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p-MODULE OF VECTOR MEASURES IN DOMAINS WITH INTRINSIC METRIC ON CARNOT GROUPS

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Abstract. We define the extremal length of horizontal vector measures on a Carnot group and study capacities associated with linear sub-elliptic equations. The coincidence between the definition of the p-module of horizontal vector measure system and two different definitions of the p-capacity is proved. We show the continuity property of a p-module generated by a family of horizontal vector measures. Reciprocal relations between the p-capacity and q-module $(1/p + 1/q = 1)$ of horizontal vector measures are obtained. A peculiarity of our approach consists of the study of the above mentioned notions in domains with an intrinsic metric.

1. Introduction. The concept of the extremal length and the module of a family of curves goes back to Grötzsch, Beurling, and Ahlfors [1, 16]. In 1957 Fuglede [14] has introduced the p-module of a measure system. These notions play an important role and have a lot of applications in analysis and potential theory. An interest to non-linear elliptic equations has inspired a more general notion of the module of a family of curves and the capacity associated with this type of equations [2, 19, 20, 21, 26].

Recently, analysis on Carnot groups (the simplest example of which is the Heisenberg group) has been developed intensively. The fundamental role of such groups in analysis was pointed out by Stein [34], in his address to the International Congress of Mathematicians in 1970, see also his monograph [35]. Briefly, a Carnot group is a simply connected nilpotent Lie group, whose Lie algebra admits a grading. There is a natural family of dilations on the group under which the metric behaves like the Euclidean metric under the Euclidean dilation [7, 13]. An analysis on homogeneous groups is a test ground for the study of general sub-elliptic problems arising from vector fields $X_1, \dots, X_k$ satisfying the Hörmander hypoellipticity condition [22]. An important motivation for the study of quasilinear sub-elliptic equations of the second order comes from the theory of quasiconformal and quasiregular mappings on stratified nilpotent groups [8, 15, 18, 31, 39]. Quasilinear sub-elliptic equations generate the interest to a concept of the capacity and extremal length, associated with this type of equations. The foundation of the theory of quasilinear sub-elliptic equations and non-linear potential theory can be found in the papers [3, 4, 5, 9, 12, 17, 28, 29] and the references therein.

In the present work, based on ideas of [2], we define a horizontal vector measure on a Carnot group. The non-Riemannian geometry of the group and the properties of sub-elliptic

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equations make us to introduce some natural modifications for the definition of measure systems. We prove the continuity property of the p-module of a family of curves, associated with the p-module of horizontal vector measures. We show the equivalence of two different definitions of the p-capacity, associated with sub-elliptic equations, and coincidence between them and the p-module of a measure system. Other relations among the extremal length of horizontal vector measures and capacity of a condenser are considered. Our approach to defining boundary values of functions on some ideal boundary, that differs from the Euclidean one, is based on results of [37, 38]. This boundary is obtained as a result of completing the domain with respect to the intrinsic metric. This approach allows us to distinguish edges of cuts and due to this fact the p-modules and p-capacities may take different values. In the next paragraph the reader finds explicit definitions and detailed formulations of main results.

2. Definitions and statement of the results. Let $\boldsymbol{G}$ be a simply connected nilpotent Lie group and $\mathcal{G}$ its Lie algebra. We identify $\mathcal{G}$ with $T_e\boldsymbol{G}$, the tangent space at the identity e, in a natural way: A tangent vector $X \in T_e\boldsymbol{G}$ corresponds to the left invariant vector field for which $X(q) = L_{q_*}X$, where L_q is the left translation by $q \in \boldsymbol{G}$. Let us denote by $[U, V]$ the subspace of $\mathcal{G}$ generated by elements $[X, Y] = XY - YX$ where $X \in U$, $Y \in V$. We suppose that the Lie algebra splits into the direct sum

$$
\begin{aligned}
&\mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_m\,, \\
&[V_1, V_k] = V_{k+1}\,, \quad k = 1, \ldots, m-1\,, \qquad [V_1, V_m] = \{0\}\,.
\end{aligned}
\tag{2.1}
$$

We call the underlying space V_1 *the horizontal space*. Let $X_{11}, \ldots X_{1n_1}$, $n_1 = \dim V_1$, be a basis of V_1. It generates a basis $\{X_{ij}\}$ of the Lie algebra $\mathcal{G}$, $X_{ij} \subset V_i$, $i = 1, \ldots, m$, $j = 1, \ldots, n_i = \dim V_i$, according to (2.1).

It is known (see, for instance [13]) that for a simply connected nilpotent Lie group $\boldsymbol{G}$ with the Lie algebra $\mathcal{G}$ the exponential map $\exp : \mathcal{G} \to \boldsymbol{G}$ is a global diffeomorphism. Thus we can identify the elements x of the group $\boldsymbol{G}$ with the elements x of the algebra $\mathcal{G}$, and so, with $x \in \boldsymbol{R}^N$, $N = \sum_{i=1}^m \dim V_i$, by the exponential map $x = \exp(\sum x_{ij}X_{ij})$. The numbers $x = (x_{ij})$, $1 \le i \le m$, $1 \le j \le \dim V_i = n_i$, are called the coordinates of the point x. There is a natural group of dilations, which is defined by the rule $\delta_r x = (r^i x_{ij})$, $1 \le i \le m$, $1 \le j \le n_i$. The quantity $Q = \sum_{i=1}^m i \cdot n_i$ is called the *homogeneous dimension* of the group $\boldsymbol{G}$. It is easy to see that $d(\delta_r x) = r^Q dx$. If we denote by dx the Lebesgue measure on $\mathcal{G}$, then $dx \circ \exp^{-1}$ is a biinvariant Haar measure on $\boldsymbol{G}$. We use the symbol $\mathrm{mes}(E)$ to denote the Haar measure of a measurable set $E \in \boldsymbol{G}$: $\mathrm{mes}(E) = \int_E dx$.

We fix a quadratic form $\langle \cdot, \cdot \rangle$ on V_1, such that $\langle X_{1i}(x), X_{1j}(x)\rangle = \delta_{ij}$ at every point $x \in \boldsymbol{G}$. For a vector $\xi \in V_1$ we shall use the notation $|\xi| = \langle \xi, \xi\rangle^{1/2}$. An absolutely continuous curve $\gamma : [0, b] \to \boldsymbol{G}$ is said to be *horizontal* if its tangent vector (if exist) $\gamma'(t)$ lies in the horizontal space, i.e., there exist functions $a_j(s)$, $s \in [0, b]$, such that $\gamma'(s) = \sum_{j=1}^{n_1} a_j(s)X_{1j}(\gamma(s))$. A result by [6] implies that one can connect two arbitrary points $x, y \in \boldsymbol{G}$ by a horizontal curve. Then the length of a horizontal curve γ is defined by the formula

$$l(\gamma) = \int_0^b \langle \gamma'(s), \gamma'(s) \rangle^{1/2} \, ds = \int_0^b \left(\sum_{j=1}^{n_1} |a_j(s)|^2 \right)^{1/2} ds \, .$$

The Carnot-Carathéodory distance $d_c(x, y)$ is the infimum of the length over all horizontal curves connecting x and $y \in \boldsymbol{G}$. In fact, the Carnot-Carathéodory distance does not give a metric, for it does not need to satisfy the triangle inequality, but satisfied only its weak form: $d_c(x, y) \le C(d_c(x, w) + d_c(w, y))$. Non-horizontal curves can be said to have infinite arc length [24]. Thus, from now on, we work only with horizontal curves.

We call any smooth function $|\cdot| : \boldsymbol{G} \setminus \{e\} \to (0, \infty)$ satisfying $|\delta_r x| = r|x|$ and $|x^{-1}| = |x|$, a *homogeneous norm* on $\mathcal{G}$. The homogeneous norm defines the distance by $d(x, y) = |x^{-1}y|$, which is equivalent to the Carnot-Carathéodory distance. We choose a homogeneous norm that satisfies the triangle inequality: $|x^{-1}y| \le |x| + |y|$ (for the construction, see [35]).

EXAMPLE 1. The Euclidean space $\boldsymbol{R}^n$ with the standard structure is an example of the Abelian group: the exponential map is the identity and the vector fields $X_i = \partial/\partial x_i$, $i = 1, \ldots, n$, have only trivial commutative relations and form the basis of the corresponding Lie algebra.

