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Π-REGULAR VARIATION

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ABSTRACT. A function U: $R^+ \to R^+$ is said to be Π-regularly varying with exponent α if $U(x)x^{-\alpha}$ is nondecreasing and there exists a positive function L such that

$$\frac{U(\lambda x)/\lambda^{\alpha} - U(x)}{x^{\alpha}L(x)} \to \log \lambda \qquad (x \to \infty) \text{ for } \lambda > 0.$$

Suppose

$$\hat{U}(t) \coloneqq \int_0^{\infty} e^{-tx}\, dU(x) \text{ exists for } t > 0.$$

We prove that U is Π-regularly varying iff $\hat{U}$ is Π-regularly varying.

1. Introduction. First we give the definition of regular variation.

DEFINITION. A function U is said to be regularly varying with exponent ρ at infinity if it is real-valued, positive and measured on $(0, \infty)$ and if for each $\lambda > 0$

$$\lim_{x\to\infty} \frac{U(\lambda x)}{U(x)} = \lambda^{\rho} \quad \text{where } \rho \in R \left(\text{notation } U(x) \in RV_{\rho}\right).$$

Regularly varying functions with exponent zero are called slowly varying. The theory of regularly varying functions has been developed by Karamata. For some basic facts see **[1]**, **[8]**, **[9]**.

A recent treatment of regular variation is also given in Seneta's book **[10]**. Karamata proved the following theorems on regular variation which are basic in this theory.

THEOREM A. *Suppose* U: $R^+ \to R^+$ *is Lebesgue summable on finite intervals.*

(i) *If U varies regularly at infinity with exponent* $\beta > -1$ *then*

$$\lim_{x\to\infty} \frac{xU(x)}{\int_0^x U(t)dt} = \beta + 1.$$

(ii) *If* $\lim_{x\to\infty}(xU(x)/\int_0^x U(t)dt) = \beta + 1$ *with* $\beta > -1$ *then* $U(x) \in RV_{\beta}$. *See, e.g.*, **[5**, *Theorem* 1.2.1**]**.

The second theorem concerns the Laplace-Stieltjes transform: $\hat{U}(t) = \int_0^{\infty} e^{-ts}\, dU(s)$ of U. For a proof of this theorem the reader is referred to **[10,** Theorem 2.3].

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THEOREM B. *Suppose* $U\colon R^+ \to R^+$ *is nondecreasing, right-continuous* $U(0+) = 0$, $\hat{U}(t)$ *is finite for* $t > 0$. *For* $\beta \geqslant 0$ *the following assertions are equivalent*:

(i) $U(x) \in RV_\beta$;

(ii) $\hat{U}(1/x) \in RV_\beta$.

Both imply

(iii) $\lim_{x\to\infty}(U(x)/\hat{U}(1/x)) = 1/\Gamma(\beta + 1)$.

For nondecreasing functions U we can combine Theorems A and B using the notion of a fractional integral:

DEFINITION. ${}_\alpha U(x) = (1/\Gamma(\alpha + 1))\int_0^x (x - t)^\alpha \, dU(t)$ where $\alpha > 0$.

THEOREM C. *Suppose* $U\colon R^+ \to R^+$ *is nondecreasing and right-continuous*, $U(0+) = 0$ *and* $\hat{U}(t)$ *is finite for* $t > 0$. *For* $\alpha > 0$ *and* $\beta \geqslant 0$ *the following assertions are equivalent*:

(i) $U(x) \in RV_\beta$;

(ii) ${}_\alpha U(x) \in RV_{\alpha+\beta}$;

(iii) $\hat{U}(1/x) \in RV_\beta$.

They imply

(iv) ${}_\alpha U(x)/x^\alpha U(x) \to \Gamma(\beta + 1)/\Gamma(\alpha + \beta + 1)$ $(x \to \infty)$;

(v) $U(x)/\hat{U}(1/x) \to 1/\Gamma(\beta + 1)$ $(x \to \infty)$.

Remark that the case $\alpha = 1$ yields Theorem A(i) with $\beta \geqslant 0$. For arbitrary $\alpha > 0$ Theorem C can be proved by using Theorems A and B and the relation

$$ {}_\alpha\hat{U}(1/x) = x^\alpha \hat{U}(1/x), $$

since ${}_\alpha U(x)$ is nondecreasing.

