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P.L.-Spheres, Convex Polytopes, and Stress*

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Abstract. We describe here the notion of generalized stress on simplicial complexes, which serves several purposes: it establishes a link between two proofs of the Lower Bound Theorem for simplicial convex polytopes; elucidates some connections between the algebraic tools and the geometric properties of polytopes; leads to an associated natural generalization of infinitesimal motions; behaves well with respect to bistellar operations in the same way that the face ring of a simplicial complex coordinates well with shelling operations, giving rise to a new proof that p.l.-spheres are Cohen–Macaulay; and is dual to the notion of McMullen's weights on simple polytopes which he used to give a simpler, more geometric proof of the g-theorem.

1. Introduction

About 25 years ago two important extremal problems for convex polytopes were solved at almost the same time. McMullen [12] proved the Upper Bound Conjecture, which predicts the maximum number of faces of each dimension that a convex d-polytope (d-dimensional polytope) with n vertices can have. Barnette [1], [2] settled the Lower Bound Conjecture, which specifies the minimum number of faces of each dimension that a simplicial convex d-polytope with n vertices can possess.

The first proofs of these results were somewhat unrelated, but in the subsequent decade Stanley developed a common algebraic perspective for recasting and ultimately reproving both of these results [21]. In fact, he established the complete characterization of face-vectors of simplicial (or dually, simple) polytopes (the g-theorem) originally

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conjectured by McMullen. This quickly led to the further development of very powerful connections between the combinatorics of convex polytopes and the algebraic geometry of associated toric varieties [6], [16]. This interplay has proved to be very fruitful and far from exhausted, and many issues are as yet unresolved. Some important progress, as well as questions, center around the extensions of the face-counting results to other classes of objects such as nonsimplicial (or nonsimple) polytopes, unbounded polyhedra, or simplicial spheres.

We describe here the notion of generalized stress, which serves several purposes: it establishes a link between two proofs of the Lower Bound Theorem; elucidates some connections between the algebraic tools and the geometric properties of polytopes; leads to an associated natural generalization of infinitesimal motions; behaves well with respect to bistellar operations in the same way that the face ring of a simplicial complex coordinates well with shelling operations, giving rise to a new proof that p.l.-spheres are Cohen–Macaulay; and is dual to the notion of McMullen's weights on simple polytopes which he used to give a simpler, more geometric proof of the g-theorem [13], [14]. Generalized stress was first introduced in [10], and a detailed overview was presented in [11].

2. The Lower Bound Theorem

For a simplicial convex d-polytope P, let

$$g_2 = f_1 - df_0 + \binom{d+1}{2}.$$

We begin by sketching two proofs of the Lower Bound Theorem, which states:

Theorem 1 (Barnette). *For all simplicial polytopes, g_2 is nonnegative.*

Here, f_j denotes the number of j-faces (j-dimensional faces) of P. The first proof is due to Stanley, the second to Kalai.

2.1. *Stanley's Proof*

Stanley's [21] proof of this result is actually an easy corollary of his proof of the more powerful g-theorem, and requires some preliminary definitions. Let Δ be a simplicial $(d-1)$-complex ($(d-1)$-dimensional complex) on the vertex set $\{1,\ldots,n\}$. The *f-vector* of Δ is the vector of nonnegative integers $f = (f_0,\ldots,f_{d-1})$, where f_j denotes the number of faces (elements) of Δ of dimension j (cardinality $j+1$). With the convention that $f_{-1} = 1$, the *h-vector* of Δ is the vector of integers $h = (h_0,\ldots,h_d)$ defined by

$$h_k = \sum_{j=0}^{k} (-1)^{j-k} \binom{d-j}{d-k} f_{j-1}, \qquad k = 0,\ldots,d.$$

As is well known, the h-vector encodes the same amount of information as the f-vector, since

$$f_j = \sum_{k=0}^{j+1} \binom{d-k}{d-j-1} h_k, \qquad j = -1, \dots, d-1.$$

Now define $g_0 = h_0 = 1$ and $g_k = h_k - h_{k-1}$, $k = 1, \dots, \lfloor d/2 \rfloor$.

The *face ring* of Δ over $\mathbf{R}$ is $A = \mathbf{R}[x_1, \dots, x_n]/I_\Delta$, where I_Δ is the ideal generated by all square-free monomials $x_{i_1} \cdots x_{i_s}$ such that $\{i_1, \dots, i_s\}$ is not a member of Δ. We grade A in a natural way by degree, $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$. For $\theta_1, \dots, \theta_d \in A_1$, define $B = B_0 \oplus B_1 \oplus \cdots = A/(\theta_1, \dots, \theta_d)$. Stanley [19], [20] proves:

Theorem 2 (Stanley). *A is Cohen–Macaulay if and only if $\theta_1, \dots, \theta_d$ exist such that $B = B_0 \oplus \cdots \oplus B_d$ and $\dim B_k = h_k$, $k = 0, \dots, d$. In this case the θ_j can be chosen generically (i.e., with coefficients that are algebraically independent over $\mathbf{R}$).*

If the ring A is Cohen–Macaulay, then Δ is called a *Cohen–Macaulay complex.* Reisner [18] gives a homological characterization of the class of Cohen–Macaulay complexes, which includes shellable simplicial complexes, simplicial balls and spheres, and boundary complexes of simplicial polytopes. The h-vectors of Cohen–Macaulay complexes are clearly nonnegative, but they must also satisfy certain nonlinear conditions.

For $\theta_1, \dots, \theta_d \in A_1$, $B = A/(\theta_1, \dots, \theta_d)$, and $\omega \in B_1$, define $C = C_0 \oplus C_1 \oplus \cdots = B/(\omega)$. Stanley exploits a connection between the face ring of a simplicial convex polytope and the cohomology of an associated toric variety, and invokes the Hard Lefschetz Theorem for such varieties to prove:

Theorem 3 (Stanley). *Suppose that A is the face ring of the boundary complex Δ of some simplicial convex d-polytope. Then $\theta_1, \dots, \theta_d \in A_1$ and $\omega \in B_1$ exist such that:*

1. *$B = B_0 \oplus \cdots \oplus B_d$ and $\dim B_k = h_k$, $k = 0, \dots, d$.*
2. *Multiplication by ω^{d-2k} is a bijection between B_k and B_{d-k}, $k = 0, \dots, \lfloor d/2 \rfloor$.*

In particular, multiplication by ω is an injection from B_k into B_{k+1}, $k = 0, \dots, \lfloor d/2 \rfloor - 1$. As a consequence, $C = C_0 \oplus \cdots \oplus C_{\lfloor d/2 \rfloor}$ and $g_k = \dim C_k$, $k = 0, \dots, \lfloor d/2 \rfloor$.

An immediate corollary is that the numbers g_k are nonnegative, $k = 0, \dots, \lfloor d/2 \rfloor$. (This was first conjectured by McMullen and Walkup [15].) In particular, $g_2 \geq 0$. We also see that $h_i = h_{d-i}$, $i = 0, \dots, \lfloor d/2 \rfloor$. These are the *Dehn–Sommerville relations*, which can be proved directly by various combinatorial methods, and hold more generally for simplicial spheres.

Stanley's theorem yields an explicit numerical characterization of the f-vectors of simplicial d-polytopes, which is expressed in terms of the h_k and the g_k (the g-theorem) [21].

Theorem 4 (Stanley). *Suppose that $h = (h_0, \dots, h_d) \in \mathbf{Z}^{d+1}$, $g_0 = h_0$, and $g_k = h_k - h_{k-1}$, $k = 1, \dots, \lfloor d/2 \rfloor$. Then h is the h-vector of a simplicial d-polytope if and only if:*

1. $h_i = d_{d-i}$, $i = 0, \dots, \lfloor d/2 \rfloor$.

2. $g_i \geq 0, i = 0, \ldots, \lfloor d/2 \rfloor$.
3. $g_0 = 1$ *and* $g_{i+1} \leq g_i^{\langle i \rangle}, i = 1, \ldots, \lfloor d/2 \rfloor - 1$.

See, for example, [21] for the definition of the pseudopower $g_i^{\langle i \rangle}$.

2.2. *Kalai's Proof*

Kalai's proof [8] that g_2 is nonnegative is quite accessible, but does not have the full force of the g-theorem. Again, we need to start with some definitions. Let $G = (V, E)$ be a graph, where $V = \{1, \ldots, n\}$. Choose a point $v_i \in \mathbf{R}^d$ for each vertex of the graph and make a *bar-and-joint structure* by placing bars connecting pairs of points corresponding to the edges of G (we do not worry about self-intersection). Often we refer to the v_i themselves as the vertices, and the bars as the edges. A *stress* on this bar-and-joint structure is an assignment of numbers λ_{ij} to edges $v_i v_j$ such that

$$\sum_{j:\ v_i v_j \in E} \lambda_{ij}(v_j - v_i) = O \tag{1}$$

holds for every vertex v_i. The vector space of all stresses is the *stress space* of the structure.

An *infinitesimal motion* of the structure is a set of vectors $\overline{v}_1, \ldots, \overline{v}_n \in \mathbf{R}^d$ such that $d(\|(v_i + t\overline{v}_i) - (v_j + t\overline{v}_j)\|^2)/dt = 0$ for all edges $v_i v_j$. Equivalently, $(v_i - v_j)^T(\overline{v}_i - \overline{v}_j) = 0$ for all edges, or the projections of $\overline{v}_i$ and $\overline{v}_j$ onto the affine span of $\{v_i, v_j\}$ agree. Some infinitesimal motions are *trivial* in the sense that they are induced by rigid motions of $\mathbf{R}^d$ itself. Motions apart from these are called *nontrivial*. If the structure admits only trivial motions, it is *infinitesimally rigid*.

Using the classical relationship between the space of infinitesimal motions and the space of stresses of a structure, and the fact that the bar-and-joint structure associated with the edge-skeleton of a simplicial convex d-polytope P, $d \geq 3$, is infinitesimally rigid (where we take the v_i to be the vertices of P itself), Kalai observes that the dimension of the stress space of P is g_2, and hence g_2 must be nonnegative.

In this striking proof of the Lower Bound Theorem Kalai speculates whether it might be possible to extend the notions of stress and rigidity appropriately to the higher-dimensional faces of P to reprove the nonnegativity of the other g_k, and possibly even find a new proof of the g-theorem. The notion of generalized stress presented below accomplishes this, but these results depend in an essential way upon McMullen's new proof of the g-theorem [13], [14] using weights on simple polytopes.

3. Generalized Stress

3.1. *Working Toward a Definition*

We could define generalized stress by starting with some analog of classical stress or infinitesimal motion, but instead we work toward the definition by following the path by which it was originally discovered. This route was primarily motivated by attempts to mimic some aspects of Kalai's algebraic shifting technique [7].

For $x = (x_1, \dots, x_n)$, and for $(r_1, \dots, r_n) \in \mathbf{Z}_+^n$, by x^r we mean $x_1^{r_1} \cdots x_n^{r_n}$. Define also $\operatorname{supp} x^r = \{i\colon r_i \neq 0\}$ (the *support* of x^r), $r! = r_1! \cdots r_n!$, and $|r| = r_1 + \cdots + r_n$. Write e_i for the vector of length n consisting of all zeros, except for a one in the ith position, and $e = (1, \dots, 1)$.

Let Δ be a simplicial complex (not necessarily of dimension $d-1$) with n vertices $\{1, \dots, n\}$, and let $R = \mathbf{R}[x_1, \dots, x_n] = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be the ring of polynomials, graded by degree. Consider any elements $\theta_1, \dots, \theta_d \in R_1$. We wish to determine information about the dimension of B_k (as a vector space over $\mathbf{R}$), where $B = B_0 \oplus B_1 \oplus B_2 \oplus \cdots$ is the result of taking R and factoring out the ideal $J = J_0 \oplus J_1 \oplus J_2 \oplus \cdots$ generated by I_Δ and $\theta_1, \dots, \theta_d$. Place an inner product on the vector space R_k by defining $\langle \sum_{r\colon |r|=k} a_r x^r, \sum_{r\colon |r|=k} b_r x^r \rangle = \sum_{r\colon |r|=k} a_r b_r$. Write $R_k = J_k \oplus J_k^\perp$. It is straightforward to see that $\sum_{r\colon |r|=k} b_r x^r$ is in $J_k^\perp$ if and only if it is orthogonal to:

1. All monomials of the form $x^s x^q$ where x^q is square-free, $\operatorname{supp} x^q \notin \Delta$, and $|s| + |q| = k$.
2. All polynomials of the form $x^s \theta_j$, where $|s| = k-1$.

