Research Article

Patrizia Pucci* and Letizia Temperini

(p, Q) systems with critical singular exponential nonlinearities in the Heisenberg group

Abstract: The paper deals with the existence of solutions for (p, Q) coupled elliptic systems in the Heisenberg group, with critical exponential growth at infinity and singular behavior at the origin. We derive existence of nonnegative solutions with both components nontrivial and different, that is solving an actual system, which does not reduce into an equation. The main features and novelties of the paper are the presence of a general coupled critical exponential term of the Trudinger-Moser type and the fact that the system is set in $\mathbb{H}^n$.

Keywords: (p,Q) Laplacian, nonlinear system, critical exponential nonlinearities, variational methods, Heisenberg group

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1 Introduction

In this paper, we study the following system in the Heisenberg group $\mathbb{H}^n$

$$\begin{cases} -\Delta_{H,p}u - \Delta_{H,Q}u + |u|^{p-2}u + |u|^{Q-2}u = \dfrac{F_u(\xi, u, v)}{r(\xi)^\beta} + g(\xi), \\[2mm] -\Delta_{H,p}v - \Delta_{H,Q}v + |v|^{p-2}v + |v|^{Q-2}v = \dfrac{F_v(\xi, u, v)}{r(\xi)^\beta} + h(\xi), \end{cases} \quad (\mathcal{S})$$

where $Q = 2n + 2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$, $1 < p < Q$, $0 \le \beta < Q$ and g, h are nontrivial nonnegative functionals in the dual space $HW^{-1,Q'}(\mathbb{H}^n)$ of $HW^{1,Q}(\mathbb{H}^n)$, which will be characterized later on. In other words, $h, g \ne 0$ and $\langle h, u \rangle_{HW^{-1,Q'},HW^{1,Q}} \ge 0$, $\langle g, u \rangle_{HW^{-1,Q'},HW^{1,Q}} \ge 0$ for all $u \in HW^{1,Q}(\mathbb{H}^n)$, with $u \ge 0$ in $\mathbb{H}^n$. For brevity in what follows the duality pairing between $HW^{1,Q}(\mathbb{H}^n)$ and $HW^{-1,Q'}(\mathbb{H}^n)$ will be denoted simply by $\langle \cdot, \cdot \rangle$. The function r in $(\mathcal{S})$ is the *Korányi norm* in the Heisenberg group $\mathbb{H}^n$, which is defined as

$$r(\xi) = r(z, t) = (|z|^4 + t^2)^{1/4},$$

with $\xi = (z, t) \in \mathbb{H}^n$, $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \in \mathbb{R}$. From here on, $\cdot$ denotes the natural inner product in any Euclidean space $\mathbb{R}^d$ for any dimension $d \ge 1$ and $|\cdot|$ the corresponding Euclidean norm.

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* Corresponding author: Patrizia Pucci, Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, 06123, Perugia, Italy, e-mail: patrizia.pucci@unipg.it
Letizia Temperini: Dipartimento di Matematica e Informatica 'Ulisse Dini', Università degli Studi di Firenze, Viale Morgagni 67/a, Firenze, 50134, Italy, e-mail: letizia.temperini@unifi.it

We denote by $D_H\varphi$ the horizontal gradient of a regular function φ, that is,

$$D_H\varphi = \sum_{j=1}^{n}[(X_j\varphi)X_j + (Y_j\varphi)Y_j],$$

where $\{X_j, Y_j\}_{j=1}^{n}$ is the standard basis of the horizontal left invariant vector fields on $\mathbb{H}^n$,

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},$$

for $j = 1,\ldots, n$. The operator $\Delta_{H,\wp}$, with $\wp \in \{p, Q\}$, appearing in system $(\mathcal{S})$ is the well-known *horizontal $\wp$-Laplacian* on the Heisenberg group, which is defined as

$$\Delta_{H,\wp}\varphi = \text{div}_H(|D_H\varphi|_H^{\wp-2}D_H\varphi),$$

for all $\varphi \in C^2(\mathbb{H}^n)$, where $|D_H\varphi|_H = \sqrt{\sum_{j=1}^{n}[(X_j\varphi)^2 + (Y_j\varphi)^2]}$.

Taking inspiration from [1], we assume that the functions F_u, F_v are partial derivatives of *a Carathéodory function F, of exponential type*, and are assumed to satisfy

(F_1) $F(\xi, \cdot, \cdot) \in C^1(\mathbb{R}^2)$ for a.e. $\xi \in \mathbb{H}^n$, $F_u(\xi, u, v) = 0$ for all $u \leq 0$ and $v \in \mathbb{R}$, $F_v(\xi, u, v) = 0$ for all $u \in \mathbb{R}$ and $v \leq 0$, $F_u(\xi, u, 0) = 0$ for all $u \in \mathbb{R}$, and $F_v(\xi, 0, v) = 0$ for all $v \in \mathbb{R}$. Furthermore, there is $\alpha_0 > 0$ with the property that for all $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that

$$|\nabla F(\xi, u, v)| \leq \varepsilon\varrho^{Q-1} + \kappa_\varepsilon\left(e^{\alpha_0\varrho^{Q'}} - S_{Q-2}(\alpha_0, \varrho)\right)$$

for a.e. $x \in \mathbb{H}^n$ and all $(u, v) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = [0, \infty)$, $\varrho = \sqrt{u^2 + v^2}$, $\nabla F = (F_u, F_v)$,

$$Q' = \frac{Q}{Q-1} \quad \text{and} \quad S_{Q-2}(\alpha, t) = \sum_{j=0}^{Q-2}\frac{\alpha^j t^{jQ'}}{j!}, \alpha > 0, t \in \mathbb{R};$$

(F_2) there exists $\nu > Q$ such that $0 < \nu F(\xi, u, v) \leq \nabla F(\xi, u, v)\cdot(u, v)$ for a.e. $\xi \in \mathbb{H}^n$ and for any pair $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$, where $\mathbb{R}^+ = (0, \infty)$.

A function satisfying (F_1)–(F_2) is given for example by $F(u, v) = u_+^Q(e^{v_+^{Q'}} - S_{Q-1}(1, v_+))$ for all $(u, v) \in \mathbb{R} \times \mathbb{R}$. Indeed, $F_u(u, v) = Qu_+^{Q-1}(e^{v_+^{Q'}} - S_{Q-1}(1, v_+))$, $F_v(u, v) = Q'v_+^{Q'-1}u_+^Q(e^{v_+^{Q'}} - S_{Q-2}(1, v_+))$ for all $(u, v) \in \mathbb{R} \times \mathbb{R}$ and it is easy to check that condition (F_1) is satisfied for any $\alpha_0 > 1$. It is not hard to see that also (F_2) is verified with $\nu = Q(1 + Q') > Q$. Of course, it is possible to produce an entire class of functions satisfying (F_1)–(F_2) by taking $F(\xi, u, v) = a(\xi)\Phi(u, v)$ for all $(\xi, u, v) \in \mathbb{H}^n \times \mathbb{R} \times \mathbb{R}$, where a is a positive measurable function, $a \in L^\infty(\mathbb{H}^n)$ and ess $\inf_{\xi\in\mathbb{H}^n} a(\xi) > 0$, while $\Phi(u, v) = u_+^Q(e^{v_+^{Q'}} - S_{Q-1}(1, v_+))$, $(u, v) \in \mathbb{R} \times \mathbb{R}$.

Let us also introduce two alternative assumptions under which it is possible to construct a solution of $(\mathcal{S})$ with different components

(H_1) if $g \not\equiv h$, assume that $F_u(\xi, u, u) \equiv F_v(\xi, u, u)$ for a.e. $x \in \mathbb{H}^n$ and all $u \in \mathbb{R}$,

(H_2) if $g \equiv h$, assume that $F_u(\xi, u, u) \neq F_v(\xi, u, u)$ for a.e. $\xi \in \mathbb{H}^n$ and all $u \in \mathbb{R}^+$.

Clearly, the example $F(u, v) = u_+^Q(e^{v_+^{Q'}} - S_{Q-1}(1, v_+))$, $(u, v) \in \mathbb{R} \times \mathbb{R}$, verifies (H_2).

The natural space where finding solutions of $(\mathcal{S})$ is

$$W = [HW^{1,p}(\mathbb{H}^n) \cap HW^{1,Q}(\mathbb{H}^n)]^2$$

endowed with the norm

$$\|(u, v)\| = \|u\|_{HW^{1,p}} + \|v\|_{HW^{1,p}} + \|u\|_{HW^{1,Q}} + \|v\|_{HW^{1,Q}},$$

where $\|u\|_{HW^{1,\wp}} = (\|u\|_\wp^\wp + \|D_H u\|_\wp^\wp)^{1/\wp}$ for all $u \in HW^{1,\wp}(\mathbb{H}^n)$ and $\|\cdot\|_\wp$ denotes the canonical $L^\wp(\mathbb{H}^n)$ norm for any $\wp \geq 1$. We say that a pair (u, v) is nonnegative if both components are nonnegative in $\mathbb{H}^n$.

Existence and multiplicity of nontrivial nonnegative solutions for equations and systems in the entire Heisenberg group $\mathbb{H}^n$, involving elliptic operators with standard Q-growth and critical Trudinger-Moser nonlinearities, have been proved in a series of papers. We refer to [2–6] and to references therein.

On the other hand, in the literature there are few contributions devoted to the study of coupled systems involving both exponential nonlinearities and nonstandard growth conditions in the Heinsenberg context. In the Euclidean setting, a similar problem has been studied in [1], where the authors consider coupled systems involving exponential nonlinearities and (p, N) growth. Other references, again in the Euclidean setting, are given by [7–11]. We also refer to the recent paper [12], which contains the proof of a non-singular version of the Moser-Trudinger inequality in the Cartesian product of Sobolev spaces.

In the Heisenberg context, existence of solutions for the equation corresponding to system $(\mathcal{S})$ has been established in [13]. Let us cite [14–16] for related problems.

In this paper, we solve for the first time in the literature a coupled exponential system in the Heisenberg group $\mathbb{H}^n$ driven by a (p, Q) operator and establish the existence of nonnegative solutions for system $(\mathcal{S})$, which have both components nontrivial and under further reasonable assumptions different.

