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$\mathfrak{p}$-RIGIDITY AND IWASAWA μ-INVARIANTS

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ABSTRACT. Let F be a totally real field with ring of integers O and p be an odd prime unramified in F. Let $\mathfrak{p}$ be a prime above p. We prove that a mod p Hilbert modular form associated to F is determined by its restriction to the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$ ($\mathfrak{p}$-rigidity). Let K/F be an imaginary quadratic CM extension such that each prime of F above p splits in K and λ a Hecke character of K. Partly based on $\mathfrak{p}$-rigidity, we prove that the μ-invariant of anticyclotomic Katz $\mathfrak{p}$-adic L-function of λ equals the μ-invariant of the full anticyclotomic Katz p-adic L-function of λ. An analogue holds for a class of Rankin-Selberg p-adic L-functions. When λ is self-dual with the root number -1, we prove that the μ-invariant of the cyclotomic derivatives of Katz $\mathfrak{p}$-adic L-function of λ equals the μ-invariant of the cyclotomic derivatives of Katz p-adic L-function of λ. Based on previous works of authors and Hsieh, we consequently obtain a formula for the μ-invariant of these $\mathfrak{p}$-adic L-functions and derivatives, in most of the cases. We also prove a $\mathfrak{p}$-version of a conjecture of Gillard, namely the vanishing of the μ-invariant of Katz $\mathfrak{p}$-adic L-function of λ.

Contents

1.	Introduction	1
2.	Hilbert modular Shimura variety	6
2.1.	Setup	6
2.2.	p-integral model	6
2.3.	Igusa tower	7
2.4.	Mod p modular forms	8
2.5.	p-adic modular forms	8
3.	$\mathfrak{p}$-rigidity	9
3.1.	Serre-Tate deformation theory	9
3.2.	Reciprocity law for the deformation space	11
3.3.	$\mathfrak{p}$-rigidity	11
4.	$\mathfrak{p}$-independence	13
4.1.	Formulation	13
4.2.	$\mathfrak{p}$-independence	14
5.	p-adic differential operators	15
5.1.	Elementary construction	15
5.2.	Action on the t-expansion	16
6.	Iwasawa μ-invariants	18
6.1.	μ-invariant of anticyclotomic $\mathfrak{p}$-adic L-functions	18
6.2.	μ-invariant of the cyclotomic derivative of Katz $\mathfrak{p}$-adic L-function	20
6.3.	$\mathfrak{p}$-version of a conjecture of Gillard	21
	References	22

1. INTRODUCTION

Zeta values seem to suggest deep phenomena in Mathematics. They seem to mysteriously encode deep arithmetic information. They also seem to suggest surprising modular and Iwasawa-theoretic phenomena.

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Sometimes, they are a sum of evaluation of modular forms at CM points. Such an expression for critical Hecke L-values and conjectural non-triviality of the corresponding anticyclotomic p-adic L-function, suggested to the second named author a linear independence of mod p Hilbert modular forms. Based on Chai's theory of Hecke-stable subvarieties of a Shimura variety (*cf.* [8], [9] and [10]), this guess was proven in [19]. Let $\mathfrak{p}$ be a prime above p (*cf.* Abstract). Recently, the expression and conjectural non-triviality of the corresponding anticyclotomic $\mathfrak{p}$-adic L-function suggested to us a rather surprising rigidity property of mod p Hilbert modular forms. Partly based on the rigidity, we obtain intriguing equalities of Iwasawa μ-invariants of seemingly independent p-adic L-functions.

Here is some geometric reason for such rigidity referred as L-rigidity in the following. Let $\widehat{S}$ be a formal torus over a field which is the residue field of a mixed characteristic ring. Suppose that we have a rational structure coming from an algebraic subscheme S over the mixed characteristic ring whose formal completion at a closed point $x \in S$ gives $\widehat{S}$. Here S is not necessarily an algebraic torus. We suppose that there exists a positive dimensional transcendental linear subvariety L of $\widehat{S}$ with strictly smaller dimension i.e., a non-trivial formal subtorus which does not equal formal completion along x of an algebraic subscheme of S. Transcendence of L implies "algebraic" Zariski closure of L in S is the entire S. This may not happen for the formal Zariski closure. Let $A = \mathcal{O}_{S,x}$ and $L = \mathrm{Spf}(B)$. Thus, the transcendence of L is equivalent to the injectivity of the natural morphism $A \to B$ given by $\phi \mapsto \phi|_L$ for $\phi \in A$ and hence the L-rigidity i.e. ϕ is determined by its restriction to the formal subtorus L. The rigidity remains true for $\phi \circ a$ for any automorphism a of $\widehat{S}$. It turns out that often the rigidity also remains true for $\sum_i \phi_i \circ a_i$ for a well chosen set of automorphisms a_i of $\widehat{S}$ and $\phi_i \in \Gamma(S, \mathcal{O}_S)$ i.e., $\sum_i \phi_i \circ a_i = 0 \Rightarrow \phi_i = 0$ for all i. As the notion of well chosen may vary from context to context, we only mention that $\{a_i\}$'s typically satisfy a transcendental property and specify it in the current context later. We study the case for the Serre-Tate deformation space $\widehat{S}$ with rational structure induced from the Hilbert modular Shimura variety. The formal subtorus we study is the partial $\mathfrak{p}$-deformation subspace of $\widehat{S}$. We consider certain $\{a_i\}_i$ such that the differences $\{a_i a_j^{-1}\}_{i \neq j}$ are "transcendetal" automorphisms of the Serre-Tate deformation space (*cf.* §4.1).

Let F be a totally real field of degree d and O the integers ring. Let p be an odd prime unramified in F. Let $\mathfrak{p}_1, ..., \mathfrak{p}_r$ be the primes above p. Fix two embeddings $\iota_\infty \colon \overline{\mathbf{Q}} \to \mathbf{C}$ and $\iota_p \colon \overline{\mathbf{Q}} \to \mathbf{C}_p$. Let v_p be the p-adic valuation of $\mathbf{C}_p$ normalised such that $v_p(p) = 1$. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$.

Let $Sh_{/\mathbb{F}}$ be the Kottwitz model of prime to p Hilbert modular Shimura variety associated to F. We refer to §2.2 for the definition. Here we only mention that in the moduli interpretation for $Sh_{/\mathbf{Z}_{(p)}}$, the full prime to p level structure appears. Let $x \in Sh$ be a closed ordinary point. From Serre-Tate deformation theory, a p^∞-level structure on x induces a canonical isomorphism

$$\mathrm{Spf}(\widehat{\mathcal{O}}_{Sh,x}) \simeq \prod_i \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}_i}. \tag{1.1}$$

Let $\mathfrak{p} = \mathfrak{p}_i$, for some i. Let f be a mod p Hilbert modular form in the sense of §2.4. In view of the irreducibility of the connected components of Sh, the form f is determined by its restriction to $\mathrm{Spf}(\widehat{\mathcal{O}}_{Sh,x})$. In fact, we have the following rigidity result.

Theorem A ($\mathfrak{p}$-rigidity) Let f be a non-zero mod p Hilbert modular form. Then, f does not vanish identically on the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$. In particular, a mod p Hilbert modular form is determined by its restriction to the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.

We now describe the results regarding the Iwasawa μ-invariants.

Let K be a totally imaginary quadratic extension of F. Let $h_?$ denote the class number of ?, for ? $= K, F$. Let c denote the complex conjugation on $\mathbf{C}$ which induces the unique non-trivial element of $\mathrm{Gal}(K/F)$ via

ι_∞. We assume the following hypothesis:

(ord) Every prime of F above p splits in K.

The condition (ord) guarantees the existence of a p-adic CM type Σ i.e. Σ is a CM type of K such that, p-adic places induced by elements in Σ via ι_p are disjoint from those induced by Σc. We fix such a CM type. We also identify it with the set of infinite places of F. Let K_∞^- (resp. K_∞^+) be the anticyclotomic $\mathbf{Z}_p^d$-extension (resp. cyclotomic $\mathbf{Z}_p$-extension) of K and and $K_{\mathfrak{p},\infty}^- \subset K_\infty^-$ be the $\mathfrak{p}$-anticyclotomic subextension i.e. the maximal subextension unramified outside the primes above $\mathfrak{p}$ in K. Let $K_{\mathfrak{p},\infty} = K_{\mathfrak{p},\infty}^- K_\infty^+$. Let $\Gamma^\pm := \mathrm{Gal}(K_\infty^\pm/K)$, $\Gamma_\mathfrak{p}^- = \mathrm{Gal}(K_{\mathfrak{p},\infty}^-/K)$ and $\Gamma_\mathfrak{p} = \mathrm{Gal}(K_{\mathfrak{p},\infty}/K)$.

Let $\mathfrak{C}$ be a prime-to-p integral ideal of K. Let λ be a Hecke character of K. Suppose that $\mathfrak{C}$ is the prime-to-p conductor of λ. Associated to this data, a natural $(d+1)$-variable Katz p-adic L-function $L_{\Sigma,\lambda} = L_{\Sigma,\lambda}(T_1,...,T_d,S) \in \overline{\mathbf{Z}}_p[\![\Gamma]\!]$ is constructed in [25] and [14]. Here, T_i's are the anticyclotomic variables and S is the cyclotomic variable. Katz p-adic L-function $L_{\Sigma,\lambda}$ interpolates critical Hecke L-values $L(0,\lambda\chi)$ as χ varies over certain Hecke characters mod $\mathfrak{C}p^\infty$ (*cf.* [14, Thm. II]). Let $L_{\Sigma,\lambda}^- \in \overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$ (resp. $L_{\Sigma,\lambda,\mathfrak{p}}^- \in \overline{\mathbf{Z}}_p[\![\Gamma_\mathfrak{p}^-]\!]$) be the anticyclotomic (resp. $\mathfrak{p}$-anticyclotomic) projection obtained from the projection $\pi^- : \overline{\mathbf{Z}}_p[\![\Gamma]\!] \twoheadrightarrow \overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$ (resp. $\pi_\mathfrak{p}^- : \overline{\mathbf{Z}}_p[\![\Gamma]\!] \twoheadrightarrow \overline{\mathbf{Z}}_p[\![\Gamma_\mathfrak{p}^-]\!]$). Let $L_{\Sigma,\lambda,\mathfrak{p}} \in \overline{\mathbf{Z}}_p[\![\Gamma_\mathfrak{p}]\!]$ be obtained from the projection $\pi_\mathfrak{p} : \overline{\mathbf{Z}}_p[\![\Gamma]\!] \twoheadrightarrow \overline{\mathbf{Z}}_p[\![\Gamma_\mathfrak{p}]\!]$. We call $L_{\Sigma,\lambda,\mathfrak{p}}$ as the Katz $\mathfrak{p}$-adic L-function to emphasise the consideration of $\mathfrak{p}$-component. This is a slightly non-traditional terminology as the construction is still under the same embedding ι_p.

