather involved it will be given in Part II of the paper.

THEOREM E (*extension theorem*). – Suppose that M is a smooth bounded region of the type described in section 1. Then there exist constants $\gamma = \gamma(p) \in (0, 1]$ and ε_0, δ, q, $\tilde{q}$, C_0 depending on the dimensions, on p and the geometry of ∂M (and Y) with the following property: Let $u \in H^{1,p}(S^{n-1}, \bar{M})$ and $u^* \in \boldsymbol{R}^N$ be given such that

$$E_p(u, S^{n-1})\, W_p(u, S^{n-1})^\gamma \leqslant \varepsilon^q \delta^{1+\gamma}$$

for some $0 < \varepsilon \leqslant \varepsilon_0$; then we find $\bar{u} \in H^{1,p}(B, \bar{M})$ with boundary values u and

$$E_p(\bar{u}, B) \leqslant C_0\{\varepsilon E_p(u, S^{n-1}) + \varepsilon^{-\tilde{q}} W_p(u, S^{n-1})\}\ ,$$

$$W_p(\bar{u}, B) \leqslant C_0\, \varepsilon^{-\tilde{q}} W_p(u, S^{n-1})\ .$$

Here and in the sequel we use the notation

$$E_p(f, A) := \int_A |Df|^p\ , \qquad W_p(f, A) := \int_A |f - u^*|^p$$

for functions $f\colon A \to \boldsymbol{R}^N$. S^{n-1} is the $(n-1)$-dimensional standard sphere and B the unit ball in $\boldsymbol{R}^n$.

By a simple scaling argument we see

COROLLARY. – If $u \in H^{1,p}(S_r^{n-1}, \overline{M})$ satisfies

$$E_p(u, S_r^{n-1})\, W_p(u, S_r^{n-1})^{\gamma} \leqslant \varepsilon^q \delta^{\gamma+1} r^{(1+\gamma)(n-1)-p},$$

then we find an extension $\bar{u} \in H^{1,p}(B_r, \overline{M})$ such that

$$E_p(\bar{u}, B_r) \leqslant C_0\{\varepsilon r E_p(u, S_r^{n-1}) + \varepsilon^{-\tilde{q}} r^{1-p} W_p(u, S_r^{n-1})\}\,,$$

$$W_p(\bar{u}, B_r) \leqslant C_0 \varepsilon^{-\tilde{q}} r W_p(u, S_r^{n-1})\,.$$

Theorem E enables us to prove

THEOREM 2.3 (*hybrid inequality*). – Assume that $D \subset \mathbf{R}^n$ is a bounded domain and that $f\colon D \times \mathbf{R}^N \times \mathbf{R}^{nN} \to \mathbf{R}$ is a Carathéodory function satisfying

$$k_0|Q|^p \leqslant f(x, y, Q) \leqslant k_1|Q|^p\,, \quad x \in D\,, \quad y \in \mathbf{R}^N\,, \quad Q \in \mathbf{R}^{nN}\,,$$

with positive constants k_0, k_1. Moreover, let M denote a region as described in section 1 and assume that $u \in H^{1,p}(D, \overline{M})$ locally minimizes the functional $G(v, D) := \int_D f(\cdot, v, Dv)\, dx$ in the class $H^{1,p}(D, \overline{M})$. Then there is a constant $C_1 := C_1(n, N, p, M, k_0, k_1)$ (and also depending on Y if $M \subset Y$) with the following property: If

$$\Phi(u, B_R(x)) := R^{p-n} E_p(u, B_R(x)) := R^p \fint_{B_R(x)} |Du|^p \leqslant C_1^{-1} \lambda^{q/(\gamma+1)}$$

for some $0 < \lambda < 1$, then

$$\Phi(u, B_{R/2}(x)) \leqslant \lambda \Phi(u, B_R(x)) + C_1 \lambda^{-\tilde{q}} \fint_{B_R(x)} |u - (u)_R|^p\, dz\,,$$

where $(u)_R$ is the mean value of u on $B_R(x)$ and $q, \tilde{q}, \gamma$ denote the constants in Theorem E.

REMARKS. – 1) Obviously Theorem 2.3 extends to locally E_p-minimizing maps $u\colon \Omega \to \overline{M}$ if Ω and M are subdomains of Riemannian manifolds. In this case we introduce local coordinates on Ω and get a functional of type G with bounds k_0, k, depending on the metric of Ω. 2) For unconstrained quadratic problems Theorem 2.3 is due to HARDT-KINDERLEHRER-LIN [H, K, L].

PROOF OF THEOREM 2.3. – For simplicity we assume $x = 0$ and denote all constants depending only on n, N, p and the geometry with the symbols $c_1, c_2, \ldots$. Moreover we suppose that

$$\Phi(u, B_R) \leqslant C_1^{-1} \lambda^{q/(\gamma+1)}$$

holds with C_1 being specified later. According to Fubini's Theorem and [Mo], Theorem 3.6.1 (c), there is a radius $r \in [R/2, R]$ such that

$$E_p(u, \partial B_r) \leqslant \frac{8}{R} E_p(u, B_R), \qquad W_p(u, \partial B_r) \leqslant \frac{8}{R} W_p(u, B_R);$$

here we calculate W_p with respect to $u^* := (u)_R$. From Poincaré's inequality we deduce

$$r^{p-(n-1)(\gamma+1)} E_p(u, \partial B_r)\, W_p(u, \partial B_r)^\gamma \leqslant c_1 R^{-1-\gamma} R^{p-(n-1)(\gamma+1)} E_p(u, B_R)\, W_p(u, B_R)^\gamma \leqslant$$
$$\leqslant c_2 \Phi(u, B_R)^{1+\gamma} \leqslant c_2 C_1^{-1-\gamma} \lambda^q .$$

Introducing $\varepsilon := \lambda\mu$ we see

$$r^{p-(n-1)(\gamma+1)} E_p(u, \partial B_r)\, W_p(u, \partial B_r)^\gamma \leqslant (c_2 C_1^{-1-\gamma} \mu^{-q})\, \varepsilon^q ,$$

and if we assume (δ being defined in Theorem E)

$$c_2 C_1^{-1-\gamma} \mu^{-q} \leqslant \delta^{\gamma+1} \Leftrightarrow C_1 \geqslant (c_2 \mu^{-q})^{1/(1+\gamma)} \delta^{-1} \tag{2.9}$$

the scaled version of Theorem E gives the existence of $\overline{u} \in H^{1,p}(B_r, \overline{M})$ with boundary values u and

$$E_p(\overline{u}, B_r) \leqslant C_0 \{\varepsilon R E_p(u, \partial B_r) + \varepsilon^{-\tilde{q}} R^{1-p} W_p(u, \partial B_r)\} .$$

Since u is locally G-minimizing we have

$$\Phi(u, B_{R/2}) \leqslant c_3 k_0^{-1} R^{p-n} G(u, B_r) \leqslant c_3 k_0^{-1} G(\overline{u}, B_r) R^{p-n} \leqslant$$
$$\leqslant c_4 \frac{k_1}{k_0} R^{p-n} E_p(\overline{u}, B_r) \leqslant C_5 \frac{k_1}{k_0} R^{p-n} \{\varepsilon E_p(u, B_R) + \varepsilon^{-\tilde{q}} R^{-p} W_p(u, B_R)\}$$
$$\leqslant c_5 \frac{k_1}{k_0} \varepsilon \Phi(u, B_R) + c_5 \frac{k_1}{k_0} \varepsilon^{-\tilde{q}} \fint_{B_R} |u - (u)_R|^p \, dz .$$