Theorem 1.1. *Let $1 < p < Q$ and $0 \leq \beta < Q$. Suppose that F verifies (F_1)–(F_2) and that g, h are nontrivial nonnegative functionals of the dual space $HW^{-1,Q'}(\mathbb{H}^n)$ of $HW^{1,Q}(\mathbb{H}^n)$. Then, there exists $\sigma > 0$ such that system $(\mathcal{S})$ admits at least one nonnegative solution $(u_{g,h}, v_{g,h})$ in W, with both components nontrivial, provided that $0 < s_{g,h} = \max\{\|g\|_{HW^{-1,Q'}(\mathbb{H}^n)}, \|h\|_{HW^{-1,Q'}(\mathbb{H}^n)}\} \leq \sigma$. Moreover,*

$$\lim_{s_{g,h}\to 0^+} \|(u_{g,h}, v_{g,h})\| = 0 \tag{1.1}$$

holds true. Furthermore, if one between (H_1) and (H_2) holds, then the constructed solution $(u_{g,h}, v_{g,h})$ has the property that $u_{g,h} \not\equiv v_{g,h}$ in $\mathbb{H}^n$.

Since the solution $(u_{g,h}, v_{g,h})$, constructed in Theorem 1.1, has both components nontrivial and different, it is evident that it solves an actual system, which does not reduce into an equation.

The essential tool when dealing with exponential nonlinearities is the celebrated Trudinger-Moser inequality. The Trudinger-Moser inequality in bounded domains Ω of the Heisenberg group was first established in [17] by Cohn and Lu, using a sharp representation formula for functions of class $C_c^\infty(\mathbb{H}^n)$ in terms of the horizontal gradient. The authors in [17] adapted Adams' idea in [18] to avoid considering the horizontal gradient of the rearrangement function, which is not available in the Heisenberg setting. The situation is more involved when concerning the Trudinger-Moser-type inequalities for unbounded domains of $\mathbb{H}^n$, since Adams' approach does not work any longer. However, Lam, Lu and Tang obtained in [3], see also [19], a sharp Trudinger-Moser inequality on the whole Heisenberg group $\mathbb{H}^n$, which is sub-critical in the sense clarified in Theorem 2.2. This inequality is crucial in the proof of Theorem 1.1.

The proof of Theorem 1.1 is obtained via an application of the Ekeland variational principle and the Trudinger-Moser inequality on the whole Heisenberg group $\mathbb{H}^n$. Even if the argument follows somehow the strategies in [1,13–15,20] and relies on standard variational methods, the extension to this more general context is pretty involved and leads to new difficulties, arising from the non-Euclidean and vectorial nature of the problem. In particular, a delicate step is the proof of the fact that both components of the constructed solution are nontrivial and different. A similar process to show that solutions of systems have both components nontrivial and different first appears, under different assumptions, in [1,15,21].

Theorem 1.1 improves in several directions previous results, not only from the Euclidean to the Heisenberg setting but also for the presence of the coupled exponential nonlinearities and the (p, Q) growth. In particular, Theorem 1.1 extends Theorem 1.1 of [1] because of the non-Euclidean context and the presence of the singularity at zero in the right hand side of $(\mathcal{S})$, and also Theorem 1.1 of [13] from the scalar to the vectorial case.

The paper is organized as follows. In Section 2, we recall some basic definitions and backgrounds related to the Heisenberg group $\mathbb{H}^n$, as well as useful properties of the solution space W and some technical

lemmas on the Trudinger-Moser inequality in the Heisenberg group. In Section 3, we prove Theorem 1.1, using a minimization argument based on the Ekeland variational principle.

2 Preliminaries

In this section, we briefly recall some useful notations and preliminaries on the Heisenberg group. For a complete treatment we refer to [22–26].

Let $\mathbb{H}^n$ be the Heisenberg group of topological dimension $2n + 1$, that is, the Lie group which has $\mathbb{R}^{2n+1}$ as a background manifold and whose group structure is given by the non-Abelian law

$$\xi \circ \xi' = \left(z + z', t + t' + 2 \sum_{i=1}^{n} (y_i x_i' - x_i y_i') \right) \tag{2.1}$$

for all $\xi, \xi' \in \mathbb{H}^n$, with

$$\xi = (z, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) \quad \text{and} \quad \xi' = (z', t') = (x_1', \ldots, x_n', y_1', \ldots, y_n', t').$$

The inverse is given by $\xi^{-1} = -\xi$ and so $(\xi \circ \xi')^{-1} = (\xi')^{-1} \circ \xi^{-1}$.

We can consider the family of dilations $(\delta_R)_R$, where $\delta_R : \mathbb{H}^n \to \mathbb{H}^n$ is defined for any $R \in \mathbb{R}$ by

$$\delta_R(\xi) = (Rz, R^2 t) \quad \text{for all} \ \xi = (z, t) \in \mathbb{H}^n. \tag{2.2}$$

It is easy to check that the Jacobian determinant of dilatations δ_R is constant and equal to R^{2n+2}, where the natural number $Q = 2n + 2$ is the *homogeneous dimension* of $\mathbb{H}^n$.

The anisotropic dilation structure on $\mathbb{H}^n$ induces the *Korányi norm*, which is given by

$$r(\xi) = r(z, t) = (|z|^4 + t^2)^{1/4} \quad \text{for all} \ \xi = (z, t) \in \mathbb{H}^n.$$

Consequently, the Korányi norm is homogeneous of degree 1, with respect to the dilations δ_R, $R > 0$, that is,

$$r(\delta_R(\xi)) = r(Rz, R^2 t) = (|Rz|^4 + R^4 t^2)^{1/4} = Rr(\xi)$$

for all $\xi = (z, t) \in \mathbb{H}^n$. The corresponding distance, the so-called *Korányi distance*, is

$$d_K(\xi, \xi') = r(\xi^{-1} \circ \xi') \quad \text{for all} \ (\xi, \xi') \in \mathbb{H}^n \times \mathbb{H}^n.$$

We denote by $B_R(\xi_0) = \{\xi \in \mathbb{H}^n : d_K(\xi, \xi_0) < R\}$ the *Korányi open ball* of radius R centered at ξ_0. For simplicity we put $B_R = B_R(O)$, where $O = (0, 0)$ is the natural origin of $\mathbb{H}^n$.

For any measurable set $U \subset \mathbb{H}^n$ let $|U|$ be the Haar measure of U, which coincides with the $(2n + 1)$-dimensional Lebesgue measure and is invariant under left translations and Q-homogeneous with respect to dilations.

The real Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields on $\mathbb{H}^n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$. This basis satisfies the Heisenberg canonical commutation relations

$$[X_j, Y_k] = -4\delta_{jk} T, \quad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

A left invariant vector field X, which is in the span of $\{X_j, Y_j\}_{j=1}^{n}$, is called *horizontal*.

We define the *horizontal gradient* of a C^1 function $u : \mathbb{H}^n \to \mathbb{R}$ by

$$D_H u = \sum_{j=1}^{n} [(X_j u) X_j + (Y_j u) Y_j].$$

Clearly, $D_H u$ is an element of the span of $\{X_j, Y_j\}_{j=1}^{n}$, denoted by $\mathrm{span}\{X_j, Y_j\}_{j=1}^{n}$. In $\mathrm{span}\{X_j, Y_j\}_{j=1}^{n} \simeq \mathbb{R}^{2n}$, we consider the natural inner product given by

$$(X, Y)_H = \sum_{j=1}^{n} (x^j y^j + \tilde{x}^j \tilde{y}^j)$$

for $X = \{x^j X_j + \tilde{x}^j Y_j\}_{j=1}^n$ and $Y = \{y^j X_j + \tilde{y}^j Y_j\}_{j=1}^n$. The inner product $(\cdot, \cdot)_H$ produces the Hilbertian norm

$$|X|_H = \sqrt{(X, X)_H}$$

for the horizontal vector field X.

For any horizontal vector field function $X = X(\xi)$, $X = \{x^j X_j + \tilde{x}^j Y_j\}_{j=1}^n$, of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$, we define the *horizontal divergence* of X by

$$\mathrm{div}_H X = \sum_{j=1}^n [X_j(x^j) + Y_j(\tilde{x}^j)].$$

Let us now review some classical facts about the first-order Sobolev spaces on the Heisenberg group $\mathbb{H}^n$. We just consider the special case in which $1 \le \wp < \infty$ and Ω is an open set in $\mathbb{H}^n$. Denote by $HW^{1,\wp}(\Omega)$ the *horizontal Sobolev space* consisting of the functions $u \in L^\wp(\Omega)$ such that $D_H u$ exists in the sense of distributions and $|D_H u|_H \in L^\wp(\Omega)$, endowed with the natural norm

$$\|u\|_{HW^{1,\wp}(\Omega)} = (\|u\|_{L^\wp(\Omega)}^\wp + \|D_H u\|_{L^\wp(\Omega)}^\wp)^{1/\wp}, \quad \|D_H u\|_{L^\wp(\Omega)} = \left| \int_\Omega D_H u|_H^\wp \, d\xi \right|^{1/\wp}.$$

When $\Omega = \mathbb{H}^n$ the notation will be simplified into $\|\cdot\|_\wp = \|\cdot\|_{L^\wp(\mathbb{H}^n)}$ and $\|\cdot\|_{HW^{1,\wp}} = \|\cdot\|_{HW^{1,\wp}(\mathbb{H}^n)}$. For a complete treatment on the topic we refer to [27–29] and we just recall that, if $1 \le \wp < Q$, then the embedding $HW^{1,\wp}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [\wp, \wp^\star]$, being $\wp^\star = \wp Q/(Q - \wp)$, while $HW^{1,Q}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [Q, \infty)$. As a consequence, we get a simple lemma for the real separable reflexive Banach space W, which is the solution space for $(\mathcal{S})$ defined in Section 1. Before stating this result, let us introduce some useful remarks concerning the vectorial Sobolev norms.