Define $v_i = (v_{i1}, \dots, v_{id})^T$, $i = 1, \dots, n$, where $\theta_j = \sum_{i=1}^n v_{ij} x_i$, $j = 1, \dots, d$. Then the first condition is equivalent to the condition

$$b_r = 0 \qquad \text{if} \quad \operatorname{supp} x^r \notin \Delta, \tag{2}$$

and the second condition is equivalent to the condition

$$\sum_{i=1}^n b_{s+e_i} v_i = O, \tag{3}$$

for every $s \in \mathbf{Z}_+^n$ such that $|s| = k-1$. Thus we have a linear equation on the vectors v_i for every such s.

The second condition can be expressed more compactly if we look at

$$b(x) = \sum_{r\colon |r|=k} b_r \frac{x^r}{r!}.$$

Define M to be the $d \times n$ matrix with columns $v_1, \dots, v_n$. Then $\sum_{r\colon |r|=k} b_r x^r$ satisfies condition (3) if and only if

$$\sum_{i=1}^n \left(\frac{\partial b}{\partial x_i}\right) v_{ij} = 0, \qquad j = 1, \dots, d,$$

or

$$\sum_{i=1}^n \left(\frac{\partial b}{\partial x_i}\right) v_i = O,$$

where the left-hand side is to be regarded as a polynomial with vector coefficients, or

$$M \nabla b = O. \tag{4}$$

This leads directly to our definition of generalized linear stress:

Definition 1. Let Δ be a simplicial complex (not necessarily of dimension $d-1$) on the set $\{1,\dots,n\}$, and let $v_1\dots,v_n\in\mathbf{R}^d$. Let M be the $d\times n$ matrix with columns $v_1,\dots,v_n$. For each $k=0,1,2,\dots$, a *linear k-stress* on Δ (with respect to $v_1,\dots,v_n$) is a polynomial of the form

$$b(x)=\sum_{r:\,|r|=k} b_r\frac{x^r}{r!}$$

that satisfies

$$b_r=0 \qquad \text{if} \quad \operatorname{supp} x^r \notin \Delta,$$

and

$$M\nabla b=O.$$

The collection of all linear k-stresses forms a vector space, which is denoted S_k^ℓ. (In [10] we used the notation $\overline{B}_k$.)

There was evidence to suggest that the Hard Lefschetz element ω in the proof of the g-theorem could be chosen to be $x_1+\cdots+x_n$. This was confirmed by McMullen [13], [14]. So we are also interested in the effect of factoring out $x_1+\cdots+x_n$ from R as well. This suggests the definition of generalized affine stress:

Definition 2. Let Δ be a simplicial complex (not necessarily of dimension $d-1$) on the set $\{1,\dots,n\}$, and let $v_1\dots,v_n\in\mathbf{R}^d$. Let $\overline{M}$ be the $(d+1)\times n$ matrix obtained by appending a final row of ones to the matrix M with columns $v_1,\dots,v_n$. For each $k=0,1,2,\dots$, an *affine k-stress* on Δ (with respect to $v_1,\dots,v_n$) is a polynomial of the form

$$b(x)=\sum_{r:\,|r|=k} b_r\frac{x^r}{r!}$$

that satisfies

$$b_r=0 \qquad \text{if} \quad \operatorname{supp} x^r \notin \Delta,$$

and

$$\overline{M}\nabla b=O.$$

The collection of all affine k-stresses forms a vector space, which is denoted S_k^{a}. (In [10] we used the notation $\overline{C}_k$.)

Equivalently, an affine k-stress is a linear k-stress that satisfies the additional condition

$$e^T\nabla b=0,$$

or

$$\sum_{i=1}^n\frac{\partial b}{\partial x_i}=0,$$

or

$$\sum_{i=1}^{n} b_{s+e_i} = 0 \tag{5}$$

for every $s \in \mathbf{Z}_+^n$ such that $|s| = k - 1$. That is, we have an affine relation on the vectors v_i for every such s.

It is obvious that $b(x)$ is an affine k-stress with respect to $v_1, \ldots, v_n$ if and only if it is a linear k-stress with respect to $\overline{v}_1, \ldots, \overline{v}_n$, where

$$\overline{v}_i = \begin{bmatrix} v_i \\ 1 \end{bmatrix}, \qquad i = 1, \ldots, n.$$

3.2. *Connections to Cohen–Macaulay Complexes*

Suppose that Δ is a simplicial complex (not necessarily of dimension $d - 1$) with n vertices. Let A be its face ring, and assume we have $\theta_1, \ldots, \theta_d \in A_1$ and $v_1, \ldots, v_n \in \mathbf{R}^d$ such that $\theta_j = \sum_{i=1}^{n} v_{ij} x_i$, $j = 1, \ldots, d$, and $v_i = (v_{i1}, \ldots, v_{id})^T$, $i = 1, \ldots, n$. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots = R/I_\Delta$, $B = B_0 \oplus B_1 \oplus B_2 \oplus \cdots = A/(\theta_1, \ldots, \theta_d)$, and $C = C_0 \oplus C_1 \oplus C_2 \oplus \cdots = B/(x_1 + \cdots + x_n)$. Simply from the way the definitions are crafted we immediately conclude:

Theorem 5. *Regardless of whether or not Δ is Cohen–Macaulay,* $\dim B_k = \dim S_k^\ell$, $k = 0, 1, 2, \ldots$, *and* $\dim C_k = \dim S_k^{\mathrm{a}}$, $k = 0, 1, 2, \ldots$.

Corollary 1. *Let Δ be any simplicial $(d-1)$-complex with n vertices.*

1. *Δ is Cohen–Macaulay if and only if $v_1, \ldots, v_n \in \mathbf{R}^d$ exist such that* $\dim S_k^\ell = h_k$, $k = 0, \ldots, d$. *In this case the v_i can be chosen generically* (*i.e., with components algebraically independent over* $\mathbf{R}$).
2. *Suppose that Δ is in fact a simplicial $(d-1)$-sphere. If $v_1, \ldots, v_n \in \mathbf{R}^d$ are chosen such that* $\dim S_k^\ell = h_k$, $k = 0, \ldots, d$, *and further* $\dim S_k^{\mathrm{a}} = g_k$, $k = 0, \ldots, \lfloor d/2 \rfloor$, *then the h-vector of Δ satisfies the numerical conditions of the g-theorem.*

3.3. *Differential Operators*

Differential operators with constant coefficients acting on the stress spaces play an important role. In particular, we can construct an operator that will provide a relationship between linear and affine stresses, and which is seen in Sections 9 and 10 to serve as the Lefschetz element in the proof of the g-theorem.

For $c \in \mathbf{R}^n$, define the function σ_c on the space of linear stresses by

$$\sigma_c(b) = c^T \nabla b = \sum_{i=1}^{n} c_i \frac{\partial b}{\partial x_i}$$

for any linear stress $b(x)$. Define in particular

$$\omega(b) = \sigma_e(b) = \sum_{i=1}^{n} \frac{\partial b}{\partial x_i}.$$

Theorem 6. *Let Δ be any simplicial complex with n vertices, and let $v_1, \ldots, v_n \in \mathbf{R}^d$. Then, for $k = 1, 2, 3, \ldots$, the function σ_c maps S_k^ℓ into S_{k-1}^ℓ, and, for $k = 0, 1, 2, \ldots$, the kernel of ω restricted to S_k^ℓ is S_k^{a}.*

Proof. Let b be a linear k-stress and $r \in \mathbf{Z}_+^n$ such that $|r| = k - 1$. The coefficient of $x^r/r!$ in $\sigma_c(b)$ is $\sum_{i=1}^n c_i b_{r+e_i}$. If $\operatorname{supp} x^r \notin \Delta$, then $\operatorname{supp} x^{r+e_i} \notin \Delta$ for $i = 1, \ldots, n$. So $b_{r+e_i} = 0$, $i = 1, \ldots, n$, and $\sigma_c(b)$ satisfies condition (2). Further, $M\nabla(c^T\nabla b) = M[(\nabla^2 b)c] = [\nabla(M\nabla b)]c = O$ since $M\nabla b = O$, and so $\sigma_c(b)$ satisfies condition (4). □

3.4. *Coning*

We conclude this section with a simple but useful result first proved by Tay *et al.* [22]. Suppose that Δ is a simplicial complex with vertex set $\{1, \ldots, n\}$, and $v_1, \ldots, v_n \in \mathbf{R}^d$. Let $a_0, \ldots, a_n \in \mathbf{R}$ such that $a_0 \neq 0$ and let $\hat{\Delta}$ be the simplicial complex $\Delta \cdot 0 = \{F \cup \{0\}\colon F \in \Delta\}$. Sometimes this operation is called *coning*. How does the linear k-stress space $S_k^\ell(\Delta)$ of Δ with respect to $v_1, \ldots, v_n$ relate to the linear k-stress space $S_k^\ell(\hat{\Delta})$ of $\hat{\Delta}$ with respect to $\hat{v}_0 = (0, \ldots, 0, a_0)^T$, $\hat{v}_1 = (v_1, a_1)^T, \ldots, \hat{v}_n = (v_n, a_n)^T$?

Theorem 7. *Let $S_k^\ell(\Delta)$ and $S_k^\ell(\hat{\Delta})$ be as above. Then$S_k^\ell(\Delta)$ is isomorphic to $S_k^\ell(\hat{\Delta})$. In particular, $S_k^\ell(\Delta)$ with respect to $v_1, \ldots, v_n$ is isomorphic to $S_k^{\mathrm{a}}(\hat{\Delta})$ with respect to $0, v_1, \ldots, v_n$.*

Proof. Let $b(x_1, \ldots, x_n) \in S_k^\ell(\Delta)$. For a polynomial expression f in $x_1, \ldots, x_n$, define $\hat{s}(f) = f(x_1 - (a_1/a_0)x_0, \ldots, x_n - (a_n/a_0)x_0)$. We claim that $\hat{b} = \hat{s}(b)$ is in $S_k^\ell(\hat{\Delta})$. For

$$\begin{aligned}\sum_{i=0}^{n} \frac{\partial \hat{b}}{\partial x_i} v_i &= \sum_{i=1}^{n} \frac{\partial \hat{b}}{\partial x_i} v_i \\ &= \sum_{i=1}^{n} \hat{s}\left(\frac{\partial b}{\partial x_i}\right) v_i \\ &= \hat{s}\left(\sum_{i=1}^{n} \frac{\partial b}{\partial x_i} v_i\right) \\ &= O\end{aligned}$$

and

$$\sum_{i=0}^{n} \frac{\partial \hat{b}}{\partial x_i} a_i = \frac{\partial \hat{b}}{\partial x_0} a_0 + \sum_{i=1}^{n} \frac{\partial \hat{b}}{\partial x_i} a_i$$

$$
\begin{aligned}
&= \hat{s}\left[\sum_{i=1}^{n}\left(-\frac{a_i}{a_0}\right)\frac{\partial b}{\partial x_i}\right]a_0 + \sum_{i=1}^{n} a_i s\left(\frac{\partial b}{\partial x_i}\right) \\
&= \hat{s}\left(\sum_{i=1}^{n}(-a_i + a_i)\frac{\partial b}{\partial x_i}\right) \\
&= 0.
\end{aligned}
$$

Conversely, suppose that $\hat{b}(x_0, x_1, \ldots, x_n) \in S_k^{\ell}(\hat{\Delta})$. For a polynomial expression f in $x_0, \ldots, x_n$, define $s(f) = f(0, x_1, \ldots, x_n)$. We can check that $b = s(\hat{b})$ is in $S_k^{\ell}(\Delta)$:

$$
\begin{aligned}
\sum_{i=1}^{n}\frac{\partial b}{\partial x_i}v_i &= \sum_{i=1}^{n} s\left(\frac{\partial \hat{b}}{\partial x_i}\right)v_i \\
&= s\left(\sum_{i=1}^{n}\frac{\partial \hat{b}}{\partial x_i}v_i\right) \\
&= s\left(\sum_{i=0}^{n}\frac{\partial \hat{b}}{\partial x_i}v_i\right) \\
&= O.
\end{aligned}
$$

□

4. Why "Stress"?

The use of the terms "linear" and "affine" in the definition has already been justified—the conditions for $b(x)$ to be a stress involve either linear or affine relations on the v_i. However, it is not yet clear why the term "stress" makes sense. This will be motivated in several stages. First we show that S_2^{a} is isomorphic to the classical stress space of a bar-and-joint structure. The higher dimensions will require some preliminary work. However, first, we consider some simple examples.