EXAMPLE 2. The simplest example of a non-Abelian group is the Heisenberg group $\boldsymbol{H}^n$. The underlying space of $\boldsymbol{H}^n$ is $\boldsymbol{R}^{2n+1}$ with the group law of multiplication defined as

$$(x, t)(x', t') = \left(x + x', t + t' + 2 \sum_{i=1}^{n} (x_{n+j} x'_j - x_j x'_{n+j}) \right), \quad x, x' \in \boldsymbol{R}^{2n}, \quad t, t' \in \boldsymbol{R}.$$

The Lie algebra $\mathcal{G}$ of the Heisenberg group $\boldsymbol{H}^n$ is generated by the left-invariant vector fields $X_j = \partial/\partial x_j + 2x_{n+j}\partial/\partial t$, $X_{n+j} = \partial/\partial x_{n+j} - 2x_j \partial/\partial t$, $1 \le j \le n$, and $T = \partial/\partial t$. There are nontrivial commutative relations $[X_j, X_{n+j}] = -4T$, $1 \le j \le n$. The vector fields X_j, $j = 1, \ldots, 2n$, form a basis of the horizontal vector space V_1, $\text{span}\{T\} = V_2$, and the Lie algebra $\mathcal{G}$ of the Heisenberg group is represented as the sum $\mathcal{G} = V_1 \oplus V_2$. The required homogeneous norm is given by $|x| = ((\sum_{j=1}^{2n} x_j^2)^2 + t^2)^{1/4}$. The homogeneous dimension Q is equal to $2n + 2$.

We define an absolutely continuous function on curves of the horizontal fibration. For this we consider a family $\mathcal{X}$ of horizontal curves that forms a smooth fibration of an open set $U \subset \boldsymbol{G}$. Usually, one can think of a curve $\varrho \in \mathcal{X}$ as an orbit of a smooth horizontal vector field $X \in V_1$. If we denote by φ_s the flow associated with this vector field, then the fiber is of the form $\varrho(s) = \varphi_s(x)$. Here the point x belongs to the surface S which is transversal to the vector field X. The parameter s ranges over an open interval $J \in \boldsymbol{R}$. One can assume that there is a measure $d\varrho$ on the fibration $\mathcal{X}$ of the set $U \subset \boldsymbol{G}$. The measure $d\varrho$ on $\mathcal{X}$ is equal to the inner product of the vector field $X \in V_1$ and a biinvariant volume form dx. The measure

$d\varrho$ satisfies the inequality

$$k_0 \operatorname{mes}(B(x,R))^{\frac{Q-1}{Q}} \leq \int\limits_{\varrho\in\mathcal{X},\ \varrho\cap B(x,R)\neq\emptyset} d\varrho \leq k_1 \operatorname{mes}(B(x,R))^{\frac{Q-1}{Q}}$$

for sufficiently small balls $B(x, R) \subset U$ with constants k_0, k_1 that do not depend on a ball $B(x, R)$ (for more information see, for instance, [25, 36]).

DEFINITION 2.1. Let D be a domain (open connected set) on $\boldsymbol{G}$. A function $u : D \to \boldsymbol{R}$, is said to be *absolutely continuous on lines* ($u \in ACL(D)$) if for any domain U, $\bar{U} \subset D$, and any fibration $\mathcal{X}$ defined by a left-invariant vector field X_{1j}, $j = 1, \dots, n_1$, the function u is absolutely continuous on $\varrho \cap U$ with respect to the $\mathcal{H}^1$-Hausdorff measure for $d\varrho$-almost all curves $\varrho \in \mathcal{X}$.

The derivatives $X_{1j}u$, $j = 1, \dots, n_1$, exist almost everywhere in D for such function u [25]. If they belong to $L_p(D)$, $p \geq 1$, for all $X_{1j} \in V_1$, then u is said to be from $ACL_p(D)$. A result from [27, 32] implies that an ACL_p-function is absolutely continuous on p-almost all horizontal curves.

A function $u : D \to \boldsymbol{R}$, $D \subset \boldsymbol{G}$, is said to belong to the Sobolev space $L^1_p(D)$ if its distributional derivatives $X_{1j}u$ along the horizontal vector fields X_{1j}, $j = 1, \dots, n_1$, exist, i.e., the equality $\int_D X_{1j}u\,\varphi\,dx = \int_D uX_{1j}\varphi\,dx$ holds for all $\varphi \in C_0^\infty(D)$ and the seminorm $\|u \mid L^1_p(D)\| = (\int_D |\nabla_0 u|^p(x)dx)^{1/p}$ is finite. Here $\nabla_0 u = (X_{11}u, \dots, X_{1n_1}u)$ is *the horizontal gradient* of u and $|\nabla_0 u| = (\sum_{j=1}^{n_1} |X_{1j}u|^2)^{1/2}$. If the function u belongs to $L^1_p(D)$, then there exists a function $v \in ACL_p(D)$ such that $u = v$ almost everywhere.

We define an intrinsic metric $d_D(x, y)$ on D, $x, y \in D$. We put $d_D(x, y) = \inf\{l(\gamma)$; where $\gamma(t)$ are horizontal curves such that $\gamma(t) \in D$ for all $t \in [0, 1]$, $\gamma(0) = x$, $\gamma(1) = y\}$. Consider the metric space $\boldsymbol{D} = (D, d_D)$ and the identical mapping $\pi : \boldsymbol{D} \to D$, $\pi(x) = x$, $x \in D$. The sequence $\pi(x_l)$, $l \in \boldsymbol{N}$, is a Cauchy sequence in D when $\{x_l\}$, $l \in \boldsymbol{N}$, is such in $\boldsymbol{D}$. Therefore, the sequence $\pi(x_l)$ converges to a point either inside D or at the boundary $\partial D = \bar{D} \setminus D$ of D ($\bar{D}$ is the closure of D). In the first case, the original sequence converges to some point $x \in \boldsymbol{D}$. In the latter case, the sequence $\{x_l\}$, $l \in \boldsymbol{N}$, has no limit in $\boldsymbol{D}$. By Hausdorff's theorem, we can complete the metric space $\boldsymbol{D}$. Let $\tilde{\boldsymbol{D}}$ be a completion; as a result, we add to D some ideal elements which are the limits of Cauchy (in $\boldsymbol{D}$) sequences corresponding to the latter case. We call the set $\partial\tilde{\boldsymbol{D}} = \tilde{\boldsymbol{D}} \setminus \boldsymbol{D}$ the *ideal boundary* of D and assume this set to be compact. For a domain Ω such that $\bar{\Omega} \subset D$, the boundaries (the closure) of Ω in the metric spaces $(\boldsymbol{G}, d(x, y))$ and $(\tilde{\boldsymbol{D}}, d_D(x, y))$ coincide.

Together with the Sobolev space on D we define the Sobolev space $L^1_p(\tilde{\boldsymbol{D}})$ on $\tilde{\boldsymbol{D}}$ as the completion of the class $C(\tilde{\boldsymbol{D}}) \cap L^1_p(\boldsymbol{D})$ with respect to the norm $\| \cdot \mid L^1_p(D)\|$. (Here $C(\tilde{\boldsymbol{D}})$ is the space of functions continuous on $\tilde{\boldsymbol{D}}$.) Obviously, the restrictions of functions in $L^1_p(\tilde{\boldsymbol{D}})$ to D belong to the Sobolev class $L^1_p(D)$. Formally this imbedding is induced by the identical mapping $i : D \to \tilde{\boldsymbol{D}}$, $i(x) = x$, $x \in D$, in accordance with the convention $i^\star = f \circ i$ (see the properties of the Sobolev spaces in [4, 5, 38]).

Let $\Omega \subset D$ be an open subset in the complete metric space $\tilde{\boldsymbol{D}}$ equipped with the intrinsic metric $d_D(x, y)$. It is possible that the closure $\bar{\Omega}$ coincides with the whole space $\tilde{\boldsymbol{D}}$. Henceforth, the closure $\bar{\Omega}$ is taken in the metric $d_D(x, y)$ and $\partial\Omega$ is the boundary of Ω in the metric space $\tilde{\boldsymbol{D}}$.