If $(u, v) \in [HW^{1,Q}(\mathbb{H}^n)]^2$, then $\varrho = |(u, v)| = \sqrt{u^2 + v^2} \in HW^{1,Q}(\mathbb{H}^n)$ and

$$\|\varrho\|_{HW^{1,Q}} \le 2^{1/Q'} \|(u, v)\|_{HW^{1,Q}}. \tag{2.3}$$

The above inequality is trivial when $\varrho \equiv 0$. Otherwise, in the case in which $\varrho(\xi) \ne 0$ at a point $\xi \in \mathbb{H}^n$, we have

$$D_H \varrho = \frac{u}{\varrho} D_H u + \frac{v}{\varrho} D_H v.$$

This implies that $|D_H \varrho| \le |D_H u|_H + |D_H v|_H$ in $\mathbb{H}^n$. Consequently,

$$\|\varrho\|_Q^Q \le 2^{(Q-2)/2}\{\|u\|_Q^Q + \|v\|_Q^Q\}, \quad \|D_H \varrho\|_Q^Q \le 2^{Q-1}\{\|D_H u\|_Q^Q + \|D_H v\|_Q^Q\}.$$

Both estimates give (2.3).

Combining the classical results in the Sobolev space theory, we easily get the next lemma, where we endow $[L^\wp(\mathbb{H}^n)]^2$, with the norm $\|(u, v)\|_\wp = \|u\|_\wp + \|v\|_\wp$, whenever $1 \le \wp < \infty$.

Lemma 2.1. *The embedding $W \hookrightarrow [L^q(\mathbb{H}^n)]^2$ is continuous for all $q \in [p, p^\star] \cup [Q, \infty)$, and*

$$\|(u, v)\|_q \le C_q \|(u, v)\| \quad \text{for all } (u, v) \in W, \tag{2.4}$$

where C_q depends on q, p and Q.

Clearly, when in particular $Q/2 \le p < Q$, the embedding (2.4) is continuous for all $q \in [p, \infty)$.

Throughout the paper, $HW^{-1,Q'}(\mathbb{H}^n)$ denotes the dual space of $HW^{1,Q}(\mathbb{H}^n)$. It is well known, see for example [30], that

$$HW^{-1,Q'}(\mathbb{H}^n) = \left\{ h^0 + \sum_{j=1}^n (h_j^1 X_j + h_j^2 Y_j) : h^0, h_j^1, h_j^2 \in L^{Q'}(\mathbb{H}^n), \ j = 1, \dots, n \right\}, \tag{2.5}$$

where the pairing between a function $u \in HW^{1,Q}(\mathbb{H}^n)$ and a distribution $h = h^0 + \sum_{j=1}^n (h_j^1 X_j + h_j^2 Y_j)$ is given as usual by

$$\langle h, u \rangle = \int\limits_{\mathbb{H}^n} \left\{ h^0 u + \sum_{j=1}^{n} (h_j^1 X_j u + h_j^2 Y_j u) \right\} \mathrm{d}\xi. \tag{2.6}$$

The corresponding norm is

$$\|h\|_{HW^{-1,Q'}} = \inf \left\{ \|h^0\|_{Q'} + \sum_{j=1}^{n} (\|h_j^1\|_{Q'} + \|h_j^2\|_{Q'}) : h = h^0 + \sum_{j=1}^{n} (h_j^1 X_j + h_j^2 Y_j) \right\}. \tag{2.7}$$

Let us now report the next result obtained by Lam et al. in [3], see also [19].

Theorem 2.2. (Trudinger-Moser inequality in $\mathbb{H}^n$ [3]) *There exists a positive constant*

$$\alpha_Q = Q \left(\frac{2\pi^n \Gamma(1/2)\Gamma((Q-1)/2)}{\Gamma(Q/2)\Gamma(n)} \right)^{Q'-1}$$

such that for β, with $0 \le \beta < Q$, and for any α, with $0 < \alpha < \alpha_Q(1 - \beta/Q) = \alpha_{Q,\beta}$, there exists a constant $C_{\alpha,\beta} > 0$ such that the inequality

$$\int\limits_{\mathbb{H}^n} \frac{e^{\alpha|u|^{Q'}} - S_{Q-2}(\alpha, u)}{r(\xi)^\beta} \mathrm{d}\xi \le C_{\alpha,\beta} \|u\|_Q^{Q-\beta}$$

holds for all $u \in HW^{1,Q}(\mathbb{H}^n)$, with $\|D_H u\|_Q \le 1$.

The Trudinger-Moser inequality in Theorem 2.2 is subcritical, since α cannot reach the sharp threshold $\alpha_{Q,\beta}$. For later purposes, let us introduce for all $q \in [Q, \infty)$, the singular eigenvalue defined by

$$\lambda_q = \inf_{\substack{\phi \in HW^{1,Q}(\mathbb{H}^n) \\ \phi \ne 0}} \frac{\|\phi\|^q}{\int\limits_{\mathbb{H}^n} \frac{|\phi|^q}{r(\xi)^\beta} \mathrm{d}\xi}, \tag{2.8}$$

see also [3,4,6]. By the continuity of the Sobolev embedding and by the Hölder inequality, we get that $\lambda_q > 0$ for any $q \in [Q, \infty)$, see [13]. Let us now state Lemma 4.1 of [4] and prove in Lemma 2.4 an extension of this result, which is crucial in the proof of Theorem 1.1.

Lemma 2.3. (Lemma 4.1 of [4]) *Let $\alpha > 0$ and $s > Q$. Assume that $u \in HW^{1,Q}(\mathbb{H}^n)$ is such that $\|u\|_{HW^{1,Q}} \le M$, with $M = M(\alpha, Q, s) > 0$ sufficiently small. Then, there exists a constant $\tilde{C} = \tilde{C}(\alpha, Q, s) > 0$ such that*

$$\int\limits_{\mathbb{H}^n} \frac{\left\{ e^{\alpha|u|^{Q'}} - S_{Q-2}(\alpha, u) \right\} |u|^s}{r(\xi)^\beta} \mathrm{d}\xi \le \tilde{C} \|u\|_{HW^{1,Q}}^s.$$

Lemma 2.4. *Let $\alpha > 0$, $s > Q$ and let $M = M(\alpha, Q, s) > 0$ be the constant determined in* Lemma 2.3. *Assume that (u, v) in $[HW^{1,Q}(\mathbb{H}^n)]^2$ is such that*

$$\|(u, v)\|_{HW^{1,Q}} \le 2^{-1/Q'} M.$$

Then, there exists a constant $C = C(\alpha, Q, s) \ge 0$ such that

$$\int\limits_{\mathbb{H}^n} \frac{\left\{ e^{\alpha \varrho^{Q'}} - S_{Q-2}(\alpha, \varrho) \right\} \varrho^s}{r(\xi)^\beta} \mathrm{d}\xi \le C \|(u, v)\|_{HW^{1,Q}}^s,$$

where $\varrho = |(u, v)| = \sqrt{u^2 + v^2}$.

Proof. From (2.3) it is clear that $\varrho \in HW^{1,Q}(\mathbb{H}^n)$ and $\|\varrho\|_{HW^{1,Q}} \leq 2^{1/Q'}\|(u,v)\|_{HW^{1,Q}} \leq M$. Then, we can apply Lemma 2.3 to the nonnegative function ϱ, getting the existence of a constant $\tilde{C}$ such that

$$\int\limits_{\mathbb{H}^n} \frac{\left\{e^{\alpha \varrho^{Q'}} - S_{Q-2}(\alpha, \varrho)\right\}\varrho^s}{r(\xi)^\beta}\,\mathrm{d}\xi \leq \tilde{C}\|\varrho\|_{HW^{1,Q}}^s \leq \tilde{C}2^{s/Q'}\|(u,v)\|_{HW^{1,Q}}^s = C\|(u,v)\|_{HW^{1,Q}}^s,$$

where $C = \tilde{C}2^{s/Q'}$, and so the proof is complete. $\qquad\qquad\square$

3 Existence of solutions

This section is devoted to the proof of the main result. From now on we assume, without further mentioning, that the structural assumptions required in Theorem 1.1 hold. From here on we adopt the notation

$$\mathcal{A}_{p,Q}(u,\varphi) = \int\limits_{\mathbb{H}^n} ((|D_H u|_H^{p-2} + |D_H u|_H^{Q-2})D_H u, D_H \varphi)_H \,\mathrm{d}\xi, \quad \mathcal{B}_{p,Q}(u,\varphi) = \int\limits_{\mathbb{H}^n} (|u|^{p-2} + |u|^{Q-2})u\varphi\,\mathrm{d}\xi$$

for all $(u,\varphi) \in W$.

We say that the couple $(u,v) \in W$ is a (weak) *solution* of system $(\mathcal{S})$ if

$$\mathcal{A}_{p,Q}(u,\varphi) + \mathcal{A}_{p,Q}(v,\psi) + \mathcal{B}_{p,Q}(u,\varphi) + \mathcal{B}_{p,Q}(v,\psi) = \int\limits_{\mathbb{H}^n} \frac{\nabla F(\xi,u,v)\cdot(\varphi,\psi)}{r(\xi)^\beta}\,\mathrm{d}\xi + \langle g,\varphi\rangle + \langle h,\psi\rangle \quad (3.1)$$

for any $(\varphi,\psi) \in W$. Evidently, system $(\mathcal{S})$ has a variational structure. Indeed, the functional $I : W \to \mathbb{R}$, defined by

$$I(u,v) = \frac{1}{p}\left\{\|u\|_{HW^{1,p}}^p + \|v\|_{HW^{1,p}}^p\right\} + \frac{1}{Q}\left\{\|u\|_{HW^{1,Q}}^Q + \|v\|_{HW^{1,Q}}^Q\right\} - \int\limits_{\mathbb{H}^n} \frac{F(\xi,u,v)}{r(\xi)^\beta}\,\mathrm{d}\xi - \langle g,u\rangle - \langle h,v\rangle$$

for all $(u,v) \in W$ is the Euler-Lagrange functional associated with $(\mathcal{S})$. Clearly, I is well defined and of class $C^1(W)$ by (F_1) and the assumptions on g and h, and the (weak) solutions of $(\mathcal{S})$ are exactly the critical points of I. Here and in the following we denote by W' the dual space of W and by $\langle\cdot,\cdot\rangle_{W',W}$ the dual pairing between W' and W.