In 1963 Bojanic and Karamata [2] studied the class of functions U for which

$$ \lim_{x\to\infty} \frac{U(\lambda x) - U(x)}{x^\sigma L(x)} $$

exists for some function $L(x)$ and showed that σ can be chosen such that $L(x)$ is slowly varying. In this paper we shall see that the Theorems A and B can be sharpened for functions U which satisfy the relation

$$ \lim_{x\to\infty} \frac{U(\lambda x)/\lambda^\sigma - U(x)}{x^\sigma L(x)} = \log \lambda $$

for some function $L(x)$ and $\sigma \geqslant 0$ fixed. For $\sigma = 0$ this relation defines the class Π.

THEOREM D. *Suppose* $\phi\colon R^+ \to R$ *is nondecreasing. Then the following three statements are equivalent*:

(i) *There exist functions* $a\colon R^+ \to R^+$ *and* $b\colon R^+ \to R$ *such that for all positive* x

$$ \lim_{t\to\infty} \frac{\phi(tx) - b(t)}{a(t)} = \log x; $$

(ii) *there exists a slowly varying function* L *such that*

$$ \phi(x) = L(x) + \int_1^x L(t)/t \, dt; $$

(iii) *there exists a slowly varying function* L_0 *such that*

$$\phi(x) = L_0(x) + \int_0^x L_0(t)/t\, dt.$$

Moreover if a function ϕ *satisfies the conditions of this theorem then*

$$a(x) \sim L(x) \sim \phi(xe) - \phi(x) \sim \frac{1}{x}\int_0^x s\, d\phi(s) \sim L_0(x) \qquad (x \to \infty)$$

(*see* [5, *Theorem* 1.4.1]).

We call the function $a(x)$ the auxiliary function of $\phi(x)$. This function is (of course) determined up to asymptotic equivalence.

DEFINITION. A function ϕ which satisfies the conditions of Theorem D is said to belong to the class Π. It can be shown that the class Π is a proper subclass of the slowly varying functions (see [5, Corollary 1.4.1]). From Theorem D we can see that if $\phi(x) \in \Pi$ with auxiliary functions $a(x)$ and $[\phi(x) - \phi_1(x)]/a(x) \to c$ $(x \to \infty)$ where $c \in R$ is a constant and $\phi_1(x)$ a nondecreasing function, then $\phi_1(x) \in \Pi$ with auxiliary function $a(x)$.

In this paper we generalize the following theorem (see [6]).

THEOREM E. *Suppose* ϕ: $R^+ \to R^+$ *is nondecreasing*, $\phi(0+) = 0$ *and* $\hat{\phi}(s)$ *is finite for* $s > 0$. *Then the following statements are equivalent*:

(i) $\phi(x) \in \Pi$;

(ii) $\hat{\phi}(1/x) \in \Pi$;

Both imply

(iii) $(\phi(x) - \hat{\phi}(1/x))/(1/x)\int_0^x s\, d\phi(s) \to \gamma$ $(x \to \infty)$.

We give a second order version of Karamata's Theorems A and B for nondecreasing functions U. A necessary and sufficient condition for a function to obey the second order relation is formulated in the following definition.

DEFINITION. $U \in \Pi RV_\alpha$ iff $U(x)/x^\alpha \in \Pi$ where $\alpha \in R$.

If $U \in \Pi RV_\alpha$ then we say that L is the auxiliary function of U if L is the auxiliary function of $U(x)/x^\alpha \in \Pi$. We call the function U Π-regularly varying with exponent α. The Π-varying functions with exponent α form a subclass of RV_α.

2. Results. Our result is the following theorem.

THEOREM 1. *Suppose* $\alpha > 0$, $\beta \geqslant 0$, U: $R^+ \to R^+$, $U(x)/x^\beta$ *nondecreasing*, $\lim_{x \downarrow 0}(U(x)/x^\beta) = 0$, *and* $\hat{U}(t)$ *exists for* $t > 0$. *Then the following statements are equivalent*:

(i) $U(x) \in \Pi RV_\beta$;

(ii) ${}_\alpha U(x) \in \Pi RV_{\alpha+\beta}$;

(iii) $\hat{U}(1/x) \in \Pi RV_\beta$.