Let $\mathcal{A}(x) = (a_{ij}(x))$, $x \in \Omega$, be a positive definite symmetric $(n_1 \times n_1)$-matrix, with measurable components $a_{ij}(x)$, such that

$$\alpha^{-1}|\xi| \leq \langle \mathcal{A}\xi, \mathcal{A}\xi \rangle^{1/2} = |\mathcal{A}\xi| \leq \alpha|\xi| \tag{2.2}$$

for any $\xi \in V_1 \subset \mathcal{G}$ and some constant $\alpha \geq 1$. Let $\mathcal{B}(x) = (b_{ij}(x))$ be the inverse matrix to $\mathcal{A}(x)$. The matrix $\mathcal{B}(x)$ satisfies the inequality (2.2). One can associate with the matrix $\mathcal{A}$ a second order sub-elliptic operator $-\operatorname{div}\mathcal{A}^2(x)\nabla_0 = -\sum_{j=1}^{n_1} X_{1j}(x)\mathcal{A}^2(x)\nabla_0$, where $\nabla_0 u = (X_{11}u, \ldots, X_{1n_1}u)$ for any smooth function u. If $\mathcal{A}$ is the unit matrix, then we obtain the sub-Laplacian on the Carnot group.

We recall the definition of the p-module of a system of measures [14]. Let f be a non-negative Borel measurable function and μ be a non-negative Borel measure. If $\int f d\mu \geq 1$, then we say that the function f is admissible for the measure μ. Let $\mathcal{E}$ be a system of non-negative Borel measures. If f is admissible for all $\mu \in \mathcal{E}$, then we denote by $\mathcal{F}M(\mathcal{E})$ the set of admissible functions for the module of the system of measures $\mathcal{E}$. The quantity

$$M_p(\mathcal{E}) = \inf\left\{ \int f^p dx\,;\ f \geq 0,\ f \in \mathcal{F}M(\mathcal{E})\right\}$$

is called the *p-module of $\mathcal{E}$*.

Now we define the p-module of a system of vector measures which is related to the stratified structure of the Lie algebra of the Carnot group. Let $\mu = (\mu_1, \ldots, \mu_{n_1})$ be a vector measure whose components μ_i are signed measures defined for sets from $\boldsymbol{G}$. We call these measures *horizontal vector measures* because the dimension of each vector measure is equal to n_1 and coincides with the dimension of horizontal vector space $V_1 \subset \mathcal{G}$. We define the total variation $|\mu|$ of μ by $|\mu|(E) = \sup \sum_j (\sum_{i=1}^{n_1} \mu_i^2(E_j))^{1/2}$ for Borel sets E, where the supremum is taken over all finite partitions of E into Borel sets E_j. The total variation $|\mu|$ is a non-negative measure. We give the definition of exceptional sets of horizontal vector measures.

DEFINITION 2.2. Let $\mathcal{M}$ be a set of vector measures μ. We put $|\mathcal{M}| = \{|\mu|\,;\ \mu \in \mathcal{M}\}$. If $M_p(|\mathcal{M}|) = 0$, then we say that $\mathcal{M}$ is *p-exceptional*. If a statement with respect to vector measures fails only for a p-exceptional system $\mathcal{M}$, then we say that it holds p-almost everywhere.

Let $D \subset \boldsymbol{G}$ and $(\tilde{\boldsymbol{D}}, d_D(x, y))$ be a complete metric space with the intrinsic metric $d_D(x, y)$. Let Ω be a domain on $\tilde{\boldsymbol{D}}$, K_0 and K_1 be closed non-empty disjoint sets such that $K_0 \cap \bar{\Omega} \neq \emptyset$ and $K_1 \cap \bar{\Omega} \neq \emptyset$. It is not excluded that $\bar{\Omega} = \tilde{\boldsymbol{D}}$. We call the triplet $(K_0, K_1; \Omega)$ the condenser. Let $\lfloor a, b \rfloor$ be an interval of one of the following types: $[a, b]$, $[a, b)$, $(a, b]$, or

(a, b). We let

$$\Gamma = \Gamma(K_0, K_1; \Omega) = \{\gamma\ ;\ \overline{\gamma([a, b])} \cap K_i \neq \emptyset,\ i = 0, 1,\ \text{and}\ \gamma(t) \in \Omega,\ t \in (a, b)\}$$

and call $\Gamma(K_0, K_1; \Omega)$ the family of curves that connect the compacts K_0 and K_1 in the domain Ω. Now we give two different definitions of the $\mathcal{A}_p$-capacity of a condenser.

DEFINITION 2.3. Denote by $\mathcal{F}C(K_0, K_1; \Omega)$ the class of admissible functions $u \in ACL_p(\Omega)$ such that $u(x) \to 0$ as $x \to K_0 \cap \bar{\Omega}$ along p-almost all curves from $\Gamma(K_0, K_1; \Omega)$ and $u(x) \to 1$ as $x \to K_1 \cap \bar{\Omega}$ along p-almost all curves from $\Gamma(K_0, K_1; \Omega)$. We define the $\mathcal{A}_p$-capacity of the condenser $(K_0, K_1; \Omega)$ to be

$$\operatorname{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) = \inf\left\{\int_\Omega |\mathcal{A}\nabla_0 u|^p\, dx\ ;\ u \in \mathcal{F}C(K_0, K_1; \Omega)\right\}.$$

DEFINITION 2.4. Let $\mathcal{F}C^\star(K_0, K_1; \Omega)$ be the class of admissible functions $u \in ACL_p(\Omega)$ such that $u(x) = 0$ on the intersection of Ω with a neighborhood of K_0 and $u(x) = 1$ on the intersection of Ω with a neighborhood of K_1. We define the $\mathcal{A}_p^\star$-capacity to be

$$\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) = \inf\left\{\int_\Omega |\mathcal{A}\nabla_0 u|^p\, dx\ ;\ u \in \mathcal{F}C^\star(K_0, K_1; \Omega)\right\}.$$

We will prove the equivalence of Definitions 2.3 and 2.4 in domains with the intrinsic metric.

THEOREM 2.1. *Let Ω be a bounded domain in a complete metric space $(\tilde{\boldsymbol{D}}, d_D(x, y))$, $D \subset \boldsymbol{G}$ equipped with the intrinsic metric $d_D(x, y)$. Then,*

$$\operatorname{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) = \operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega)\,.$$

Capacities associated with sub-elliptic equations were studied in [4, 5, 9, 10, 28, 29, 30].

Now we give the definition of the $\mathcal{A}_p$-module of a system of horizontal vector measures correlated with Definitions 2.3 and 2.4. Let $\zeta(x) = (\zeta_1(x), \ldots, \zeta_{n_1}(x))$ be a vector valued function. If $\int |\zeta_i| d|\mu_i| < \infty$ for all i, then we define $\int \zeta\, d\mu = \sum_{i=1}^{n_1} \int \zeta_i \mu_i$. We denote by $\mathcal{F}M(\mu)$ a class of functions $\zeta(x)$ such that $\int \zeta\, d\mu \geq 1$. If $\zeta \in \mathcal{F}M(\mu)$ for all $\mu \in \mathcal{M}$, then we write $\zeta \in \mathcal{F}M(\mathcal{M})$ and call $\zeta(x)$ an admissible function for the system $\mathcal{M}$.

DEFINITION 2.5. Let $\xi = (\xi_1, \ldots, \xi_{n_1})$ be a vector valued function and let $\mathcal{M}$ denote a family of complete horizontal vector measures on $\Omega \subset \tilde{\boldsymbol{D}}$. We define the $\mathcal{A}_p$-module by

$$M_{\mathcal{A}_p}(\mathcal{M}) = \inf\left\{\int_\Omega |\mathcal{A}\xi|^p\, dx;\ \xi \in \mathcal{F}M(\mathcal{M}) \quad p\text{-almost everywhere}\right\}.$$

We put the condition p-almost everywhere to avoid nonsense. For example, let us choose some horizontal vector field X_{1j}, with orbit β_i, and the Lebesgue measure $d\beta_i$ on β_i. We fix an arc $C \subset \beta_i$ of finite length. Let us consider the horizontal vector measure system $\mathcal{M} = \{(0, \ldots, d\beta_i|_C, \ldots, 0), (0, \ldots, -d\beta_i|_C, \ldots, 0)\}$. There is no admissible vector-valued function ξ for $\mathcal{M}$. However, since $M_p(|\mathcal{M}|) = 0$, the p-exceptional set coincides with $\mathcal{M}$, and therefore $M_{\mathcal{A}_p}(\mathcal{M}) = 0$.