The μ-invariant of $L_{\Sigma,\lambda,\mathfrak{p}}^-$ is given by the following Theorem.

Theorem B

$$\mu(L_{\Sigma,\lambda}^-) = \mu(L_{\Sigma,\lambda,\mathfrak{p}}^-).$$

In most of the cases, $\mu(L_{\Sigma,\lambda}^-)$ has been explicitly determined (*cf.* [19] and [23]). Thus, we obtain a formula for $\mu(L_{\Sigma,\lambda,\mathfrak{p}}^-)$.

A result analogous to Theorem B also holds for a class of Rankin-Selberg anticyclotomic $\mathfrak{p}$-adic L-functions.

When λ is self-dual with the root number -1, all the Hecke L-values appearing in the interpolation property of $L_{\Sigma,\lambda}^-$ vanish. Accordingly, $L_{\Sigma,\lambda}^-$ and $L_{\Sigma,\lambda,\mathfrak{p}}^-$ identically vanish. The anticyclotomic arithmetic information contained in $L_{\Sigma,\lambda}^-$ and $L_{\Sigma,\lambda,\mathfrak{p}}^-$ may seem to have disappeared. However, we can look at the cyclotomic derivatives

$$L_{\Sigma,\lambda}^{'} = (\frac{\partial}{\partial S} L_{\Sigma,\lambda}(T_1,...,T_d,S))|_{S=0} \tag{1.2}$$

and $L_{\Sigma,\lambda,\mathfrak{p}}^{'}$ (defined analogously).

The μ-invariant of $L_{\Sigma,\lambda,\mathfrak{p}}'$ is given by the following Theorem.

Theorem C Suppose that $p \nmid h_K^-$, where h_K^- is the relative class number given by $h_K^- = h_K/h_F$. Then,

$$\mu(L_{\Sigma,\lambda}') = \mu(L_{\Sigma,\lambda,\mathfrak{p}}').$$

In most of the cases, $\mu(L_{\Sigma,\lambda}')$ has been explicitly determined (*cf.* [3]). Thus, we obtain a formula for $\mu(L_{\Sigma,\lambda,\mathfrak{p}}')$.

We have the following $\mathfrak{p}$-version of a conjecture of Gillard regarding the vanishing of the μ-invariant of Katz

p-adic L-function (*cf.* [12, Conj. (i)]).

Theorem D

$$\mu(L_{\Sigma,\lambda,\mathfrak{p}}) = 0.$$

We now describe the strategy of the proof of Theorem A. Some of the notation used here is not followed in the rest of the article.

Let $G = \mathrm{Res}_{O/\mathbf{Z}}(\mathrm{GL}_2)$. The group $G(\mathbf{Z}_{(p)})$ acts on the prime to p Hilbert modular Shimura variety $Sh_{/\mathbf{Z}_{(p)}}$. We refer to §3.2 for the action. Here we only mention that in terms of the moduli interpretation, the action corresponds to the one on the level structure. Let V be an irreducible component of Sh containing x. Let $H_x(\mathbf{Z}_{(p)})$ be the stabiliser of x in $G(\mathbf{Z}_{(p)})$. It acts on $\mathrm{Spec}\,(\mathcal{O}_{V,x})$ and thus on the Serre-Tate deformation space $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$. In view of the description of the action on the Serre-Tate co-ordinates, we observe that the formal subtorus $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$ is stable under the action of $H_x(\mathbf{Z}_{(p)})$. Chai and the second named author have proven that a positive dimensional closed irreducible subvariety of V containing x and stable under $H_x(\mathbf{Z}_{(p)})$ equals V itself. In this sense, the formal subtorus $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$ is transcedental in the Shimura variety. Recall that the Igusa tower is *étale* over V. As a mod p Hilbert modular form is an algebraic function on the Igusa tower, we prove $\mathfrak{p}$-rigidity based on the transcendence.

We now describe the strategy of the proof of Theorem B. Some of the notation used here is not followed in the rest of the article.

Let us briefly recall the second named author's strategy to determine $\mu(L^{-}_{\Sigma,\lambda})$ (*cf.* [19]). Let $O_p = O \otimes \mathbb{Z}_p$. Let $G_{\Sigma,\lambda} \in \overline{\mathbf{Z}}_p[\![T_1, ..., T_d]\!]$ be the power series expansion of the measure $L^{-}_{\Sigma,\lambda}$ regarded as a p-adic measure on O_p with support in $1 + pO_p$, given by

$$G_{\Sigma,\lambda} = \int_{1+pO_p} t^y dL^{-}_{\Sigma,\lambda}(y) = \sum_{(k_1,...,k_d)\in(\mathbf{Z}_{\geq 0})^d} \Big(\int_{1+pO_p} \binom{y}{k_1, ..., k_d} dL^{-}_{\Sigma,\lambda}(y)\Big) T_1^{k_1} ... T_d^{k_d}. \tag{1.3}$$

The starting point is the observation that there are classical Hilbert modular Eisenstein series $(f_{\lambda,i})_i$ such that

$$G_{\Sigma,\lambda} = \sum_i a_i \circ (f_{\lambda,i}(t)), \tag{1.4}$$

where $f_{\lambda,i}(t)$ is the t-expansion of $f_{\lambda,i}$ around a well chosen CM point y with the CM type (K, Σ) on the Hilbert modular Shimura variety Sh and a_i is an automorphism of the the Serre-Tate deformation space $\mathrm{Spf}(\widehat{\mathcal{O}}_{Sh,y})$ i.e. $a_i \in H_y(\mathbf{Z}_p)$ (*cf.* §3.2). Based on Chai's study of Hecke-stable subvarieties of a Shimura variety, the second named author has proven the linear independence of $(a_i \circ f_{\lambda,i})_i$ modulo p. It follows that $\mu(L^{-}_{\Sigma,\lambda}) = \min_i \mu(f_{\lambda,i}(t))$.

Let $G_{\Sigma,\lambda,\mathfrak{p}} \in \overline{\mathbf{Z}}_p[\![\Gamma^{-}_{\mathfrak{p}}]\!]$ be the analogous power series expansion of the measure $L^{-}_{\Sigma,\lambda,\mathfrak{p}}$. Based on the action of p-adic differential operators on the t-expansion of a p-adic Hilbert modular form around an ordinary point in terms of the partial Serre-Tate co-ordinates, we show that

$$G_{\Sigma,\lambda,\mathfrak{p}} = \sum_i a_{i,\mathfrak{p}} \circ (f_{\lambda,i}(t_{\mathfrak{p}})), \tag{1.5}$$

where $a_{i,\mathfrak{p}}$ is the projection of a_i to $H_y(\mathbf{Z}_p)_{\mathfrak{p}}$ (*cf.* §3.2) and $f_{\lambda,i}(t_{\mathfrak{p}})$ is the $\mathfrak{p}$-adic Serre-Tate expansion of f around y (*cf.* §6.1). Based on $\mathfrak{p}$-rigidity and Chai's study of Hecke-stable subvarieties of a Shimura variety, we prove $\mathfrak{p}$-independence i.e. the linear independence of $(a_{i,\mathfrak{p}} \circ (f_{\lambda,i}(t_{\mathfrak{p}})))_i$ modulo p. It follows that $\mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}}) = \min_i \mu(f_{\lambda,i}(t_{\mathfrak{p}}))$. From $\mathfrak{p}$-rigidity, we have $\mu(f_{\lambda,i}(t)) = \mu(f_{\lambda,i}(t_{\mathfrak{p}}))$ and this concludes the proof of Theorem B.

The strategy of the proof of Theorem C is similar to the above strategy. Finally, Theorem D is proven based on Theorem B and the proof of results in [20] and [2]. As in [19], we would like to emphasise that Chai's theory plays an underlying role in all of the above results.

In this article, we give an elementary construction of p-adic differential operators on the space of p-adic Hilbert modular forms avoiding the use of Gauss-Manin connection in Katz's construction. This is based on the ideas of the second named author in the early 90's. Based on this construction, we determine the action of p-adic differential operators on the t-expansion of a p-adic Hilbert modular form around an ordinary point in terms of the partial Serre-Tate co-ordinates.

Along with p-adic Gross–Zagier formula, Theorem B and Theorem C have application towards generic non-vanishing of p-adic heights on CM abelian varieties (*cf.* [7]). This provides an evidence for Schneider's conjecture on the non-vanishing of p-adic heights in the CM case.

Let $\mathfrak{p}_i$ and $\mathfrak{p}_j$ be primes above p as above. Theorem B implies an intriguing equality $\mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}_i}) = \mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}_j})$ of Iwasawa μ-invariants. This is rather surprising as theses μ-invariants could be non-zero and one in general does not expect any relation between the p-adic L-functions $L_{\Sigma,\lambda,\mathfrak{p}_i}$'s. These p-adic L-functions correspond to independent variables whose number may vary with i. As per as we know, Theorem B is a first phenomena possibly suggesting a relation. It would be interesting to see whether an analogue holds for Iwasawa λ-invariants. One can perhaps first collect experimental data. In the case of self-duality and the root number being -1, the equality of the μ-invariants persists even after taking the cyclotomic derivative. It seems tempting that a deeper phenomena mediates the relation.

In view of the anticyclotomic main conjectures, Theorem B would imply an equality of the corresponding algebraic μ-invariants. Note that the underlying Selmer groups correspond to rather different local conditions. In many cases, the anticyclotomic main conjecture has been proven (*cf.* [16] and [18]). It would be interesting to prove the equality of the algebraic μ-invariants directly. The equality of this type does not seem to be conjectured in the literature.