We now define $\mu := c_5^{-1}(k_0/k_1)$ (by enlarging c_5 we may assume $\mu \leqslant \varepsilon_0$ with ε_0 taken from Theorem E, especially $\varepsilon \leqslant \varepsilon_0$); this gives

$$\Phi(u\, B_{R/2}) \leqslant \lambda \Phi(u, B_R) + \left[c_5 \frac{k_1}{k_0}\right]^{\tilde{q}+1} \lambda^{-\tilde{q}} \fint_{B_R} |u - (u)_R|^p \, dz .$$

Observing (2.9) the statement of the theorem follows if we take

$$C_1 := \max \left\{ \delta^{-1} (c_2 \mu^{-q})^{1/(\gamma+1)}, \left(c_5 \frac{k_1}{k_0}\right)^{\tilde{q}+1} \right\} . \quad \blacksquare$$

We finish this section with

THEOREM 2.4 (*monotonicity formula*). – Assume that M is a domain in Euclidean space and that $u \in H^{1,p}(\Omega, \bar{M})$ is locally F_p-minimizing. Then there is a constant C_2 depending on n, N, p, the modulus of ellipticity of the coefficients and their Lipschitz constants on $\bar{\Omega} \times \overline{B_L^N(0)}$, $L := \sup\{|z| : z \in \bar{M}\}$, such that for all balls $B_r(x) \subset B_R(x) \subset \Omega$, $R \leqslant 1$

$$\Phi(u, B_r(x)) := r^p \fint_{B_r(x)} |Du|^p \leqslant C_2 \Phi(u, B_R(x)) \,. \tag{2.10}$$

The proof of Theorem 2.4 is a direct consequence of [F, H2], Lemma 8.2, since all variations used by Fusco-Hutchinson respect the obstacle. For a Riemannian version of Theorem 2.4 one has to introduce local coordinates on Ω, in this case the constant C_2 also depends on the metric of Ω.

3. – First estimate on the singular set.

We start with the proof of Theorem A for the Euclidean case: Assume that $u \in H^{1,p}(\Omega, \bar{M})$ is locally F_p-minimizing and that there is a ball $B_{R_0}(x_0) \subset\subset \Omega$, $R_0 \leqslant 1$, such that

$$\Phi(u, B_{R_0}(x_0)) := R_0^p \fint_{B_{R_0}(x_0)} |Du|^p \, dx < \varepsilon^p \tag{3.1}$$

for some positive ε being determined later. The monotonicity formula (2.10) combined with (3.1) gives for $x \in B_{R_0/2}(x_0)$ and $0 < r \leqslant R_0/2$:

$$\Phi(u, B_r(x)) \leqslant C_2 \Phi(u, B_{R_0/2}(x)) \leqslant 2^{n-p} C_2 \Phi(u, B_{R_0}(x_0)) \leqslant 2^{n-p} C_2 \varepsilon^p \,.$$

With $q, \tilde{q}, C_1, \gamma$ from Theorem E we let

$$\lambda := \{C_1 C_2 2^{n-p} \varepsilon^p\}^{(\gamma+1)/q};$$

then the above inequality rereads

$$\Phi(u, B_r(x)) \leqslant C_1^{-1} \lambda^{q/(\gamma+1)}$$

and from Theorem 2.3 we infer

$$\Phi(u, B_{r/2}(x)) \leqslant \lambda \Phi(u, B_r(x)) + C_1 \lambda^{-\tilde{q}} \fint_{B_r(x)} |u - (u_r)|^p \Rightarrow$$

$$\Rightarrow \fint_{B_{r/2}(x)} |Du|^p \leqslant c(n, p)\, \lambda \fint_{B_r(x)} |Du|^p + C_1 c(n, p)\, \lambda^{-\tilde{q}} r^{-p} \fint_{B_r(x)} |u - (u)_r|^p \,.$$

We fix $\lambda := \frac{1}{2} c(n, p)^{-1}$ or equivalently

$$\varepsilon := 2^{1-n/p}(C_1 C_2)^{-1/p}(\tfrac{1}{2} c(n, p)^{-1})^{q/(p\gamma+p)} \tag{3.2}$$

and arrive at

$$\left\{\begin{array}{l} \displaystyle\fint_{B_{r/2}(x)} |Du|^p \leqslant \frac{1}{2} \fint_{B_r(x)} |Du|^p + c_1 r^{-p} \fint_{B_r(x)} |u-(u)_r|^p , \\ x \in B_{R_0/2}(x_0) , \quad r \in (0, R_0/2] \end{array}\right. \tag{3.3}$$

for a suitable absolute constant c_1. The second integral on the right-hand-side of (3.3) can be handled by the Sobolev-Poincaré inequality, and according to [G], V, Proposition 1.1, there is an exponent $t > p$ (depending on absolute data) and a constant c_2 such that

$$Du \in L^t_{\text{loc}}(B_{R_0}(x_0), \mathbf{R}^{nN})$$

and

$$\left\{\fint_{B_{r/2}(x)} |Du|^t \, dz\right\}^{1/t} \leqslant c_2 \left\{\fint_{B_r(x)} |Du|^p \, dz\right\}^{1/p} \tag{3.4}$$

or balls $B_r(x)$ as in (3.3). Thus we have shown a *partial higher integrability* result: Near points with sufficiently small scaled p-energy the gradient of a minimizer is integrable for some exponent $> p$.

In order to proceed further we fix a ball $B_r(\tilde{x})$, $r \leqslant R_0/2$, $\tilde{x} \in B_{R_0/2}(x_0)$, and abbreviate

$$f(x, y, Q) := \left(a_{\alpha\beta}(x) B^{ij}(x, y) Q^i_\alpha Q^j_\beta\right)^{p/2} ,$$

$$f_0(Q) := \left(a_{\alpha\beta}(\tilde{x}) B^{ij}(\tilde{x}, (u)_{r/2}) Q^i_\alpha Q^j_\beta\right)^{p/2} .$$

According to (2.1) the Euler system for u can be written as

$$\int_\Omega D_p f(\cdot, u, Du) \cdot D\varphi \, dx = \int_\Omega g(\cdot, u, Du) \cdot \varphi \, dx \tag{315}$$

for all $\varphi \in \mathring{H}^{1,p} \cap L^\infty(\Omega, \mathbf{R}^N)$ with right-hand-side $g\colon \Omega \times \mathbf{R}^N \times \mathbf{R}^{nN} \to \mathbf{R}^N$ satisfying the growth condition

$$|g(x, y, Q)| \leqslant c_3(|Du|^p + |Du|^{p-1}) \tag{3.6}$$

Let $v \in H^{1,p}(B_{r/2}(\tilde{x}), \mathbf{R}^N)$ denote the solution of

$$\int_{B_{r/2}(\tilde{x})} f_0(Dw) \, dx \to \min \text{ in } u + \mathring{H}^{1,p}(B_{r/2}(\tilde{x}), \mathbf{R}^N) .$$

Since u is bounded a simple transformation argument gives $v \in L^\infty(B_{r/2}(\tilde{x}), \boldsymbol{R}^N)$; moreover v is a solution of

$$(3.7)\qquad \int_{B_{r/2}(\tilde{x})} D_P f_0(Dv)\cdot D\varphi = 0\,, \qquad \varphi \in \mathring{H}^{1,p}(B_{r/2}(\tilde{x}), \boldsymbol{R}^N)\,.$$

Combining (3.5) with (3.7) and choosing $\varphi := u - v$ we arrive at

$$\int_{B_{r/2}(\tilde{x})} \{D_P f_0(Du) - D_P f_0(Dv)\}\cdot(Du - Dv)\,dx = \\ = \int_{B_{r/2}(\tilde{x})} [D_P f_0(Du) - D_P f(\cdot, u, Du)]\cdot(Du - Dv)\,dx - \int_{B_{r/2}(\tilde{x})} g(\cdot, u, Du)\cdot(u - v)\,dx\,.$$