4.1. *Examples*

The first example is an easy but important one that will resurface later in this paper.

Example 1. Consider a geometric d-simplex in $\mathbf{R}^d$ and let Δ be its boundary complex. Choose $v_1, \ldots, v_{d+1}$ to be the vertices of the simplex itself. Assume further that the simplex is positioned such that no proper subset of the vertices is linearly dependent. Then nonzero $c_i \in \mathbf{R}$ exist such that

$$
\sum_{i=1}^{d+1} c_i v_i = O,
$$

and all linear relations on the v_i are nonzero scalar multiples of this one. We claim that for all $k = 0, \dots, d$, S_k^ℓ is one-dimensional and is spanned by

$$\sum_{r:\ |r|=k} c^r \frac{x^r}{r!}.$$

We can verify this by using the fact that

$$\sum_{i=1}^{d+1} c^{s+e_i} v_i = c^s \sum_{i=1}^{d+1} c_i v_i = O$$

for all $s \in \mathbf{Z}_+^{d+1}$ such that $|s| = k - 1$. Observe that c^r is nonzero for all r. On the other hand, $\dim S_k^\ell = 0$ for all $k > d$, $\dim S_0^{\mathrm{a}} = 1$, and $\dim S_k^{\mathrm{a}} = 0$ for all $k \geq 1$, since the v_i are affinely independent and so $\sum_{i=1}^{d+1} c_i \neq 0$.

Example 2. Suppose that P is the standard octahedron in $\mathbf{R}^3$ with vertices $v_1 = (1,0,0)^T$, $v_2 = (-1,0,0)^T$, $v_3 = (0,1,0)^T$, $v_4 = (0,-1,0)^T$, $v_5 = (0,0,1)^T$, and $v_6 = (0,0,-1)^T$. Then, for the boundary complex Δ of P, the stress spaces with respect to $v_1, \dots, v_6$ are given by:

1. $S_0^\ell = \mathbf{R}$.
2. S_1^ℓ is three-dimensional and has a basis $\{x_1 + x_2, x_3 + x_4, x_5 + x_6\}$.
3. S_2^ℓ is three-dimensional and has a basis $\{(x_1 + x_2)(x_3 + x_4), (x_1 + x_2)(x_5 + x_6), (x_3 + x_4)(x_5 + x_6)\}$.
4. S_3^ℓ is one-dimensional and has a basis $\{(x_1 + x_2)(x_3 + x_4)(x_5 + x_6)\}$.
5. $S_k^\ell = \{0\}$ if $k > 3$.
6. $S_0^{\mathrm{a}} = \mathbf{R}$.
7. S_1^{a} is two-dimensional and has a basis $\{x_1 + x_2 - x_3 - x_4, x_1 + x_2 - x_5 - x_6\}$.
8. $S_k^{\mathrm{a}} = \{0\}$ if $k > 1$.

4.2. *Connection with Classical Stress*

Turning now to general simplicial complexes, we can describe the low-dimensional stress spaces and clarify the connection with classical stress:

Theorem 8. *Let Δ be any simplicial complex with n vertices, and let $v_1, \dots, v_n \in \mathbf{R}^d$. Then*:

1. $S_0^\ell = S_0^{\mathrm{a}} = \mathbf{R}$.
2. S_1^ℓ *is isomorphic to the space of all linear relations on the vectors $v_1, \dots, v_n$.*
3. S_1^{a} *is isomorphic to the space of all affine relations on the vectors $v_1, \dots, v_n$.*
4. S_2^{a} *is isomorphic to the classical stress space on the bar-and-joint structure where the vertices are placed at the points $v_1, \dots, v_n$, under the correspondence $\lambda_{ij} = b_{e_i} + b_{e_j}$ for all $i \neq j$.*

Proof. Only the fourth part requires any explanation. Assume $b \in S_2^a$. Set $\lambda_{ij} = b_{e_i+e_j}$ for all $i, j = 1, \dots, n$. Of course, $\lambda_{ij} = \lambda_{ji}$, and $\lambda_{ij} = 0$ if $\{i, j\}$ is not an edge of Δ. From conditions (3) and (5) we find that, for all $j = 1, \dots, n$,

$$\begin{aligned} O &= \sum_{i=1}^{n} \lambda_{ij} v_i \\ &= \sum_{i:\, i \neq j} \lambda_{ij} v_i + \lambda_{jj} v_j \\ &= \sum_{i:\, i \neq j} \lambda_{ij} v_i + \sum_{i:\, i \neq j} (-\lambda_{ij}) v_j \\ &= \sum_{i:\, \{i,j\} \in E} \lambda_{ij} (v_i - v_j), \end{aligned}$$

where E is the set of edges of Δ. Therefore the λ_{ij} satisfy the equilibrium condition (1).

Conversely, assume we have numbers λ_{ij} for each $\{i, j\} \in E$ that satisfy condition (1). For $j = 1, \dots, n$ define

$$b_{jj} = -\sum_{i:\, \{i,j\} \in E} \lambda_{ij},$$

and for $i \neq j$ define

$$b_{e_i+e_j} = \begin{cases} \lambda_{ij} & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Reversing the previous calculations shows that the resulting quadratic polynomial $b(x)$ is an affine 2-stress. □

4.3. *Coefficients of Square-Free Terms*

Our next step is to show that under suitable conditions the coefficients of the square-free monomials of a linear or affine k-stress uniquely determine the remaining coefficients of the polynomial. We then concentrate our attention on the square-free terms, regarding the coefficients as assignments of numbers to various faces of the simplicial complex, and give a geometric necessary condition on these numbers that turns out to be a natural generalization of classical stress. We are, in fact, able to give explicit formulas for the coefficients of the non-square-free monomials in terms of the coefficients of the square-free monomials, and in the process show that the above necessary condition is also sufficient and thus characterizes the coefficients of the square-free terms.

For a simplicial complex Δ with n vertices and for $v_1, \dots, v_n \in \mathbf{R}^d$, we say that the v_i are in *linearly general position with respect to* Δ if $\{v_{i_1}, \dots, v_{i_s}\}$ is linearly independent for every face $\{i_1, \dots, i_s\}$ of Δ.

Theorem 9. *Let Δ be any simplicial complex with n vertices and assume that $v_1, \dots, v_n$ are in linearly general position with respect to Δ. If $b(x)$ is a linear stress, then the coefficients of the non-square-free monomials in $b(x)$ are linear combinations of the coefficients of the square-free monomials and hence are uniquely determined by them.*

Proof. Let $b(x) \in S_k^\ell$. We use reverse induction on $q = \operatorname{card}(\operatorname{supp} x^r)$. The result is trivially true if $q = k$, so assume the result is true for some q such that $2 \leq q \leq k$ and suppose that $\operatorname{card}(\operatorname{supp} x^r) = q - 1$. Choose j such that $r_j > 1$ and let $s = r - e_j$. Condition (3) implies

$$\sum_{i=1}^{n} b_{s+e_i} v_i = O.$$

However, by the induction hypothesis the coefficients b_{s+e_i} are linear combinations of the coefficients of the square-free monomials when $r_i = 0$, since $\operatorname{card}(\operatorname{supp} x^{s+e_i}) = q$ in this case. This leaves the $q-1$ coefficients b_{s+e_i} for $i \in \operatorname{supp} x^r$ to be uniquely determined since the corresponding v_i are linearly independent by assumption. In particular, $b_{s+e_j} = b_r$ is a linear combination of the coefficients of the square-free monomials. □

The above proof shows how conditions (2) and (3) can be used in a systematic way to find all the coefficients of $b(x)$ if the coefficients of the square-free terms are given.

Corollary 2. *Let Δ be any simplicial complex with n vertices and let $v_1, \dots, v_n \in \mathbf{R}^d$ be chosen in a linearly general position with respect to Δ. Then* $\dim S_k^\ell = 0$ *for all* $k > \dim \Delta + 1$.

Proof. In the case that $k > \dim \Delta + 1$ there are no faces of cardinality k, so all coefficients of square-free monomials of a linear k-stress must be zero. □

4.4. *A Geometrical Interpretation of Stress*

For $F = \{i_1, \dots, i_s\} \in \Delta$, define $\operatorname{conv} F$ (with respect to $v_1, \dots, v_n$) to be $\operatorname{conv}\{v_{i_1}, \dots, v_{i_s}\}$. In an analogous way, define $\operatorname{aff} F$ and $\operatorname{span} F$. We sometimes abuse notation and write b_F and x^F for b_r and x^r, respectively, where $r_i = 1$ if $i \in F$ and $r_i = 0$ if $i \notin F$. We also use the notation $F + i$ for $F \cup \{i\}$ and $F - i$ for $F \backslash \{i\}$. Finally, if $i \in F$, by b_{F+i} we mean b_{r+e_i}, where r is as above.

Theorem 10. *Let Δ be any simplicial $(d-1)$-complex with n vertices and let $v_1, \dots, v_n \in \mathbf{R}^d$. Let $b(x)$ be a linear (resp. affine) k-stress, $k \geq 1$. Choose any face F of Δ of cardinality $k - 1$ and any point v in* $\operatorname{span} F$ *(resp.* $\operatorname{aff} F$*). Then*

$$v + \sum_{i \in \operatorname{lk} F} b_{F+i}(v_i - v)$$

lies in $\operatorname{span} F$ *(resp.* $\operatorname{aff} F$*). Equivalently, if w_i is the vector joining the projection of v_i onto* $\operatorname{span} F$ *(resp.* $\operatorname{aff} F$*) to v_i, then*

$$\sum_{i \in \operatorname{lk} F} b_{F+i} w_i = O. \tag{6}$$

Proof. Suppose that $v \in \operatorname{span} F$. Then, using condition (3),

$$\begin{aligned} v + \sum_{i \in \operatorname{lk} F} b_{F+i}(v_i - v) &= v + \sum_{i \in \operatorname{lk} F} b_{F+i} v_i - \sum_{i \in \operatorname{lk} F} b_{F+i} v \\ &= v - \sum_{i \in F} b_{F+i} v_i - \sum_{i \in \operatorname{lk} F} b_{F+i} v \end{aligned}$$

which is in span F. If b is an affine stress, then by condition (5) the sum of the coefficients in the above expression is

$$1 - \sum_{i \in F} b_{F+i} - \sum_{i \in \operatorname{lk} F} b_{F+i} = 1.$$

So we have an element of aff F. □

Note that for linear k-stress, w_i is the altitude vector for the point v_i in the simplex $\operatorname{conv}(\{O\} \cup (F + i))$, and for affine k-stress, w_i is the altitude vector for the point v_i in the simplex $\operatorname{conv}(F + i)$. In particular, condition (6) for affine 2-stress is identical to condition (1) defining classical stress. So affine k-stress generalizes classical stress in a natural way, and could in fact have been defined by condition (6) in the first place. This is the definition that Kalai was thinking of (personal communication). Linear k-stress seems less natural at first sight since it is dependent upon choice of origin. In the case of simplicial polytopes, however, we see in Section 10 that linear stress becomes invariant under rigid motions when dualized and interpreted as McMullen's weights on simple polytopes.