In [6], the first named author proves analogue of $\mathfrak{p}$-rigidity and $\mathfrak{p}$-independence for quaternionic modular forms over totally real fields. In the near future, the first named author hopes to consider an analogue of Theorem B for a class of quaternionic Rankin-Selberg p-adic L-functions. In [13], a construction of p-adic L-function for Unitary Shimura varieties is announced (*cf.* [11]). In such a case, the number of variables of the p-adic L-function is typically less than the dimension of the Shimura variety. The variables of the p-adic L-function may correspond to an analgoue of the $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$-variables. In the future, we hope to consider this question starting with the case of $U(n,1)$ Shimura varieties.

Formulating $\mathfrak{p}$-rigidity type statement for a PEL Shimura variety seems to be an interesting question. In characteristic zero, we hope to approach rigidity based on the approach in [5].

The article is organised as follows. In §2, we recall basic facts about Hilbert modular Shimura variety Sh. In §3, we prove Theorem A. In §3.1-3.2, we firstly recall some facts about Serre-Tate deformation theory of an ordinary closed point in Sh. In §3.3, we prove the Theorem. In §4, we prove $\mathfrak{p}$-rigidity i.e. the linear independence of mod p Hilbert modular forms restricted to the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$. In §5.1, we give an elementary construction of p-adic differential operators on the space of p-adic Hilbert modular forms. In §5.2, we use it to compute the action of the p-adic differential operators on the t-expansion of a p-adic Hilbert modular form around an ordinary point in terms of the partial Serre-Tate co-ordinates. In §6, we consider Iwasawa μ-invariants as in Theorems B-D. In §6.1, we determine the μ-invariant of certain anticyclotomic $\mathfrak{p}$-adic L-functions (*cf.* Theorem B). In §6.2, we determine the μ-invariant of the cyclotomic derivative $L'_{\Sigma,\lambda,\mathfrak{p}}$ of Katz $\mathfrak{p}$-adic L-function, when the branch character λ is self-dual with the root number -1 (*cf.* Theorem C). In §6.3, we prove a $\mathfrak{p}$-version of a conjecture Gillard regarding the vanishing of the μ-invariant of Katz p-adic L-function (*cf.* Theorem D).

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Notation We use the following notation unless otherwise stated.

For a number field L, let $\mathbf{A}_L$ be the the adele ring and $\mathbf{A}_L^f$ the finite adeles of L. Let G_L be the absolute Galois group of L and G_L^{ab} the maximal abelian quotient. Let $\mathrm{rec}_L : \mathbf{A}_L^\times \to G_L^{ab}$ be the geometrically normalized reciprocity law.

2. Hilbert modular Shimura variety

In this section, we recall basic facts about Hilbert modular Shimura variety. We follow [15].

2.1. Setup.

In this subsection, we recall a basic setup regarding Hilbert modular Shimura variety.

Let $G = Res_{F/\mathbf{Q}}GL_2$ and $h_0 : Res_{\mathbf{C}/\mathbf{R}}\mathbb{G}_m \to G_{/\mathbf{R}}$ be the morphism of real group schemes arising from

$$a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a + bi \in \mathbf{C}^\times$. Let X be the set of $G(\mathbf{R})$-conjugacy classes of h_0. We have a canonical isomorphism $X \simeq (\mathbf{C} - \mathbf{R})^I$, where I is the set of real places of F. The pair (G, X) satisfies Deligne's axioms for a Shimura variety. It gives rise to a tower $(Sh_K = Sh_K(G, X))_K$ of quasi-projective smooth varieties over $\mathbf{Q}$ indexed by open compact subgroups K of $G(\mathbf{A}^f)$. The pro-algebraic variety $Sh_{/\mathbf{Q}}$ is the projective limit of these varieties. The complex points of these varieties are given as follows

$$Sh_K(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}^f)/K, Sh(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}^f)/\overline{Z(\mathbf{Q})}. \tag{2.1}$$

Here, $\overline{Z(\mathbf{Q})}$ is the closure of the center $Z(\mathbf{Q})$ in $G(\mathbf{A}^f)$ under the *adélic* topology. From (2.1), it follows that $Sh_{/\mathbf{Q}}$ is endowed with an action of $G(\mathbf{A}^f)$ (*cf.* [15, §4.2]). This gives rise to the Hecke action.

2.2. p-integral model.

In this subsection, we briefly recall a canonical p-integral smooth model $Sh^{(p)}_{/\mathbf{Z}_{(p)}}$ of the Shimura variety $Sh/G(\mathbf{Z}_p)_{/\mathbf{Q}}$.

Hilbert modular Shimura variety $Sh_{/\mathbf{Q}}$ represents a functor $\mathcal{F}$ classifying abelian schemes having multiplication by O along with additional structure, where O is the ring of integers of F (*cf.* [15, §4.2] and [27]). When p is unramified in F, a p-integral interpretation $\mathcal{F}^{(p)}$ of $\mathcal{F}$ leads to a p-integral smooth model of $Sh/G(\mathbf{Z}_p)_{/\mathbf{Q}}$.

The functor $\mathcal{F}^{(p)}$ is given by

$$\mathcal{F}^{(p)} : SCH_{/\mathbf{Z}_{(p)}} \to SETS$$

$$S \mapsto \{(A, \iota, \bar{\lambda}, \eta^{(p)})_{/S}\}/\sim. \tag{2.2}$$

Here,

(PM1) A is abelian scheme over S of dimension of d.
(PM2) $\iota : O \hookrightarrow \mathrm{End}_S A$ is an algebra embedding.
(PM3) $\bar{\lambda}$ is the polarisation class of a homogeneous polarisation λ up to scalar multiplication by $\iota(O^{\times}_{(p),+})$, where $O_{(p),+} := \{a \in O_{(p)} | \sigma(a) > 0, \forall \sigma \in I\}$. Also, the Rosati involution of $\mathrm{End}_S A$ takes $\iota(l)$ to $\iota(l^*)$, for $l \in O$.
(PM4) Let $\mathcal{T}^{(p)}(A)$ be the prime-to-p Tate module $\varprojlim_{p \nmid N} A[N]$. $\eta^{(p)}$ is a prime-to-p level structure given by an O-linear isomorphism $\eta^{(p)} : O^2 \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}^{(p)} \simeq \mathcal{T}^{(p)}(A)$, where $\widehat{\mathbf{Z}}^{(p)} = \prod_{l \neq p} \mathbf{Z}_l$.
(PM5) Let $\mathrm{Lie}_S(A)$ be the relative Lie algebra of A. There exists an $O \otimes_{\mathbf{Z}} \mathcal{O}_S$-module isomorphism $\mathrm{Lie}_S(A) \simeq O \otimes_{\mathbf{Z}} \mathcal{O}_S$, locally under the Zariski topology of S.

The notation $\sim$ denotes up to a prime-to-p isogeny.

Theorem 2.1 (Kottwitz)**.** *The functor $\mathcal{F}^{(p)}$ is represented by a pro-algebraic scheme $Sh^{(p)}(G,X)_{/\mathbf{Z}_{(p)}}$. Moreover, there exists an isomorphism given by*

$$Sh^{(p)} \times \mathbf{Q} \simeq Sh/G(\mathbf{Z}_p)_{/\mathbf{Q}}.$$

(cf. [15, §4.2.1]*).*

The pro-algebraic scheme $Sh^{(p)}(G,X)_{/\mathbf{Z}_{(p)}}$ is usually referred as the Kottwitz model. In what follows, we let $Sh^{(p)}_{/\mathbf{Z}_{(p)}}$ denote $Sh^{(p)}(G,X)_{/\mathbf{Z}_{(p)}}$ for simplicity of notation.

2.3. **Igusa tower.** In this subsection, we briefly recall the notion of p-ordinary Igusa tower over the p-integral model $Sh^{(p)}$.

Let $\overline{\mathbf{Q}}$ be an algebraic closure of $\mathbf{Q}$ and $\overline{\mathbf{Q}}_p$ be an algebraic closure of $\mathbf{Q}_p$. We fix a complex embedding $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and a p-adic embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$.

Let $\mathcal{W}$ be the strict Henselisation inside $\overline{\mathbf{Q}}$ of the local ring of $\mathbf{Z}_{(p)}$ corresponding to ι_p. Let $\mathbb{F}$ be the residue field of $\mathcal{W}$. Note that $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_p$.

Let $Sh^{(p)}_{/\mathcal{W}} = Sh^{(p)} \times_{\mathbf{Z}_{(p)}} \mathcal{W}$ and $Sh^{(p)}_{/\mathbb{F}} = Sh^{(p)}_{/\mathcal{W}} \times_{\mathcal{W}} \mathbb{F}$.

From now, let Sh denote $Sh^{(p)}_{/\mathbb{F}}$. Let $\mathcal{A}$ be the universal abelian scheme over Sh.

Let Sh^{ord} be the subscheme of Sh on which the Hasse-invariant does not vanish. It is an open dense subscheme. Over Sh^{ord}, the connected part $\mathcal{A}[p^m]^\circ$ of $\mathcal{A}[p^m]$ is *étale*-locally isomorphic to $\mu_{p^m} \otimes_{\mathbf{Z}_p} O^*$ as an O_p-module, where $O^* = O^{-1}\mathfrak{d}_F^{-1}$, $\mathfrak{d}_F$ is the different of $F/\mathbf{Q}$ and $O_p = O \otimes \mathbf{Z}_p$.

We now define the Igusa tower. For $m \in \mathbb{N}$, the m^{th}-layer of the Igusa tower over Sh^{ord} is defined by

$$Ig_m = \underline{\mathrm{Isom}}_{O_p}(\mu_{p^m} \otimes_{\mathbf{Z}_p} O^*, \mathcal{A}[p^m]^\circ). \tag{2.3}$$

Note that the projection $\pi_m : Ig_m \to Sh^{ord}$ is finite and *étale*. The full Igusa tower over Sh^{ord} is defined by

$$Ig = Ig_\infty = \varprojlim Ig_m = \underline{\mathrm{Isom}}_{O_p}(\mu_{p^\infty} \otimes_{\mathbf{Z}_p} O^*, \mathcal{A}[p^\infty]^\circ). \tag{2.4}$$

(*Ét*) Note that the projection $\pi : Ig \to Sh^{ord}$ is *étale*.