The integral on the left-hand-side is controlled with the help of [F, F], Lemma 3.2, estimate (3.2); this gives using (3.6):

$$(3.8)\qquad c_4 \int_{B_{r/2}(\tilde{x})} |Du - Dv|^p\,dx \leqslant \int_{B_{r/2}(\tilde{x})} [D_P f_0(Du) - D_P f(\cdot, u, Du)]\cdot(Du - Dv)\,dx + \\ + c_3 \int_{B_{r/2}(\tilde{x})} (|Du|^p + |Du|^{p-1})\cdot|u - v|\,dx =: \mathrm{I} + \mathrm{II}\,.$$

For the integral II we observe the following estimates:

$$\int_{B_{r/2}(\tilde{x})} |Du|^p|u - v|\,dx \leqslant \left\{\int_{B_{r/2}(\tilde{x})} |Du|^t\,dx\right\}^{p/t} \left\{\int_{B_{r/2}(\tilde{x})} |u - v|^{t/(t-p)}\,dx\right\}^{1-p/t} \leqslant \\ \leqslant (3.4) \leqslant c_5 \int_{B_r(\tilde{x})} |Du|^p\,dx \left\{ \fint_{B_{r/2}(\tilde{x})} |u - v|^p\,dx\right\}^{1-p/t} \leqslant c_6 \int_{B_r(\tilde{x})} |Du|^p\,dx \left\{r^p \fint_{B_{r/2}(\tilde{x})} |Du|^p\,dx\right\}^{1-p/t}$$

(using the fact that obviously $E_p(v, B_{r/2}(\tilde{x})) \leqslant c_7 E_p(u, B_{r/2}(\tilde{x}))$ by the minimality of the function v).

According to Young's inequality we have for arbitrary $\tau > 0$

$$\int_{B_{r/2}(\tilde{x})} |Du|^{p-1}|u - v|\,dx \leqslant c_8 \left\{\tau \int_{B_{r/2}(\tilde{x})} |Du|^p\,dx + \tau^{1-p} \int_{B_{r/2}(\tilde{x})} |u - v|^p\,dx\right\} \leqslant \\ \leqslant c_9 \left\{\tau \int_{B_{r/2}(\tilde{x})} |Du|^p\,dx + \tau^{1-p} r^p \int_{B_{r/2}(\tilde{x})} |Du - Dv|^p\,dx\right\}.$$

If we choose $\tau = r$, the second integral on the right-hand-side is absorbed in the left-hand-side of (3.8) provided we require $r \leqslant c_{10}$ for a suitable positive constant. This shows

$$(3.9)\qquad \int_{B_{r/2}(\tilde{x})} |Du - Dv|^p\,dx \leqslant c_{11} \left[r + \left(r^p \fint_{B_{r/2}(\tilde{x})} |Du|^p\,dx\right)^{1-p/t}\right] \int_{B_r(\tilde{x})} |Du|^p\,dx + I\,.$$

Using the boundedness of the first derivatives of the coefficients on compact subsets of $\bar{\Omega}\times\mathbf{R}^N$, a simple calculation gives for the remaining integral I:

$$|\mathrm{I}|\leqslant c_{12}\int_{B_{r/2}(\tilde{x})}(r+|u-(u)_{r/2}|)|Du|^{p-1}|Du-Dv|\,dx\leqslant$$
$$\leqslant c_{13}\Big\{\tau\int_{B_{r/2}(\tilde{x})}|Du-Dv|^p\,dx+\tau^{1-p}\int_{B_{r/2}(\tilde{x})}(r+|u-(u)_{r/2}|)^{p/(p-1)}|Du|^p\,dx\Big\};$$

inserting this estimate in (3.9) and choosing τ small enough we deduce:

$$\int_{B_{r/2}(\tilde{x})}|Du-Dv|^p\,dx\leqslant c_{14}\Big\{\Big[r^{p/(p-1)}+\Big(r^p\fint_{B_{r/2}(\tilde{x})}|Du|^p\,dx\Big)^{1-p/t}\Big]\int_{B_r(\tilde{x})}|Du|^p\,dx+$$
$$+\int_{B_{r/2}(\tilde{x})}|u-(u)_{r/2}|^{p/(p-1)}|Du|^p\,dx\Big\}.$$

Finally we use the reverse Hölder inequality (3.4) one more time to estimate the last integral on the right-hand-side in a standard way. Collecting our results we arrive at

$$\fint_{B_{r/2}(\tilde{x})}|Du-Dv|^p\,dx\leqslant c_{15}\Big[r^p\fint_{B_r(\tilde{x})}(1+|Du|^p)\,dx\Big]^s\fint_{B_r(\tilde{x})}|Du|^p\,dx \tag{3.10}$$

for a certain positive exponent s depending on p and t. Let us recall that (3.10) is valid for all balls $B_r(\tilde{x})$, $r\leqslant\min(R_0/2, c_{10})$, $\tilde{x}\in B_{R_0/2}(x_0)$ provided (3.1) is valid with ε being defined in (3.2).

With the help of the comparison inequality (3.10) we can now follow the arguments of [F, H2], proof of Theorem 6.1:

for $0<\tau<1/4$ we see

$$\Phi(u,B_{\tau r}(\tilde{x}))\leqslant c_{16}\Big\{(\tau r)^{p-n}\int_{B_{\tau r}(\tilde{x})}|Dv|^p\,dx+(\tau r)^{p-n}\int_{B_{r/2}(\tilde{x})}|Du-Dv|^p\,dx\Big\}\leqslant$$
$$\leqslant c_{17}\tau^p\Big\{1+\tau^{-n}\Big[r^p\fint_{B_r(\tilde{x})}(1+|Du|^p)\,dx\Big]^s\Big\}\Phi(u,B_r(\tilde{x}))\,.$$

Here we have used the Uhlenbeck estimate

$$\sup_{B_{\tau r}(\tilde{x})}|Dv|^p\leqslant c_{18}\fint_{B_{r/2}(\tilde{x})}|Dv|^p\,dx$$

which is an easy consequence of the results obtained in [U] (compare also [F, H2] and [G, M], Theorem 3.1; in order to apply these references one has to transform $\int f_0(Dw)\,dx$ into the functional $\int |Dw|^p\,dx$ by a suitable coordinate change in the domain of definition and in the image space; this is possible since the coefficients of $\int f_0(Dw)\,dx$ are constat, symmetric and elliptic.) Next we fix $\tau := (\frac{1}{4}c_{17}^{-1})^{1/p}$ and require in addition to (3.1)

$$R_0^p \fint_{B_{R_0}(x_0)} (1 + |Du|^p)\,dx \leqslant \tau^{n/s} 2^{p-n} C_2^{-1}\,, \qquad R_0 \leqslant c_{10}\,, \tag{3.11}$$

C_2 being defined in Theorem 2.4. This gives on account of (2.10)

$$\Phi(u, B_{\tau r}(\tilde{x})) \leqslant \tfrac{1}{2}\Phi(u, B_r(\tilde{x}))\,.$$

We apply this result inductively to $r_k := \frac{1}{2}\tau^k R_0$, $k \in \boldsymbol{N}$, and get

$$\Phi(u, B_{r_k}(\tilde{x})) \leqslant 2^{-k}\Phi(u, B_{R_0/2}(\tilde{x})) \leqslant 2^{-k}\tau^{n/s} C_2^{-1}\,.$$

Now, if $0 < r \leqslant R_0/2$ is given then the last inequality immediately implies the growth estimate

$$\Phi(u, B_r(\tilde{x})) \leqslant \tau^{n/s} 2 \left\{\frac{2r}{R_0}\right\}^{-(\log 2)/\log \tau}.$$

Recalling Morrey's Dirichlet-Growth-Theorem (see [Mo]) we have shown:

$$\left\{\begin{array}{l} u \in C^{0,\alpha}(B_{R_0/2}(x_0), \boldsymbol{R}^N)\,, \qquad \alpha := -\dfrac{1}{p}\dfrac{\log 2}{\log \tau}\,, \\ \text{and} \\ |u(x) - u(y)| \leqslant \text{const}\,(R_0)|x - y|^\alpha\,, \qquad x, y \in B_{R_0/2}(x_0) \end{array}\right. \tag{3.12}$$

for some constant which also depends on the radius. All our calculations are justified under the assumptions (3.1) and (3.11).