They imply

$$\text{(iv)} \qquad \frac{(\Gamma(\beta+1)/\Gamma(\alpha+\beta+1))U(x) - {}_\alpha U(x)/x^\alpha}{x^{\beta-1}\int_0^x s\, d(U(s)/s^\beta)} \to -\frac{\partial}{\partial\beta}\left(\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\right)$$

$(x \to \infty)$,

(v) $$\frac{U(x) - (1/\Gamma(\beta+1))\hat{U}(1/x)}{x^{\beta-1}\int_0^x s\, d(U(s)/s^{\beta})} \to -\psi(\beta+1) \qquad (x\to\infty)$$

where $\psi(x) = (d/dx)\log\Gamma(x)$.

Conversely if (iv) *with* $\alpha \in (0, 1]$, $\beta > 0$ *then* (i) *and if* (v) *with* $\beta > 1$ *then* (i).

PROOF. (i) → (iv) and (i) → (ii). We write

$$U(x) = x^{\beta}\left(L(x) + \int_0^x \frac{L(t)}{t}\,dt\right)$$

with

$$L(x) = \frac{1}{x}\int_0^x s\, d\frac{U(s)}{s^{\beta}} \in RV_0^{(\infty)}.$$

Then

$$\frac{({}_{\alpha}U(x)/x^{\alpha}) - (\Gamma(\beta+1)/\Gamma(\alpha+\beta+1))U(x)}{x^{\beta}L(x)}$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}t^{\beta}\frac{(U(tx)/t^{\beta}x^{\beta}) - (U(x)/x^{\beta})}{L(x)}\,dt$$
$$\to \frac{1}{\Gamma(\alpha)}\int_0^1 \log t(1-t)^{\alpha-1}t^{\beta}\,dt \qquad (x\to\infty).$$

The last step is justified since by substituting the expression for $U(x)$ we find

$$\frac{1}{\Gamma(\alpha)}\left[\int_0^1 (1-t)^{\alpha-1}t^{\beta}\left\{\frac{L(tx)}{L(x)} - 1\right\}dt - \int_0^1 (1-t)^{\alpha-1}t^{\beta}\int_t^1 \frac{L(sx)}{L(x)}\frac{ds}{s}\,dt\right]$$

and

$$\frac{s^{\varepsilon}x^{\varepsilon}L(sx)}{x^{\varepsilon}L(x)} \to s^{\varepsilon} \qquad (x\to\infty)$$

uniformly on (0, 1) where $\varepsilon > 0$ (see de Haan [5]). Now (iv) and (i) imply (ii) as mentioned in the introduction.

(i) → (v) and (i) → (iii). We write $U(x) = x^{\beta}L(x) + K(x)$ where $K(x) = x^{\beta}\int_0^x (L(t)/t)dt$. By Karamata's Theorem B we have

$$x^{\beta}L(x) - \frac{1}{\Gamma(\beta+1)}\int_0^{\infty} e^{-t/x}\,d(t^{\beta}L(t)) = o(x^{\beta}L(x)) \qquad (x\to\infty).$$

Substituting the expression for $K(x)$ we find

$$\frac{K(x) - \hat{K}(1/x)/\Gamma(\beta+1)}{x^{\beta}L(x)} = \int_0^1 \frac{L(tx)}{L(x)}\frac{dt}{t}$$
$$- \frac{1}{\Gamma(\beta+1)}\int_0^{\infty} e^{-t}t^{\beta}\int_0^t \frac{L(ux)}{L(x)}\frac{du}{u}\,dt$$
$$= -\frac{1}{\Gamma(\beta+1)}\int_0^1 e^{-t}t^{\beta}\int_1^t \frac{u^{\varepsilon}x^{\varepsilon}L(ux)}{x^{\varepsilon}L(x)}\frac{du}{u^{1+\varepsilon}}\,dt$$
$$- \frac{1}{\Gamma(\beta+1)}\int_1^{\infty} e^{-t}t^{\beta}\int_1^t \frac{u^{-\varepsilon}x^{-\varepsilon}L(ux)}{x^{-\varepsilon}L(x)}\frac{du}{u^{1-\varepsilon}}\,dt = (*)$$

since $\Gamma(\beta+1)=\int_0^\infty e^{-t}t^\beta\,dt$. Since

$$\frac{u^\varepsilon x^\varepsilon L(ux)}{x^\varepsilon L(x)} \to u^\varepsilon\ (x\to\infty) \quad \text{uniformly on } (0,1)$$

and

$$\frac{u^{-\varepsilon} x^{-\varepsilon} L(ux)}{x^{-\varepsilon} L(x)} \to u^{-\varepsilon}\ (x\to\infty) \quad \text{uniformly on } (1,\infty)$$

(see [5, Corollary 1.2.1.4]) we find

$$(*) \to -\frac{1}{\Gamma(\beta+1)}\int_0^\infty e^{-t}t^\beta \log t\,dt = -\psi(\beta+1) \qquad (x\to\infty).$$

This proves (i) → (v). Now we have analogously that (v) and (i) imply (iii).