Let x be a closed ordinary point in Sh. We have the following description of the level p^∞-structure on

$A_x[p^\infty]$.

(PL) Let η_p° be a level p^∞-structure on $A_x[p^\infty]^\circ$. For the primes $\mathfrak{p}$ in O dividing p, it is a collection of level $\mathfrak{p}^\infty$ structures $\eta_\mathfrak{p}^\circ$, given by isomorphisms $\eta_\mathfrak{p}^\circ : O_\mathfrak{p}^* \simeq A_x[\mathfrak{p}^\infty]^\circ$, where $O_\mathfrak{p}^* = O^* \otimes O_\mathfrak{p}$. The Cartier duality and the polarisation $\bar{\lambda}_x$ induces an isomorphism $\eta_\mathfrak{p}^{ét} : O_\mathfrak{p} \simeq A_x[\mathfrak{p}^\infty]^{ét}$. Thus, we get a level p^∞-structure $\eta_p^{ét}$ on $A_x[p^\infty]^{ét}$ from η_p°.

Let V be an irreducible component of Sh and V^{ord} be $V \cap Sh^{ord}$. Let I be the inverse image of V^{ord} under π. In [15, Ch.8] and [17], it has been shown that

(Ir) I is an irreducible component of Ig.

2.4. **Mod p modular forms.** In this subsection, we briefly recall the notion of mod p modular forms on an irreducible component of the Hilbert modular Shimura variety Sh.

Let V and I be as in §2.3. Let B be an $\mathbb{F}$-algebra. The space of mod p modular forms on V over B is defined by

$$M(V, B) = H^0(I_{/B}, \mathcal{O}_{I_{/B}}), \tag{2.5}$$

where $I_{/B} := I \times_\mathbb{F} B$. In view of §2.2-2.3, we have the following geometric interpretation of mod p modular forms.

A mod p modular form is a function f of isomorphism classes of $\tilde{x} = (x, \eta_p^\circ)_{/B'}$ where B' is a B-algebra, $x = (A, \iota, \bar{\lambda}, \eta^{(p)})_{/B'} \in \mathcal{F}^{(p)}(B')$ and $\eta_p^\circ : \mu_{p^\infty} \otimes_{\mathbf{Z}_p} O^* \simeq A[p^\infty]^\circ$ is an O_p-linear isomorphism, such that the following conditions are satisfied.

(G1) $f(\tilde{x}) \in B'$.
(G2) If $\tilde{x} \simeq \tilde{x}'$, then $f(\tilde{x}) = f(\tilde{x}')$, where $\tilde{x} \simeq \tilde{x}'$ means $x \simeq x'$ and the corresponding isomorphism between A and A' induces an isomorphism between η_p° and $\eta_p'^\circ$.
(G3) $f(\tilde{x} \times_{B'} B'') = h(f(\tilde{x}))$ for any B-algebra homomorphism $h : B' \to B''$.

We also have the notion of q-expansion and q-expansion principle for mod p modular forms (*cf.* [15, Thm. 4.21]).

2.5. **p-adic modular forms.** In this subsection, we briefly recall the notion of p-adic modular forms on an irreducible component of the Hilbert modular Shimura variety Sh.

Let W denote the Witt ring $W(\mathbb{F})$. The construction of the Igusa tower in §2.3 is well defined for the base W. Let $I_{/W}$ be the irreducible component of the Igusa tower $Ig_{/W}$ over an irreducible component $V_{/W}$ of the Shimura variety $Sh_{/W}$. Let C be a p-adically complete local W-algebra with maximal ideal $\mathfrak{m}_C$. The space of p-adic modular forms on V over C is defined by

$$M(V, C) = H^0(I_{/C}, \mathcal{O}_{I_{/C}}), \tag{2.6}$$

where $I_{/C} := I_{/W} \times_{/W} C$.

By definition, a p-adic modular form over C modulo $\mathfrak{m}_C$ is a mod p modular form.

We have an analogous moduli interpretation as in §2.4 and also the q-expansion principle, for p-adic modular forms (*cf.* [19, §4.1]).

3. $\mathfrak{p}$-RIGIDITY

In this section, we prove the rigidity property of mod p modular forms (*cf.* Theorem A). In §3.1-3.2, we firstly recall basic facts about Serre-Tate deformation theory of an ordinary closed point in Sh. In §3.3, we prove the rigidity property.

3.1. **Serre-Tate deformation theory.** In this subsection, we briefly recall Serre-Tate deformation theory of an ordinary closed point in Sh. We follow [15, §8.2], [19, §2] and [21, §1].

Let x be a closed point in Sh^{ord} carrying $(A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/\mathbb{F}}$. Let V be the irreducible component of Sh containing x.

Let CL_W be the category of complete local W-algebras with residue field $\mathbb{F}$. Let $\mathcal{D}_{/W}$ be the fiber category over CL_W of deformations of $x_{/\mathbb{F}}$ defined as follows. Let $R \in CL_W$. The objects of $\mathcal{D}_{/W}$ over R consist of $x'^{*} = (x', \iota_{x'})$, where $x' \in \mathcal{F}^{(p)}(R)$ and $\iota_{x'} : x' \times_R \mathbb{F} \simeq x$. Let x'^{*} and x''^{*} be in $\mathcal{D}_{/W}$ over R. By definition, a morphism ϕ between x'^{*} and x''^{*} is a morphism (still denoted by) ϕ between x' and x'' satisfying [15, (7.3)] and the following condition.

(M) Let ϕ_0 be the special fiber of ϕ. The automorphism $\iota_{x''} \circ \phi_0 \circ \iota_{x'}^{-1}$ of x equals the identity.

Let $\widehat{\mathcal{F}}_x$ be the deformation functor given by

$$\widehat{\mathcal{F}}_x : CL_{/W} \to SETS$$

$$R \mapsto \{x'^{*}_{/R} \in \mathcal{D}\}/\simeq . \tag{3.1}$$

The notation $\simeq$ denotes up to an isomorphism.

Recall, $R \in CL_W$. As R is a projective limit of local W-algebras with nilpotent maximal ideal, we can (and do) suppose that R is a local Artinian W-algebra with nilpotent maximal ideal m_R. Let $x'^{*}_{/R} \in \mathcal{D}$ and A denote $A_{x'}$. By Drinfeld's theorem (*cf.* [15, §8.2.1]), $A[p^\infty]^\circ(R)$ is killed by p^{n_0} for sufficiently large n_0. Let $y \in A(\mathbb{F})$ and $\tilde{y} \in A(R)$ such that $\tilde{y}_0 = y$, where $\tilde{y}_0$ denotes the special fiber of $\tilde{y}$ (as $A_{/R}$ is smooth, such a lift always exists). By definition, $\tilde{y}$ is determined modulo $\ker(A(R) \mapsto A(\mathbb{F})) = A[p^\infty]^\circ(R)$. Thus, for $n \geq n_0$, "p^n"$y_0 := p^n \tilde{y}$ is well defined. From now, we suppose that $n \geq n_0$. If $y \in A[\mathfrak{p}^n](\mathbb{F})$, then "$p^n$"$y \in A[\mathfrak{p}^\infty]^\circ(R)$. Strictly speaking, we apply idempotent $e_{\mathfrak{p}}$ corresponding to $\mathfrak{p}$ so that $e_{\mathfrak{p}}$"p^n"$y \in A[\mathfrak{p}^\infty]^\circ(R)$. We let "$p^n$"$y$ denote $e_{\mathfrak{p}}$"p^n"y for simplicity of notation.

Thus, we have a homomorphism

$$\text{“}p^n\text{”} : A[\mathfrak{p}^n](\mathbb{F}) \to A[\mathfrak{p}^\infty]^\circ(R). \tag{3.2}$$

We also have the commutative diagram

$$\begin{tikzcd}
A[\mathfrak{p}^{n+1}]^{\acute{e}t}(R) \arrow[r, "\simeq"] \arrow[d, "p"] & A[\mathfrak{p}^{n+1}](\mathbb{F}) \arrow[r, "\text{“}p^{n+1}\text{”}"] \arrow[d, "p"] & A[\mathfrak{p}^\infty]^\circ(R) \arrow[d, "="] \\
A[\mathfrak{p}^n]^{\acute{e}t}(R) \arrow[r, "\simeq"] & A[\mathfrak{p}^n](\mathbb{F}) \arrow[r, "\text{“}p^n\text{”}"] & A[\mathfrak{p}^\infty]^\circ(R).
\end{tikzcd}$$

Passing to the projective limit, this gives rise to a homomorphism

$$\text{“}p^{\infty}\text{”} : A[\mathfrak{p}^{\infty}](\mathbb{F}) \to A[\mathfrak{p}^{\infty}]^{\circ}(R). \tag{3.3}$$

(CC) Let $y = \varprojlim y_n \in A[\mathfrak{p}^{\infty}](\mathbb{F}) \simeq A_x[\mathfrak{p}^{\infty}]^{ét}$, where $y_n \in A[\mathfrak{p}^n](\mathbb{F})$ and the later isomorphism is induced by $\iota_{x'}$. Let $q_{n,\mathfrak{p}}(y_n) = \text{“}p^n\text{”} y_n$ and $q_{\mathfrak{p}}(y) = \lim q_{n,\mathfrak{p}}(y_n)$. By definition, $q_{\mathfrak{p}}(y) \in A[\mathfrak{p}^{\infty}]^{\circ}(R) \simeq \mathrm{Hom}(A_x^{\vee}[\mathfrak{p}^{\infty}]^{ét}, \widehat{\mathbb{G}}_m(R))$ for $A_x^{\vee}$ being the dual of the abelian variety A_x. Let $q_{A,\mathfrak{p}}$ be the pairing given by

$$\begin{aligned} q_{A,\mathfrak{p}} : A_x[\mathfrak{p}^{\infty}]^{ét} A_x^{\vee}[\mathfrak{p}^{\infty}]^{ét} &\to \widehat{\mathbb{G}}_m(R) \\ q_{A,\mathfrak{p}}(y,z) &= q_{\mathfrak{p}}(y)(z). \end{aligned} \tag{3.4}$$

We have the following fundamental result.