Example 3. Let P be a simplicial convex d-polytope in $\mathbf{R}^d$, Δ its boundary complex, and $v_1, \ldots, v_n$ its vertices. Then the above theorem shows that $\dim S_d^{\mathrm{a}} = 0$. Take any $b(x) \in S_d^{\mathrm{a}}$ and consider any subfacet F (i.e., of cardinality $d-1$). There are exactly two facets containing F and hence only two altitude vectors w_i with respect to aff F, where $i \in \operatorname{lk} F$. By convexity these two vectors are not collinear and we know

$$\sum_{i \in \operatorname{lk} F} b_{F+i} w_i = O,$$

from which it follows that $b_{F+i} = 0$ for $i \in \operatorname{lk} F$. Thus all the coefficients of the square-free monomials of $b(x)$ are zero, and so all of the remaining coefficients must likewise be zero.

4.5. *Formulas for the Coefficients*

Condition (6) is a nice geometrical necessary condition for the coefficients of the square-free terms of generalized stress. However (again with suitably general v_i), this condition is also sufficient, as we now show.

Assume Δ is a simplicial complex of dimension at most $d-1$ with vertices $1, \ldots, n$, and that $v_1, \ldots, v_n \in \mathbf{R}^d$ are in linearly general position with respect to Δ. Assume further that $u_1, \ldots, u_d \in \mathbf{R}^d$.

Suppose that $G = \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\}$, where $s \leq d$. Fix an ordering of the elements of G and define

$$[G] = \det \begin{bmatrix} v_{i_1} & \cdots & v_{i_s} & u_{s+1} & \cdots & u_d \end{bmatrix}.$$

If $i \in G$, we compute $[G - i]$ using the ordering induced by G and multiplying by $+1$ (resp. -1) if i is in an odd (resp. even) position with respect to this ordering, and we compute $[G - i + j]$ by replacing the column corresponding to v_i with the column corresponding to v_j.

We say that $u_1, \ldots, u_d$ are in *linearly general position with respect to* Δ *and* $v_1, \ldots, v_n$ if $[G]$ is nonsingular for every face G of Δ.

Theorem 11. *Let Δ be a simplicial complex on n vertices of dimension at most $d - 1$, let $v_1, \ldots, v_n$ be in linearly general position with respect to Δ, and let $u_1, \ldots, u_d$ be in linearly general position with respect to Δ and $v_1, \ldots, v_n$. Suppose that we have numbers b_F assigned to each $(k-1)$-face F of Δ that satisfy condition* (6). *For each $r \in \mathbf{Z}_+^n$ such that $|r| = k$ and $S = \operatorname{supp} x^r \in \Delta$, define*

$$b_r = \sum_{(k-1)\text{-faces } F \text{ containing } S} b_F \prod_{i \in F} [F - i]^{r_i - 1}.$$

Then $b(x) = \sum_{r:\ |r|=k} b_r (x^r / r!)$ is a linear k-stress.

Proof. We already know that there can be at most one linear k-stress $b(x)$ with the given coefficients of the square-free terms. We must show that in fact there is one, and that it is given by the formula above. Consider one instance of condition (3):

$$\sum_{j=1}^{n} b_{s+e_j} v_j = O,$$

where $|s| = k - 1$. Let $S = \operatorname{supp} x^s$. The coefficients b_{s+e_j} appearing in the expression correspond to monomials with support size either $\operatorname{card}(\operatorname{supp} x^s)$ (if $j \in S$) or $\operatorname{card}(\operatorname{supp} x^s) + 1$ (if $j \in \operatorname{lk} S$). So we can contemplate the possibility of using these conditions repeatedly to determine the coefficients of monomials with smaller supports from the coefficients of monomials with larger supports. In the process we need to verify that:

(i) For a given instance of the condition it is possible to solve for the unknown coefficients, i.e., that

$$\sum_{j \in \operatorname{lk} S} b_{s+e_j} v_j \in \operatorname{span}\{v_j\colon\ j \in S\}. \tag{7}$$

(ii) If the same coefficient is determined from two different applications of condition (3) in this manner, that we do not get contradictory values.

The proof will therefore be by reverse induction on $p = \operatorname{card}(\operatorname{supp} x^r)$. The formula stated in the theorem is trivially true if $k = 1$ or if $p = k$ so we assume that $k \geq 2$ and $1 \leq p < k$. Choose any m for which $r_m > 1$. Let $s = r - e_m$ and $S = \operatorname{supp} x^s = \operatorname{supp} x^r$.

If $p = k - 1$, then (7) is true by the assumption that the coefficients satisfy condition (6). So suppose that $p < k - 1$. Find q such that $s_q > 1$.

Fix an ordering of the elements of S, $(v_{i_1}, \ldots, v_{i_p})$. Let v_S denote $v_{i_1} \wedge \cdots \wedge v_{i_p}$, let v_{S-q} denote the same wedge product with v_q removed, and let v_{S-q+j} denote $v_j \wedge v_{S-q}$.

For each $j \notin S$, define $s(j) = s - e_q + e_j$. Note that $\operatorname{card}(\operatorname{supp} x^{s(j)}) = \operatorname{card}(\operatorname{supp} x^s) + 1$. So by the induction hypothesis, the formula gives coefficients such that

$$\sum_{i=1}^{n} b_{s(j)+e_i} v_i = O.$$

Wedge this with v_{S-q+j} and sum over all $j \notin S$:

$$\sum_{j \notin S} \sum_{i=1}^{n} (b_{s(j)+e_i} v_i \wedge v_{S-q+j}) = O.$$

If $i = j$ or if $i \in S - q$, then $v_i \wedge v_{s-q+j} = O$. If $i \notin S$, then one of the terms in the above expression is

$$b_{s(j)+e_i} v_i \wedge v_j \wedge v_{S-q}.$$

However, interchanging the roles of i and j also yields the term

$$b_{s(i)+e_j} v_j \wedge v_i \wedge v_{S-q}.$$

These terms cancel since $s(j) + e_i = s(i) + e_j$. Looking at the remaining terms (where $i = q$) we see

$$\sum_{j \notin S} (b_{s(j)+e_q} v_q \wedge v_{S-q+j}) = O.$$

However, $s(j) + e_q = r - e_q + e_j + e_q = r + e_j$, so

$$\begin{aligned} O &= \sum_{j \notin S} (b_{r+e_j} v_j \wedge v_S) \\ &= \left(\sum_{j \notin S} b_{r+e_j} v_j \right) \wedge v_S. \end{aligned}$$

Therefore

$$\sum_{j \notin S} b_{r+e_j} v_j \in \operatorname{span} S,$$

and we have confirmed (7) since $b_{r+e_j} = 0$ if $j \notin S$ and $j \notin \operatorname{lk} S$.

We proceed to find b_r, using the same s and m. Since

$$\sum_{j \in S} b_{s+e_j} v_j = - \sum_{j \in \operatorname{lk} S} b_{s+e_j} v_j,$$

Cramer's rule can be used to solve for b_r (which equals b_{s+e_j} when $j = m$) in the system

$$\sum_{j \in S} b_{s+e_j} v_j + \sum_{j=p+1}^{d} c_j u_j = - \sum_{j \in \operatorname{lk} S} b_{s+e_j} v_j$$

giving

$$b_r = \frac{[S - m + (-\sum_{j\in \mathrm{lk}\, S} b_{s+e_j} v_j)]}{[S]}$$

$$= -\sum_{j\in \mathrm{lk}\, S} \frac{[S-m+j]}{[S]} b_{r-e_m+e_j}.$$

Applying induction and using some Grassman–Plücker relations,

$$b_r = -\sum_{j\in \mathrm{lk}\, S} \frac{[S-m+j]}{[S]} \sum_{(k-1)\text{-faces } F \text{ containing } S+j} b_F \prod_{i\in F} [F-i]^{(r-e_m+e_j)_i - 1}$$

$$= -\sum_{(k-1)\text{-faces } F \text{ containing } S} \sum_{j\in F\setminus S} b_F \frac{[S-m+j]}{[S]} \prod_{i\in F\setminus j} [F-i]^{(r-e_m)_i-1}$$

$$= -\sum_{(k-1)\text{-faces } F \text{ containing } S} \frac{b_F}{[S]} \prod_{i\in F} [F-i]^{(r-e_m)_i-1} \sum_{j\in F\setminus S} [S-m+j][F-j]$$

$$= \sum_{(k-1)\text{-faces } F \text{ containing } S} \frac{b_F}{[S]} \prod_{i\in F} [F-i]^{(r-e_m)_i-1} [S][F-m]$$

$$= \sum_{(k-1)\text{-faces } F \text{ containing } S} b_F \prod_{i\in F} [F-i]^{r_i-1}.$$

The fact that this final formula is independent of the choice of m shows that we will not get contradictory values for b_r from different choices of s. □

Filliman [4] and Tay *et al.* [22] have also shown the sufficiency of condition (6), but without the explicit formula above. (Unfortunately, however, the proof of the g-theorem for p.l.-spheres in [4] is incorrect.)

5. Generalized Infinitesimal Motion

The relationship between the space of classical stresses (affine 2-stress) and the space of classical infinitesimal motions of bar-and-joint structures is straightforward: they are the left and right nullspaces of a common matrix. This suggests a natural way to define generalized infinitesimal motions associated with affine k-stress.

Suppose that $b(x)$ is an affine k-stress of a simplicial complex Δ with respect to $v_1, \ldots, v_n \in \mathbf{R}^d$. Looking at the equivalent condition on coefficients of square-free terms, we have a number b_F assigned to each $(k-1)$-face F, such that, for every $(k-2)$-face G,

$$\sum_{i\in \mathrm{lk}\, G} b_{G+i} w_i = O,$$

where w_i is the altitude vector of v_i in the simplex $\operatorname{conv}(G+i)$. We can express these conditions in terms of a matrix R whose rows are indexed by the $(k-1)$-faces F and whose columns occur in groups of d with one group for each $(k-2)$-face G. In the row corresponding to F and the group of columns corresponding to G we place the row vector of length d:

$$\begin{cases} O^T & \text{if} \quad G \not\subset F, \\ w_{(G,F)}^T & \text{if} \quad G \subset F. \end{cases}$$

Here, $w_{(G,F)}$ is the altitude vector of the simplex conv F with respect to conv G. Then the left nullspace of R is S_k^{a}. An *infinitesimal k-motion* will then be defined to be an element of the right nullspace of R, and is described by an assignment of a vector $\overline{v}_G \in \mathbf{R}^d$ to each $(k-2)$-face G such that, for every $(k-1)$-face F,

$$\sum_{i \in F} w_i \cdot \overline{v}_i = 0.$$

We have condensed the notation, writing w_i for $w_{(F-i,F)}$ and $\overline{v}_i$ for $\overline{v}_{F-i}$.

We can reformulate this condition by writing G_i for $F-i$ and u_i for the unit outer normal vector of conv G_i with respect to conv F in aff F. This yields

$$\sum_{i \in F} u_i \|w_i\| \cdot \overline{v}_i = 0,$$

which upon dividing by $\operatorname{vol}_{k-1}(F)$ implies

$$\sum_{i \in F} u_i \cdot \frac{\overline{v}_i}{\operatorname{vol}_{k-2}(G_i)} = 0.$$

Therefore

$$\sum_{i \in F} \operatorname{vol}_{k-2}(G_i)(u_i \cdot \overline{m}_i) = 0,$$

where $\overline{m}_i = \overline{v}_i / \operatorname{vol}_{k-2}^2(G_i)$.

So an infinitesimal k-motion is a choice of vector $\overline{m}_G \in \mathbf{R}^d$ for each $(k-2)$-face G such that

$$\sum_{G \subset F} \operatorname{vol}_{k-2}(G)(u_G \cdot \overline{m}_G) = 0 \tag{8}$$

for every $(k-1)$-face F, where u_G is unit outer normal of conv G with respect to conv F in aff F.