According to (3.12) we see that all points with small scaled p-energy belong to the regular set $\operatorname{Reg}(u)$. Finally we show that for $\tilde{x} \in \operatorname{Reg}(u)$ the condition

$$\liminf_{\varrho \to 0} \varrho^{p-n} \int_{B_\varrho(\tilde{x})} |Du|^p\,dx = 0 \tag{3.13}$$

holds which gives the final result:

$$\operatorname{Reg}(u) = \left\{x \in \Omega : \liminf_{\varrho \to 0} \varrho^{p-n} \int_{B_\varrho(x)} |Du|^p\,dz = 0\right\}$$

$\tilde{x} \in \operatorname{Reg}(u)$ means by definition

$$\operatorname*{osc}_{B_r(\tilde{x})} u := \operatorname{ess\,sup} \{|u(x) - u(y)| : x, y \in B_r(\tilde{x})\} \xrightarrow[r \to 0]{} 0 .$$

Proceeding as before we introduce the solution v of the frozen problem

$$\int_{B_{r/2}(\tilde{x})} f_0(Dw)\, dx \to \min \quad \text{in} \quad u + \overset{\circ}{H}{}^{1,p}(B_{r/2}(\tilde{x}), \boldsymbol{R}^N)$$

which satisfies (3.8). But now the integrals I, II are estimated as follows:

$$|\mathrm{I}| \leqslant c_{19}\Big\{\lambda \int_{B_{r/2}(\tilde{x})} |Du - Dv|^p\, dx + \lambda^{1-p} \int_{B_{r/2}(\tilde{x})} (r + \operatorname*{osc}_{B_{r/2}(\tilde{x})} u)^{p/(p-1)} |Du|^p\, dx\Big\},$$

$$|\mathrm{II}| \leqslant c_{20}\Big\{\operatorname*{osc}_{B_{r/2}(\tilde{x})} (u - v) \int_{B_{r/2}(\tilde{x})} |Du|^p\, dx + \mu \int_{B_{r/2}(\tilde{x})} |Du|^p\, dx + r^p \mu^{1-p} \int_{B_{r/2}(\tilde{x})} |Du - Dv|^p\, dx\Big\}$$

for arbitrary λ, $\mu > 0$. Since the coefficients of $\int_{B_{r/2}(\tilde{x})} f_0(Dw)\, dx$ are constant, symmetric and elliptic, we have by transformation

$$\operatorname*{osc}_{B_{r/2}(\tilde{x})} (u - v) \leqslant c_{21} \operatorname*{osc}_{B_{r/2}(\tilde{x})} u ,$$

and the proper choice of λ, μ immediately implies:

$$\int_{B_{r/2}(\tilde{x})} |Du - Dv|^p\, dx \leqslant c_{22}\Big[r + \operatorname*{osc}_{B_{r/2}(\tilde{x})} u\Big] \int_{B_{r/2}(\tilde{x})} |Du|^p\, dx .$$

Next we use the Uhlenbeck estimate ($\varrho \leqslant r/2$)

$$\int_{B_\varrho(\tilde{x})} |Dv|^p\, dx \leqslant c_{23}(\varrho/r)^n \int_{B_{r/2}(\tilde{x})} |Dv|^p\, dx \leqslant c_{24}(\varrho/r)^n \int_{B_{r/2}(\tilde{x})} |Du|^p\, dx$$

to get

$$\int_{B_\varrho(\tilde{x})} |Du|^p\, dx \leqslant c_{24}(r + \operatorname*{osc}_{B_{r/2}(\tilde{x})} u + (\varrho/r)^n) \int_{B_{r/2}(\tilde{x})} |Du|^p\, dx .$$

Since the oscillation of u becomes arbitrarily small if r goes to 0, we can apply [G], III, Lemma 2.1, to deduce: If $n - p < \beta < n$ is given then

$$\int_{B_\varrho(\tilde{x})} |Du|^p\, dx \leqslant c_{25}(\varrho/r)^\beta \int_{B_{r/2}(\tilde{x})} |Du|^p\, dx$$

for all $\varrho < r \leqslant r(\beta)$. This proves (3.13).

We summarize our results in

THEOREM 3.1. – Assume that M and F_p are as described in section 1. Then there exists $\varepsilon_1 > 0$, $R_1 \leqslant 1$ depending on the dimensions, on M, p and on boundo for the coefficients with the following property: If $u \in H^{1,p}(\Omega, \bar{M})$ is locally F_p-minimizing and

$$R^{p-n} \int_{B_R(x)} |Du|^p \, dz < \varepsilon_1^p$$

for some ball $B_R(x) \subset\subset \Omega$, $R \leqslant R_1$, then $x \in \operatorname{Reg}(u)$ and we find $\alpha \in (0, 1)$, $c(R) > 0$ depending on absolute data such that $|u(y) - u(z)| \leqslant c(R)|y - z|^\alpha$ on $B_{R/2}(x)$. Moreover

$$\operatorname{Reg}(u) = \left\{x \in \Omega : \liminf_{r \to 0} \Phi(u, B_r(x) = 0\right\}$$

and $u \in C^{0,\beta}(\operatorname{Reg}(u))$ for all $\beta < 1$.

REMARK. – The Riemannian version of Theorem A follows directly from the preceeding arguments by choosing local coordinates on Ω in which the p-energy takes the form $\int (a_{\alpha\beta} D_\alpha u \cdot D_\beta u)^{p/2} dx$ with smooth elliptic coefficients $a_{\alpha\beta}$.

The higher regularity theorem C can now be derived using the arguments of Fusco-Hutchinson [F, H2], Theorem 7.1, but for later purposes we need a more explicit description of the modulus of continuity of Du near regular points.

So assume that $u \in H^{1,p}(\Omega, \bar{M})$ is locally F_p-minimizing and that

$$\Phi(u, B_{R_0}(x_0)) < \varepsilon_1^p \tag{3.14}$$

holds for some ball $B_{R_0}(x_0)$ in Ω with ε_1 taken from Theorem 3.1. For $0 < \beta < 1$ we find a small radius $R_0(\beta)$ such that

$$\Phi(u, B_r(\tilde{x})) \leqslant c(\beta, R_0)(r/R)^{p\beta} \Phi(u, B_R(\tilde{x})) \tag{3.15}$$

is valid for points $\tilde{x} \in B_{R_0/4}(x_0)$ and radii $0 < r \leqslant R \leqslant R_0(\beta)$. (Compare the calculations after (3.13).) We introduce the quantity

$$\varphi(u, B_R(\tilde{x})) := |(Du)_R|^{p-2} \fint_{B_R(\tilde{x})} |Du - (Du)_R|^2 dx + \fint_{B_R(\tilde{x})} |Du - (Du)_R|^p dx$$

and let v denote the solution of

$$\int_{B_{r/2}(\tilde{x})} [a_{\alpha\beta}(\tilde{x}) B^{ij}(\tilde{x}, (u)_{R/2}) D_\alpha w^i D_\beta w^j]^{p/2} dx \to \min$$

in the class $u+\overset{\circ}{H}{}^{1,p}(B_{R/2}(\tilde{x}), \mathbf{R}^N)$. According to [F, H2], Theorem 4.2, v satisfies (with α_0 depending on absolute data)