Theorem 3.1 (Serre-Tate). *(1). There exists a canonical isomorphism*

$$\widehat{\mathcal{F}}_x(R) \simeq \prod_{\mathfrak{p}} \mathrm{Hom}_{\mathbf{Z}_p}(A_x[\mathfrak{p}^{\infty}]^{ét} \times A_x^{\vee}[\mathfrak{p}^{\infty}]^{ét}, \widehat{\mathbb{G}}_m(R)) \tag{3.5}$$

given by $x'^{*} \mapsto q_A := \prod_{\mathfrak{p}} q_{A_{x'},\mathfrak{p}}$.
(2). The deformation functor $\widehat{\mathcal{F}}_x$ *is represented by the formal scheme* $\widehat{S}_{/W} := \mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$. *A level* p^{∞}*-structure as in (PL), gives rise to a canonical isomorphism of the deformation space* $\widehat{S}_{/W}$ *with the formal torus* $\prod_{\mathfrak{p}} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} O_{\mathfrak{p}}$ *(cf.* [21, Prop. 1.2]*).*

Let x_{ST} be the universal deformation.

We now recall a couple of definitions.

Definition 3.2. Recall that a level p^{∞}-structure as in (PL) gives rise to a canonical isomorphism of the deformation space $\widehat{S}_{/W}$ with the formal torus $\prod_{\mathfrak{p}} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} O_{\mathfrak{p}}$ (*cf.* part (2) of Theorem 3.1). Under this identification, let $t = (t_{\mathfrak{p}})_{\mathfrak{p}}$ be the co-ordinates of the deformation space $\widehat{S}_{/W}$, where $t_{\mathfrak{p}}$ is the co-ordinate of $\widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} O_{\mathfrak{p}}$. We call $t = (t_{\mathfrak{p}})_{\mathfrak{p}}$ the Serre-Tate co-ordinates of the deformation space $\widehat{S}_{/W}$.

We have $\widehat{S} = \mathrm{Spf}(\widehat{W[O]})$, where $S = \mathbb{G}_m \otimes O^*$, $W[O] = W[X(S)]$ and $\widehat{W[O]}$ is the completion at the augmentation ideal. Here, $X(S)$ is the character group of S. Note that $W[O]$ is the ring consisting of formal finite sums $\sum_{\xi \in O} a(\xi) t^{\xi}$, where $a(\xi) \in W$ and t is the co-ordinate of $\mathbb{G}_m$. Here, t^{ξ} is the character given by $t^{\xi}(t \otimes u) = t^{\mathrm{Tr}_{F/\mathbf{Q}}(\xi u)}$, for $u \in O^*$.

Definition 3.3. Let f be a mod p modular form over $\mathbb{F}$ (*cf.* §2.4). A level p^{∞}-structure η_p° of x gives rise to a canonical level p^{∞}-structure $\eta_{p,ST}^{\circ}$ of the universal deformation x_{ST}. We call $f((x_{ST}, \eta_{p,ST}^{\circ})) \in \widehat{\mathbb{F}[O]}$ as the t-expansion of f around x.

We have the following t-expansion principle.

(t-expansion principle) The t-expansion of f around x determines f uniquely (*cf.* (Ir)).

We have an analogous t-expansion principle for p-adic modular forms (*cf.* [19, §4.1]).

3.2. **Reciprocity law for the deformation space.** In this subsection, we recall the action of the local algebraic stabiliser of a closed ordinary point x on the Serre-Tate co-ordinates of the deformation space of x. This can be considered as an infinitesimal analogue of Shimura's reciprocity law.

Let $g \in G(\mathbf{Z}_{(p)})$ acts on Sh through the right multiplication on the prime-to-p level structure i.e.,

$$(A, \iota, \bar{\lambda}, \eta^{(p)})_{/S} \mapsto (A, \iota, \bar{\lambda}, \eta^{(p)} \circ g)_{/S}$$

(*cf.* §2.2).

Recall that x is a closed ordinary point in Sh with a p^∞-level structure η_p^{ord}. Let (K_x, Σ_x) be the CM-type of x. We suppose that $\iota : O \hookrightarrow \mathrm{End}(A)$ extends to $\iota_x : \mathcal{O} \hookrightarrow \mathrm{End}(A)$, where $\mathcal{O}$ is the ring of integers of K_x. Let $H_x(\mathbf{Z}_{(p)})$ be the stabiliser of x in $G(\mathbf{Z}_{(p)})$. In fact,

$$H_x(\mathbf{Z}_{(p)}) = (\mathrm{Res}_{\mathcal{O}_{(p)}/\mathbf{Z}_{(p)}} \mathbb{G}_m)(\mathbf{Z}_{(p)}) = \mathcal{O}_{(p)}^\times, \tag{3.6}$$

where $\mathcal{O}_{(p)} = \mathcal{O} \otimes \mathbf{Z}_{(p)}$ (*cf.* [19, §3.2] and [28]). We call $H_x(\mathbf{Z}_{(p)})$ the local algebraic stabiliser of x.

Let c_x be the complex conjugation of K_x.

As x is ordinary, Σ_x is a p-ordinary CM type. When considered as a p-adic CM type, we denote it by $\Sigma_{x,p}$. Let $\mathbf{p} = \prod_{v \in \Sigma_{x,p}} \mathfrak{p}_v$, for the primes $\mathfrak{p}_v$ associated to the valuation $v \in \Sigma_{x,p}$. Note that $\mathcal{O}_{\mathbf{p}} = O_p = \prod_{\mathfrak{p}} O_{\mathfrak{p}}$ and $\mathcal{O}_p = \mathcal{O}_{\mathbf{p}} \times \mathcal{O}_{\mathbf{p}^{c_x}}$. Let $H_x(\mathbf{Z}_p)_{\mathfrak{p}}$ be the $\mathfrak{p}$-component $O_{\mathfrak{p}}^\times$ of $H_x(\mathbf{Z}_p) = \mathcal{O}_p^\times$. We have a natural inclusion $\mathcal{O}_{(p)} \subset \mathcal{O}_p$. Thus, we regard $H_x(\mathbf{Z}_{(p)}) \subset H_x(\mathbf{Z}_p)$. Let $H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$ be the projection of $H_x(\mathbf{Z}_{(p)})$ to $H_x(\mathbf{Z}_p)_{\mathfrak{p}}$. Note that we have an isomorphism $H_x(\mathbf{Z}_{(p)}) \simeq H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$. Let $\alpha \in H_x(\mathbf{Z}_{(p)})$. Let $\alpha_{\mathfrak{p}}$ be the projection of α to the $\mathfrak{p}$-component $O_{\mathfrak{p}}^\times$ of $O_{\mathbf{p}}^\times$. As $H_x(\mathbf{Z}_{(p)})$ stabilises x, it follows that α acts on $\mathrm{Spec}\,(\mathcal{O}_{V,x})$ and thus on $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$. In particular, it acts on the Serre-Tate co-ordinates (*cf.* Definition 3.2). The action is given by the following Lemma.

Lemma 3.4. *The endomorphism α acts on the Serre-Tate co-ordinates $t = (t_{\mathfrak{p}})_{\mathfrak{p}}$ by $t \mapsto t^{\alpha^{1-c_x}}$, where $t^{\alpha^{1-c_x}} = (t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}^{1-c_x}})_{\mathfrak{p}}$ (cf.* [19, Lem. 3.3]*).*

We have the following immediate corollary.

Corollary 3.5. *The partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$ is stable under the action of the local algebraic stabiliser $H_x(\mathbf{Z}_{(p)})$.*

This simple corollary is crucial for $\mathfrak{p}$-rigidity. It seems to be overlooked in the literature and may lead to rigidity-style phenomena with different flavour.

3.3. **$\mathfrak{p}$-rigidity.** In this subsection, we prove the rigidity of mod p modular forms (*cf.* Theorem A).

Recall that x is a closed ordinary point in V with p^∞-level structure η_p° and $V = \varprojlim V_K$, where V_K is the projection of V to $Sh_K = Sh/K$ for K small and maximal at p. This gives rise to a closed point $\tilde{x} = (x, \eta_p^\circ)$ in the Igusa tower I over x. In view of ($\acute{E}t$), there exists a canonical isomorphism $\widehat{\mathcal{O}}_{V,x} \simeq \widehat{\mathcal{O}}_{I,\tilde{x}}$. Thus, a mod p modular form can be thought of as a function on $\mathrm{Spec}\,(\mathcal{O}_{V,x})$.

Let f be a non-zero mod p modular form. We suppose that f is a non-unit in $\mathcal{O}_{V,x}$. Let $\mathfrak{b} \subset \mathcal{O}_{V,x}$ be the zero ideal of f i.e. $\mathfrak{b} = (f) \cap \mathcal{O}_{V,x}$. As f is an algebraic function on I, it follows that $V(\mathfrak{b})$ is non-empty not only in $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$ but also in $\mathrm{Spec}\,(\mathcal{O}_{V,x})$. In particular, we have $\mathfrak{b} \neq 0$. Let X be the Zariski closure of $\mathrm{Spec}\,(\mathcal{O}_{V,x}/\mathfrak{b})$ in V. Note that $X \subset V$ is a closed irreducible pro-subscheme containing x and $X = \varprojlim X_K$, where X_K is the projection of X to V_K.

We start with a preparatory lemma.

Lemma 3.6. *Let Y_i be a family of closed subschemes of an irreducible noetherian scheme Y such that there exists a closed point $y \in Y_i$, for all i i.e., $y \in \bigcap_i Y_i$. Suppose that the intersection of $\mathrm{Spf}(\widehat{\mathcal{O}}_{Y_i,y})$'s (viewed inside $\mathrm{Spf}(\widehat{\mathcal{O}}_{Y,y})$) is positive dimensional. Then, the intersection of Y_i's is positive dimensional.*

Proof. It suffices to consider the affine case.