Theorem 12. *Suppose that vectors $\overline{m}_G$ are given for each $(k-2)$-face G. Then the following two conditions are equivalent*:

1. *The $\overline{m}_G$ constitute an infinitesimal k-motion.*
2. *For each $(k-1)$-face F a vector $\overline{m}_F \in \mathbf{R}^d$ parallel to F exists such that $\overline{m}_F \cdot u_G = \overline{m}_G \cdot u_G$ for all $G \subset F$.*

Proof. One direction is easy. Suppose that (2) holds. Then, for each $(k-1)$-face F,

$$\begin{aligned}\sum_{G\subset F} \mathrm{vol}_{k-2}(G)(u_G\cdot \overline{m}_G) &= \sum_{G\subset F} \mathrm{vol}_{k-2}(G)(u_G\cdot \overline{m}_F)\\ &= \left(\sum_{G\subset F} \mathrm{vol}_{k-2}(G)u_G\right)\cdot \overline{m}_F\\ &= O\cdot \overline{m}_F\\ &= 0\end{aligned}$$

by Minkowski's theorem. So (1) holds.

On the other hand, suppose that (1) holds. Fix F and regard the u_G and the $\overline{m}_G$ as sitting naturally in $\mathbf{R}^{k-1}$ and constructthe $(k-1)\times k$ matrix U whose columns are the u_G. The rows of U are linearly independent, so the left nullspace of U has dimension one. Now the vector y whose entries are the $\mathrm{vol}_{k-2}(G)$ is in the nullspace of U, and the vector z whose entries are the $u_G\cdot\overline{m}_G$ is orthogonal to y. Therefore z is in the rowspace of U, and in particular a single vector $\overline{m}_F$ exists such that $\overline{m}_F\cdot U = z$. □

This gives us equivalent formulations of condition (8). Suppose that vectors $\overline{m}_G$ are given for each $(k-2)$-face G. For a $(k-1)$-face F and $G\subset F$, let m_G be the projection of $\overline{m}_G$ onto the $(k-1)$-dimensional linear space V parallel to F. For a real number t let $F(t)$ be the $(k-1)$-simplex determined by translating aff G by the vector tm_G.

Corollary 3. *The following conditions are each equivalent to condition* (8):

1. $\sum_{G\subset F}\mathrm{vol}_{k-2}(G)(u_G\cdot m_G)=0$.
2. $F(t)$ *is congruent to* F.
3. $F(1)$ *is congruent to* F.
4. $(d/dt)\,\mathrm{vol}^2_{k-1}(F(t))=0$.

Proof. (1) is clear since $u_G\cdot\overline{m}_G = u_G\cdot m_G$, (2) and (3) hold since by the theorem we are equivalently translating each aff G by the same vector m_F. (4) then follows immediately. □

The last condition is a very natural generalization of the definition of classical infinitesimal motion (infinitesimal 2-motion) and was also observed by Filliman [4]. See [22] for a deeper study of the relationship between generalized stress and skeletal rigidity of cell complexes (not necessarily simplicial).

6. Bistellar Operations

In this section we examine how the various stress spaces change under the action of certain local changes in a simplicial $(d-1)$-complex Δ. Let F and G be disjoint nonempty subsets of $\{1,\ldots,n\}$ of cardinality p and q, respectively, such that $p+q=d+1$, $F\in\Delta$, $G\notin\Delta$, and $\mathrm{lk}\,F=\partial G=\{G'\colon G'$ is a proper subset of $G\}$, the boundary of G.

The simplicial complex $\Delta' = (\Delta \backslash F) \cup (G \cdot \partial F)$ is the result of performing a *bistellar operation* on Δ. During this operation we remove all faces containing F and introduce all sets of the form $G \cup F'$ where $F' \in \partial F$. The faces of Δ' that are new are those faces of Δ' that contain G, and the faces of Δ which are lost are the faces of Δ that contain F. The local change in the structure of Δ induces a corresponding simple "local" change in the linear stress spaces.

Theorem 13. *Assume Δ and Δ' are as above and that $v_1, \dots, v_n$ are in linearly general position with respect to both Δ and Δ'. Then*

$$\dim S_s^\ell(\Delta) = \begin{cases} \dim S_s^\ell(\Delta) + 1 & \textit{if} \quad p > q \quad \textit{and} \quad q \le s \le d - q, \\ \dim S_s^\ell(\Delta) - 1 & \textit{if} \quad p < q \quad \textit{and} \quad p \le s \le d - p, \\ \dim S_s^\ell(\Delta) & \textit{otherwise.} \end{cases}$$

Proof. The main idea is to use an intermediate simplicial complex Δ'' to mediate the changes in the stress spaces. Define $\Delta'' = \Delta \cup (G \cdot \partial F)$ and observe that Δ'' also equals $\Delta' \cup (F \cdot \partial G)$. We will show that

$$\dim S_s^\ell(\Delta'') = \begin{cases} \dim S_s^\ell(\Delta) & \text{if} \quad 0 \le s \le q - 1, \\ \dim S_s^\ell(\Delta) + 1 & \text{if} \quad q \le s \le d, \end{cases}$$

and by symmetry

$$\dim S_s^\ell(\Delta'') = \begin{cases} \dim S_s^\ell(\Delta') & \text{if} \quad 0 \le s \le p - 1, \\ \dim S_s^\ell(\Delta') + 1 & \text{if} \quad p \le s \le d. \end{cases}$$

Since Δ and Δ'' share the same faces of cardinality s when $0 \le s \le q - 1$, then $S_s^\ell(\Delta)$ must be the same as $S_s^\ell(\Delta'')$ for these values of s. Assume that $q \le s \le d$ and take $S = F \cup G$, a subset of cardinality $d + 1$. All of the proper faces of S are in Δ''. Define $c(x)$ to be the unique (up to scalar multiple) linear s-stress on the simplicial complex consisting of all subsets of S as constructed in Example 1. Each face of Δ is also a face of Δ'', so $S_s^\ell(\Delta) \subseteq S_s^\ell(\Delta'')$. Suppose that $b(x)$ is a linear s-stress that is in Δ'' but not in Δ. This implies that b_r is nonzero for some r such that $\operatorname{supp} x^r \in \operatorname{openstar} G$ (the set of all faces of Δ'' that contain G). We claim that the restriction of b to the faces of S must be a nonzero multiple of $c(x)$; i.e., that there is a nonzero real number t such that $b_r = tc_r$ for all x^r supported on $\operatorname{openstar} G$, and hence $b(x) - tc(x)$ is in $S_s^\ell(\Delta)$. Since G is the only face of cardinality q in $\operatorname{openstar} G$, this is clearly true if $s = q$. So assume $q + 1 \le s \le d$. Choose any r such that b_r is nonzero and $\operatorname{supp} x^r \in \operatorname{openstar} G$. Since $s > q$, there is a j such that $\operatorname{supp} x^{r-e_j} \in \operatorname{openstar} G$. Condition (3) implies that $\sum_{i=1}^n b_{r-e_j+e_i} v_i = O$. As this sum involves only the $d + 1$ vectors v_i such that $i \in S$, the coefficients in this sum must be a common multiple of the corresponding coefficients of $c(x)$. Using this procedure repeatedly to determine the other coefficients b_r verifies the claim. The resulting direct sum decomposition of $S_s^\ell(\Delta'')$ establishes the change in dimension. □

7. Simplicial Spheres, P.L.-Spheres, and Pseudomanifolds

7.1. *P.L.-Spheres*

Bistellar operations are ideally suited for proofs by induction on p.l.-manifolds, especially considering Pachner's result [17] that any two p.l.-manifolds that are p.l.-homeomorphic can be transformed into each other by a sequence of bistellar operations. In particular, every p.l.-sphere can be obtained from the boundary of a simplex by such operations, and the discussion in the previous section almost immediately proves:

Corollary 4. *If Δ is a simplicial p.l.-sphere, then Δ is Cohen–Macaulay.*

Proof. Assume that Δ is a $(d-1)$-dimensional simplicial p.l.-sphere with vertices $\{1, \ldots, n\}$. Choose $v_1, \ldots, v_n \in \mathbf{R}^d$ in linearly general position with respect to all subsets of $\{1, \ldots, n\}$ of cardinality d. The boundary of a $(d-1)$-simplex is Cohen–Macaulay by part (1) of Corollary 1 and Example 1, since its h-vector equals $(1, \ldots, 1)$. It is well known that the components of the h-vector of a simplicial complex change under the action of a bistellar operation in exactly the same way as the changes in the dimensions of the linear stress spaces described in Theorem 13. So since Δ can be obtained from the boundary of a simplex by a sequence of bistellar operations, we have $h_s(\Delta) = \dim S_s^\ell$ for all s. Therefore Δ is Cohen–Macaulay by Corollary 1. □

7.2. *Pseudomanifolds*

We now turn to a larger class of simplicial complexes which includes simplicial manifolds. A simplicial $(d-1)$-complex is said to be a *pseudomanifold* if:

(i) Every maximal face has dimension $d-1$.
(ii) Every $(d-2)$-face is contained in exactly two faces of dimension $d-1$.
(iii) Any two $(d-1)$-faces can be connected by a path of $(d-1)$-faces, each two succeeding faces of which are adjacent (share a common $(d-2)$-face).

Theorem 14. *If Δ is an orientable $(d-1)$-pseudomanifold on n vertices and $v_1, \ldots, v_n$ are in a linearly general position with respect to Δ, then* $\dim S_d^\ell(\Delta) = 1$.

Actually, Tay *et al.* [23] prove the stronger result that the dimension of $S_d^\ell(\Delta)$ equals the dimension of the homology $H_d(\Delta, \mathbf{R})$, and the proof of the above theorem hints why this is true.

Proof. Let $b(x)$ be a linear d-stress on Δ. By Theorem 9 it suffices to study the square-free coefficients of $b(x)$. Choose a consistent orientation of all the facets ($(d-1)$-faces) of Δ and use this to induce an ordering of the elements of each facet. Let G be a subfacet ($(d-2)$-face) of Δ, and let F_1 and F_2 be the two facets containing G. Theorem 10 implies that $[F_1]b_{[F_1]} = [F_2]b_{[F_2]}$, so a constant t exists such that $b_F = t[F]^{-1}$ for every facet F. The coefficients of the non-square-free terms are then uniquely determined. So, up to scalar multiple, there is only one element in $S_d^\ell(\Delta)$. □

In Section 9 the geometrical significance of this canonical linear d-stress (the one for which $b_F = [F]^{-1}$ for each facet), in the case that Δ is the boundary complex of a simplicial convex polytope, will become apparent.

Suppose that Δ is a simplicial $(d-1)$-complex on $\{1, \ldots, n\}$ and $v_1, \ldots, v_n \in \mathbf{R}^d$. For $G = \{i_1, \ldots, i_s\}$, a subset of $\{1, \ldots, n\}$, define the function τ_G on the space of linear stresses by

$$\tau_G(b) = \frac{\partial^s b}{\partial x_{i_1} \cdots \partial x_{i_s}}.$$

In particular, write

$$\tau_i(b) = \frac{\partial b}{\partial x_i}.$$

Theorem 15. *Let Δ be a simplicial orientable $(d-1)$-pseudomanifold on $\{1, \ldots, n\}$ and let $v_1, \ldots, v_n \in \mathbf{R}^d$ be in linearly general position with respect to Δ. Suppose that $b(x)$ is the canonical linear d-stress of Δ and G is a face of Δ of cardinality s. Then $\tau_G(b)$ is a linear $(d-s)$-stress supported on* clstar G. *In the special case that* lk G *is a $(d-s-1)$-sphere, then up to scalar multiple $\tau_G(b)$ is the unique nonzero linear $(d-s)$-stress supported on* clstar G.