$$\varphi(v, B_{\delta R}(\tilde{x}))\leqslant c_{26}\delta^{\alpha_0}\varphi(v, B_{R/2}(\tilde{x}))\,, \qquad \delta\in(0, /4)\,,$$

and we get using

$$|(Dv)_{\delta R}|^p\leqslant c_{27}\fint_{B_{R/2}(\tilde{x})}|Dv|^p\,dx\leqslant c_{28}\fint_{B_{R/2}(\tilde{x})}|Du|^p\,dx:$$

$$\varphi(u, B_{\delta R}(\tilde{x}))\leqslant c_{29}\left\{\varphi(v, B_{\delta R}(x)) + |(Dv)_{\delta R}|^{p-2}\fint_{B_{\delta R}(\tilde{x})}|Du-Dv|^2\,dx + \fint_{B_{\delta R}(\tilde{x})}|Dv-Dv|^p\,dx\right\}\leqslant$$

$$\leqslant c_{30}\left\{\delta^{\alpha_0}\varphi(v, B_{R/2}(\tilde{x})) + \delta^{-n}\left[|(Dv)_{\delta R}|^{p-2}\fint_{B_{R/2}(\tilde{x})}|Du-Dv|^2\,dx + \fint_{B_{R/2}(\tilde{x})}|Du-Dv|^p\,dx\right]\right\}.$$

Using (3.14), (3.15) and the monotonicity formula we have

$$|(Dv)_{\delta R}|^{p-2}\leqslant c_{31}\left(\fint_{B_R(\tilde{x})}|Du|^p\,dx\right)^{1-2/p}\leqslant(3.15)\leqslant$$

$$\leqslant C_{32}(\beta)\{R^{p(\beta-1)}R_0(\beta)^{-p\beta}\Phi(u, B_{R_0(\beta)}(\tilde{x}))\}^{1-2/p}\leqslant(2.10),\,(3.14)\leqslant c_{33}(\beta)R^{(\beta-1)(p-2)}.$$

Collecting the estimates we deduce

$$\varphi(u, B_{\delta R}(\tilde{x}))\leqslant c_{34}(\beta)\cdot$$

$$\cdot\left\{\delta^{\alpha_0}\varphi(v, B_{R/2}(\tilde{x})) + \delta^{-n}\left[R^{(\beta-1)(p-2)}\fint_{B_{R/2}(\tilde{x})}|Du-Dv|^2\,dx + \fint_{B_{R/2}(\tilde{x})}|Du-Dv|^p\,dx\right]\right\}.$$

The quantity $\varphi(v, B_{R/2}(\tilde{x}))$ can be controlled in terms of $\varphi(u, B_R(\tilde{x}))$:

$$\varphi(v, B_{R/2}(\tilde{x}))\leqslant c_{35}\left\{\varphi(u, B_R(\tilde{x})) + \fint_{B_{R/2}(\tilde{x})}|Du-Dv|^p\,dx + |(Du)_R|^{p-2}\fint_{B_{R/2}(\tilde{x})}|Du-Dv|^2\,dx\right\}.$$

Next we recall inequality (3.10)

$$\fint_{B_{R/2}(\tilde{x})}|Du-Dv|^p\,dx\leqslant c_{36}\left\{R^p\fint_{B_R(\tilde{x})}(1+|Du|^p)\,dx\right\}^s\fint_{B_R(\tilde{x})}|Du|^p\,dx\leqslant$$

$$\leqslant c_{37}R^{\alpha_1}\fint_{B_R(\tilde{x})}|Du|^p\,dx\leqslant(3.15)\leqslant c_{38}(\beta)R^{p(\beta-1)+\alpha_1}$$

where we have used the inequality in front of (3.12) to estimate

$$\{\ldots\}^s \leqslant c_{39} R^{\alpha_1}.$$

Collecting our results we arrive at

$$\varphi(u, B_{\delta R}(\tilde{x})) \leqslant c_{40}(\beta)\left\{\delta^{\alpha_0}\varphi(u, B_R(\tilde{x})) + \delta^{-n} R^{\alpha_2+(\beta-1)q}\right\} \tag{3.16}$$

with positive exponents α_2, q. We finally fix β such that $\alpha_2 + (\beta-1)q > 0$ and choose δ with the property $c_{40}(\beta)\delta^{\alpha_0} \leqslant 1/2$. Then (3.16) rereads

$$\varphi(u, B_{\delta R}(\tilde{x})) \leqslant \tfrac{1}{2}\varphi(u, B_R(\tilde{x})) + c_{41} R^{\alpha_3}$$

and we get by iteration

$$\varphi(u, B_r(x)) \leqslant c_{42}\{(r/R)^{\alpha_4}\varphi(u, B_R(x)) + R^{\alpha_3}\}$$

for all $r \leqslant R \leqslant R_0(\beta)$, $\tilde{x} \in B_{R_0/2}(x_0)$ with positive exponents α_3, α_4. Quoting [G], III, Lemma 2.1, we find an exponent $\gamma' \in (0, 1)$ such that

$$\varphi(u, B_r(\tilde{x})) \leqslant c_{43} r^{\gamma'}\left(R^{-\gamma'}\varphi(u, B_R(\tilde{x})) + 1\right)$$

or by the definition of φ:

$$\fint_{B_r(\tilde{x})} |Du - (Du)_r|^p \, dx \leqslant c_{43} r^{\gamma'}\left(R^{-\gamma'}\varphi(u, B_R(\tilde{x})) + 1\right). \tag{3.17}$$

We choose $R = R_0(\beta)$ on the right-hand-side of (3.17) and observe

$$\varphi(u, B_{R_0(\beta)}(\tilde{x})) \leqslant c_{44} \fint_{B_{R_0(\beta)}(\tilde{x})} |Du|^p \, dx \leqslant \text{(assuming } R_0(\beta) \leqslant R_0/2 \text{ and using (2.10))}$$
$$c_{45} R_0(\beta)^{-p} \Phi(u, B_{R_0/2}(\tilde{x})) \leqslant c_{46} R_0(\beta)^{-p} \Phi(u, B_{R_0}(x_0)) \leqslant c_{46} R_0(\beta)^{-p} \varepsilon_1^p$$

which finally gives the growth condition

$$\fint_{B_r(\tilde{x})} |Du - (Du)_r| \, dx \leqslant \text{const}\,(R_0, R_0(\beta))\, r^{\gamma'}.$$

Thus Du is Hölder continuous on the ball $B_{R_0/4}(x_0)$ with exponent $\gamma := \gamma'/p$. Our calculations are summarized in the next theorem.

THEOREM 3.2. – Suppose that the assumptions of Theorem 3.1 are satisfied for some ball $B_R(x) \subset\subset \Omega$. Then there are constants $\gamma \in (0, 1)$, $b = b(R) > 0$ depending on absolute data (as n, N, p, the coefficients and the geometry of M, b also depending on the radius R) such that

$$Du \in C^{0,\gamma}(B_{R/4}(x), \boldsymbol{R}^{nN}) , \qquad |Du(y) - Du(z)| \le b|y - z|^\gamma .$$

Theorem 3.2 is the Euclidean version of Theorem C. The Riemannian case is a direct consequence of the preceeding calculations.

4. – Optimal interior partial regularity.