Let A be a noetherian integral domain and $R_i = A/a_i$, for ideals a_i. Take non-units $t_1,\dots, t_r$ in A and let $(t) = (t_1, \dots, t_r)$. Suppose that $V(t) = \mathrm{Spec}\,(R/(t)) \subset V_i = \mathrm{Spec}\,(R/a_i)$, for all i. This implies $a_i \subset (t)$. In particular, $\sum_i a_i \subset (t)$.

If A is local noetherian with maximal ideal m, by faithful flatness of $\widehat{A} = \varprojlim A/m^n$ over A, $\widehat{\sum_i a_i} = \sum_i \widehat{a}_i$. So, the above argument can be applied to $(t) \in \widehat{m}$, $\widehat{A}$ and $\widehat{(A/a_i)}$.

This finishes the proof as

$$\dim(\bigcap_i \mathrm{Spec}\,(A/a_i)) = \dim(\mathrm{Spec}\,(A/\sum_i a_i)) = \dim(\mathrm{Spec}\,(\widehat{A}/\sum_i \widehat{a}_i)) = \dim(\bigcap_i \mathrm{Spec}\,(\widehat{A}/\widehat{a}_i)) \geq \dim(\widehat{A}/(t)).$$

□

We are now ready to prove Theorem A.

Theorem 3.7 ($\mathfrak{p}$-rigidity)**.** *The partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$ is not a formal subscheme of $\mathrm{Spf}(\widehat{\mathcal{O}}_{X,x})$ (viewed inside $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$). Thus, Theorem A holds.*

Proof. Suppose that it is a formal subscheme.

We now consider $Z = \bigcap_{\alpha \in H_x(\mathbb{Z}_{(p)})} \alpha(X)$ (viewed inside V). As above, recall that $Z = \varprojlim Z_K$.

Note that $\mathrm{Spf}(\widehat{\mathcal{O}}_{\alpha(X),x}) = \alpha(\mathrm{Spf}(\widehat{\mathcal{O}}_{X,x}))$ (viewed inside $\mathrm{Spf}(\widehat{\mathcal{O}}_{X,x})$). It follows that if $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{X,x})$, then $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{\alpha(X),x})$ (*cf.* Corollary 3.5). In particular, we have $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{\alpha(X)_K,x})$. In the last equality, $\alpha(X)_K$ is the projection of $\alpha(X)$ to V_K. By Lemma 3.6, Z_K is positive dimensional. As the projection $\pi_K : V \to V_K$ is *étale*, we conclude that Z itself is positive dimensional.

Thus, Z is a closed irreducible pro-subscheme of V containing x and stable under $H_x(\mathbb{Z}_{(p)})$. We conclude that $Z = V$ (*cf.* [19, Prop. 3.8]). It follows that $\mathfrak{b} = 0$. This is a contradiction for f being non-zero as noted in the paragraph before Lemma 3.6.

□

We have the following immediate consequence.

Corollary 3.8. *Let g be a p-adic modular form. Then,*

$$\mu(g) = \mu(g|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}).$$

PROOF. Let g be non-zero and defined over a p-adically complete local W-algebra C.

If g is a unit in $\mathcal{O}_{V,x}$, the corollary follows instantly. We thus suppose that g is a non-unit in $\mathcal{O}_{V,x}$.

In view of the t-expansion principle, we have

$$\mu(g) = \mu(g|_{\prod_{\mathfrak{p}} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} O_{\mathfrak{p}}}).$$

Let $c \in C$ such that $\mu(g/c) = 0$. By definition, g/c modulo the maximal ideal $\mathfrak{m}_C$ is a non-zero mod p modular form. In view of Theorem 3.7, it thus follows that

$$\mu(g/c|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}) = 0.$$

This finishes the proof. □

4. $\mathfrak{p}$-INDEPENDENCE

In this section, we consider $\mathfrak{p}$-independence i.e. a linear independence of mod p modular forms restricted to the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$. In §4.1, we firstly state the formulation and in §4.2, we prove the independence.

4.1. **Formulation.** In this subsection, we give a formulation of the linear independence of mod p modular forms restricted to the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.

Recall that x is a closed ordinary point in V with p^∞-level structure η_p°. This gives rise to a closed point $\tilde{x} = (x, \eta_p^\circ)$ in the Igusa tower I over x and a canonical isomorphism

$$\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x}) \simeq \prod_{\mathfrak{p}} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} O_{\mathfrak{p}} \tag{4.1}$$

(*cf.* Theorem 3.1).

Recall that we have a canonical isomorphism $\widehat{\mathcal{O}}_{V,x} \simeq \widehat{\mathcal{O}}_{I,\tilde{x}}$ (*cf.* (*Ét*)). Thus, the p-completed stabiliser $H_x(\mathbf{Z}_p)$ acts on $\widehat{\mathcal{O}}_{I,\tilde{x}}$.

Let f be a mod p modular form and $a \in H_x(\mathbf{Z}_p)$. Note that $(a(f))|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}} = a_{\mathfrak{p}}(f|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}})$ (*cf.* Corollary 3.5). Here $a_{\mathfrak{p}}$ is as in §3.2.

To provide context for the following, note that there exists $a \in H_x(\mathbf{Z}_p)$ such that $a_{\mathfrak{p}_i} \in H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}_i}$ and $a_{\mathfrak{p}_j} \notin H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}_j}$, for $j \neq i$. Indeed, $H_x(\mathbf{Z}_p) = O_p^\times = \prod_{\mathfrak{p}} O_{\mathfrak{p}}^\times$ and we may choose a to be non-identity precisely at the $\mathfrak{p}_i$'th component.

For $1 \leq i \leq n$, let $a_i \in H_x(\mathbf{Z}_p)$ such that $(a_i a_j^{-1})_{\mathfrak{p}} \notin H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$ for all $i \neq j$ (*cf.* §3.2). Let $f_1, ..., f_n$ be n non-constant mod p modular forms on V (*cf.* §3.4).

Our formulation of the linear independence is the following.

Theorem 4.1 ($\mathfrak{p}$-independence)**.** *Suppose that $(a_i a_j^{-1})_{\mathfrak{p}} \notin H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$ for all $i \neq j$. Then, $(a_{i,\mathfrak{p}} \circ (f_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}))_i$ are linearly independent in the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.*

Note that $a_{i,\mathfrak{p}} \circ (f_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}})$ is not necessarily the restriction of a mod p modular form to $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.

4.2. **$\mathfrak{p}$-independence.** In this subsection, we prove the linear independence of mod p modular forms restricted to the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.

The approach is based on $\mathfrak{p}$-rigidity and Chai's theory of Hecke-stable subvarieties of a mod p Shimura variety adopted for local algebraic stabilisers in [19, §3]. For a detailed treatment of the latter, we refer to [8], [9], [10] and [19].

Let n be a positive integer. In this subsection, any tensor product is taken n-times.

We consider an $\mathbb{F}$-algebra homomorphism

$$\phi_I : \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}} \to \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \tag{4.2}$$

given by

$$f_1 \otimes ... \otimes f_n \mapsto \prod_{i=1}^{i=n} a_{i,\mathfrak{p}} \circ (f_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}). \tag{4.3}$$

As we are interested in the linear independence of $(a_{i,\mathfrak{p}}(f_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}))_i$, we consider $\mathfrak{b}_I := ker(\phi_I)$.

Similarly, we consider an $\mathbb{F}$-algebra homomorphism

$$\phi = \phi_V : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \tag{4.4}$$

given by

$$h_1 \otimes ... \otimes h_n \mapsto \prod_{i=1}^{i=n} a_{i,\mathfrak{p}} \circ (h_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}). \tag{4.5}$$

In view of Theorem 3.7, it follows that ϕ_I and ϕ_V are both non-trivial.

(EQ) We note that ϕ is equivariant with the $H_x(\mathbf{Z}_{(p)})$-action.

Let $\mathfrak{b}_I = ker(\phi_V)$ and $\mathfrak{b} = ker(\phi_V)$.

Lemma 4.2. *We have $\mathfrak{b}_I = 0$ if and only if $\mathfrak{b} = 0$.*

Proof. In view of ($\acute{E}t$), we have an *étale* morphism $\pi^m : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}$. Note that $\mathfrak{b}_I$ is the unique prime ideal of $\mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}$ over $\mathfrak{b}$. This finishes the proof. $\square$

As ϕ is equivariant with the $H_x(\mathbf{Z}_{(p)})$-action (*cf.* (EQ)), it follows that $\mathfrak{b}$ is a prime ideal of $\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}$ stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$. Let Y be the Zariski closure of Spec$(\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}/\mathfrak{b})$ in V^n . Thus, Y is a closed irreducible subscheme of V^n containing x^n stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$. We also have an analogue of the commutative diagram [19, (3.22)] with $\widehat{O}_S$ replaced by $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$. For $n \geq 2$, the subscheme Y thus satisfies the hypothesis in [19, Cor. 3.19].

Theorem 4.3. *The subscheme Y equals V^n.*

Proof. When $n = 1$, this is nothing but [19, Prop. 3.8].

Now, suppose that $n \geq 2$. From [19, Cor. 3.19], we have two possibilities namely $Y = V^{n-2} \times \Delta_{\alpha,\beta}$ (up to a permutation of the factors), for some $\alpha, \beta \in H_x(\mathbf{Z}_{(p)})$ or $Y = V^n$. The skewed diagonal $\Delta_{\alpha,\beta}$ is given by

$$\Delta_{\alpha,\beta} = \big\{(\alpha(v), \beta(v))\big| v \in V\big\}.$$

Suppose that $Y = V^{n-2} \times \Delta_{\alpha,\beta}$ (up to a permutation of the factors). Let $(t, t^{'})$ be the Serre-Tate co-ordinates of the last two factors $\Delta_{\alpha,\beta}$ at (x, x), respectively. It follows that $t^{\beta^{1-c_x}} = t^{'\alpha^{1-c_x}}$. On the other hand, from the definition of Y it follows that $t_{\mathfrak{p}}^{a_{n,\mathfrak{p}}} = t_{\mathfrak{p}}^{'a_{n-1,\mathfrak{p}}}$. Thus, $(a_n a_{n-1}^{-1})_{\mathfrak{p}} = (\beta\alpha^{-1})_{\mathfrak{p}}^{1-c_x} \in H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$. This contradicts the hypothesis on a_i's. Thus, we conclude $Y = V^n$.