We first prove in Lemma 3.1 the geometric properties of the functional I, necessary to apply a minimization argument. The next lemma is an extension of Lemma 3.1 of [1].

Lemma 3.1. (Geometry of the functional I).
 (i) *Any solution $(u,v) \in W$ of $(\mathcal{S})$ is nonnegative.*
 (ii) *There exists $\rho \in (0,1]$ and two positive numbers σ_* and j, depending on ρ, such that $I(u,v) \geq j$ for all $(u,v) \in W$, with $\|(u,v)\| = \rho$, and for all nontrivial nonnegative functionals $h, g \in HW^{-1,Q'}(\mathbb{H}^n)$ such that $0 < s_{g,h} = \max\left\{\|g\|_{HW^{-1,Q'}}, \|h\|_{HW^{-1,Q'}}\right\} \leq \sigma_*$.*
 (iii) *For all nontrivial nonnegative functionals $g, h \in HW^{-1,Q'}(\mathbb{H}^n)$ with $0 < s_{g,h} \leq \sigma_*$, it results*

$$m_{g,h} = \inf\{I(u,v) : (u,v) \in \overline{B_\rho}\} < 0,$$

where $B_\rho = \{(u,v) \in W : \|(u,v)\| < \rho\}$. Finally, there exist in B_ρ a sequence $\{(u_k,v_k)\}_k$ of nonnegative pairs and some nonnegative pair $(u_{g,h}, v_{g,h})$ such that for all $k \in \mathbb{N}$

$$\|(u_k,v_k)\| < \rho, \quad m_{g,h} \leq I(u_k,v_k) \leq m_{g,h} + \frac{1}{k},$$
$$(u_k,v_k) \rightharpoonup (u_{g,h}, v_{g,h}) \quad in \ W, \quad (u_k,v_k) \to (u_{g,h}, v_{g,h}) \quad a.e. \ in \ \mathbb{H}^n \ and \ I'(u_k,v_k) \to 0$$

(3.2)

as $k \to \infty$.

Proof. Let g, h be fixed nontrivial nonnegative elements of $HW^{-1,Q'}(\mathbb{H}^n)$ and let (u, v) be any solution of (S) in W. Since $u = u_+ - u_-$, $v = v_+ - v_-$, we have that (u_+, v_+) and (u_-, v_-) are in W and that

$$\mathcal{A}_{p,Q}(u, u_-) + \mathcal{A}_{p,Q}(v, v_-) + \mathcal{B}_{p,Q}(u, u_-) + \mathcal{B}_{p,Q}(v, v_-) = -\|u_-\|_{HW^{1,p}}^p - \|u_-\|_{HW^{1,Q}}^Q - \|v_-\|_{HW^{1,p}}^p - \|v_-\|_{HW^{1,Q}}^Q.$$

Thus, by the definition of solution for (S), we get taking as test pair (u_-, v_-)

$$0 \geq -\|u_-\|_{HW^{1,p}}^p - \|u_-\|_{HW^{1,Q}}^Q - \|v_-\|_{HW^{1,p}}^p - \|v_-\|_{HW^{1,Q}}^Q = \langle g, u_- \rangle + \langle h, v_- \rangle + \int_{\mathbb{H}^n} \frac{\nabla F(\xi, u, v) \cdot (u_-, v_-)}{r(\xi)^\beta} d\xi \geq 0,$$

by (F_1) and the fact that g and h are nonnegative. Hence, $u_- = 0$ and $v_- = 0$ a.e. in $\mathbb{H}^n$ and so (u, v) is a nonnegative pair in W, as stated.

Fix $\varepsilon > 0$. Assumption (F_1) gives

$$0 \leq F(\xi, u, v) \leq \frac{\varepsilon}{Q}\varrho^Q + \kappa_\varepsilon \varrho\left(e^{\alpha_0 \varrho^{Q'}} - S_{Q-2}(\alpha_0, \varrho)\right) \tag{3.3}$$

for a.e. $\xi \in \mathbb{H}^n$ and all $(u, v) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$, where $\varrho = \sqrt{u^2 + v^2}$. Indeed, by (F_1) arguing as in [1] we get

$$\begin{aligned}
F(\xi, u, v) &= \int_0^1 \frac{d}{dt} F(\xi, tu, tv) dt = \int_0^1 \nabla F(\xi, tu, tv) \cdot (u, v) dt \\
&\leq \int_0^1 \left\{ \varepsilon t^{Q-1}\varrho^Q + \kappa_\varepsilon \varrho(e^{\alpha_0(t\varrho)^{Q'}} - S_{Q-2}(\alpha_0, t\varrho)) \right\} dt \\
&= \frac{\varepsilon}{Q}\varrho^Q + \kappa_\varepsilon \int_0^1 \sum_{j=Q-1}^\infty \frac{\alpha_0^j t^{jQ'} \varrho^{jQ'+1}}{j!} dt,
\end{aligned}$$

from which (3.3) follows directly.

Furthermore, (F_1) yields

$$\lim_{\varrho \to 0^+} \varrho^{1-Q}|\nabla F(\xi, u, v)| = 0 \quad \text{uniformly in } \xi \in \mathbb{H}^n. \tag{3.4}$$

Thus, using the argument above, we get that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 \leq F(\xi, u, v) \leq \frac{\varepsilon}{Q}\varrho^Q \quad \text{for a.e. } \xi \in \mathbb{H}^n \text{ and all } (u, v), \text{ with } |(u, v)| = \varrho \in [0, \delta]. \tag{3.5}$$

Take now $s > Q$ and $\alpha > \alpha_0$. Then by (F_1) there exists $\tilde{\kappa}_\varepsilon = \tilde{\kappa}(\alpha_0, s, \varepsilon) > 0$ such that

$$F(\xi, u, v) \leq \tilde{\kappa}_\varepsilon \varrho^s(e^{\alpha \varrho^{Q'}} - S_{Q-2}(\alpha, \varrho)) \quad \text{for a.e. } \xi \in \mathbb{H}^n \text{ and all } (u, v), \text{ with } |(u, v)| = \varrho \in [\delta, \infty). \tag{3.6}$$

In conclusion, (3.5) and (3.6) give

$$F(\xi, u, v) \leq \frac{\varepsilon}{Q}\varrho^Q + \tilde{\kappa}_\varepsilon \varrho^s(e^{\alpha \varrho^{Q'}} - S_{Q-2}(\alpha, \varrho)) \quad \text{for a.e. } \xi \in \mathbb{H}^n \text{ and all } (u, v) \in \mathbb{R}^2. \tag{3.7}$$

By (2.8) and (3.7) and taking $\delta \in (0, 1]$ sufficiently small to apply Lemma 2.4, we get for all $(u, v) \in W$, with $\|(u, v)\| \leq \delta$,

$$\begin{aligned}
I(u, v) &\geq \frac{1}{Q}\left(\|u\|_{HW^{1,p}}^Q + \|v\|_{HW^{1,p}}^Q + \|u\|_{HW^{1,Q}}^Q + \|v\|_{HW^{1,Q}}^Q\right) - s_{g,h}\left(\|u\|_{HW^{1,Q}} + \|v\|_{HW^{1,Q}}\right) \\
&\quad - \frac{\varepsilon}{Q\lambda_Q}\|\varrho\|_{HW^{1,Q}}^Q - \tilde{\kappa}_\varepsilon C\|(u, v)\|_{HW^{1,Q}}^s \\
&\geq \frac{4^{1-Q}}{Q}\|(u, v)\|^Q - s_{g,h}\|(u, v)\|_{HW^{1,Q}} - \frac{\varepsilon 2^{Q-1}}{Q\lambda_Q}\|(u, v)\|_{HW^{1,Q}}^Q - \tilde{\kappa}_\varepsilon C\|(u, v)\|_{HW^{1,Q}}^s \\
&\geq \frac{1}{Q}\left(4^{1-Q} - \frac{\varepsilon 2^{Q-1}}{\lambda_Q}\right)\|(u, v)\|^Q - s_{g,h}\|(u, v)\| - \tilde{C}\|(u, v)\|^s,
\end{aligned}$$

where $\tilde{C} = \kappa_\varepsilon C$ and $s_{g,h} = \max\{\|g\|_{HW^{-1,Q'}}, \|h\|_{HW^{-1,Q'}}\}$. Finally, choosing $\varepsilon = 3\lambda_Q/2^{3Q-1}$ we obtain

$$I(u, v) \geq \frac{1}{2^{2Q}Q}\|(u, v)\|^Q - s_{g,h}\|(u, v)\| - \tilde{C}\|(u, v)\|^s.$$

Consider now the function

$$\psi(t) = \frac{1}{2^{1+2Q}Q}t^Q - \tilde{C}t^s, \quad t \in [0, 2^{-1/Q'}\delta].$$

Then ψ admits a positive maximum $\jmath$ in $[0, 2^{-1/Q'}\delta]$ at a point $\rho \in (0, 2^{-1/Q'}\delta]$, since $s > Q$ and $\delta \leq 1$. Consequently, for all $(u, v) \in W$, with $\|(u, v)\| = \rho$, we get

$$I(u, v) \geq \frac{1}{2^{2Q}Q}\rho^Q - s_{g,h}\rho - \tilde{C}\rho^s \geq \frac{1}{2^{1+2Q}Q}\rho^Q - \tilde{C}\rho^s = \psi(\rho) = \jmath > 0,$$

for all $g, h \in HW^{-1,Q'}(\mathbb{H}^n)$ such that $s_{g,h} \leq \sigma_\star$, where $\sigma_\star = \dfrac{\rho^{Q-1}}{2^{1+2Q}Q}$.

This completes the proof of the geometry of the functional I.