In [F, F] we showed that for sets M diffeomorphic to a ball weak limits of minimizing maps are again minimizing and that the limit point of a sequence of singular points is singular for the limit function. Both facts entered the dimension reduction argument but unfortunately their proofs made use of Caccioppoli's inequality. For example, if Caccioppoli's inequality is valid, then it is trivial to check that Theorem 3.1 implies the regularity criterion

$$\text{(4.1)} \qquad \fint_{B_r(x)} |u - (u)_r|^p \, dz < \varepsilon^p \Rightarrow x \in \operatorname{Reg}(u) .$$

For general sets M the proof of (4.1) requires more work: we have to make use of the Extension Theorem E, the details are given in Lemma 4.1 below. Obviously (4.1) is stable under weak convergence and this enables us to derive the following compactness property of a sequence of minimizing maps: the weak limit is not necessary minimizing but nevertheless of class C^1 up to a set of vanishing $\boldsymbol{H}^{n-p}$-measure. With the help of this statement it is then possible to carry out the dimension reduction and to prove Theorem B. We wish to remark that our arguments follow ideas of Schoen-Uhlenbeck [S, U1], the case of quadratic obstacle problems is treated in [D, F2].

In order to avoid notational difficulties we assume from now on:

$$\text{(4.2)} \qquad \begin{cases} M \text{ is a smooth subset of } \boldsymbol{R}^N \text{ (as described in section 1)} \\ \text{and } a_{\alpha\beta} = \delta_{\alpha\beta}, \; B^{ij} = \delta^{ij}, \end{cases}$$

so that $E_p(u, \Omega) = F_p(u, \Omega) = \int_\Omega |Du|^p \, dx$. The minor changes which are necessary to handle the more general functional or the Riemannian case are left to the reader.

LEMMA 4.1. – Let (4.2) hold and assume that $B > 0$ and $u^* \in \mathbf{R}^N$ are given. Then there exist constants

$$\varepsilon_2 = \varepsilon_2(B, n, N, p, M), \quad c = c(n, N, M, p) \quad \text{and } \alpha = \alpha(n, N, M, p)$$

with the following property: If $u \in H^{1,p}(B_1, \bar{M})$ is locally minimizing with

$$E_p(u, B_1) \leqslant B, \quad W_p(u, B_1) := \int_{B_1} |u - u^*|^p \, dx < \varepsilon_2^p,$$

then $u \in C^{0,\alpha}(B_{1/2}, \mathbf{R}^N)$ and $|u(x) - u(y)| \leqslant c|x - y|^\alpha$ on $B_{1/2}$.

PROOF. – According to Theorem 3.1 we know that the statement of the lemma is correct if we require

$$E_p(u, B_{3/4}) < \varepsilon_1^p \tag{4.3}$$

for a certain $\varepsilon_1 = \varepsilon_1(n, N, M, p)$. As usually we use $c_1, c_2, \ldots$ to denote positive constants depending only on n, N, M, p. By Fubini's Theorem there is a radius $r \in [3/4, 1]$ such that

$$E_p(u, S_r) \leqslant 8E_p(u, B_1), \quad W_p(u, S_r) \leqslant 8W_p(u, B_1),$$

S_r being the standard sphere of radius r. Recalling Theorem E we find $\bar{u} \in H^{1,p}(B_r, \bar{M})$ such that

$$E_p(\bar{u}, B_r) \leqslant C_0\{\varepsilon E_p(u, S_r) + \varepsilon^{-\tilde{q}} W_p(u, S_r)\}$$

provided we know

$$E_p(u, S_r)\, W_p(u, S_r)^\gamma \leqslant \varepsilon^q \delta^{\gamma+1}.$$

By the choice of r we have

$$E_p(u, S_r)\, W_p(u, S_r)^\gamma \leqslant c_1 B \varepsilon_2^{p\gamma}$$

and the smallness condition in Theorem E is satisfied if

$$c_1 B \varepsilon_2^{p\gamma} \leqslant \varepsilon^q \delta^{1+\gamma}. \tag{4.4}$$

Now, since u is minimal we get

$$E_p(u, B_r) \leqslant E_p(\bar{u}, B_r) \leqslant C_0(8B\varepsilon + 8\varepsilon^{-\tilde{q}}\varepsilon_2^p) \leqslant c_2(B\varepsilon + \varepsilon^{-\tilde{q}}\varepsilon_2^p).$$

We choose $\varepsilon = \varepsilon(n, N, M, p, B)$ to satisfy $c_2 B\varepsilon \leqslant \frac{1}{2}\varepsilon_1^p$ and fix ε_2 according to (4.4) by setting

$$\varepsilon_2 \leqslant \min\{(c_1^{-1}B^{-1}\varepsilon^q\delta^{\gamma+1})^{1/p\gamma}, (c_2^{-1}\varepsilon^{\bar q}\tfrac{1}{2}\varepsilon_1^p)^{1,p}\}\,.$$

Then all our calculations are justified, especially $E_p(u, B_r) \leqslant \varepsilon_1^p$, and the statement of the lemma follows from (4.3). □

The next lemma characterizes the behaviour of weak limits of minimizing maps:

LEMMA 4.2. – Under the hypothesis (4.2) let $(u_i) \subset H^{1,p}(B_1, \overline{M})$ denote a sequence of locally minimizing maps such that $u_i \to u$ weakly in $H^{1,p}(B_1, \boldsymbol{R}^N)$ for some function u in this space. Then there is a closed (relative to B_1) subset Σ of B_1 such that $\boldsymbol{H}^{n-p}(\Sigma) = 0$ and the property that u is locally Hölder continuous on $B_1 \diagdown \Sigma$. Moreover, we have strong convergence $u_i \to u$ in $H^{1,p}_{\mathrm{loc}}(B_1, \boldsymbol{R}^N)$ and uniform convergence on compact subsets of $B_1 \diagdown \Sigma$.

REMARK. – It is trivial to show that the limit function u is locally minimizing on the regular set $B_1 \diagdown \Sigma$ but we do not know if u is also locally minimizing on the whole ball (compare [S, U1], remark before Lemma 5.2).

PROOF. – We have $\sup\limits_{i\in \boldsymbol{N}} \|Du_i\|_{L^p(B_1)} =: B < \infty$ and after passing to a subsequence we may assume: $u_i \to u$ a.e. and in $L^p(B_1, \boldsymbol{R}^N)$ so that u belongs to the space $H^{1,p}(B_1, \overline{M})$. Let us set $\Sigma := B_1 \diagdown \mathrm{Reg}\,(u)$ and consider a point x_0 in $B_{1/2}$ with the property

$$\fint_{B_r(x_0)} |u - (u)_r|^p\,dx < \varepsilon_2^p \tag{4.5}$$

for a ball $B_r(x_0) \subset B_1$, $\varepsilon_2 = \varepsilon_2(B)$ being defined in Lemma 4.1. The L^p-convergence and (4.5) imply

$$\fint_{B_r(x_0)} |u_i - (u_i)_r|^p\,dx < \varepsilon_2^p \tag{4.6}$$

for i sufficiently large. On the other hand we have by (2.10) assuming $r \leqslant 1/2$:

$$r^{p-n}\int_{B_r(x_0)} |Du_i|^p\,dx \leqslant C_2 2^{n-p}\int_{B_1} |Du_i|^p\,dx \leqslant C_2 2^{n-p} B\,.$$

Combining this inequality with (4.6) we see that the scaled functions $U_i(z) := u_i(x_0 + rz)$, $z \in B_1$, satisfy the hypothesis of Lemma 4.1 (with ε_2 being defined for $BC_2 2^{n-p}$ instead of B). Thus the functions u_i are Hölder continuous on $B_{r/2}(x_0)$

with Hölder exponent and Hölder constant independent of the index i. From Arcela's Theorem we immediately get $u \in C^{0,\alpha}(B_{r/2}(x_0), \boldsymbol{R}^N)$ and uniform convergence $u_i \to u$ on $B_{r/2}(x_0)$. Similiar calculations give:

If $\liminf_{r\to 0} \fint_{B_r(y)} |u - (u)_r|^p \, dx = 0$ for some point y in B_1, then u is Hölder continuous in a neighborhood of y.