□

We have the following immediate consequence.

Corollary 4.4. *Theorem 4.1 holds.*

PROOF. In view of Theorem 4.3, it follows that $\mathfrak{b} = 0$. Thus, $\mathfrak{b}_I = 0$ (*cf.* Lemma 4.2).

□

5. p-ADIC DIFFERENTIAL OPERATORS

In this section, we consider p-adic operators on the space of p-adic Hilbert modular forms. This is a p-adic analogue of the Maass-Shimura differential operators on complex Hilbert modular forms. In §5.1, we give an elementary construction of these operators. In §5.2, we compute their action on the t-expansion of a p-adic Hilbert modular form around an ordinary point in terms of the partial Serre-Tate co-ordinates. For a more detailed account of the elliptic modular case, we refer to [22, 1.3.6]. In §6, we will use these results to compute the power series expansion of anticyclotomic Katz $\mathfrak{p}$-adic L-function.

5.1. **Elementary construction.** In this subsection, we give an elementary construction of p-adic differential operators on the space of p-adic Hilbert modular forms.

For the geometric definition of classical and p-adic modular forms on Sh, we refer to [15, §4.2] and [19, §4.1].

Recall that $\mathbb{F}$ denotes an algebraic closure of $\mathbb{F}_p$, W the Witt ring $W(\mathbb{F})$, $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ a p-adic embedding and $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ a complex embedding. Let $\mathcal{W}$ denote $\iota_p^{-1}(W)$. Possibly enlarging $\mathcal{W}$, we suppose that $\tau(O) \subset \mathcal{W}$, for all $\tau \in \Sigma$.

In this subsection, for simplicity we suppose that the prime to p level of classical or p-adic Hilbert modular forms is one.

Let $G_\kappa(\Gamma_1(p^m), \mathcal{W})$ be the space of classical Hilbert modular forms of weight κ and level $\Gamma_1(\mathfrak{n}p^m)$ over $\mathcal{W}$, where $\kappa \in \mathbf{Z}_{\geq 0}[\Sigma]$ and m is a non-negative integer. Let $f \in G_\kappa(\Gamma_1(p^m), \mathcal{W})$. Via ι_∞, we regard f as a Hilbert modular form over $\mathbf{C}$. Let $z = (z_\tau)_{\tau\in\Sigma}$ be the complex variables of the Hilbert modular Shimura variety or those of $\mathfrak{H}^\Sigma$, where $\mathfrak{H}$ is the upper half plane (*cf.* [19, §4.1]). Let $(\mathfrak{a}, \mathfrak{b})$ be a pair giving rise to a cusp of the Hilbert modular Shimura variety (*cf.* [loc. cit.,§4.1]). The Fourier expansion of f at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$ is given by $f(z) = \sum_{\xi\in O} a(\xi) e_F(\xi z)$, where $e_F(\xi z) = \exp\left(2\pi i \sum_{\tau\in\Sigma} \tau(\xi) z_\tau\right)$.

Let $\phi : O/p^r O \to \mathcal{W}$ be an arbitrary function with the normalised Fourier transform ϕ^* given by

$$\phi^*(y) = \frac{1}{p^{rd/2}} \sum_{u\in O/p^r O} \phi(u) e_F(yu/p^r), \tag{5.1}$$

where $y \in O/p^r O$ and $e_F(w) = \exp(2\pi i \operatorname{Tr}_{F/\mathbf{Q}}(w))$, for $w \in F$.

Let $f|\phi$ be the classical Hilbert modular form given by

$$f|\phi(z) = \sum_{u'=(\sigma_1(u),\ldots,\sigma_d(u)),u\in O/p^r O} \phi^*(-u)f(z+u'/p^r), \tag{5.2}$$

where $z + u'/p^r = (z_\tau + \tau(u)/p^r)_\tau$.

Note that $f|\phi \in G_\kappa(\Gamma_1(p^s), \mathcal{W})$, where $s = \max(m, 2r)$. In view of the Fourier inversion formula, it follow that the Fourier expansion of $f|\phi$ at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$ is given by

$$f|\phi(z) = \sum_{\xi\in O} \phi(\xi)a(\xi)e_F(\xi z). \tag{5.3}$$

Let $(\phi_n^\sigma : O/p^n O \to \mathcal{W})_n$ be a sequence of functions such that $\phi_n^\sigma(\xi) \equiv \sigma(\xi) \pmod{p^n\mathcal{W}}$. In view of the q-expansion principle, it follow that the sequence $(f|\phi_n^\sigma)_n$ of classical Hilbert modular forms converges p-adically to a p-adic Hilbert modular form $d^\sigma f$ whose formal Fourier expansion at the the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$ is given by

$$d^\sigma f(z) = \sum_{\xi\in O} \sigma(\xi)a(\xi)e_F(\xi z). \tag{5.4}$$

In other words, the operator d^σ equals the Maass-Shimura differential operator $\delta_0^\sigma = \frac{1}{2\pi i}\frac{\partial}{\partial z_\sigma}$ on $G_\kappa(\Gamma_1(p^m), \mathcal{W})$.

This construction extends to the space of p-adic Hilbert modular forms over W as follows. Let $V(W)$ be the space of p-adic Hilbert modular forms of prime-to-p level one over W. Via ι_p, we can regard $G_\kappa(\Gamma_1(p^m), \mathcal{W})$ as a subspace of $V(W)$ (*cf.* [15, §8.1]). The space $G_\kappa(\Gamma_1(p^\infty), \mathcal{W}) = \bigcup_m G_\kappa(\Gamma_1(p^m), \mathcal{W})$ is p-adically dense in $V(W)$ (*cf.* [15, §8.1])). Thus, the differential operator d^σ extends to $V(W)$.

Remark. In [25, Ch. II], the operator d^σ is constructed based on the Gauss-Manin connection of the universal abelian scheme over the Shimura variety. The above approach can be generalised for a class of PEL Shimura varieties.

5.2. **Action on the t-expansion.** In this subsection, we compute the action of the differential operator d^σ on the t-expansion of a p-adic Hilbert modular form around a p-ordinary point in terms of the partial Serre-Tate co-ordinates.

Let $(\zeta_{p^n} = \exp(2\pi i/p^n))_n \in \overline{\mathbf{Q}}$ be a compatible system of p-power roots of unity. Via ι_p, we regard it as a compatible system in $\mathbf{C}_p$.

Let $\mathfrak{p}$ be the prime corresponding to σ via ι_p and Σ_p be the set of places above p in F. For $\mathfrak{q} \in \Sigma_p$, let $\Sigma_\mathfrak{q} \subset \Sigma$ be the subset giving rise to $\mathfrak{q}$ under ι_p.

Let $W_n = W[\mu_{p^n}]$ and m_n be the maximal ideal. We have

$$(\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p}^*)(W_n) = (1 + m_n) \otimes O_\mathfrak{p}^* \tag{5.5}$$

and

$$\widehat{\mathbb{G}}_m \otimes O_\mathfrak{q}^* = \prod_{\tau\in\Sigma_\mathfrak{q}} \widehat{\mathbb{G}}_m. \tag{5.6}$$

Let $u \in O$ and $\alpha(u/p^m) \in G(\mathbf{A}^f)$ such that

$$\alpha(u/p^m)_\mathfrak{p} = \begin{bmatrix} 1 & u/p^m \\ 0 & 1 \end{bmatrix}$$

and $\alpha(u/p^m)_{\mathfrak{l}} = 1$, for $\mathfrak{l} \neq \mathfrak{p}$.

Let us recall some notation. Let $\pi : Ig \to Sh$ be the Igusa tower over W. Let $x \in Sh^{ord}_{/\mathbb{F}}$ be a closed point and $\tilde{x}$ be a point above it in Sh. Let $\widehat{S}_{/W}$ be the deformation space of $\tilde{x}$. For $q \in \Sigma_p$, recall $t_{\mathfrak{q}} = t \otimes 1_{\mathfrak{q}} \in \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{q}}$ is the Serre-Tate co-ordinate of the partial deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{q}}$ (*cf.* §3.1). We regard $1 \otimes u \in \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{q}}$ via the image $u \in O_{\mathfrak{q}}$.

We start with a preparatory lemma.

Lemma 5.1. *The isogeny action of $\alpha(u/p^m)$ on the Igusa tower $\pi : Ig \to Sh_{/W_m}$ preserves the deformation space $\widehat{S}$ and induces $t_{\mathfrak{p}} \mapsto \zeta_{p^m} t \otimes u$ (cf. (5.5)) and $t_{\mathfrak{p}'} \mapsto t_{\mathfrak{p}'}$, for $\mathfrak{p}' \neq \mathfrak{p}$.*

Proof. Let $x_{ST} = (\mathcal{A}_x, -)$ be the universal deformation and $x_0 = (A_0, -)$ be the origin of the deformation space $\widehat{S}$. In particular, we have

$$A_0[\mathfrak{p}^m] = \mu_{p^m} \otimes O_{\mathfrak{p}}^* \oplus O_{\mathfrak{p}}/\mathfrak{p}^m O_{\mathfrak{p}}. \tag{5.7}$$

By the universal level structure, we have an exact sequence

$$0 \longrightarrow \mu_{p^m} \otimes O_{\mathfrak{p}}^* \to \mathcal{A}_x[\mathfrak{p}^m] \xrightarrow{h} O_{\mathfrak{p}}/\mathfrak{p}^m O_{\mathfrak{p}} \longrightarrow 0.$$

A section of the morphism h determines an $O_{\mathfrak{p}}$-cyclic subgroup C_u isomorphic to $O_{\mathfrak{p}}/\mathfrak{p}^m O_{\mathfrak{p}}$ defined over W_m, which specialises at the origin A_0 to a cyclic subgroup generated by $(\zeta_{p^m} \otimes u, \gamma_0)$ in $A_0[\mathfrak{p}^m]$ (*cf.* (5.7)). Here γ_0 denotes the image of 1 under the identification $A_0[\mathfrak{p}^m]^{\acute{e}t} = O_{\mathfrak{p}}/\mathfrak{p}^m O_{\mathfrak{p}}$.