Fix $g, h \in HW^{-1,Q'}(\mathbb{H}^n)$ nontrivial and nonnegative such that $s_{g,h} = \max\{\|g\|_{HW^{-1,Q'}}, \|h\|_{HW^{-1,Q'}}\} \leq \sigma_\star$, and a pair of functions $(u, v) \in C_c^\infty(\mathbb{H}^n) \times C_c^\infty(\mathbb{H}^n)$, with $\|(u, v)\| = 1$. Thus,

$$I(\tau u, \tau v) \leq \frac{1}{p}\tau^Q - \tau\langle g, u\rangle - \tau\langle h, v\rangle$$

for all $\tau \in (0, 1]$ sufficiently small. Hence,

$$m_{g,h} = \inf\{I(u, v) : (u, v) \in \overline{B_\rho}\} < 0.$$

An application of the Ekeland variational principle in $\overline{B_\rho}$ yields the existence of a sequence $\{(u_k, v_k)\}_k \subset B_\rho$ such that

$$m_{g,h} \leq I(u_k, v_k) \leq m_{g,h} + \frac{1}{k} \quad \text{and} \quad I(u, v) \geq I(u_k, v_k) - \frac{1}{k}\|(u, v) - (u_k, v_k)\| \tag{3.8}$$

for all $k \in \mathbb{N}$ and for any $(u, v) \in \overline{B_\rho}$. Set $S_W = \{(U, V) \in W : \|(U, V)\| = 1\}$. Fixed $k \in \mathbb{N}$, for all $(U, V) \in S_W$ and for all $\tau > 0$ so small that $(u_k + \tau U, v_k + \tau V) \in \overline{B_\rho}$, we have

$$I(u_k + \tau U, v_k + \tau V) - I(u_k, v_k) \geq -\frac{\tau}{k}$$

by (3.8). Since I is Gâteaux differentiable in W, we get

$$\langle I'(u_k, v_k), (U, V)\rangle_{W',W} = \lim_{\tau \to 0}\frac{I(u_k + \tau U, v_k + \tau V) - I(u_k, v_k)}{\tau} \geq -\frac{1}{k}$$

for all $(U, V) \in S_W$. Consequently, $|\langle I'(u_k, v_k), (U, V)\rangle_{W',W}| \leq 1/k$, being $(U, V) \in S_W$ arbitrary. Thus, $I'(u_k, v_k) \to 0$ in W' as $k \to \infty$. Of course, the sequence $\{(u_k, v_k)\}_k$ is bounded in W so that, up to a subsequence, it weakly converges to some $(u_{g,h}, v_{g,h}) \in \overline{B_\rho}$ and $(u_k, v_k) \to (u_{g,h}, v_{g,h})$ a.e. in $\mathbb{H}^n$. In particular, $u_{k,-} \to u_{g,h,-}, v_{k,-} \to v_{g,h,-}, u_{k,+} \to u_{g,h,+}$ and $v_{k,+} \to v_{g,h,+}$ a.e. in $\mathbb{H}^n$. Moreover, as shown in the first part of the lemma, by (F_1) and the nonnegativity of g and h, as $k \to \infty$

$$\begin{aligned}
o(1) &= -\langle I'(u_k, v_k), (u_{k,-}, v_{k,-})\rangle_{W',W} \\
&= -\mathcal{A}_{p,Q}(u_k, u_{k,-}) - \mathcal{A}_{p,Q}(v_k, v_{k,-}) - \mathcal{B}_{p,Q}(u_k, u_{k,-}) - \mathcal{B}_{p,Q}(v_k, v_{k,-}) + \langle g, u_{k,-}\rangle + \langle h, v_{k,-}\rangle \\
&\geq \|u_{k,-}\|^p_{HW^{1,p}} + \|u_{k,-}\|^Q_{HW^{1,Q}} + \|v_{k,-}\|^p_{HW^{1,p}} + \|v_{k,-}\|^Q_{HW^{1,Q}}.
\end{aligned}$$

Therefore, $\{(u_{k,-}, v_{k,-})\}_k$ strongly converges to $(0, 0)$ in W and so $u_{k,-} \to 0$, $v_{k,-} \to 0$ a.e. in $\mathbb{H}^n$. Thus, $u_{g,h,-} = 0$ and $v_{g,h,-} = 0$ a.e. in $\mathbb{H}^n$. Hence, $(u_{g,h}, v_{g,h})$ is a nonnegative pair in $\mathbb{H}^n$. Consequently, without loss of generality, we can assume that $(u_k, v_k) = (u_{k,+}, v_{k,+})$, since $(u_{k,-}, v_{k,-}) \to (0, 0)$ in W. This completes the proof of (3.2). $\qquad\square$

Taking inspiration from [1], we are now ready to prove the main result of the paper.

Proof of Theorem 1.1. Let g and h be two fixed nontrivial nonnegative functionals in $HW^{-1,Q'}(\mathbb{H}^n)$ such that $0 < s_{g,h} = \max\{\|g\|_{HW^{-1,Q'}}, \|h\|_{HW^{-1,Q'}}\} \le \sigma_\star$, where $\sigma_\star$ is given in Lemma 3.1. Then, again Lemma 3.1 gives the existence of the sequence $\{(u_k, v_k)\}_k$ of nonnegative pairs in B_ρ, satisfying (3.2). Consequently,

$$
\begin{aligned}
(u_k, v_k) &\to (u_{g,h}, v_{g,h}) \text{ in } L^q_{\text{loc}}(\mathbb{H}^n) \times L^q_{\text{loc}}(\mathbb{H}^n), \quad q \in [1, \infty), \\
|(u_k, v_k)| &\le \mathfrak{g}_R \quad \text{a.e. in } \mathbb{H}^n, \text{ for some } \mathfrak{g}_R \in L^Q(B_R) \text{ and all } R > 0.
\end{aligned}
\tag{3.9}
$$

Take $R > 0$ and $\varphi \in C^\infty_c(\mathbb{H}^n)$ such that $0 \le \varphi \le 1$ in $\mathbb{H}^n$, $\varphi \equiv 1$ in B_R and $\varphi \equiv 0$ in B^c_{2R}. Now, by convexity $(|D_H u_k|^{p-2}_H D_H u_k - |D_H u_{g,h}|^{p-2}_H D_H u_{g,h}, D_H u_k - D_H u_{g,h})_H \ge 0$ a.e. in $\mathbb{H}^n$ and for all k. Then, since $Q = 2n + 2 > 2$, an elementary inequality, see [31], yields the existence of a constant $c_Q > 0$ such that

$$
\begin{aligned}
c_Q \int_{B_R} |D_H u_k - D_H u_{g,h}|^Q_H d\xi \le{}& \int_{B_R} (|D_H u_k|^{Q-2}_H D_H u_k - |D_H u_{g,h}|^{Q-2}_H D_H u_{g,h}, D_H u_k - D_H u_{g,h})_H d\xi \\
\le{}& \int_{B_R} (|D_H u_k|^{p-2}_H D_H u_k - |D_H u_{g,h}|^{p-2}_H D_H u_{g,h}, D_H u_k - D_H u_{g,h})_H d\xi \\
&+ \int_{B_R} (|D_H u_k|^{Q-2}_H D_H u_k - |D_H u_{g,h}|^{Q-2}_H D_H u_{g,h}, D_H u_k - D_H u_{g,h})_H d\xi \\
\le{}& \int_{\mathbb{H}^n} (|D_H u_k|^{p-2}_H D_H u_k - |D_H u_{g,h}|^{p-2}_H D_H u_{g,h}, D_H u_k - D_H u_{g,h})_H \varphi d\xi \\
&+ \int_{\mathbb{H}^n} (|D_H u_k|^{Q-2}_H D_H u_k - |D_H u_{g,h}|^{Q-2}_H D_H u_{g,h}, D_H u_k - D_H u_{g,h})_H \varphi d\xi \\
={}& \int_{\mathbb{H}^n} \varphi\{|D_H u_k|^p_H + |D_H u_k|^Q_H\} d\xi \\
&- \int_{\mathbb{H}^n} \varphi((|D_H u_k|^{p-2}_H + |D_H u_k|^{Q-2}_H) D_H u_k, D_H u_{g,h})_H d\xi + o(1)
\end{aligned}
\tag{3.10}
$$

as $k \to \infty$, since $(u_k, v_k) \to (u_{g,h}, v_{g,h})$ in W. Similarly, we can obtain (3.10) also in the v variable, that is, as $k \to \infty$

$$
\begin{aligned}
c_Q \int_{B_R} |D_H v_k - D_H v_{g,h}|^Q_H d\xi \le{}& \int_{\mathbb{H}^n} \varphi\{|D_H v_k|^p_H + |D_H v_k|^Q_H\} d\xi \\
&- \int_{\mathbb{H}^n} \varphi((|D_H v_k|^{p-2}_H + |D_H v_k|^{Q-2}_H) D_H v_k, D_H v_{g,h})_H d\xi + o(1).
\end{aligned}
\tag{3.11}
$$

Now, (3.2) gives

$$
\langle I'(u_k, v_k), \varphi(u_k, v_k)\rangle_{W',W} - \langle I'(u_k, v_k), \varphi(u_{g,h}, v_{g,h})\rangle_{W',W} = o(1) \quad \text{as } k \to \infty.
\tag{3.12}
$$

Moreover,

$$
\begin{aligned}
&\int_{\mathbb{H}^n} \varphi\{|D_H u_k|^p_H + |D_H u_k|^Q_H\} d\xi - \int_{\mathbb{H}^n} \varphi((|D_H u_k|^{p-2}_H + |D_H u_k|^{Q-2}_H) D_H u_k, D_H u_{g,h})_H d\xi \\
&\quad + \int_{\mathbb{H}^n} \varphi\{|D_H v_k|^p_H + |D_H v_k|^Q_H\} d\xi - \int_{\mathbb{H}^n} \varphi((|D_H v_k|^{p-2}_H + |D_H v_k|^{Q-2}_H) D_H v_k, D_H v_{g,h})_H d\xi \\
&= \langle I'(u_k, v_k), \varphi(u_k, v_k)\rangle_{W',W} - \langle I'(u_k, v_k), \varphi(u_{g,h}, v_{g,h})\rangle_{W',W} \\
&\quad - \int_{\mathbb{H}^n} (|D_H u_k|^{p-2}_H + |D_H u_k|^{Q-2}_H)(u_k - u_{g,h})(D_H u_k, D_H \varphi)_H d\xi \\
&\quad - \int_{\mathbb{H}^n} (|D_H v_k|^{p-2}_H + |D_H v_k|^{Q-2}_H)(v_k - v_{g,h})(D_H v_k, D_H \varphi)_H d\xi
\end{aligned}
\tag{3.13}
$$

$$- \int_{\mathbb{H}^n} \varphi \{ (|u_k|^{p-2} + |u_k|^{Q-2}) u_k (u_k - u_{g,h}) + (|v_k|^{p-2} + |v_k|^{Q-2}) v_k (v_k - v_{g,h}) \} d\xi$$

$$+ \int_{\mathbb{H}^n} \varphi \frac{\nabla F(\xi, u_k, v_k) \cdot (u_k - u_{g,h}, v_k - v_{g,h})}{r(\xi)^\beta} d\xi + \langle g, \varphi(u_k - u_{g,h}) \rangle + \langle h, \varphi(v_k - v_{g,h}) \rangle.$$