For a family Γ of horizontal curves γ we naturally have horizontal vector measures $d\gamma$, and measures $|d\gamma| = \langle d\gamma, d\gamma\rangle^{1/2}$. We write $d\Gamma = \{d\gamma;\ \gamma \in \Gamma\}$ and $|d\Gamma| = \{|d\gamma|;\ \gamma \in \Gamma\}$. More generally, for a positive definite $(n_1 \times n_1)$-matrix $Q(x) = (q_{ij}(x))$ we put $|Qd\gamma| = \langle Qd\gamma, Qd\gamma\rangle^{1/2}$ and $|Qd\Gamma| = \{|Qd\gamma|;\ \gamma \in \Gamma\}$.

We prove the next relations between the $\mathcal{A}_p$-capacity and the $\mathcal{A}_p$-module.

THEOREM 2.2. *Let Ω be a domain in a complete metric space $(\tilde{\boldsymbol{D}}, d_D(x, y))$, $D \subset \boldsymbol{G}$ equipped with the intrinsic metric $d_D(x, y)$. Then,*

$$\operatorname{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) = M_{\mathcal{A}_p}(d\Gamma) = M_p(|\mathcal{B}\, d\Gamma|) < \infty \quad \textit{for} \quad p \in [1, \infty)\,.$$

We consider another family of horizontal vector measures. Let us denote by $\nabla_0 C^\star = \nabla_0 C^\star(K_0, K_1; \Omega) = \{\nabla_0 u;\ u \in \mathcal{F}C^\star(K_0, K_1; \Omega)\}$.

THEOREM 2.3. *Let $1/p + 1/q = 1$. If $\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) > 0$, then*

$$\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega)^{1/p} M_{\mathcal{B}_q}(\nabla_0 C^\star)^{1/q} = 1\,.$$

In the case when $\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) = 0$ we have $M_{\mathcal{B}_q}(\nabla_0 C^\star) = \infty$.

Later we will use the following notation. Let K_0 and K_1 be compact sets from $\tilde{\Omega}$, and let K_0^j and K_1^j be sequences of compact sets such that $K_0^0 \cap K_1^0 = \emptyset$, $K_0^j \subset \operatorname{int} K_0^{j-1}$, $K_1^j \subset \operatorname{int} K_1^{j-1}$, $K_0 = \bigcap_{j=0}^\infty K_0^j$, and $K_1 = \bigcap_{j=0}^\infty K_1^j$.

THEOREM 2.4. *Suppose that $\mathcal{B}(x)$ is uniformly continuous in a bounded domain Ω. Then $M_p(|\mathcal{B}\, d\Gamma|)$ possesses the continuity property. Namely, if $\Gamma_j = \Gamma(K_0^j, K_1^j; \Omega)$, then*

$$\lim_{j\to\infty} M_p(|\mathcal{B}\, d\Gamma_j|) = M_p(|\mathcal{B}\, d\Gamma|)\,.$$

3. Auxiliary lemmas. Here and in Sections 5 and 6 we will be working under the assumption that K_0 and K_1 are disjoint non-empty compacts in the closure $\tilde{\Omega}$ of a domain Ω. Moreover, let K_0^j and K_1^j be sequences of closed sets such that $K_0^0 \cap K_1^0 = \emptyset$, $K_0^j \subset \operatorname{int} K_0^{j-1}$, $K_1^j \subset \operatorname{int} K_1^{j-1}$, $K_0 = \bigcap_{j=0}^\infty K_0^j$, and $K_1 = \bigcap_{j=0}^\infty K_1^j$. We recall that notions of closure and inner points are considered in the topology of the complete metric space $(\tilde{\boldsymbol{D}}, d_D)$, $D \subset \boldsymbol{G}$.

The next lemma in the case of $\tilde{\boldsymbol{D}} = \boldsymbol{R}^n$ goes back to the work [33] and then has been revised by Ohtsuka (see for instance [2]).

LEMMA 3.1. *Let $\rho \in L_p(\tilde{\boldsymbol{D}})$ be a positive lower semicontinuous function which is continuous in $\Omega \setminus (K_0 \cup K_1)$, $\Omega \subset \tilde{\boldsymbol{D}}$. For each $\varepsilon > 0$ we can construct a function ρ' on Ω, $\rho' \geq \rho$, with the following properties*:

(i) $\int_\Omega \rho'^p\, dx \leq \int_\Omega \rho^p\, dx + \varepsilon$.

(ii) *Suppose that for each j there is $\gamma_j \in \Gamma(K_0^j, K_1^j; \Omega)$ such that $\int_{\gamma_j} \rho'\, |\mathcal{B}\, d\gamma| \leq \alpha$. Then there exists $\tilde{\gamma} \in \Gamma(K_0, K_1; \Omega)$ that satisfies the inequality $\int_{\tilde{\gamma}} \rho\, |\mathcal{B}\, d\gamma| \leq \alpha + \varepsilon$.*

The proof of Lemma 3.1 on Carnot groups for $\mathcal{B}$ which is equal to the unit matrix I can be found in [27]. For the case $\mathcal{B} \neq I$ and for domains with intrinsic metric the proof is essentially the same.

LEMMA 3.2. *Suppose that U is a bounded domain in $\boldsymbol{G}$. Let $f \in L_p(U)$ and $\varepsilon > 0$. Then there exists a continuous function $\tilde{f}$ such that*

$$\|f - \tilde{f} \mid L_p(U)\| < \varepsilon .$$

PROOF. Set $U_n \subset \bar{U}_n \subset U_{n+1} \subset \bar{U}_{n+1} \subset \cdots \subset U$ the sequence of open sets that exhaust the domain U. We put $U_{-1} = U_0 = \emptyset$. For each n we find a positive function $h_n(x)$, such that $h_n \in C_0^\infty(U_{n+1} \setminus U_{n-2})$, $|\nabla h_n| \leq 1/6$, $|h_n(x)| \leq 1/6 \min\{1, \operatorname{dist}(x, \partial U)\}$. Then the function $\eta(x) = \sum_{n=1}^{\infty} h_n(x)$ has the following properties

1. $|\nabla \eta(x)| \leq 1/2$,
2. $0 < \eta(x) \leq 1/2 \min\{1, \operatorname{dist}(x, \partial U)\}$.

Let $y \in \boldsymbol{G}$, $|y| \leq 1$, and $0 < t < \min\{1, C, \hat{C}\}$ where the constants C, $\tilde{C}$ will be made more precise later. We define a C^∞-map of the domain U onto itself by $T_{t,y}(x) = x \cdot \delta_{t\eta(x)} y$. We claim that $T_{t,y}$ is a homeomorphism. If $y = 0$, then $T_{t,0}$ is identity map. Let $y \neq 0$. Since $0 < \eta(x) \leq 1/2 \operatorname{dist}(x, \partial U)$ the map $T_{t,y}$ transforms U to U. Let us show that $T_{t,y}$ is injective. Suppose that for some x and x' in U we have $T_{t,y}(x) = T_{t,y}(x')$. Applying the left translation and dilatation for the domain U we can assume that $|x| = 1$ and $x' = 0$, where 0 denotes the unity of $\boldsymbol{G}$. In this case we get $x\delta_{t\eta(x)} y = \delta_{t\eta(0)} y$ or $x = \delta_{t\eta(0)} y (\delta_{t\eta(x)} y)^{-1}$. A homogeneous norm $|\cdot|$ and the Euclidean norm $\|\cdot\|$ are connected by the inequality $C_1\|x\| \leq |x| \leq C_2 \|x\|^{1/m}$, $x \in U$, where C_1, C_2 some positive constants (see, for instance [17]). We deduce that

$$\begin{aligned} |x| &= |\delta_{t\eta(0)} y (\delta_{t\eta(x)} y)^{-1}| \leq C_2 \|\delta_{t\eta(0)} y - \delta_{t\eta(x)} y\|^{1/m} \\ &\leq C_2 t^{1/m} |\eta(0) - \eta(x)|^{1/m} |P_{m-1}(\eta(0), \eta(x), y, t)|^{1/m} . \end{aligned} \tag{3.1}$$

Here P_{m-1} is a polynomial of the order $m - 1$, that depends on $\eta(x)$, t, and coordinates of the point y. Since $|y| \leq 1$, $0 < t \leq 1$ and $0 < \eta(x) \leq 1/2$, we have $|P_{m-1}| \leq C_3$, where the constant C_3 depends only on m. We estimate $|\eta(0) - \eta(x)| \leq |x|/2$ by the first property of the function η. Here $|x|$ is the homogeneous norm of x. Taking into account these estimates we conclude that $|x| \leq C_4 t^{1/m} |x|^{1/m}$ from (3.1). Since $|x| = 1$ for $t < C_0 = C_4^{-m}$, we obtain the contradiction.