Thus we have shown:

$$\Sigma \subset \left\{ y \in B_1 : \liminf_{r\to 0} \fint_{B_r(y)} |u - (u)_r|^p \, dx > 0 \right\}$$

so that $\boldsymbol{H}^{n-p}(\Sigma) = 0$. We cover $\Sigma \cap B_{1/2}$ with balls $B_i := B_{r_i}(x_i)$ having the property $\sum_{i=1}^{\infty} r_i^{n-p} < \varepsilon$ for some positive ε. We let $U := \bigcup_{i=1}^{\infty} B_i$. Using the monotonicity formula we can control the energy of u_j on the set U:

$$\int_U |Du_j|^p \, dx \leqslant \sum_{i=1}^{\infty} \int_{B_i} |Du_j|^p \, dx \leqslant \sum_{i=1}^{\infty} \int_{B_1} C_2 2^{n-p} r_i^{n-p} |Du_j|^p \, dx \leqslant \varepsilon C_2 B 2^{n-p} .$$

In order to show L^p-convergence $Du_j \to Du$ on $B_{1/2} \setminus U$ we make use of the Euler system Theorem 2.1:

$$\int_{B_1} |Du_j|^{p-2} Du_j \cdot D\Phi \, dx = \int_{B_1} R(\cdot, u_j, Du_j) \cdot \Phi \, dx , \qquad \Phi \in \mathring{H}^{1,p} \cap L^\infty;$$

by formula (2.1) the right-hand-side grows of order p in the derivative Du_j, the growth constant being independent of j. Using the system for j and k, inserting the test vector $\Phi := \varphi^p(u_j - u_k)$ for $\varphi \in C_0^1(B_1, [0, 1])$, $\varphi = 1$ on $\overline{B_{1,2}} \setminus U$, $\operatorname{spt} \varphi \cap \Sigma = \emptyset$, subtraction of the results gives after a short calculation:

$$\int_{B_1} \varphi^p |Du_j - Du_k|^p \, dx \leqslant \operatorname{const} (\varphi, D\varphi, \|Du_k\|_p, \|Du_k\|_p) \sup_{\operatorname{spt}(\varphi)} |u_j - u_k|$$

for a constant which can be controlled independent of j and k.

Combining our results we see that (Du_j) is a Cauchy sequence in $L^p(B_{1/2})$ so that $u_j \to u$ in $H^{1,p}(B_{1/2}, \boldsymbol{R}^N)$. A trivial modification of the arguments shows $H^{1,p}(B_r, \boldsymbol{R}^N)$-convergence for $r < 1$. □

In the proof of the next lemma we make use of the following fact:

PROPOSITION. – Let (4.2) hold and assume that (w_i) is a sequence of local minimizers in $H^{1,p}(B_1, \overline{M})$ converging strongly in $H^{1,p}_{\mathrm{loc}}(B_1, \boldsymbol{R}^N)$ to a function w.

(i) If $\Phi(w, B_r(y)) := r^p \fint_{B_r(y)} |Dw|^p dx < \varepsilon_1^p$ for some ball $B_r(y)$ in B_1 (with ε_1 taken from Theorem 3.1), then there exist absolute constants α, $\ell(r)$ such that

$$|w(x) - w(x')| \leqslant \ell |x - x'|^\alpha \quad \text{on } B_{r/2}(y) .$$

(ii) $\operatorname{Reg}(w) = \left\{ y \in B_1 : \lim_{r \to 0} \Phi(\omega, B_j(y)) = 0 \right\}$.

PROOF. – (i) is a direct consequence of Theorem 3.1 and the local L^p-convergence $Dw_i \to Dw$.

(ii) It remains to show that for regular points y the scaled p-energy vanishes of one shrinks the radius of the ball. For a fixed test vector $\varphi \in H^{1,p} \cap L^\infty(B_1, \mathbf{R}^N)$ with compact support we have on account of the Euler system Theorem 2.1

$$\int_{B_1} |Dw_i|^{p-2} Dw_i \cdot D\varphi \, dx \leqslant C \int_{B_1} |Dw_i|^p |\varphi| \, dx$$

with C independent of i; by strong convergence this inequality is valid for the limit function w and we can proceed as in the proof of Theorem 3.1. □

In a next step we use Lemma 4.2 to produce homogeneous blow up limits. For $B > 0$ we let $\mathbf{H}_B$ denote the $H^{1,p}$-closure of all locally E_p-minimizing maps $u \in H^{1,p}(B_1, \overline{M})$ with $E_p(u, B_1) \leqslant B$.

LEMMA 4.3. – Let (4.2) hold and suppose that $u \in \mathbf{H}_B$, $x_0 \in B_{r/2}$ and a sequence $r_i \to 0$ are given. Then there is a subsequence (r_i') such that the scaled functions $u_i(x) := u(x_0 + r_i' x)$, $x \in B_1$, converge weakly to some u_0 in $\mathbf{H}_B$. The limit u_0 is radially independent and for the singular set of u_0 we have $\mathbf{H}^{n-p}(\Sigma_0) = 0$. The convergence $u_i \to u_0$ is uniform on compact subsets of $B_1 \setminus \Sigma_0$ and $u_i \to u_0$ in $H^{1,p}_{\mathrm{loc}}(B_1, \mathbf{R}^N)$.

PROOF. – We first observe that for functions in the class $\mathbf{H}_B$ the monotonicity formula (2.10) is valid. Therefore

$$E_p(u_i, B_1) = \Phi(u, B_{r_i}(x_0)) \leqslant C_2 \Phi(u, B_{1/2}(x_0)) \leqslant C_2 2^{n-p} E_p(u, B_1) \leqslant C_2 2^{n-p} B$$

and we may assume (replacing (r_i) by a subsequence if necessary) that the scaled functions u_i have a weak limit u_0 in $H^{1,p}(B_1, \mathbf{R}^N)$.

Let U^j denote a sequence of local minimizers such that $E_p(U^j, B_1) \leqslant B$ and $U^j \to u$ in $H^{1,p}(B_1, \mathbf{R}^N)$. We abbreviate $U_i^j(z) := U^j(x_0 + r_i z)$; for i fixed obviously

$$\| U_i^j - u_i \|_{H^{1,p}} \to 0 , \quad j \to \infty ,$$

$$E_p(U_i^j, B_1) \leqslant C_2 2^{n-p} B$$

and U_i^j is locally $E_p(\cdot, B_1)$-minimizing. Finally, for each integer i we fix $j(i)$ such that

$$\| U_i^{j(i)} - u_i \|_{H^{1,p}} \leqslant 1/i .$$

This gives $U_i^{j(i)} \to u_0$ weakly in $H^{1,p}(B_1, \mathbf{R}^N)$ and we may apply Lemma 4.2 to see strong convergence in $H^{1,p}_{\mathrm{loc}}$ so that especially $u_i \to u_0$ strongly in $H^{1,p}_{\mathrm{loc}}(B_1, \mathbf{R}^N)$. We show uniform convergence $u_i \to u_0$ on compact subsets of $B_1 \setminus \Sigma_0$. Suppose that y is regular for u_0; then according to the Proposition $\Phi(u_0, B_r(y)) < \varepsilon_1^p$ for a small radius r and $Du_i \to Du_0$ in L^p_{loc} gives the same inequality for u_i provided i is large enough. We recall that u_i is the strong limit of the sequence $(U_i^j)_{j\in N}$ of minimizers U_i^j so that by part (i) of the Proposition

$$|u_i(x) - u_i(z)| \leqslant \mathscr{C} |x - z|^{\alpha}$$

for points x, z in $B_{r/2}(y)$ and i sufficiently large. Quoting Arcela's Theorem we arrive at $u_i \to u_0$ uniformly on $B_{r/2}(y)$.