The Hölder inequality gives

$$\left| \int_{\mathbb{H}^n} (|D_H u_k|_H^{p-2} + |D_H u_k|_H^{Q-2})(u_k - u_{g,h})(D_H u_k, D_H \varphi)_H d\xi \right|$$

$$\leq \|D_H \varphi\|_\infty \left\{ \|D_H u_k\|_p^{p-1} \left(\int_{B_{2R}} |u_k - u_{g,h}|^p d\xi \right)^{1/p} + \|D_H u_k\|_Q^{Q-1} \left(\int_{B_{2R}} |u_k - u_{g,h}|^Q d\xi \right)^{1/Q} \right\},$$

and similarly the v component. Thus, the above inequality yields, by (3.2) and (3.9), that

$$\lim_{k \to \infty} \int_{\mathbb{H}^n} \{ (|D_H u_k|_H^{p-2} + |D_H u_k|_H^{Q-2})(u_k - u_{g,h})(D_H u_k, D_H \varphi)_H \tag{3.14}$$
$$+ (|D_H v_k|_H^{p-2} + |D_H v_k|_H^{Q-2})(v_k - v_{g,h})(D_H v_k, D_H \varphi)_H \} d\xi = 0,$$

being $\{(u_k, v_k)\}_k$ bounded in W. Likewise, again by the Hölder inequality

$$\left| \int_{\mathbb{H}^n} \varphi (|u_k|^{p-2} + |u_k|^{Q-2}) u_k (u_k - u_{g,h}) d\xi \right| \leq \left\{ \|u_k\|_p^{p-1} \left(\int_{B_{2R}} |u_k - u_{g,h}|^p d\xi \right)^{1/p} + \|u_k\|_Q^{Q-1} \left(\int_{B_{2R}} |u_k - u_{g,h}|^Q d\xi \right)^{1/Q} \right\},$$

which implies, also in the v component, by (3.2) and (3.9) that

$$\lim_{k \to \infty} \int_{\mathbb{H}^n} \varphi \{ (|u_k|^{p-2} + |u_k|^{Q-2}) u_k (u_k - u_{g,h}) + (|v_k|^{p-2} + |v_k|^{Q-2}) v_k (v_k - v_{g,h}) \} d\xi = 0. \tag{3.15}$$

Clearly, by (3.2) the sequence $(\varphi(u_k - u_{g,h}))_k$ in W weakly converges to 0. Indeed, $(\varphi(u_k - u_{g,h}))_k$ is bounded in the reflexive Banach space $HW^{1,Q}(\mathbb{H}^n)$ and converges to 0 a.e. in $\mathbb{H}^n$. Consequently, the entire sequence $(\varphi(u_k - u_h))_k$ weakly converges to 0 in $HW^{1,Q}(\mathbb{H}^n)$. Similarly, $(\psi(v_k - v_{g,h}))_k$ weakly converges to 0 in $HW^{1,Q}(\mathbb{H}^n)$. Thus, as $k \to \infty$

$$\langle g, \varphi(u_k - u_{g,h}) \rangle \to 0, \quad \langle h, \varphi(v_k - v_{g,h}) \rangle \to 0. \tag{3.16}$$

Therefore, (3.10)–(3.15) imply at once that as $k \to \infty$

$$c_Q \int_{B_R} \{ |D_H u_k - D_H u_{g,h}|^Q + |D_H v_k - D_H v_{g,h}|^Q d\xi \leq \int_{\mathbb{H}^n} \varphi \frac{\nabla F(\xi, u_k, v_k) \cdot (u_k - u_{g,h}, v_k - v_{g,h})}{r(\xi)^\beta} d\xi + o(1). \tag{3.17}$$

Now, using the notation of Lemma 3.1, thanks to (3.2) and (F_2) we get as $k \to \infty$

$$0 > m_{g,h} = I(u_k, v_k) - \frac{1}{\nu} \langle I'(u_k, v_k), (u_k, v_k) \rangle_{W',W} + o(1)$$

$$\geq \left(\frac{1}{p} - \frac{1}{\nu} \right) (\|u_k\|_{HW^{1,p}}^p + \|v_k\|_{HW^{1,p}}^p) + \left(\frac{1}{Q} - \frac{1}{\nu} \right) (\|u_k\|_{HW^{1,Q}}^Q + \|v_k\|_{HW^{1,Q}}^Q)$$

$$- s_{g,h} \left(1 - \frac{1}{\nu} \right) (\|u_k\|_{HW^{1,Q}} + \|v_k\|_{HW^{1,Q}}) + o(1)$$

$$\geq 2^{1-Q} \left(\frac{1}{Q} - \frac{1}{\nu} \right) \|(u_k, v_k)\|_{HW^{1,Q}}^Q - s_{g,h} \left(1 - \frac{1}{\nu} \right) \|(u_k, v_k)\|_{HW^{1,Q}} + o(1),$$

where as usual $s_{g,h} = \max\{\|g\|_{HW^{-1,Q'}}, \|h\|_{HW^{-1,Q'}}\}$. Therefore, as $k \to \infty$ we obtain

$$2^{1-Q}\left(\frac{1}{Q} - \frac{1}{\nu}\right)\|(u_k, v_k)\|_{HW^{1,Q}}^Q - s_{g,h}\left(1 - \frac{1}{\nu}\right)\|(u_k, v_k)\|_{HW^{1,Q}} + o(1) < 0,$$

and so

$$\limsup_{k \to \infty} \|(u_k, v_k)\|_{HW^{1,Q}} \leq 2\left[\frac{Q(\nu-1)}{\nu-Q}s_{g,h}\right]^{1/(Q-1)}.$$

Then, choosing $\sigma = \min\{\sigma_\star, \tilde{\sigma}\}$, where

$$\tilde{\sigma} = 2^{Q-1}\left(\frac{\alpha_{Q,\beta}}{2\alpha_0}\right)^{Q-1/Q'}\frac{\nu-Q}{Q(\nu-1)},$$

we obtain for any $g, h \in HW^{-1,Q'}(\mathbb{H}^n)$ with $s_{g,h} < \sigma$ that

$$\limsup_{k \to \infty} \|\varrho_k\|_{HW^{1,Q}} \leq 2^{1/Q'}\limsup_{k \to \infty} \|(u_k, v_k)\|_{HW^{1,Q}} < 2^{1/Q'}\left(\frac{\alpha_{Q,\beta}}{2\alpha_0}\right)^{1/Q'} = \left(\frac{\alpha_{Q,\beta}}{\alpha_0}\right)^{1/Q'},$$

where again $\varrho_k = |(u_k, v_k)|$. Therefore, passing if necessary to a subsequence, still labeled $\{(u_k, v_k)\}_k$ for simplicity, we obtain

$$\sup_{k \in \mathbb{N}} \|\varrho_k\|_{HW^{1,Q}} \leq 2^{1/Q'}\sup_{k \in \mathbb{N}} \|(u_k, v_k)\|_{HW^{1,Q}} < \left(\frac{\alpha_{Q,\beta}}{\alpha_0}\right)^{1/Q'}.$$

Thus, arguing as in the proof of Lemma 2.3 of [13], we get the existence of a constant $C = C(Q, \alpha, \beta) > 0$ and an exponent $s > Q$ such that

$$\int_{\mathbb{H}^n} \frac{\left\{e^{\alpha\varrho_k^{Q'}} - S_{Q-2}(\alpha, \varrho_k)\right\}}{r(\xi)^\beta}|\varphi(u_k - u_{g,h}, v_k - v_{g,h})|\,\mathrm{d}\xi \leq C\left(\int_{\mathbb{H}^n} |\varphi(u_k - u_{g,h}, v_k - v_{g,h})|^s\,\mathrm{d}\xi\right)^{1/s}$$

$$= C\left(\int_{B_{2R}} |(u_k - u_{g,h}, v_k - v_{g,h})|^s\,\mathrm{d}\xi\right)^{1/s}. \tag{3.18}$$

Now, by (F_1) there exists $\tilde{\kappa} = \tilde{\kappa}(\alpha_0, 1) > 0$ such that

$$|\nabla F(\xi, u, v)| \leq \varrho^{Q-1} + \tilde{\kappa}(e^{\alpha\varrho^{Q'}} - S_{Q-2}(\alpha, \varrho)) \tag{3.19}$$

for a.e. $\xi \in \mathbb{H}^n$ and all $(u, v) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$, with $\rho = |(u, v)|$.