Proof. The first part of the theorem is obvious: if the coefficient of x^r is nonzero in $\tau_G(b)$, then the coefficient of $x_{i_1} \cdots x_{i_s} x^r$ must be nonzero in $b(x)$. Hence $G \cup (\operatorname{supp} x^r)$ is a face of Δ and so $\operatorname{supp} x^r \in \operatorname{clstar} G$.

For the second part, note that clstar G can be obtained by starting with lk G and successively joining it to the vertices of G. Since lk G is a $(d-s-1)$-sphere, it has a unique linear $(d-s)$-stress (up to scalar multiple). By repeated application of Theorem 7, so does clstar G. Now it is easy to see that $\tau_G(b)$ is nonzero since $b(x)$ is, and so $\tau_G(b)$ must be a generator of the linear $(d-s)$-stresses on clstar G. □

8. Shellings

Consider a simplicial $(d-1)$-complex on $\{1, \ldots, n\}$ such that every maximal face has dimension $d-1$ (is a facet). Then the complex is said to be *shellable* if the facets can be ordered $F_1, \ldots, F_m$ in such a way that, for $k = 1, \ldots, m$, there is a unique minimal face G_k that is in F_k but is not in $\bigcup_{i=1}^{k-1} \overline{F}_i$. Here, $\overline{F}_i$ denotes the simplicial complex consisting of all subsets of F_i. It is well known that, as each facet F_k is added, precisely one component h_s of the h-vector increases by one, the remaining components being unchanged; specifically, $s = \operatorname{card} G_k$. Using this and understanding the changes in the face ring during the shelling, Kind and Kleinschmidt [9] give an inductive proof that shellable simplicial complexes are Cohen–Macaulay.

It is also possible to use generalized stress to prove this result by showing that the dimension of S_s^ℓ increases by one when F_k is added, while the dimensions of the other linear stress spaces do not change. However, we content ourselves with considering the special case when Δ is a simplicial $(d-1)$-sphere. Assume that $v_1, \ldots, v_n \in \mathbf{R}^d$ are in linearly general position with respect to Δ, and let $b(x)$ be the canonical linear d-stress on Δ. When F_k is added, the closed star of $G'_k = F_k \backslash G_k$ is completed. If $\operatorname{card} G_k = s$, then $\operatorname{card} G'_k = d-s$, and Theorem 15 implies that up to scalar multiple there is a unique linear s-stress $\tau_{G'_k}(b)$ supported on clstar G'_k. The coefficient of this stress associated with the face G_k is nonzero, so this stress was not present before F_k was added. So we can use the shelling of Δ to derive a basis for the stress spaces.

Theorem 16. *If Δ is a shellable simplicial $(d-1)$-sphere whose n vertices, $v_1, \ldots, v_n \in \mathbf{R}^d$, are in linearly general position with respect to Δ, and $F_1, \ldots, F_m, G'_1, \ldots, G'_m$ and $b(x)$ are as above, then $\{\tau_{G'_k}(b)\colon \operatorname{card} G'_k = d-s\}$ is a basis for S^ℓ_s. Hence the collection $\{\tau_G(b)\colon$ G is a face of Δ of cardinality $d-s\}$ spans S^ℓ_s.*

9. Simplicial Convex Polytopes

In this section we specialize further and consider the case when Δ is the boundary complex of some simplicial convex d-polytope $P \subset \mathbf{R}^d$. This was the motivating case for defining generalized stress in the first place and trying to understand the g-theorem.

9.1. *Canonical Stress and Volume*

Assume that P contains the origin in its interior. Then the vertices $v_1, \ldots, v_n$ of P are in linearly general position with respect to Δ. Since Δ is shellable, we know $\dim S^\ell_i = h_i$, $i = 0, \ldots, d$.

The definition of affine stress seems more geometrically natural for simplicial complexes since affine stress is invariant under translation. The linear stress spaces, while also geometrically definable, depend upon the choice of origin and change with translation. It turns out, however, that this situation changes entirely when we turn to the polar P^* of P and describe the linear stresses in terms of conditions on P^*. This will become clearer in Section 10, but already in this section we begin to see the significance of using the polar to understand stress.

For $x \in \mathbf{R}^n$, consider the polytope $Q(x) = \{y \in \mathbf{R}^d\colon\ y^T v_i \le x_i,\ i = 1, \ldots, n\}$. Of course, $Q(e)$ is the polar P^* of P. Since P^* is simple, for values of x_i near 1, $Q(x)$ and P^* are strongly isomorphic. It is well known that the volume of $Q(x)$ as a function of the x_i is a homogeneous polynomial $V(x) = \sum_{r\colon |r|=d} b_r(x^r/r!)$ of degree d, $b_r = 0$ whenever $\operatorname{supp} x^r \notin P$, and $b_F = [F]^{-1}$ for every facet F of P. The canonical linear d-stress $b(x)$ on Δ also shares these properties, and so perhaps the following result is not completely unexpected:

Theorem 17. *Let P be as above. Then the canonical linear d-stress is precisely $V(x)$.*

Proof. For every $u \in \mathbf{R}^d$, $Q(x_1, \ldots, x_n) + u = Q(x_1 + u^T v_1, \ldots, x_n + u^T v_n)$ (we are just translating $Q(x)$ by the vector u). So $V(x_1,\ldots,x_n)-V(x_1+u^Tv_1,\ldots,x_n+u^Tv_n)=0$. Fix r such that $|r|=d-1$. Then

$$\begin{aligned} O &= \frac{\partial^{d-1}}{(\partial x_1)^{r_1}\cdots(\partial x_n)^{r_n}}[V(x_1,\ldots,x_n)-V(x_1+u^Tv_1,\ldots,x_n+u^Tv_n)] \\ &= \sum_{i=1}^{d} b_{r+e_i}x_i-\sum_{i=1}^{n} b_{r+e_i}(x_i+u^Tv_i) \\ &= \sum_{i=1}^{n} b_{r+e_i}u^Tv_i \\ &= u^T\left(\sum_{i=1}^{n} b_{r+e_i}v_i\right). \end{aligned}$$

However, this is true for every u, so $\sum_{i=1}^{n} b_{r+e_i}v_i=O$ and $V(x)$ is a linear d-stress. That $V(x)$ is the same as the canonical linear d-stress follows from the fact that the coefficients of the square-free terms of $V(x)$ agree with those of the canonical linear d-stress. □

9.2. *Lower-Dimensional Canonical Stresses*

The above proof mimics the proof of Minkowski's theorem that

$$\sum_{i=1}^{n}\mathrm{vol}_{d-1}(F_i)\frac{v_i}{\|v_i\|}=O,$$

which we already used in Section 5. ($F_1,\ldots,F_n$ are the facets of P^* corresponding to the vertices $v_1,\ldots,v_n$ of P.) In fact, the relationship between Minkowski's theorem and stress is quite strong, as we will see.

One way to prove the g-theorem would be to show that the application of ω^{d-2i} induces a bijection between S_{d-i}^{ℓ} and S_i^{ℓ}, $i=0,\ldots,\lfloor d/2\rfloor$. Actually, it would suffice to show that ω: $S_i^{\ell}\to S_{i-1}^{\ell}$ is surjective for $i=1,\ldots,\lfloor d/2\rfloor$. McMullen's new proof of the g-theorem shows that the bijections proposed here are valid.

Given the canonical linear d-stress $V(x)$, we might consider applying ω repeatedly to get canonical linear i-stresses $\omega^{d-i}(V(x))$, $i=0,\ldots,d-1$. Let $W(x)=V(x_1+1,\ldots,x_n+1)$. Then, for small x, $W(x)$ is the volume of a polytope near P^*. Write $W(x)=\sum_{i=1}^{d}W_i(x)$, where $W_i(x)$ is a homogeneous polynomial of degree i, $i=0,\ldots,d$. It is clear that the constant $W_0(x)$ is the volume of P^* and $W_d(x)=V(x)$. It is also easy to see that $W_1(x)=\sum_{i=1}^{n}(\mathrm{vol}_{d-1}(F_i)/\|v_i\|)x_i$.

Theorem 18. *Let P be as above. Then $\omega^{d-i}(V(x))=(d-i)!\,W_i(x)$, $i=0,\ldots,d$.*

Proof. We calculate the contribution of $b_r(x^r/r!)$ in $V(x)$ to the coefficient of x^s in $W_i(x)$, where $x^s|x^r$. Expanding

$$b_r\frac{(x_1+1)^{r_1}\cdots(x_n+1)^{r_n}}{r_1!\cdots r_n!}$$

we see that the contribution is

$$b_r \frac{\binom{r_1}{s_1} \cdots \binom{r_n}{s_n}}{r_1! \cdots r_n!} = \frac{b_r}{s_1!(r_1 - s_1)! \cdots s_n!(r_n - s_n)!}.$$

On the other hand, the contribution of

$$b_r \frac{x^r}{r!} = b_r \frac{x_1^{r_1} \cdots x_n^{r_n}}{r_1! \cdots r_n!}$$

in $V(x)$ to the coefficient of x^s in $\omega^{d-i}(V(x))$, where $i = d - |s|$, is

$$b_r \frac{\binom{d-i}{r_1 - s_1, \ldots, r_n - s_n}}{r_1! \cdots r_n!} = \frac{(d-i)!\, b_r}{s_1!\,(r_1 - s_1)! \cdots s_n!\,(r_n - s_n)!}.$$ □

Corollary 5. *Let P be as above.*

1. *The canonical linear* 0*-stress* $\omega^d(V(x))$ *equals* $d!\,\mathrm{vol}(P^*)$.
2. *The canonical linear* 1*-stress* $\omega^{d-1}(V(x))$ *equals*

$$(d-1)! \sum_{i=1}^{n} \frac{\mathrm{vol}_{d-1}(F_i)}{\|v_i\|} x_i.$$

That is, the canonical linear combination of the v_i induced by ω is (up to scalar multiple) the same as that induced by Minkowski's theorem.

We can find the coefficient of the square-free term of W_i corresponding to an $(i-1)$-face F of P by looking at the corresponding $(d-i)$-face F^* of P^* and computing the contribution to the local change in the volume of P^* due to the translations of the facets containing F^*. This change depends upon the $(d-i)$-volume of F^* and the size of the cone of the associated normal vectors, rescaled to account for the fact that they may not be of unit length.

Theorem 19. *The coefficient of the square-free term of W_i corresponding to F is*

$$\frac{\mathrm{vol}_{d-i}(F^*)}{\mathrm{vol}_i(\mathrm{conv}(\{O\} \cup \{v_i \colon i \in F\}))}.$$

See also [4].

So each W_i is associated in a very natural way with the $(d-i)$-volumes of the $(d-i)$-faces of P^*.

Notice that we can write

$$\begin{aligned} W(x) &= \sum_{i=0}^{d} W_i(x) \\ &= \sum_{i=0}^{d} \frac{\omega^{d-i}(V(x))}{(d-i)!} \\ &= \sum_{i=0}^{d} \frac{\omega^i(V(x))}{i!}. \end{aligned}$$

9.3. *Easy Bijections*

Some of the bijections associated with ω can now be confirmed.