It remains to show that u_0 is homogeneous of degree zero: from the proof of [F, H2], Lemma 8.2, we infer the following strong version of (2.10): if $w \in H^{1,p}(B_1, \bar{M})$ is locally minimizing, then

$$\Phi(w, B_t) - \Phi(w, B_s) \geqslant \int_{B_t \setminus B_s} |D_r w|^p |x|^{p-n} \, dx , \qquad t > s ,$$

(as already remarked the proof in [F, H2] extends to obstacle problems). Since u_0 is the strong limit of minimizers the formula is valid for u_0. Let $L := \lim_{t\to 0} \Phi(u, B_r(x_0))$; then we have for $t > 0$:

$$\Phi(u_0, B_t) = t^{p-n} \lim_{i\to\infty} \int_{B_t} |Du_i|^p \, dx = \lim_{i\to\infty} \Phi(u, B_{tr_i}(x_0)) = L$$

and $D_r u_0 = 0$ follows from the above formula. □

We are now in the position to prove Theorem B: since the arguments appear in [S, U1] in a slightly different form we restrict ourselves to a

SKETCH OF THE PROOF OF THEOREM B. – For simplicity we assume that (4.2) is true and that u is a local minimizer in $H^{1,p}(B_1, \bar{M})$. We consider two cases:

Case 1: $n \leqslant p + 1$. Let Σ denote the singular set of u and assume $\Sigma \ni y_i \to 0$ for a sequence of singular points. $x_i := y_i/(4|y_i|)$ is singular for $u_i(z) := u(4|y_i|z)$, $z \in B_1$, and after passing to subsequences we may assume $x_i \to x$, $|x| = 1/4$, and $u_i \to u_0$ for some function u_0 having the properties described in Lemma 4.3. x is not in the singular set Σ_0 of u_0 since otherwise $\overline{Ox} \subset \Sigma_0$ and therefore $\mathbf{H}^1(\Sigma_0) > 0$ contradicting $\mathbf{H}^{n-p}(\Sigma_0) = 0$. But $x \in \operatorname{Reg}(u_0)$ implies $\lim_{t\to 0} \Phi(u_0, B_t(x)) = 0$ and using the strong convergence $Du_i \to Du_0$ we can arrange $\Phi(u_i, B_r(x_i)) < \varepsilon_1^p$ for small radii r and i large enough. Thus $x_i \in \operatorname{Reg}(u_i)$.

Case 2: $n > p + 1$. We assume that $\boldsymbol{H}^s_\infty(\Sigma) > 0$ for some $s \in [0, n - [p])$ (for a definition of the measure $\boldsymbol{H}^s_\infty$ see [Fe], 2.10.2) and that the origin is a singular point of u with the property

$$\theta^{*s}(0, \boldsymbol{H}^s_\infty \llcorner \Sigma) := \limsup_{r\to 0} r^{-s} \boldsymbol{H}^s_\infty(B_r(0) \cap \Sigma) > 0 .$$

Blowing up at 0 and using Lemma 4.3 we find a radially independent function u_0 with singular set Σ_0 and

$$\boldsymbol{H}^s_\infty(\Sigma_0 \cap B_1^n(0)) > 0 .$$

There are two possibilities: either we have $s \leqslant 0$ or there exists a point $x_1 \in S_1^{n-1} \cap \Sigma_0$ with

$$\theta^{*s}(x_1, \boldsymbol{H}^s_\infty \llcorner \Sigma_0) > 0 .$$

Blowing up u_0 at $x_1 = (1, 0, \ldots, 0)$ (after a coordinate transformation) we find a radially independent map u_1 with singular set Σ_1, $D_1 u_1 = 0$ and

$$\boldsymbol{H}^s_\infty(\Sigma_1 \cap B_1^n(0)) > 0 .$$

If we repeat this procedure m times we find maps $u_1, \ldots, u_m$ such that $u_{j|B_1^n} \in \boldsymbol{H}_B$ for a suitable $B > 0$, $D_r u_j = D_\alpha u_j = 0$, $\alpha = 1, \ldots, j$, $j = 1, \ldots, m$ and

$$\boldsymbol{H}^s_\infty(\Sigma_j \cap B_1^n(0)) > 0 .$$

We can repeat this argument until we have $s - m \leqslant 0$. In order to construct u_m we need $s - m + 1 > 0$. This implies $m \leqslant n - [p]$. For $m = n - [p]$ it would follow

$$\Sigma_m \supset \boldsymbol{R}^{n-[p]} \times \tilde{\Sigma}_m , \qquad \boldsymbol{H}^{s-(n-[p])}_\infty(\tilde{\Sigma}_m \cap B_1^m(0)) > 0$$

and in conclusion $\boldsymbol{H}^{n-[p]}_\infty(\Sigma_m) = \infty$ contradicting $\boldsymbol{H}^{n-p}(\Sigma_m) = 0$ (on account of Lemma 4.3). Thus $m \leqslant n - [p] - 1$ and therefore $s \leqslant m \leqslant n - [p] - 1$. This shows:

$$\boldsymbol{H}^t(\Sigma) = 0 \quad \text{for all } t > n - [p] - 1 \Leftrightarrow \boldsymbol{H}-\dim(\Sigma) \leqslant n - [p] - 1 . \qquad \square$$

We finish this section with the

PROOF OF THEOREM D. – For simplicity we assume (4.2) and consider a local E_p-minimizer $u \in H^{1,p}(B_1, \bar{M})$ such that $0 \in \operatorname{Sing}(u)$, but

$$\lim_{i\to\infty} |x_i|\,|Du(x_i)| = \infty$$

for a sequence (x_i) converging to 0. We let $y_i := x_i/(2|x_i|)$ and $u_i(z) := u(2|x_i|z)$, $z \in B_1$. Then the functions u_i are locally minimizing and

$$E_p(u_i, B_1) = \Phi(u, B_{2|x_i|}(0)) \leqslant C_2 \Phi(u, B_R(0))$$

if $B_R(0)$ denotes a fixed ball such that $2|x_i| \leqslant R$. Quoting Lemma 4.3 we can arrange: $u_i \to u_0$ weakly in $H^{1,p}(B_1, \mathbf{R}^N)$, strongly in $H^{1,p}_{\text{loc}}(B_1, \mathbf{R}^N)$ and uniformly on compact subsets of $B_1 \setminus \Sigma_0$ where Σ_0 denotes the singular set of the limit function u_0. Consider the point $y := \lim_{i\to\infty} y_i$; since $\mathbf{H}^1(\Sigma_0) = 0$ (recall $n-1 \leqslant p < n$) y belongs to the regular set of u_0 and from the Proposition we infer:

$$\lim_{r\to 0} \Phi(u_0, B_r(y)) = 0\,,$$

hence: $\Phi(u_i, B_r(y)) < \varepsilon_1^p$ for a small ball $B_r(y)$ and i large enough. Here ε_1 is defined in Theorem 3.1. Now we apply Theorem 3.2 to see that the sequence (Du_i) is uniformly Hölder continuous on $B_{r/4}(y)$, and by Arzelà's Theorem we get $Du_i \to Du_0$ on $B_{r/4}(y)$, especially: $Du_i(y_i) \to Du_0(y)$ contradicting the choice of (x_i). □

Note added in proof. – After having finished the manuscript the author was informed that similar results have been obtained independently by HARDT-LIN (*Mappings minimizing the L^p-norm of the gradient*, Comm. Pure Appl. Math., **11** (1987), 555-588) and LUCKHAUS (*Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold*, Ind. Univ. Math. J., **37**, no. 2 (1988)). HARDT-LIN discuss the case of targets without boundary, the work of Luckhaus also includes the obstacle problem.

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