Let us show that $T_{t,y}$ is surjection. We denote by $\omega(t)$ the curve $\delta_t y$. The intersection $\omega(t) \cap U$ is invariant under the map $T_{t,y}$ because of the second property of $\eta(x)$. This shows that the map is surjection.

The Jacobian matrix of $T_{t,y}(x)$ is equal to $I + t\hat{T}$, where I is the identity matrix and elements of the matrix $\hat{T}$ depend on $t, x, y, \nabla\eta(x), \eta(x)$. Thus the Jacobian $J(T_{t,y})$ is of the form $1 + tH(t, x, y, \nabla\eta(x), \eta(x))$, where H is a polynomial. The properties of function η, the choice of y, t, and the boundedness of the domain U, imply that $\max_{x \in U} |H| \leq C_5$, where the constant C_5 depends only on m, and on the diameter of U. If we choose $\tilde{C} = 1/(2C_5)$, then we have $J(T_{t,y}) \geq 1 - tC_5 > 0$ for $t < \tilde{C}$. This shows that the inverse map $T_{t,y}^{-1}$ is defined and smooth.

Let $\varphi(y)$ be a nonnegative C^∞-function supported in the unit ball $|y| < 1$ such that $\int_{|y|<1} \varphi(y)\, dy = 1$. For $f \in L_p(U)$ we define

$$f_t(x) = \int_{|y|<1} f(x\delta_{t\eta(x)}y)\varphi(y)dy = \int_{\boldsymbol{G}} f(z)\varphi(\delta_{(t\eta(x))^{-1}}(x^{-1}z))\frac{dz}{(t\eta(x))^Q}\,.$$

The function $f_t(x)$ is a C^∞-function in the domain U.

We show that $\| f_t - f \mid L_p(U)\| \to 0$ as $t \to 0$. Using the fact that continuous functions with compact support are dense in $L_p(U)$ we obtain

$$\| f(xy) - f(x) \mid L_p(U)\| \to 0 \quad \text{as} \quad |y| \to 0\,. \tag{3.2}$$

Since $f_t(x) - f(x) = \int_{|y|<1}(f(x\delta_{t\eta(x)}y) - f(x))\varphi(y)dy$ and applying the Minkowski inequality, we deduce

$$\| f_t - f \mid L_p(U)\| \leq \int_{|y|<1} \| f(x\delta_{t\eta(x)}y) - f(x) \mid L_p(U)\|\varphi(y)dy\,.$$

It follows that $\| f_t - f \mid L_p(U)\| \to 0$ as $t \to 0$ from the property (3.2), the inequality $\| f(x\delta_{t\eta(x)}y) - f(x) \mid L_p(U)\| \leq 2\| f(x) \mid L_p(U)\|$, and the dominated convergence theorem. □

THEOREM 3.1. *Let Ω be a bounded domain in $\tilde{\boldsymbol{D}}$, $D \subset \boldsymbol{G}$. Let $\mathcal{B}(x)$ be uniformly continuous on $\Omega \setminus (K_0 \cup K_1)$ and $\mathcal{C} \subset \mathcal{F}M(|\mathcal{B}\, d\Gamma|)$ consist of continuous functions on $\Omega \setminus (K_0 \cup K_1)$. Then,*

$$M = \inf_{\hat{\rho}\in\mathcal{C}} \int\limits_{\Omega\setminus(K_0\cup K_1)} \hat{\rho}^p(x)dx = M_p(|\mathcal{B}d\Gamma|)\,. \tag{3.3}$$

PROOF. We denote by U the domain $\Omega \setminus (K_0 \cup K_1)$. Let $\varepsilon \in (0, 1/2)$. We choose a function $\rho \in \mathcal{F}M(|\mathcal{B}\, d\Gamma|)$ with

$$\int_U \rho^p(x)dx < \varepsilon + M_p(|\mathcal{B}d\Gamma|)\,. \tag{3.4}$$

Then, by Lemma 3.2, we can find a continuous function ρ_t in the domain U such that

$$\int_U \rho_t^p(x)dx < \varepsilon + \int_U \rho^p(x)dx\,. \tag{3.5}$$

We claim that for a sufficiently small t the function $(1 + \varepsilon)^2\rho_t(x)$ is admissible for $M(|\mathcal{B}\, d\Gamma|)$.

The matrix $\mathcal{B}(x)$ is uniformly continuous. If $x, y \in \Omega \setminus (K_0 \cup K_1)$ and $d(x, y) \leq d_D(x, y) < \varsigma(\varepsilon)$, then $|\mathcal{B}(x) - \mathcal{B}(y)| < \alpha^{-1}\varepsilon$. Hence, we obtain

$$|\mathcal{B}(y)\xi| \leq |\mathcal{B}(x)\xi| + |\mathcal{B}(x)\xi - \mathcal{B}(y)\xi| \leq |\mathcal{B}(x)\xi| + \alpha^{-1}\varepsilon|\xi| \leq (1 + \varepsilon)|\mathcal{B}(x)\xi| \tag{3.6}$$

from the property (2.2) for the matrix $\mathcal{B}$.

We estimate

$$\begin{aligned}\int_\gamma \rho_t(x)\,|\mathcal{B}(x)d\gamma| &= \int_\gamma \int_{|y|<1} \rho(x\delta_{t\eta(x)}y)\varphi(y)dy\,|\mathcal{B}(x)d\gamma| \\ &= \int_{|y|<1} \varphi(y)dy \int_\gamma \rho(x\delta_{t\eta}y)\,|\mathcal{B}(x)d\gamma|\,. \end{aligned} \tag{3.7}$$

Let us fix y for a moment and consider the integral $\int_\gamma \rho(x\delta_{t\eta}y)\,|\mathcal{B}(x)d\gamma|$. We denote by $\tilde{\gamma}$ the image of the curve γ under the map $T_{t,y}(x)|_\gamma$. We recall that the map $T_{t,y}$ has the Jacobian matrix of the form $I + t\hat{T}$, where I is the identity matrix and elements of the matrix $\hat{T}$ depend on $t, x, y, \nabla\eta(x), \eta(x)$. The properties of the function η, the choice of $|y| < 1$, $|t| < 1$, and the boundedness of the domain U imply that the norm of $\hat{T}$ is bounded by a constant C that depends only on m, and on the diameter of U. It is obvious, that the curve $\tilde{\gamma}$ connects the compacts K_0 and K_1. If the curve $\tilde{\gamma}$ is not horizontal and therefore it is not locally rectifiable, then

$$\int_\gamma \rho(x\delta_{t\eta}y)\,|\mathcal{B}(x)d\gamma| \geq \alpha^{-1}\int_\gamma \rho(x\delta_{t\eta}y)\,|d\gamma| \geq \frac{1}{\alpha(1+tC)}\int_{\tilde{\gamma}} \rho(\tilde{\gamma})\,|d\tilde{\gamma}| = \infty\,.$$

Here α is the constant from (2.2). If the curve $\tilde{\gamma}$ is horizontal, then $\tilde{\gamma} \in \Gamma(K_0, K_1, \Omega)$. We choose t sufficiently small to satisfy $d_D(x, x\delta_{t\eta(x)}y) = |x^{-1}x\delta_{t\eta(x)}y| = t\eta(x) \leq \varsigma(\varepsilon)$ and $t < \varepsilon/C$. Then we deduce

$$\begin{aligned}\int_\gamma \rho(x\delta_{t\eta}y)\,|\mathcal{B}(x)d\gamma| &\geq \frac{1}{1+tC}\int_{\tilde{\gamma}} \rho(z)\,|\mathcal{B}(T^{-1}_{t,y}(z))d\tilde{\gamma}| \\ &\geq \frac{1}{(1+\varepsilon)^2}\int_{\tilde{\gamma}} \rho(\tilde{\gamma})\,|\mathcal{B}((z))d\tilde{\gamma}| \geq \frac{1}{(1+\varepsilon)^2}\end{aligned}$$

from (3.6).