Choose $1 < t < Q/\beta$ and $1 < \mathfrak{p} < Q/t\beta$ and put $\mathfrak{q} = (Q-1)ts'$. Then, by the Hölder and the choices of the exponents t, $\mathfrak{p}$ and $\mathfrak{q}$ we have

$$\left(\int_{B_{2R}} \frac{\varrho_k^{(Q-1)t}}{r(\xi)^{t\beta}}\,\mathrm{d}\xi\right)^{1/t} \leq \|\varrho_k\|_{L^{\mathfrak{q}}(B_{2R})}^{1/s'}\left(\int_{B_{2R}} \frac{\mathrm{d}\xi}{r(\xi)^{ts\beta}}\right)^{1/s} \leq d_1, \tag{3.20}$$

since $\{(u_k, v_k)\}_k$ is bounded in W, and so $(\rho_k)_k$ is bounded in every $L^{\mathfrak{q}}(B_{2R})$. Hence, the Hölder inequality, (3.9) and (3.18)–(3.20) yield, as $k \to \infty$

$$\left|\int_{\mathbb{H}^n} \varphi\frac{\nabla F(\xi, u_k, v_k)\cdot(u_k - u_{g,h}, v_k - v_{g,h})}{r(\xi)^\beta}\,\mathrm{d}\xi\right|$$

$$\leq \int_{\mathbb{H}^n} \frac{\varrho_k^{Q-1} + \tilde{\kappa}(e^{\alpha\varrho_k^{Q'}} - S_{Q-2}(\alpha, \varrho_k))}{r(\xi)^\beta}|\varphi(u_k - u_{g,h}, v_k - v_{g,h})|\,\mathrm{d}\xi$$

$$\leq \left(\int_{B_{2R}} \frac{\varrho_k^{(Q-1)t}}{r(\xi)^{t\beta}} d\xi \right)^{1/t} \left(\int_{B_{2R}} |(u_k, v_k) - (u_{g,h}, v_{g,h})|^{t'} d\xi \right)^{1/t'} + \tilde{\kappa} C \left(\int_{B_{2R}} |(u_k, v_k) - (u_{g,h}, v_{g,h})|^s d\xi \right)^{1/s}$$

$$\leq d_1 \left(\int_{B_{2R}} |(u_k, v_k) - (u_{g,h}, v_{g,h})|^{t'} d\xi \right)^{1/t'} + d_2 \left(\int_{B_{2R}} |(u_k, v_k) - (u_{g,h}, v_{g,h})|^s d\xi \right)^{1/t'},$$

where d_1 is defined in (3.20), while $d_2 = \tilde{\kappa} C$. Therefore,

$$\lim_{k \to \infty} \int_{\mathbb{H}^n} \varphi \frac{\nabla F(\xi, u_k, v_k) \cdot (u_k - u_{g,h}, v_k - v_{g,h})}{r(\xi)^\beta} d\xi = 0. \tag{3.21}$$

Thus, combining (3.17) with (3.21), we obtain

$$\int_{B_R} (|D_H u_k - D_H u_{g,h}|_H^Q + |D_H v_k - D_H v_{g,h}|_H^Q) d\xi \leq o(1) \quad \text{as } k \to \infty.$$

Consequently, $D_H u_k \to D_H u_{g,h}$ and $D_H v_k \to D_H v_{g,h}$ in $L^Q(B_R, \mathbb{R}^{2n})$ for all $R > 0$. Therefore, up to a subsequence, not relabeled, we get that

$$D_H u_k \to D_H u_{g,h} \quad \text{and} \quad D_H v_k \to D_H v_{g,h} \quad \text{a.e. in } \mathbb{H}^n, \tag{3.22}$$

and for all $R > 0$ there exists a function $\mathfrak{h}_R \in L^Q(B_R)$ such that $|D_H u_k| \leq \mathfrak{h}_R$ and $|D_H v_k| \leq \mathfrak{h}_R$ a.e. in B_R and for all $k \in \mathbb{N}$.

Fix ϕ and ψ in $C_c^\infty(\mathbb{H}^n)$ and let $R > 0$ be so large that supp $\phi \subset B_R$ and supp $\psi \subset B_R$. By the above construction we have a.e. in B_R

$$|((|D_H u_k|_H^{p-2} + |D_H u_k|_H^{Q-2}) D_H u_k, D_H \phi)_H + ((|D_H v_k|_H^{p-2} + |D_H v_k|_H^{Q-2}) D_H u_k, D_H \phi)_H|$$
$$\leq (\mathfrak{h}_R^{p-1} + \mathfrak{h}_R^{Q-1})(|D_H \phi|_H + |D_H \psi|_H) = \mathfrak{H}_R \in L^1(B_R).$$

Thus, the dominated convergence theorem and (3.2) yield as $k \to \infty$

$$\mathcal{A}_{p,Q}(u_k, \phi) + \mathcal{A}_{p,Q}(v_k, \psi) \to \mathcal{A}_{p,Q}(u_{g,h}, \phi) + \mathcal{A}_{p,Q}(v_{g,h}, \psi).$$

Likewise, (3.9) yields a.e. in B_R that

$$|(u_k^{p-1} + u_k^{Q-1}) \phi + (v_k^{p-1} + v_k^{Q-1}) \psi| \leq (2\mathfrak{g}_R^{p-1} + 2\mathfrak{g}_R^{Q-1})(|\phi| + |\psi|) = \mathfrak{G}_R \in L^1(B_R),$$

and so the dominated convergence theorem and (3.2) give at once as $k \to \infty$

$$\mathcal{B}_{p,Q}(u_k, \phi) + \mathcal{B}_{p,Q}(v_k, \psi) \to \mathcal{B}_{p,Q}(u_{g,h}, \phi) + \mathcal{B}_{p,Q}(v_{g,h}, \psi).$$

Similarly, by Theorem 2.2 and by (3.2), (3.18) and (3.19)

$$\left| \frac{\nabla F(\xi, u_k, v_k) \cdot (\phi, \psi)}{r(\xi)^\beta} \right| \leq \mathfrak{F}_R \in L^1(B_R).$$

Thus, a further application of the dominated convergence theorem gives as $k \to \infty$

$$\int_{\mathbb{H}^n} \frac{\nabla F(\xi, u_k, v_k) \cdot (\phi, \psi)}{r(\xi)^\beta} d\xi \to \int_{\mathbb{H}^n} \frac{\nabla F(\xi, u_{g,h}, v_{g,h}) \cdot (\phi, \psi)}{r(\xi)^\beta} d\xi$$

and so, since $\langle I'(u_k, v_k), (\phi, \psi) \rangle_{W', W} = o(1)$ as $k \to \infty$ by (3.2), eventually we have

$$\mathcal{A}_{p,Q}(u_k, \phi) + \mathcal{A}_{p,Q}(v_k, \psi) + \mathcal{B}_{p,Q}(u_k, \phi) + \mathcal{B}_{p,Q}(v_k, \psi) = \int_{\mathbb{H}^n} \frac{\nabla F(\xi, u_k, v_k) \cdot (\phi, \psi)}{r(\xi)^\beta} d\xi + \langle g, \phi \rangle + \langle h, \psi \rangle + o(1).$$

Finally, letting $k \to \infty$ and using the above arguments together with (3.2), we get at once that

$$\mathcal{A}_{p,Q}(u_{g,h}, \phi) + \mathcal{A}_{p,Q}(v_{g,h}, \psi) + \mathcal{B}_{p,Q}(u_{g,h}, \phi) + \mathcal{B}_{p,Q}(v_{g,h}, \psi)$$
$$= \int_{\mathbb{H}^n} \frac{\nabla F(\xi, u_{g,h}, v_{g,h}) \cdot (\phi, \psi)}{r(\xi)^\beta} d\xi + \langle g, \phi \rangle + \langle h, \psi \rangle \qquad (3.23)$$

for all (ϕ, ψ) in $C_c^\infty(\mathbb{H}^n) \times C_c^\infty(\mathbb{H}^n)$. Now, fix a function $\zeta \in C_c^\infty(\mathbb{H}^n)$ such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ in B_1 and $\mathrm{supp}\,\zeta \subset B_2$. Define now the sequence of cutoff functions

$$\zeta_k(\xi) = \zeta(\delta_{1/k}(\xi)), \quad \xi \in \mathbb{H}^n, \qquad (3.24)$$

where $\delta_{1/k}$ is the dilation of parameter $1/k$, as introduced in (2.2). Now fix (Φ, Ψ) in W. Then the sequences $(\phi_k)_k$ and $(\psi_k)_k$ in $C_c^\infty(\mathbb{H}^n)$, defined by $\phi_k = \zeta_k(\rho_k \star \Phi)$ and $\psi_k = \zeta_k(\rho_k \star \Psi)$, where $(\rho_k)_k$ is a sequence of mollifiers constructed as in [13] and $(\zeta_k)_k$ is the sequence of cutoff functions defined in (3.24), have the properties that $\phi_k \to \Phi$ and $\psi_k \to \Psi$ in W and, up to subsequences, $\phi_k \to \Phi$, $\psi_k \to \Psi$, $D_H \phi_k \to D_H \Phi$, $D_H \psi_k \to D_H \Psi$ a.e. in $\mathbb{H}^n$ as $k \to \infty$, and there exist functions $\mathfrak{g} \in L^p(\mathbb{H}^n)$ and $\mathfrak{h} \in L^Q(\mathbb{H}^n)$ such that $|\phi_k| \le \mathfrak{g}$, $|\psi_k| \le \mathfrak{g}$, $|D_H \phi_k| \le \mathfrak{g}$, $|D_H \psi_k| \le \mathfrak{g}$ and $|\phi_k| \le \mathfrak{h}$, $|\psi_k| \le \mathfrak{h}$, $|D_H \phi_k| \le \mathfrak{h}$, $|D_H \psi_k| \le \mathfrak{h}$ a.e. in $\mathbb{H}^n$ and for all k. Clearly, (3.23) holds along $\{(\phi_k, \psi_k)\}_k$ for all $k \in \mathbb{N}$. Hence, passing to the limit as $k \to \infty$ under the sign of integrals by the dominated convergence theorem, we obtain the validity of (3.23) for all $(\Phi, \Psi) \in W$. In conclusion, $\langle I'(u_{g,h}, v_{g,h}), (\Phi, \Psi) \rangle_{W',W} = 0$ for all $(\Phi, \Psi) \in W$, that is, $(u_{g,h}, v_{g,h})$ is a solution of $(\mathcal{S})$.