Theorem 20. *Let P be as above. Then ω^d: $S_d^\ell \to S_0^\ell$ is a bijection. Further, if $d \geq 3$, then ω^{d-2}: $S_{d-1}^\ell \to S_1^\ell$ is a bijection.*

Proof. The first statement is trivially true by part (1) of the previous corollary simply because P^* has positive (and hence nonzero) volume. From the Dehn–Sommerville relations we know that $\dim S_{d-1}^\ell = h_{d-1} = h_1 = \dim S_1^\ell$. So it suffices to show that ω^{d-2}: $S_{d-1}^\ell \to S_1^\ell$ is a surjection. From Theorem 16 we know that $\{\tau_1(V(x)), \dots, \tau_n(V(x))\}$ spans S_{d-1}^ℓ. We need to show $\{\omega^{d-2}\tau_1(V(x)), \dots, \omega^{d-2}\tau_n(V(x))\}$ spans S_1^ℓ. Since $\dim S_1^\ell = h_1 = n-d$, it is sufficient to demonstrate that the given subset of S_1^ℓ has rank $n-d$. However, since ω and τ_i commute, this subset equals $(d-2)!\,\{\tau_1(W_2(x)), \dots, \tau_n(W_2(x))\}$. It is straightforward to check that $\sum_{i=1}^n x_i \tau_i(W_2(x)) = 2W_2(x)$. It is known from the Brunn–Minkowski theory that the quadratic form $W_2(x)$ has d zero eigenvalues (associated with the space of translations $x = (u^T v_1, \dots, u^T v_n)$), one positive eigenvalue, and $n-d-1$ negative eigenvalues; see, for example, [5]. So the rank of the quadratic form is $n-d$, as is required. Note that this is the case $r = 1$ of the Hodge–Riemmann–Minkowski inequalities developed by McMullen [13]. $\square$

This theorem implies that ω: $S_2^\ell \to S_1^\ell$ is a surjection when $d \geq 3$. Therefore $\dim S_2^\ell = h_2 - h_1 = g_2 \geq 0$, and we have bound together the main ideas of Stanley's and Kalai's different proofs of the Lower Bound Theorem.

9.4. *Simplicial 3-Polytopes*

What we have done so far essentially gives a complete description of the situation for simplicial 3-polytopes. If P above is three-dimensional, then:

1. The canonical linear 0-stress is $3!$ times the volume of P^*.
2. The canonical linear 1-stress is equivalent to the linear relation induced by Minkowski's theorem.
3. The canonical linear 2-stress is the classical Maxwell stress (shown by Filliman [5]).
4. $S_0^\ell = \mathbf{R}$.
5. S_1^ℓ has dimension $n-3$ and is isomorphic to the space of all linear relations on the v_i.
6. S_2^ℓ has dimension $n-3$, is spanned by the $\tau_i V(x)$, and is isomorphic to S_1^ℓ under the bijection induced by multiplication by ω.
7. S_3^ℓ is spanned by $V(X)$.
8. $S_0^{\mathrm{a}} = \mathbf{R}$.

9. S_1^{a} has dimension $n-4$ and is isomorphic to the space of all affine relations on the v_i.
10. S_2^{a} and S_3^{a} are trivial.
11. That ω^3: $S_3^\ell \to S_0^\ell$ is a bijection is equivalent to P^* having nonzero volume.
12. That ω: $S_2^\ell \to S_1^\ell$ is a bijection is equivalent to infinitesimal rigidity of the edge-skeleton of P and and is a consequence of the Brunn–Minkowski theory.

So already in dimension three there is a striking confluence ofgeometric and algebraic results.

10. Relationship to Weights

10.1. *Ring of Differential Operators*

Let Δ be the boundary complex of a simplicial convex d-polytope P containing the origin in its interior, and let $v_1, \ldots, v_n$ be the vertices of P. Consider the ring $\mathbf{R}[\partial/\partial x_1, \ldots, \partial/\partial x_n]$ of all differential operators with constant coefficients in the variables $x_1, \ldots, x_n$. Define the polynomial $V(x)$ as before and factor out the ideal of operators that annihilate the polynomial $V(x)$. Khovanskii (personal communication) observes that the resulting ring D is isomorphic to the cohomology ring of the projective toric variety associated with P. This implies the following result, which can be proved directly.

Theorem 21. *Let Δ be as above. Then D is isomorphic to $B = A/(\theta_1, \ldots, \theta_d)$ where the coefficients of the θ_i are related to the coefficients of the vertices of P as in Section* 3.2.

Proof. Clearly, $\tau_S(V(x))$ equals zero for any subset $S \notin \Delta$. However, the invariance of the polynomial $V(x)$ under translation (see the proof of Theorem 17) implies that

$$\sum_{i=1}^{n} v_{ij} \frac{\partial V(X)}{\partial x_i} = 0$$

for each $j = 1, \ldots, d$. Finally, Theorem 16 implies that the image of $V(x)$ under the homogeneous differential operators of degree k spans S_{d-k}^ℓ, hence has dimension $h_{d-k} = h_k$. Using Theorem 2, this suffices to prove that D is isomorphic to B under the map $\partial/\partial x_i \to x_i$. □

This viewpoint allows us to define a multiplication on stresses. Let $a(x)$ and $b(x)$ be linear stresses. Find operators a' and b' such that $a'(V(x)) = a(x)$ and $b'(V(x)) = b(x)$. Define $a(x) \cdot b(x)$ to be the linear stress $(a'b')(V(x))$. The multiplication is well defined, for if $a''(V(x)) = a(x)$ and $b''(V(x)) = b(x)$, then

$$\begin{aligned}(a''b'' - a'b')(V(x)) &= (a''b'' - a'b'' + a'b'' - a'b')(V(x)) \\ &= b''(a'' - a')(V(x)) + a'(b'' - b')(V(x)) \\ &= 0.\end{aligned}$$

It is also clear that the product of a $(d-i)$-stress and a $(d-j)$-stress is a $(d-i-j)$-stress. Writing $\hat{S}_i^\ell = S_{d-i}^\ell$, we then regard the space of all linear stresses as a graded algebra $\hat{S}_0^\ell \oplus \cdots \oplus \hat{S}_d^\ell$. This algebra is isomorphic to D, hence also to B.

It sometimes helps to take a slightly schizophrenic viewpoint, on the one hand thinking of a linear stress $a(x)$ as a polynomial, and on the other identifying it with the operator a' for which $a'(V(x)) = a(x)$.

10.2. *Weights*

McMullen reproved the g-theorem using the notion of weights on polytopes. An *i-weight* on a convex polytope P is a real-valued function a on the i-faces of P which satisfies the Minkowski relation

$$\sum_{F \subset G} a(F) u_{F,G} = O$$

for each $(i+1)$-face G of P. Here the sum is taken over all i-faces F contained in G, and $u_{F,G}$ is the unit outer normal vector of F with respect to G within aff G. Clearly, one natural i-weight is given by $a(F) = \mathrm{vol}_i(F)$ for each i-face F. We call this the *canonical i-weight*. The real vector space of i-weights on P is denoted $\Omega_i(P)$, and we denote $\bigoplus_{i=0}^d \Omega_i(P)$ by $\Omega(P)$. McMullen [14] defines a multiplication on $\Omega(P)$ that endows $\Omega(P)$ with a graded algebra structure.

Theorem 22. *Let Δ be the boundary complex of a simplicial convex d-polytope $P \subset \mathbf{R}^d$ containing the origin in its interior, and take $v_1, \ldots, v_n$ to be the vertices of P. Then $S_i^\ell(\Delta)$ is isomorphic to $\Omega_{d-i}(P^*)$ as vector spaces.*

Proof. Suppose that $b(x)$ is a linear k-stress. Then, by condition (6), for every face G of cardinality $k-1$ we have the condition

$$\sum_{i \in \mathrm{lk}\, G} b_{G+i} w_i = O.$$

For a particular $i \in \mathrm{lk}\, G$ and $F = G+i$, w_i is in the same direction as the corresponding u_{G^*,F^*}. So, writing $\langle G \rangle$ for $\mathrm{vol}_{d-k}(\mathrm{conv}(\{O\} \cup \{v_i \colon i \in G\}))$ and similarly for $\langle F \rangle$, we have

$$\begin{aligned} \sum_{i \in \mathrm{lk}\, G} b_{G+i} w_i &= O, \\ \sum_{F=G+i \supset G} b_F u_{G^*,F^*} \|w_i\| &= O, \\ \sum_{F \supset G} b_F u_{G^*,F^*} \frac{\langle F \rangle}{\langle G \rangle} &= O, \\ \sum_{F^* \subset G^*} b_F u_{G^*,F^*} \langle F \rangle &= O. \end{aligned}$$

Hence taking $a(F^*) = b_F \langle F \rangle$ for each $(d-k)$-face F^* yields a $(d-k)$-weight on P^*. □

Although we do not give the details here, it can be shown that the above map is an algebra isomorphism from $\hat{S}_0^\ell \oplus \cdots \oplus \hat{S}_d^\ell$ to $\Omega(P^*)$, and that the ring D of differential operators is isomorphic to the polytope subalgebra $\Pi(P^*)$ defined by McMullen [13].

10.3. *The Logarithm of a Polytope*

In the next few sections we sketch some relationships between weights and stresses.

If we rescale the vectors v_i so that they become unit vectors, then the matrix

$$\overline{M} = \begin{bmatrix} v_1 & \cdots & v_n \\ 1 & \cdots & 1 \end{bmatrix}$$

becomes

$$\begin{bmatrix} u_1 & \cdots & u_n \\ \pi_1 & \cdots & \pi_n \end{bmatrix},$$

where $\pi_i = \|v_i\|^{-1}$, $i = 1, \ldots, n$. We can define linear stress spaces $\overline{S}_i^\ell$ with respect to $u_1, \ldots, u_n$. The role of $\omega = \sum_{i=1}^n (\partial/\partial x_i)$ is replaced by that of $\overline{\omega} = \sum_{i=1}^n \pi_i(\partial/\partial x_i)$ in the results we have presented so far. McMullen proves that multiplication by $\overline{\omega}^{d-2i}$ is a bijection from Ω_i to Ω_{d-i}, $i = 0, \ldots, \lfloor d/2 \rfloor$. Note that P^* is now given by $\{y \in \mathbf{R}^d\colon\ y^T \cdot u_i^T \leq \pi_i\}$. So we replace $Q(x)$ by $\overline{Q}(x) = \{y \in \mathbf{R}^d\colon\ y^T u_i \leq x_i\}$ and define $\overline{V}(x) = \operatorname{vol}\ Q(x)$. In a similar way, we define $\overline{W}_i(x)$, $i = 0, \ldots, d$. We still have

$$\overline{W}(x) = \sum_{i=0}^d \frac{\overline{\omega}^i(\overline{V}(x))}{i!}.$$

Let $p = \overline{\omega}$, and $p_i = \overline{W}_i$, which corresponds to the canonical i-weight on P^*. Then $p_i = p^i/i!$ (keeping in mind our willingness to confuse at our convenience a differential operator with the image of $\overline{V}(x)$ under the action of the operator). Formally writing $[P^*] = \sum_{i=1}^d p_i$, we have

$$[P^*] = \sum_{i=0}^d \frac{p^i}{i!}.$$

This corresponds to the result of McMullen [13], [14] that $[P^*] = \exp p$ and $p = \log[P^*]$.

10.4. *Restrictions*

The dual interpretation of Theorem 15 is interesting. Let G be a face of P of cardinality s, and let $a(x)$ be the unique (up to scalar multiple) $(d-s)$-stress supported on clstar G. Then $a(x)$ is dual to the necessarily unique s-weight a^* supported on the set of all s-faces that meet G^*. We call a^* the weight *associated with* G^*. In particular, the weight associated with a facet F^* of P^* is a 1-weight that is supported on the edges of P^* that meet F^*, and so this must be the same weight described by McMullen in [14].

What is dual to the notion of the restriction of an i-weight a^* on P^* to a facet F^*? Let $a(x)$ be the dual $(d-i)$-stress on P and let v_k be the vertex of P corresponding to F^*. We could simply truncate $a(x)$, eliminating the terms not supported on openstar v_k. This would not necessarily be a stress on openstar v_k, but it would directly correspond to the restriction of a^* to F^*. On the other hand, we can apply $\partial/\partial x_k$ to $a(x)$, which depends only on the terms of $a(x)$ that are supported on openstar v_k. This gives a $(d-i-1)$-stress on clstar v_k. Projecting clstar v_k onto a hyperplane orthogonal to v_k, deleting v_k, and applying Theorem 7 yields a simplicial $(d-1)$-polytope F dual to F^* and a $(d-i-1)$-stress dual to the restriction of a^* to F^*.