Since $\int_{|y|<1}\varphi(y)dy = 1$ in (3.7), we conclude that $(1+\varepsilon)^2\rho_t(x) \in \mathcal{FM}(|\mathcal{B}d\Gamma|)$. We obtain

$$M = \inf_{\hat{\rho}\in\mathcal{C}}\int_\Omega \hat{\rho}^p(x)dx \leq (1+\varepsilon)^{2p}\int_\Omega \rho_t^p(x)dx \leq (1+\varepsilon)^{2p}(2\varepsilon + M_p(|\mathcal{B}d\Gamma|))$$

from (3.4) and (3.5). Since ε and $\rho \in \mathcal{FM}(|\mathcal{B}d\Gamma|)$ were arbitrary, we get $M \leq M_p(|\mathcal{B}d\Gamma|)$.

The reverse inequality is obvious and we have (3.3) as desired. □

4. Proof of Theorem 2.2. We split the proof into two steps.

Step 1. We show the inequalities

$$M_p(|\mathcal{B}d\Gamma|) \leq M_{\mathcal{A}_p}(d\Gamma) \leq \operatorname{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) < \infty\,. \tag{4.1}$$

The set $\mathcal{FC}(K_0, K_1; \Omega)$ is not empty and hence, $\operatorname{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) < \infty$. Let us choose u from $\mathcal{FC}(K_0, K_1; \Omega)$. Since the p-module of a family of non-rectifiable curves vanishes [14], we can assume that curves connecting compacts K_0 and K_1 are parameterized by the arc length parameter $s \in I \subset \boldsymbol{R}$. We claim $du(\gamma(s))/ds = \langle\nabla_0 u(\gamma(s)), \dot{\gamma}(s)\rangle$. Since γ is horizontal, we have the equality:

$$\dot\gamma(s) = \sum_{j=1}^{n_1} a_j(s) X_{1j}(\gamma(s))\,. \tag{4.2}$$

In [23] one can find the following representation: $X_{ij}(y) = \partial/\partial x_{ij} + \sum_{l,k} P_{ij,lk}(y)\partial/\partial x_{lk}$, $i = 1,\dots,m$, $j = 1,\dots,n_i$. Here $P_{ij,lk}(y)$ are homogeneous polynomials of order $l - i$, that possess the following properties:

1) $P_{ij,lk}(0) = 0$, 2) $P_{ij,lk}(y) = 0$ for $l \le i$, 3) $P_{ij,lk}(y)$ does not depend on $y_{l'k'}$ for $l' \ge l$.

Hence, for horizontal vector fields we have

$$X_{1j}(y) = \frac{\partial}{\partial x_{1j}} + \sum_{l\ge 2,k} P_{1j,lk}(y)\frac{\partial}{\partial x_{lk}}\,. \tag{4.3}$$

Let us substitute in (4.2) the expression for X_{1j} from (4.3). We then obtain

$$\dot\gamma(s) = \sum_{1\le p\le m, 1\le q\le n_p} \dot\gamma_{pq}(s)\frac{\partial}{\partial x_{pq}} = \sum_{j=1}^{n_1} a_j(s)\left(\frac{\partial}{\partial x_{1j}} + \sum_{l\ge 2,k} P_{1j,lk}(\gamma(s))\frac{\partial}{\partial x_{lk}}\right).$$

Comparing the coefficients at $\partial/\partial x_{pq}$, we deduce $a_j(s) = \dot\gamma_{1j}(s)$, $j = 1,\dots,n_1$, and $\dot\gamma_{pq}(s) = \sum_{j=1}^{n_1} a_j(s)P_{1j,pq}(\gamma_1(s),\dots,\gamma_{p-1}(s))$ for $p \ge 2$. Here $\gamma_i = (\gamma_{i1},\dots,\gamma_{in_i}) \in V_i$. Hence the tangent vector $\dot\gamma(s)$ has the form $\dot\gamma(s) = (\dot\gamma_1(s), 0,\dots,0)$ in the left-invariant basis of the vector fields X_{ij}, $i = 1,\dots,m$, $j = 1,\dots,n_i$. We get

$$\begin{aligned}\frac{du(\gamma(s))}{ds} &= \sum_{j=1}^{n_1}\frac{\partial u}{\partial x_{1j}}\dot\gamma_{1j}(s) + \sum_{p\ge 2,q}\frac{\partial u}{\partial x_{pq}}\dot\gamma_{pq}(s)\\ &= \sum_{j=1}^{n_1}\frac{\partial u}{\partial x_{1j}}\dot\gamma_{1j}(s) + \sum_{p\ge 2,q}\frac{\partial u}{\partial x_{pq}}\cdot\sum_{j=1}^{n_1}\dot\gamma_{1j}(s)P_{1j,pq}(\gamma(s))\\ &= \sum_{j=1}^{n_1}\left(\frac{\partial u}{\partial x_{1j}} + \sum_{p\ge 2,q} P_{1j,pq}(\gamma(s))\frac{\partial u}{\partial x_{pq}}\right)\dot\gamma_{1j}(s) = \langle\nabla_0 u(\gamma(s)), \dot\gamma(s)\rangle\,.\end{aligned}$$

Hence we have $\int_\gamma \nabla_0 u\, d\gamma = \int_I \langle\nabla_0 u(\gamma(s)), \dot\gamma(s)\rangle ds = u(x_1) - u(x_0) = 1$, where $x_0 \in K_0$, $x_1 \in K_1$, and the equality holds except for some exceptional family of p-module zero. Thus, $\nabla_0 u \in \mathcal{F}M(d\Gamma)$ for p-almost all curves of $d\Gamma$, and $M_{\mathcal{A}_p}(d\Gamma) \le \int_\Omega |\mathcal{A}\nabla_0 u|^p\, dx$. Taking the infimum with respect to u, we obtain the second inequality of (4.1).

Now, let $\xi \in \mathcal{F}M(d\Gamma)$ p-almost everywhere. Then we have $1 \le \int_\gamma \xi\, d\gamma = \int_\gamma \mathcal{A}\xi\mathcal{B}\, d\gamma$ $\le \int_\gamma |\mathcal{A}\xi||\mathcal{B}d\gamma|$. So $|\mathcal{A}\xi| \in \mathcal{F}M(|\mathcal{B}d\Gamma|)$. Finally, we obtain $M_p(|\mathcal{B}d\Gamma|) \le \int_\Omega |\mathcal{A}\xi|^p\, dx$. Since $\xi \in \mathcal{F}M(d\Gamma)$ was arbitrary, we have proved the first inequality of (4.1).

Step 2. To show

$$\operatorname{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) \le M_p(|\mathcal{B}d\Gamma|)\,, \tag{4.4}$$

we choose a function $\rho \in \mathcal{FM}(|\mathcal{B}d\Gamma|)$. For each $x \in \Omega$ we assume Γ_0^x to be the family of curves starting at K_0 and terminating at x. Let us define

$$u(x) = \inf_{\gamma \in \Gamma_0^x} \int_\gamma \rho|\mathcal{B}d\gamma|\,. \tag{4.5}$$

We will construct an admissible function from $\mathcal{FC}(K_0, K_1; \Omega)$ making use of (4.5). First, we prove that u possesses the following properties:

(i) $u \in ACL_p(\Omega)$.

(ii) The inequality

$$|\mathcal{A}\nabla_0 u(x)| \leq \rho(x) \tag{4.6}$$

holds for almost all points $x \in \Omega$.

(iii) $\lim u(x) = 0$ as $x \to K_0$ along p-almost all curves $\gamma \in \Gamma(K_0, K_1; \Omega)$.