Let us now prove that the constructed solution $(u_{g,h}, v_{g,h})$ is nontrivial. By (3.2), up to a further subsequence, there exist $\ell_p \ge 0$, $\ell_Q \ge 0$, $\iota_p \ge 0$, $\iota_Q \ge 0$ such that as $k \to \infty$

$$\|u_k\|_{HW^{1,p}} \to \ell_p, \quad \|u_k\|_{HW^{1,Q}} \to \ell_Q, \quad \|v_k\|_{HW^{1,p}} \to \iota_p, \quad \|v_k\|_{HW^{1,Q}} \to \iota_Q,$$
$$\langle g, u_k \rangle \to \langle g, u_{g,h} \rangle, \quad \langle h, v_k \rangle \to \langle h, v_{g,h} \rangle. \qquad (3.25)$$

Moreover, (3.2), (F_2) and the Fatou lemma yield

$$\liminf_{k \to \infty} \int_{\mathbb{H}^n} \frac{\nabla F(\xi, u_k, v_k) \cdot (u_k, v_k) - \nu F(\xi, u_k, v_k)}{r(\xi)^\beta} d\xi \ge \int_{\mathbb{H}^n} \frac{\nabla F(\xi, u_{g,h}, v_{g,h}) \cdot (u_{g,h}, v_{g,h}) - \nu F(\xi, u_{g,h}, v_{g,h})}{r(\xi)^\beta} d\xi. \quad (3.26)$$

Consequently, the weak lower semicontinuity of the norms, (3.2), (3.25) and (3.26) give at once that

$$\begin{aligned}
m_{g,h} &= \lim_{k \to \infty} \left(I(u_k, v_k) - \frac{1}{\nu} \langle I'(u_k, v_k), (u_k, v_k) \rangle_{W',W} \right) \\
&= \left(\frac{1}{p} - \frac{1}{\nu} \right)(\ell_p^p + \iota_p^p) + \left(\frac{1}{Q} - \frac{1}{\nu} \right)(\ell_Q^Q + \iota_Q^Q) - \left(1 - \frac{1}{\nu} \right)(\langle g, u_{g,h} \rangle + \langle h, v_{g,h} \rangle) \\
&\quad + \liminf_{k \to \infty} \frac{1}{\nu} \int_{\mathbb{H}^n} \frac{\nabla F(\xi, u_k, v_k) \cdot (u_k, v_k) - \nu F(\xi, u_k, v_k)}{r(\xi)^\beta} d\xi \\
&\ge \left(\frac{1}{p} - \frac{1}{\nu} \right)(\|u_{g,h}\|_{HW^{1,p}}^p + \|v_{g,h}\|_{HW^{1,p}}^p) + \left(\frac{1}{Q} - \frac{1}{\nu} \right)(\|u_{g,h}\|_{HW^{1,Q}}^Q + \|v_{g,h}\|_{HW^{1,Q}}^Q) \\
&\quad - \left(1 - \frac{1}{\nu} \right)(\|g\|_{HW^{-1,Q'}} \|u_{g,h}\|_{HW^{1,Q}} + \|h\|_{HW^{-1,Q'}} \|v_{g,h}\|_{HW^{1,Q}}) \\
&\quad + \frac{1}{\nu} \int_{\mathbb{H}^n} \frac{\nabla F(\xi, u_{g,h}, v_{g,h}) \cdot (u_{g,h}, v_{g,h}) - \nu F(\xi, u_{g,h}, v_{g,h})}{r(\xi)^\beta} d\xi \\
&= I(u_{g,h}, v_{g,h}) - \frac{1}{\nu} \langle I'(u_{g,h}, v_{g,h}), (u_{g,h}, v_{g,h}) \rangle_{W',W} = I(u_{g,h}, v_{g,h}) \ge m_{g,h},
\end{aligned}$$

since $(u_{g,h}, v_{g,h}) \in \overline{B_\rho}$. Therefore, the solution $(u_{g,h}, v_{g,h})$ is also a minimizer of the functional I in $\overline{B_\rho}$ and $I(u_{g,h}, v_{g,h}) = m_{g,h} < 0 < j \le I(u, v)$ for all $(u, v) \in \partial B_\rho$ by Lemma 3.1. This implies that $(u_{g,h}, v_{g,h}) \in B_\rho$ and that $(u_{g,h}, v_{g,h})$ is a nontrivial nonnegative solution of $(\mathcal{S})$, since the couple of nontrivial nonnegative functionals g, h in $HW^{-1,Q'}(\mathbb{H}^n)$ has the property that $s_{g,h} = \max\{\|g\|_{HW^{-1,Q'}}, \|h\|_{HW^{-1,Q'}}\} \in (0, \sigma]$.

In order to prove the asymptotic property (1.1), let us observe that from Lemma 3.1 it is evident that $\rho > 0$ is independent of σ. Therefore,

$$\{(u_{g,h}, v_{g,h})\}_{s_{g,h} \in (0,\sigma]} \subset B_\rho \tag{3.27}$$

is uniformly bounded in W. Thus, by (3.2) and (F_2) we have

$$
\begin{aligned}
m_{g,h} &\geq \left(\frac{1}{p} - \frac{1}{v}\right)\left(\|u_{g,h}\|_{HW^{1,p}}^p + \|v_{g,h}\|_{HW^{1,p}}^p\right) + \left(\frac{1}{Q} - \frac{1}{v}\right)\left(\|u_{g,h}\|_{HW^{1,Q}}^Q + \|v_{g,h}\|_{HW^{1,Q}}^Q\right) \\
&\quad - s_{g,h}\left(1 - \frac{1}{v}\right)\left(\|u_{g,h}\|_{HW^{1,Q}} + \|v_{g,h}\|_{HW^{1,Q}}\right) + o(1) \\
&\geq \left(\frac{1}{p} - \frac{1}{v}\right)\left(\|u_{g,h}\|_{HW^{1,p}}^p + \|v_{g,h}\|_{HW^{1,p}}^p\right) + \left(\frac{1}{Q} - \frac{1}{v}\right)\left(\|u_{g,h}\|_{HW^{1,Q}}^Q + \|v_{g,h}\|_{HW^{1,Q}}^Q\right) - s_{g,h}C_\rho,
\end{aligned}
$$

where $C_\rho = \rho(1 - 1/v)$ and the last inequality holds by (3.27) and the fact that ρ is independent of σ. Therefore,

$$0 \geq \limsup_{s_{g,h} \to 0^+} m_{g,h} \geq \limsup_{s_{g,h} \to 0^+}\left\{\left(\frac{1}{p} - \frac{1}{v}\right)\left(\|u_{g,h}\|_{HW^{1,p}}^p + \|v_{g,h}\|_{HW^{1,p}}^p\right) + \left(\frac{1}{Q} - \frac{1}{v}\right)\left(\|u_{g,h}\|_{HW^{1,Q}}^Q + \|v_{g,h}\|_{HW^{1,Q}}^Q\right)\right\} \geq 0.$$

This implies at once

$$\lim_{s_{g,h} \to 0^+} \|u_{g,h}\| = \lim_{s_{g,h} \to 0^+} \|v_{g,h}\| = 0, \tag{3.28}$$

and so the validity of (1.1).

Of course by (3.28), possibly shrinking σ, we can assume that $\|u_{g,h}\| < \rho/2$ and $\|v_{g,h}\| < \rho/2$. Then, we claim that $(u_{g,h}, v_{g,h}) \in W$ has both nontrivial components. Otherwise, if for example $v_{g,h} = 0$, then (F_1) and (F_2) give that $F(\xi, u_{g,h}, 0) = F(\xi, 0, u_{g,h}) = 0$ a.e. in $\mathbb{H}^n$. Therefore, $m_{g,h} = I(u_{g,h}, 0) = I(0, u_{g,h})$, since by construction $(u_{g,h}, u_{g,h}) \in B_\rho$, and so

$$
\begin{aligned}
m_{g,h} \leq I(u_{g,h}, u_{g,h}) &= I(u_{g,h}, 0) + I(0, u_{g,h}) - \int_{\mathbb{H}^n} \frac{F(\xi, u_{g,h}, u_{g,h})}{r(\xi)^\beta}\mathrm{d}\xi \\
&= 2m_{g,h} - \int_{\mathbb{H}^n} \frac{F(\xi, u_{g,h}, u_{g,h})}{r(\xi)^\beta}\mathrm{d}\xi \leq 2m_{g,h} < m_{g,h},
\end{aligned}
$$

since $m_{g,h} < 0$. This contradiction proves the claim.

Finally, assume that one between (H_1) and (H_2) hold and suppose by contradiction that $u_{g,h} \equiv v_{g,h}$, that is, the solution is $(u_{g,h}, u_{g,h})$, with $u_{g,h} \geq 0$ a.e. in $\mathbb{H}^n$ and $u_{g,h} \not\equiv 0$. Let us divide the proof in the two cases covered by the theorem.

First, assume that (H_1) holds, that is, $F_u(\xi, u, u) = F_v(\xi, u, u)$ for a.e. $\xi \in \mathbb{H}^n$ and any $u \in \mathbb{R}$, but $g \not\equiv h$. Then (10), with $(\varphi, \psi) = (\varphi, -\varphi)$ and $\varphi \in C_c^\infty(\mathbb{H}^n)$, gives $\langle g - h, \varphi \rangle = 0$. In other words, $g \equiv h$ since φ is arbitrary. This is impossible, as required.

Let us finally consider the remaining case (H_2) in which $g \equiv h$, but $F_u(\xi, u, u) \neq F_v(\xi, u, u)$ for a.e. $\xi \in \mathbb{H}^n$ and any $u \in \mathbb{R}^+$. Again (3.1), with $(\varphi, \psi) = (\varphi, -\varphi)$ and $\varphi \in C_c^\infty(\mathbb{H}^n)$, gives

$$\int_{\mathbb{H}^n} \varphi \frac{F_u(\xi, u_{g,h}, u_{g,h}) - F_v(\xi, u_{g,h}, u_{g,h})}{r(\xi)^\beta}\mathrm{d}\xi = 0.$$

Consequently, $F_u(\xi, u_{g,h}, u_{g,h}) = F_v(\xi, u_{g,h}, u_{g,h})$ a.e. in $\mathbb{H}^n$, since again $\varphi \in C_c^\infty(\mathbb{H}^n)$ is arbitrary. But this is impossible by (F_1), since $u_{g,h} \geq 0$ and $u_{g,h} \not\equiv 0$. The proof is now complete. $\qquad\square$

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