(ii) → (iii) follows immediately since ${}_\alpha\hat{U}(1/x)=x^\alpha\hat{U}(1/x)$ and we can use (i) → (iii).

(i) ↔ (iii). Writing $V(x)=U(x)/x^\beta$ we have by Proposition P4 in [7]

$$V(x)\in\Pi \text{ iff } \int_0^x t^\beta\,dV(t)\in RV_\beta \quad \text{where } \beta>0.$$

Or

$$U(x)\in\Pi RV_\beta \text{ iff } U(x)-\int_0^x \frac{U(t)}{t}\,dt\in RV_\beta.$$

This is equivalent to

$$\hat{U}(1/x)-\beta x\hat{K}(1/x)\in RV_\beta \quad \text{where } K(x)=U(x)/x.$$

The last statement is equivalent to $\hat{U}(1/x)\in\Pi RV_\beta$, since $x\hat{K}(1/x)=\int_0^x(\hat{U}(1/t)/t)dt$. (Both sides have the same derivative.) The case $\beta=0$ is the result of Theorem E.

(iv) → (i). As in the proof of (i) → (iv) we write

$$U(x)=x^\beta\left\{L(x)+\int_0^x\frac{L(t)}{t}\,dt\right\}.$$

Substituting this expression in (iv) and rearranging we see that (iv) is equivalent to

$$\frac{1}{\Gamma(\alpha)}\int_0^x\left(1-\frac{t}{x}\right)^{\alpha-1}\left(\frac{t}{x}\right)^{\beta+1}L(t)\frac{dt}{t}+\frac{1}{\Gamma(\alpha)}\int_0^x\int_{t/x}^1(1-u)^{\alpha-1}u^\beta\,du L(t)\frac{dt}{t}$$
$$-\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\int_0^x L(t)\frac{dt}{t}\sim\xi L(x) \qquad (x\to\infty),$$

where $\xi=(\Gamma(\beta+1)/\Gamma(\alpha+\beta+1))+(d/d\beta)\{\Gamma(\beta+1)/\Gamma(\alpha+\beta+1)\}$. Or $\int_0^\infty L(t)k(x/t)\,dt/t\sim\xi L(x)$ $(x\to\infty)$ where the kernel k is defined by

$$k(1/x)=\frac{x}{\Gamma(\alpha)}\left\{(1-x)^{\alpha-1}x^\beta-\frac{1}{x}\int_0^x(1-u)^{\alpha-1}u^\beta\,du\right\}$$

for $x<1$ and 0 for $x\geqslant 1$. For $\alpha\in(0,1]$ and $\beta>0$ the kernel is nonnegative since $(1-x)^{\alpha-1}x^\beta$ is increasing on (0, 1). Moreover we have $\lim_{x\to\infty;t\to1+}\inf(L(tx)/L(x))\geqslant 1$ since $xL(x)$ is nondecreasing. Application of Theorem 6.2 in [3] then gives the result since $\hat{k}(\rho)=\int_0^1 k(1/t)t^{\rho-1}\,dt$ is decreasing for $\rho>-\beta-1$ and so $\hat{k}(\rho)=\xi$ only if $\rho=0$.

(v) $\to$ (i). We define $L(x)$ as in the proof of (iv) $\to$ (i). Here we can reformulate (v) as follows:

$$\int_0^\infty k\left(\frac{x}{t}\right)L(t)\frac{dt}{t} \sim \xi L(x) \qquad (x \to \infty),$$

where $\xi = 1 + \psi(\beta + 1)$ and the kernel k is given by

$$k\left(\frac{1}{x}\right) = \frac{1}{\Gamma(\beta+1)}x^\beta e^{-x} - \frac{1}{x}\int_0^x u^\beta e^{-u}\,du + 1 - I_{(0,1)}(x).$$

If $\beta > 1$ this kernel is positive for all $x > 0$, since the term $x^\beta e^{-x}$ is increasing on $(0, \beta)$. Here we can also apply Theorem 6.2 in [3].

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