(iv) $\liminf u(x) \geq 1$ as $x \to K_1$ along p-almost all curves $\gamma \in \Gamma(K_0, K_1; \Omega)$.

We have

$$\int_\gamma \rho|\mathcal{B}d\gamma| \leq \alpha \int_\gamma \rho\,|d\gamma| < \infty \tag{4.7}$$

from the property (2.2) for the matrix $\mathcal{B}$. The finiteness of the last integral for p-almost all curves follows from the properties of the measure system [14]. By the definition of $u(x)$ we get

$$|u(x) - u(y)| \leq \int_\gamma \rho|\mathcal{B}d\gamma| \leq \alpha \int_\gamma \rho\,|d\gamma| \tag{4.8}$$

for arbitrary points $x, y \in \gamma$. Let us fix a horizontal vector field $X_{1j} \in V_1$, $j = 1, \ldots, n_1$, and denote by β_j an orbit of X_{1j}. If we apply (4.8) to β_j, we obtain that u is absolutely continuous along p-almost all curves of horizontal fibration. Thus the horizontal derivatives $X_{1j}u$, $j = 1, \ldots, n_1$, exist for almost all points in Ω and satisfy the inequality $|X_{1j}u| \leq \alpha\rho$. The assumption $\rho \in L_p(\Omega)$ implies that $\nabla_0 u \in L_p(\Omega)$.

To show (ii) we take $x \in \Omega$, where $\nabla_0 u(x)$ exists, and a horizontal vector field $Y(x)$, $|Y(x)| = 1$. Then (4.8) implies

$$\begin{aligned} \langle \nabla_0 u(x), Y(x)\rangle &= \lim_{h\to 0} \frac{u(x \exp hY(x)) - u(x)}{h} \\ &\leq \lim_{h\to 0} \frac{1}{h} \int_0^h \rho(x \exp tY(x))|\mathcal{B}(x \exp tY(x))Y(x)|dt \\ &= \rho(x)|\mathcal{B}(x)Y(x)| \end{aligned} \tag{4.9}$$

for almost all $x \in \Omega$. Now, choosing $Y(x) = \mathcal{A}^2\nabla_0 u(x)/|\mathcal{A}^2\nabla_0 u(x)|$, we get

$$\left\langle \nabla_0 u(x), \frac{\mathcal{A}^2\nabla_0 u(x)}{|\mathcal{A}^2\nabla_0 u(x)|}\right\rangle \leq \rho(x)\left|\mathcal{B}(x)\frac{\mathcal{A}^2\nabla_0 u(x)}{|\mathcal{A}^2\nabla_0 u(x)|}\right| = \rho(x)\left|\frac{\mathcal{A}\nabla_0 u(x)}{|\mathcal{A}^2\nabla_0 u(x)|}\right|.$$

Since $\langle \mathcal{A}^2\nabla_0 u(x), \nabla_0 u(x)\rangle = |\mathcal{A}\nabla_0 u(x)|^2$, we have the property (ii).

Using the arc length parameter s, we deduce $0 \leq u(\gamma(s)) \leq \int_\gamma \rho|\mathcal{B}d\gamma| \leq \alpha \int_0^s \rho\,|d\gamma| \to 0$ as $s \to 0$ from (4.5) and (4.7). Thus, $\lim u(x) = 0$ as $x \to K_0$ along γ.

We prove (iv) by contradiction. Suppose that there exists a curve γ_1 such that $c = \liminf_{s\to l_{\gamma_1}} u(\gamma_1(s)) < 1$, where l_{γ_1} is the length of γ_1. We fix $\varepsilon = 1 - c > 0$. By definition, there is $s_0 \in (0, l_{\gamma_1})$ such that $|u(\gamma_1(s_0)) - c| < \varepsilon/3$, and $\int_{s_0}^{l_{\gamma_1}} \rho ds < \varepsilon/3\alpha$. We consider the family Γ_0^x with $x = \gamma_1(s_0)$. The definition of the function $u(x)$ implies that we can find $\gamma_2 \in \Gamma_0^x$ with $\int_{\gamma_2} \rho|\mathcal{B}d\gamma| < u(x) + \varepsilon/3$. Let us denote by γ_3 the arc of the curve γ_1 between the points $\gamma_1(s_0)$ and $\gamma_1(l_{\gamma_1})$. Then, $\gamma_2 \cup \gamma_3 \in \Gamma(K_0, K_1; \Omega)$, and by (4.8) we get

$$\int_{\gamma_2\cup\gamma_3} \rho|\mathcal{B}d\gamma| < u(x) + \frac{\varepsilon}{3} + \alpha\int_{s_0}^{l_{\gamma_1}} \rho ds < c + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = 1\,,$$

which contradicts to $\rho \in \mathcal{F}M(|\mathcal{B}d\Gamma|)$. Hence (iv) holds.

To complete the proof of the Step 2, we denote by $\tilde{u}(x) = \min(u(x), 1)$. Then $\tilde{u} \in \mathcal{F}C(K_0, K_1; \Omega)$ and we have $\mathrm{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) \le \int_\Omega |\mathcal{A}\nabla_0\tilde{u}|^p\,dx \le \int_\Omega |\mathcal{A}\nabla_0 u|^p\,dx \le \int_\Omega \rho^p\,dx$ by the definition of the $\mathcal{A}_p$-capacity and property (ii). Taking the infimum with respect to ρ, we obtain (4.4).

Theorem 2.2 follows from (4.1) and (4.4). □

5. Proof of Theorem 2.4. Let $\varepsilon \in (0, 1/2)$. By definition, there is a non-negative function ρ such that $\rho \in \mathcal{F}M(|\mathcal{B}d\Gamma|)$ and $\|\rho \mid L_p(\Omega)\|^p \le M_p(|\mathcal{B}d\Gamma|)+\varepsilon$. We may assume that ρ is strictly positive on $\Omega \setminus K_0 \cup K_1$. If this were not this case, we could consider the cut-off-function $\max(\rho, 1/m)$ instead of ρ. Moreover, we can suppose that ρ is continuous on $\Omega \setminus K_0 \cup K_1$ by Theorem 3.1.

Let ρ' be as in Lemma 3.1. We show that $\int_\gamma \rho'\,|\mathcal{B}d\Gamma| > 1 - 2\varepsilon$ for $\gamma \in \Gamma(K_0^j, K_1^j; \Omega)$ with sufficiently big j. In fact, suppose the contrary. Then there would be a sequence $\{j_k\}$ and curves $\gamma_{j_k} \in \Gamma(K_0^{j_k}, K_1^{j_k}; \Omega)$ such that $\int_{\gamma_{j_k}} \rho'\,|\mathcal{B}d\Gamma| \le 1-2\varepsilon$. By Lemma 3.1 we would find $\gamma \in \Gamma(K_0, K_1; \Omega)$ with $\int_{\tilde{\gamma}} \rho\,|\mathcal{B}d\Gamma| \le 1 - 2\varepsilon + \varepsilon = 1 - \varepsilon$, contradicting $\rho \in \mathcal{F}M(|\mathcal{B}d\Gamma|)$.