We now have another way of interpreting McMullen's alternative formula [13], [14] for multiplying by a 1-weight a^*. Remembering that we can view a linear i-stress on P as either a polynomial of degree i or an operator of degree $(d-i)$ as it suits us, we choose to let a^* correspond to an operator $a = \sum_{j=1}^{n} \eta_j(\partial/\partial x_j)$ of degree 1, and to let any other given weight y^* correspond to a polynomial y. Then

$$ya = \sum_{i=1}^{n} \eta_j \frac{\partial y}{\partial x_i},$$

which can be regarded as a dual recasting of McMullen's formula

$$y^* a^* = \sum_{i=1}^{n} \eta_i a^*(F_i^*),$$

where $a^*(F^*)$ denotes the restriction of a^* to the facet F^*.

10.5. *Shellings and Flips*

McMullen [13] uses a shelling argument directly on the simple polytope P^* to find a basis for Ω and to prove that $\dim \Omega_i = h_i, i = 0, \ldots, d$. A general hyperplane is moved "upward" through P^*. When a vertex of type i is encountered (i.e., a vertex with exactly i edges "below" the hyperplane), an arbitrary i-weight can be assigned to the i-face F determined by these edges. The Minkowski relations defining the weights can be used to find the unique i-weights that must be assigned to the i-faces that do not have a vertex of type i at the top.

It happens that this basis is not dual to the closed star stress basis constructed in Section 8. For the dual basis, again a general hyperplane is moved upward through P^*. When a vertex of type i is encountered, we take F^* to be the accompanying i-face and add into the basis for Ω_{d-i} the $(d-i)$-weight associated with F^* as in the previous section. In some sense the elements of this basis are more local than those of McMullen's basis.

We conclude with some comments on McMullen's flips [13]. As he points out, flips are dual to bistellar operations. Even though we are considering bistellar operations in a more general context, it is straightforward to verify that our Theorem 13 is dual to McMullen's Theorem 11.3 in [13], and that the justifications of these two theorems are essentially the same in a combinatorial sense.

11. Generalized Circulations

In Section 5 we considered a matrix whose left nullspace defined affine k-stress. The right nullspace then turned out to be an appropriate generalization of infinitesimal motions. We can try the same procedure with k-weights on a simple d-polytope P. For each $(k+1)$-face G, consider a rigid motion φ_G that maps aff G onto $(\mathbf{R}^{k+1}, 0, \ldots, 0) \subset \mathbf{R}^d$ and then projects this space naturally onto $\mathbf{R}^{k+1}$. Construct a matrix R with one row for each k-face F of P and columns occurring in groups of $k+1$, one group for each $(k+1)$-face G. The row vector of length $k+1$ in row F, group G, is

$$\begin{cases} O^T & \text{if } F \not\subset G, \\ \varphi_G(u_{F,G}^T) & \text{if } F \subset G, \end{cases}$$

where $u_{F,G}$ is the unit outer normal vector of F with respect to G in aff G.

Define $m = \varphi^{-1}(m')$ (where φ^{-1} is interpreted in the obvious way) to be a $(k+1)$-*circulation* when m' is a member of the right nullspace of R. So m is an assignment to each $(k+1)$-face G of a vector parallel to G that satisfies the conditions

$$\sum_{G \supset F} m_G \cdot u_{F,G} = 0$$

for every k-face F. Denote the space of $(k+1)$-circulations by C_{k+1}.

In the case that $k = 0$, we have a vector, or flow, associated with each edge of P and a condition on each vertex of P that forces flow conservation. For higher values of k we can interpret the $(k+1)$-circulation as a translation of the $(k+1)$-dimensional content of the $(k+1)$-dimensional faces in directions parallel to these faces with flow conservation across every bounding k-face.

Theorem 23. *Let P be a simple d-polytope. Then* $\dim C_{k+1} = h_k - f_k + (k+1) f_{k+1}$.

Proof. This is an immediate consequence of the fact that R is an $f_k \times (k+1) f_{k+1}$ matrix with a left nullspace of dimension h_k. □

It is clear that $\dim C_1 = h_0 - f_0 + f_1 = f_1 - f_0 + 1$, which is the dimension of the space of ordinary circulations on a graph with f_0 vertices and f_1 edges. In general, in terms of the f-vector of the simple d-polytope P (the reverse of the f-vector of the dual simplicial polytope)

$$\begin{aligned} h_k &= \sum_{j=k}^{d} (-1)^{j-k} \binom{j}{k} f_j \\ &= f_k - (k+1) f_{k+1} + \sum_{j=k+2}^{d} (-1)^{j-k} \binom{j}{k} f_j. \end{aligned}$$

So

$$\dim C_{k+1} = \sum_{j=k+2}^{d} (-1)^{j-k} \binom{j}{k} f_j.$$

12. Unbounded Simple Polyhedra

Consider the boundary complex Δ of a simplicial convex d-polytope P with vertex set $v_1, \ldots, v_n$. Consider the ring $B = B_0 \oplus \cdots \oplus B_d = A/(\theta_1, \ldots, \theta_n)$, where the θ_i are constructed from the v_j as in Section 3.2. As we have already mentioned, the g-theorem implies that $h_i = h_{d-i}$, $i = 0, \ldots, \lfloor d/2 \rfloor$ and also that $g_i \geq g_{i-1}$, $i = 1, \ldots, \lfloor d/2 \rfloor$, and this is proved by showing that multiplication by ω^{d-2i} is a bijection between B_i and B_{d-i}, $i = 0, \ldots, \lfloor d/2 \rfloor$.

Now let v be any vertex of P and consider the simplicial complex $\Sigma = \Delta \backslash v$. In [3] it is proved that $h_i(\Sigma) \geq h_{d-i}(\Sigma)$, $i = 0, \ldots, \lfloor d/2 \rfloor$, and also that $h_i(\Sigma) \geq h_{i+1}(\Sigma)$, $i = \lfloor d/2 \rfloor, \ldots, d$. This is a consequence of the g-theorem, but now we can view this as a consequence of a weakened Lefschetz-type theorem on the face ring of Σ. Let $A' = \mathbf{R}[x_1, \ldots, x_n]/I_\Sigma$ and $B' = B'_0 \oplus \cdots \oplus B'_d = A'/(\theta_1, \ldots, \theta_d)$. Take $\omega = x_1 + \cdots + x_n$ as before.

Theorem 24. *Multiplication by ω^{d-2i} is a surjection from B_i to B_{d-i}, $i = 0, \ldots, \lfloor d/2 \rfloor$.*

It is more convenient to prove this with weights instead of stress. Let P^* be the simple d-polytope dual to P and let F^* be the facet of P^* corresponding to v. It can be arranged (for example, by choosing the origin suitably close to v) that discarding the inequality defining the facet F^* results in an unbounded simple polyhedron Q^* which is dual to the simplicial complex Σ. We can define weights on Q^* in the natural way, even though Q^* is unbounded. So $\dim \Omega_i(Q^*) = h_{d-i}(\Sigma)$, $i = 0, \ldots, d$. What we actually prove is:

Theorem 25. *Multiplication by p^{d-2i} is an injection from $\Omega_i(Q^*)$ to $\Omega_{d-i}(Q^*)$, $i = 0, \ldots, \lfloor d/2 \rfloor$.*

Proof. Use McMullen's construction to consider a basis of $\Omega(P^*)$ determined by a hyperplane. Choose this hyperplane so that it first moves past the vertices in F^* before it encounters the remaining vertices of P^*; i.e., arrange for F^* to be at the "bottom" of P^*.

Consider a vertex v_j of type i in P and the associated basis element a of $\Omega_i(P^*)$. In the case that v_j is also in F^*, v_j is also a vertex of type i in F^* and the restriction of a to F^* is an element of the basis of $\Omega(F^*)$. Reversing this restriction gives an injection of $\Omega_i(F^*)$ into $\Omega_i(P^*)$. In the case that v_j is not in F^*, the restriction of a to F^* is zero. Notice that the weights of Q^* correspond naturally to the weights of P^* that are zero on F^*. Hence we have an injection of $\Omega_i(Q^*)$ into $\Omega_i(P^*)$. Identifying $\Omega_i(F^*)$ and $\Omega_i(Q^*)$ with their images in $\Omega_i(P^*)$ yields the direct sum decomposition $\Omega_i(P^*) = \Omega_i(F^*) \oplus \Omega_i(Q^*)$.

Looking at the description of the multiplication of weights in McMullen [13], it is seen that multiplying p by a weight that is zero on F^* results in a weight that is also zero on F^*. Therefore, since multiplication by p^{d-2i} is a bijection from $\Omega_i(P) = \Omega_i(F^*) \oplus \Omega_i(Q^*)$ to $\Omega_{d-i}(P) = \Omega_{d-i}(F^*) \oplus \Omega_{d-i}(Q^*)$, it must be an injection from $\Omega_i(Q^*)$ to $\Omega_{d-i}(Q^*)$. $\square$

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References

1. D. W. Barnette, The minimum number of vertices of a simple polytope, *Israel J. Math.*, **10** (1971), 121–125.
2. D. W. Barnette, A proof of the lower bound conjecture for convex polytopes, *Pacific J. Math.*, **46** (1973), 349–354.
3. L. J. Billera and C. W. Lee, The numbers of faces of polytope pairs and unbounded polyhedra, *European J. Combin.*, **2** (1981), 307–322.
4. P. Filliman, Face numbers of pl-spheres, Manuscript, 1991.
5. P. Filliman, Rigidity and the Alexandrov–Fenchel inequality, *Monatsh. Math.*, **113** (1992), 1–22.
6. W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies, Vol. 131, Princeton University Press, Princeton, NJ, 1993.
7. G. Kalai, Characterization of f-vectors of families of convex sets in $\mathbf{R}^d$. Part I: Necessity of Eckhoff's conditions, *Israel J. Math.*, **48** (1984), 175–195.
8. G. Kalai, Rigidity and the lower bound theorem. I, *Invent. Math.*, **88** (1987), 125–151.
9. B. Kind and P. Kleinschmidt, Schälbare Cohen–Macauley-Komplexe und ihre Parametrisierung, *Math. Z.*, **167** (1979), 173–179.
10. C. W. Lee, Some recent results on convex polytopes, *Contemp. Math.*, **114** (1990), 3–19.
11. C. W. Lee, Generalized stress and motions, in *Polytopes: Abstract, Convex and Computational*, T. Bistriczky, P. McMullen, R. Schneider, and A. I. Weiss, eds., NATO ASI Series C, Vol. 440, Kluwer, Dordrecht, 1994, pp. 249–271.
12. P. McMullen, The maximum numbers of faces of a convex polytope, *Mathematika*, **17** (1970), 179–184.
13. P. McMullen, On simple polytopes, *Invent. Math.*, **113** (1993), 419–444.
14. P. McMullen, Weights on polytopes, *Discrete Comput. Geom.*, this issue, pp. 363–388.
15. P. McMullen and D. W. Walkup, A generalized lower-bound conjecture for simplicial polytopes, *Mathematika*, **18** (1971), 264–273.

16. T. Oda, *Convex Bodies and Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 15 [Results in Mathematics and Related Areas (3)], Springer-Verlag, Berlin, 1988.
17. U. Pachner, Shellings of simplicial balls and p.l. manifolds with boundary, *Discrete Math.*, **81** (1990), 37–47.
18. G. Reisner, Cohen–Macaulay quotients of polynomial rings, *Adv. in Math.*, **21** (1976), 30–49.
19. R. P. Stanley, The upper bound conjecture and Cohen–Macaulay rings, *Stud. Appl. Math.*, **54** (1975), 135–142.
20. R. P. Stanley, Hilbert functions of graded algebras, *Adv. in Math.*, **28** (1978), 57–83.
21. R. P. Stanley, The number of faces of a simplicial convex polytope, *Adv. in Math.*, **35** (1980), 236–238.
22. T.-S. Tay, N. White, and W. Whiteley, Skeletal rigidity of simplicial complexes, I, II, *European J. Combin.*, **16** (1995), 381–403, 503–523.
23. T.-S. Tay, N. White, and W. Whiteley, A homological interpretation of skeletal rigidity, Manuscript, 1993.

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