Now we can finish the proof. Since $(1-2\varepsilon)^{-1}\rho' \in \mathcal{F}M_p(|\mathcal{B}d\Gamma_j|)$, $\Gamma_j = \Gamma(K_0^j, K_1^j; \Omega)$, for sufficiently big j, we have

$$M_p(|\mathcal{B}d\Gamma_j|) \le \int_{\Omega\setminus K_0\cup K_1} [(1-2\varepsilon)^{-1}\rho']^p\,dx \le (1-2\varepsilon)^{-p}(M_p(|\mathcal{B}d\Gamma|) + \varepsilon)\,.$$

Letting $j \to \infty$ and $\varepsilon \to 0$, we obtain $\limsup_{j\to\infty} M_p(|\mathcal{B}d\Gamma_j|) \le M_p(|\mathcal{B}d\Gamma|)$. Since $M_p(|\mathcal{B}d\Gamma|) \le M_p(|\mathcal{B}d\Gamma_j|)$ for arbitrary j, we obtain the statement of Theorem 2.4. □

6. Proof of Theorem 2.1. Let K_0^j and K_1^j be sequences of compacts which were described at the beginning of Section 3. We take $u \in \mathcal{F}C(K_0^j, K_1^j; \Omega)$ and put

$$\bar{u} = \begin{cases} 0 & \text{on } K_0^j \cap \Omega\,, \\ 1 & \text{on } K_1^j \cap \Omega\,, \\ u & \text{on } \Omega \setminus (K_0^j \cup K_1^j)\,. \end{cases}$$

We have $u = 0$ on p-almost all curves in $K_0^j \cap \Omega$ and $u = 1$ on p-almost all curves in $K_1^j \cap \Omega$ by the definition of $\mathcal{F}C(K_0^j, K_1^j; \Omega)$. Hence $u = \bar{u}$ along p-almost all curves in Ω and $\bar{u} \in \mathcal{F}C^\star(K_0, K_1; \Omega)$. Therefore, $\mathrm{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) \leq \int_\Omega |\mathcal{A}\nabla_0 \bar{u}|^p\, dx = \int_\Omega |\mathcal{A}\nabla_0 u|^p\, dx$, and taking the infimum with respect to u, we obtain $\mathrm{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) \leq \mathrm{cap}_{\mathcal{A}_p}(K_0^j, K_1^j; \Omega)$. Theorem 2.2 implies the equalities $\mathrm{cap}_{\mathcal{A}_p}(K_0^j, K_1^j; \Omega) = M_p(|\mathcal{B}d\Gamma_j|)$ and $\mathrm{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) = M_p(|\mathcal{B}d\Gamma|)$. $M_p(|\mathcal{B}d\Gamma_j|)$ tends to $M_p(|\mathcal{B}d\Gamma|)$ as $j \to \infty$ by Theorem 2.4. Finally, $\mathrm{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) \leq \mathrm{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega)$. The reverse inequality is obtained from the inclusion $\mathcal{F}C^\star(K_0, K_1; \Omega) \subset \mathcal{F}C(K_0, K_1; \Omega)$. □

7. Proof of Theorem 2.3. We use the idea of the proof in [2]. Let $\mathcal{F}C^\star = \mathcal{F}C^\star(K_0, K_1; \Omega)$ be as in Definition 2.4, $\Upsilon = \{\xi(x) = (\xi_1(x), \ldots, \xi_{n_1}(x));\ \int_\Omega |\mathcal{B}\xi|^q\, dx \leq 1\}$. We introduce the bilinear functional $\Psi(u, \xi) = \int_\Omega \langle \nabla_0 u, \xi\rangle dx$, which is defined on $\mathcal{F}C^\star \times \Upsilon$. Since $\mathcal{F}C^\star$ is convex, Υ is a weakly compact convex set, and $\xi \mapsto \Psi(u, \xi)$ is continuous with respect to weak topology of Υ. The minimax theorem [11] implies that

$$\inf_{u \in \mathcal{F}C^\star} \sup_{\xi \in \Upsilon} \int_\Omega \langle \nabla_0 u, \xi\rangle dx = \sup_{\xi \in \Upsilon} \inf_{u \in \mathcal{F}C^\star} \int_\Omega \langle \nabla_0 u, \xi\rangle dx\,. \tag{7.1}$$

We will show that the left-hand side is equal to $\big(\mathrm{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega)\big)^{1/p}$ and the right-hand side is equal to $\big(M_{\mathcal{B}_q}(\nabla_0 C^\star)\big)^{-1/q}$. Hölder's inequality and the definition of Υ imply that

$$\int_\Omega \langle \nabla_0 u, \xi\rangle dx \leq \left(\int_\Omega |\mathcal{A}\nabla_0 u|^p\, dx\right)^{1/p} \left(\int_\Omega |\mathcal{B}\xi|^q\, dx\right)^{1/q} \leq \left(\int_\Omega |\mathcal{A}\nabla_0 u|^p\, dx\right)^{1/p}$$

for all $\xi \in \Upsilon$. The vector $\zeta = (\int_\Omega |\mathcal{A}\nabla_0 u|^p\, dx)^{(1-p)/p} \cdot |\mathcal{A}\nabla_0 u|^{p-2} \cdot \mathcal{A}^2 \nabla_0 u$ belongs to Υ, because $\int_\Omega |\mathcal{B}\zeta|^q dx = 1$. Since $\int_\Omega \langle \nabla_0 u, \zeta\rangle\, dx = (\int_\Omega |\mathcal{A}\nabla_0 u|^p\, dx)^{1/p}$, we have

$$\sup_{\xi \in \Upsilon} \int_\Omega \langle \nabla_0 u, \xi\rangle dx = \left(\int_\Omega |\mathcal{A}\nabla_0 u|^p\, dx\right)^{1/p}.$$

Taking infimum over all $u \in \mathcal{F}C^\star(K_0, K_1; \Omega)$, we obtain the equality

$$\inf_{u \in \mathcal{F}C^\star} \sup_{\xi \in \Upsilon} \int_\Omega \langle \nabla_0 u, \xi\rangle dx = (\mathrm{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega))^{1/p}\,. \tag{7.2}$$

If $\mathrm{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) = 0$, then $\inf_{u \in \mathcal{F}C^\star} \int_\Omega \langle \nabla_0 u, \xi\rangle dx \leq 0$ for all $\xi \in \Upsilon$. This means that the set $\mathcal{F}M(\nabla_0 C^\star)$ is empty and $M_{\mathcal{B}_q}(\nabla_0 C^\star) = \infty$. Now, we assume that the capacity $\mathrm{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega)$ is strictly positive. Let us observe that

$$\begin{aligned}(M_{\mathcal{B}_q}(\nabla_0 C^\star))^{-1/q} &= \left(\inf\left\{\int_\Omega |\mathcal{B}\xi|^q dx\,;\ \inf_{u \in \mathcal{F}C^\star} \int_\Omega \langle \nabla_0 u, \xi\rangle dx \geq 1\right\}\right)^{-1/q} \\ &= \sup\left\{\left(\int_\Omega |\mathcal{B}\xi|^q dx\right)^{-1/q};\ \inf_{u \in \mathcal{F}C^\star} \int_\Omega \langle \nabla_0 u, \xi\rangle dx \geq 1\right\}.\end{aligned} \tag{7.3}$$

We claim

$$\sup\left\{\inf_{u\in\mathcal{F}C^\star}\int_\Omega\langle\nabla_0 u,\xi\rangle dx;\ \int_\Omega|\mathcal{B}\xi|^q dx\le 1\right\} = \sup\left\{\left(\int_\Omega|\mathcal{B}\xi|^q dx\right)^{-1/q};\ \inf_{u\in\mathcal{F}C^\star}\int_\Omega\langle\nabla_0 u,\xi\rangle dx\ge 1\right\}. \tag{7.4}$$

Let us denote by α and β the left-hand side and the right-hand side of (7.4), respectively. If

$$\alpha=\sup\left\{\inf_{u\in\mathcal{F}C^\star}\int_\Omega\langle\nabla_0 u,\xi\rangle dx\,;\ \int_\Omega|\mathcal{B}\xi|^q dx\le 1\right\},$$

then the implication

$$\int_\Omega|\mathcal{B}\xi|^q dx\le 1\ \Rightarrow\ \inf_{u\in\mathcal{F}C^\star}\int_\Omega\langle\nabla_0 u,\xi\rangle dx\le\alpha \tag{7.5}$$

require the inequality $\beta\ge 1$. The negation of (7.5)

$$\inf_{u\in\mathcal{F}C^\star}\int_\Omega\langle\nabla_0 u,\xi\rangle dx>\alpha\ \Rightarrow\ \int_\Omega|\mathcal{B}\xi|^q dx>1$$

implies that $\beta=(\int_\Omega|\mathcal{B}\xi|^q dx)^{-1/q}\le 1$. We deduce that $\beta=1$. In the same way we show that $\alpha=1$. We conclude that (7.4) holds.

Finally, we get

$$\sup_{\xi\in\Upsilon}\inf_{u\in\mathcal{F}C^\star}\int_\Omega\langle\nabla_0 u,\xi\rangle dx=(M_{\mathcal{B}_q}(\nabla_0 C^\star))^{-1/q} \tag{7.6}$$

from (7.4) and (7.3). Now Theorem 2.3 follows from (7.6), (7.2) and (7.1).

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