The isogeny action $\alpha(u/p^m)$ corresponds to the isogeny $\mathcal{A}_x \to \mathcal{A}_{x,u} := \mathcal{A}_x/C_u$. By an argument similar to the proof of [1, Lem. 7.1 and Lem. 7.2], it follows that the $\mathfrak{p}$-Serre-Tate co-ordinates of $\mathcal{A}_{x,u}$ is given by $\zeta_{p^m} t \otimes u$. For $\mathfrak{p}' \neq \mathfrak{p}$, in view of the construction of the Serre-Tate co-ordinates (*cf.* §3.1), it follows that the $\mathfrak{p}'$-Serre-Tate co-ordinate is given by $t_{\mathfrak{p}'}$.

This finishes the proof.

□

For $\tau \in \Sigma_{\mathfrak{q}}$, let $t_{\mathfrak{q}}^{\tau}$ be the τ-component of $t_{\mathfrak{q}}$ (*cf.* (5.6)).

Proposition 5.2. *The p-adic differential operator d^{σ} acts as $t_{\mathfrak{p}}^{\sigma} \frac{\partial}{\partial t_{\mathfrak{p}}^{\sigma}}$ on the deformation space $\widehat{S}$.*

Proof. As a p-adic Hilbert modular form is a p-adic limit of classical Hilbert modular forms, it suffices to verify the proposition for classical Hilbert modular forms.

Let $f \in G_{\kappa}(\Gamma_1(p^m), \mathcal{W})$ and the t-expansion of f around x be given by

$$f(t) = \sum_{\omega} a(\omega) \prod_{\mathfrak{q} \in \Sigma_p} t_{\mathfrak{q}}^{\omega_{\mathfrak{q}}}. \tag{5.8}$$

Here by $t_{\mathfrak{q}}^{\omega_{\mathfrak{q}}}$, we mean the character $t^{\omega_{\mathfrak{q}}}$ (*cf.* §3.1) and the summation is over $\omega \in O$ (*cf.* [19, pp. 106-107]). To emphasise the notion of t-expansion around a point, we use the indexing notation ω instead of the traditional notation ξ for q-expansion around a cusp.

We have

$$f|\phi = \sum_{u \in O/p^r O} \phi^*(-u) f|\alpha(u/p^r),$$

as $\alpha(u/p^m)$ acts on $\mathfrak{H}^\Sigma$ by $z \mapsto z + u'/p^r$ (*cf.* (5.2)).

In view of Lemma 5.1, it follows that

$$f|\phi_n^\sigma(t) = \sum_\omega a(\omega)\Big(\sum_{u \in O/p^nO} (\phi_n^\sigma)^*(-u)\zeta_{p^n}^{\mathrm{Tr}_{F/\mathbf{Q}}(u\omega_\mathfrak{p})}\Big)t_\mathfrak{p}^{\omega_\mathfrak{p}} \prod_{\mathfrak{p}' \neq \mathfrak{p}} t_{\mathfrak{p}'}^{\omega_{\mathfrak{p}'}} \tag{5.9}$$
$$= \sum_\omega \phi_n^\sigma(\omega_\mathfrak{p})a(\omega) \prod_{\mathfrak{q} \in \Sigma_p} t_\mathfrak{q}^{\omega_\mathfrak{q}}.$$

The last equality follows from the Fourier inversion formula.

Thus, we have

$$d^\sigma f(t) = \lim_n f|\phi_n^\sigma(t) = \sum_\omega \sigma(\omega_\mathfrak{p})a(\omega) \prod_{\mathfrak{q} \in \Sigma_p} t_\mathfrak{q}^{\omega_\mathfrak{q}} = t_{\mathfrak{p},\sigma}\frac{\partial f}{\partial t_{\mathfrak{p},\sigma}}. \tag{5.10}$$

□

We have the following immediate consequence.

Corollary 5.3. *The p-adic differential operator d^σ is $\widehat{S}$-invariant.*

Remark. The above corollary is proven in [26, §4.3] based on the computation of the Gauss-Manin connection in terms of the Serre-Tate co-ordinates. The above approach can be generalised for a class of PEL Shimura varieties.

6. Iwasawa μ-invariants

In this section, we consider Iwasawa μ-invariants as in Theorems B-D. In §6.1, we determine the μ-invariant of certain anticyclotomic $\mathfrak{p}$-adic L-functions (*cf.* Theorem B). In §6.2, we determine the μ-invariant of the cyclotomic derivative $L'_{\Sigma,\lambda,\mathfrak{p}}$ of Katz $\mathfrak{p}$-adic L-function, when the branch character λ is self-dual with the root number -1 (*cf.* Theorem C). In §6.3, we prove a $\mathfrak{p}$-version of a conjecture Gillard regarding the vanishing of the μ-invariant of Katz p-adic L-function (*cf.* Theorem D).

6.1. **μ-invariant of anticyclotomic $\mathfrak{p}$-adic L-functions.** In this subsection, we first obtain a formula for the μ-invariant anticyclotomic Katz $\mathfrak{p}$-adic L-function (*cf.* Theorem B). Towards the end, we remark about a similar formula for a class of Rankin-Selberg anticyclotomic $\mathfrak{p}$-adic L-function.

Let $\mathfrak{C}$ be a prime-to-p ideal of K. Let $Z(\mathfrak{C})$ be the Ray class group of K modulo $\mathfrak{C}p^\infty$. Let $Z(\mathfrak{C})^-$ be the anticyclotomic quotient. The reciprocity law $\mathrm{rec}_K : (\mathbf{A}_K^f)^\times \to Z(\mathfrak{C})^-$ induces the isomorphism

$$\mathrm{rec}_K : \varprojlim_n K^\times(\mathbf{A}_F^f)^\times \backslash (\mathbf{A}_K^f)^\times / U_K(\mathfrak{C}p^n) \xrightarrow{\sim} Z(\mathfrak{C})^-,$$

where $U_K = (\mathcal{O}_K \otimes \widehat{\mathbf{Z}})^\times$ and $U_K(\mathfrak{C}p^n) = \{u \in U_K \mid u \equiv 1 \,(\mathrm{mod}\ \mathfrak{C}p^n)\}$. Let Γ^- be the maximal $\mathbf{Z}_p$-free quotient of $Z(\mathfrak{C})^-$ and $\Gamma^-_\mathfrak{p}$ be the $\mathfrak{p}$-part of Γ^- (*cf.* §1).

Let Γ' (resp. $\Gamma'_\mathfrak{p}$) be the open subgroup of Γ^- generated by the image of $O_p^\times \times \prod_{v|D_{K/F}} K_v^\times$ (resp. $O_\mathfrak{p}^\times \times \prod_{v|D_{K/F}} K_v^\times$) via rec_K. Let Σ_p be the places above p in K induced by the p-ordinary CM type Σ. The reciprocity law rec_K at Σ_p induces an injective map

$$\mathrm{rec}_{\Sigma_p} : 1 + pO_p \hookrightarrow O_p^\times = \oplus_{w \in \Sigma_p} \mathcal{O}_{K_w}^\times \xrightarrow{\mathrm{rec}_K} Z(\mathfrak{C})^-$$

with finite cokernel as $p \nmid D_F$. It induces isomorphisms $\mathrm{rec}_{\Sigma_p} : 1 + pO_p \xrightarrow{\sim} \Gamma'$ and $\mathrm{rec}_{\Sigma_p} : 1 + \mathfrak{p}O_{\mathfrak{p}} \xrightarrow{\sim} \Gamma'_{\mathfrak{p}}$. Via these isomorphisms, we identify Γ' (resp. $\Gamma'_{\mathfrak{p}}$) with the subgroup $\mathrm{rec}_{\Sigma_p}(1+pO_p)$ (resp. $\mathrm{rec}_{\Sigma_p}(1+\mathfrak{p}O_{\mathfrak{p}})$) of the anticyclotomic quotient $Z(\mathfrak{C})^-$.

In [25] and [14], a $\overline{\mathbf{Z}}_p$-valued p-adic measure $\mathcal{L}_{\mathfrak{C},\Sigma}$ on $Z(\mathfrak{C})$ is constructed, interpolating critical Hecke L-values (*cf.* [4]). Let λ be Hecke character over K with prime to p conductor $\mathfrak{C}$. Let $\mathcal{L}^-_{\Sigma,\lambda}$ (resp. $\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}$) be the p-adic measure on Γ^- (resp. $\Gamma^-_{\mathfrak{p}}$) obtained by the push-forward of $\mathcal{L}_{\mathfrak{C},\Sigma}$ along λ.

Recall that the μ-invariant $\mu(\varphi)$ of a $\overline{\mathbf{Z}}_p$-valued p-adic measure φ on a p-adic group H is defined by

$$\mu(\varphi) = \inf_{U \subset H \text{ open}} v_p(\varphi(U)).$$

The μ-invariants of the above measures are related by the following theorem.

Theorem 6.1.

$$\mu(\mathcal{L}^-_{\Sigma,\lambda}) = \mu(\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}).$$

Proof. Let $L^-_{\Sigma,\lambda}$ (resp. $L^-_{\Sigma,\lambda,\mathfrak{p}}$) be the power series of $\mathcal{L}^-_{\Sigma,\lambda}$ (resp. $\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}$) (*cf.* (1.3)).

We first suppose that $p \nmid h^-_K$, where h^-_K is the relative class number given by $h^-_K = h_K/h_F$ and $h_?$ is the class number of ?, for ? $= F, K$.

Under the hypothesis, there are a finite number of classical Hilbert modular Eisenstein series $(f_{\lambda,i})_i$ such that

$$L^-_{\Sigma,\lambda} = \sum_i a_i \circ (f_{\lambda,i}(t)), \tag{6.1}$$

up to an automorphism of $\overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$, where $f_{\lambda,i}(t)$ is the t-expansion of $f_{\lambda,i}$ around a well chosen CM point x with the CM type (K, Σ) on Sh and a_i is an automorphism of the deformation space of x in Sh (*cf.* [19, Thm. 5.1] and [23, §5.2]). Moreover, a_i's satisfy the hypothesis in Theorem 4.1.

For $\kappa \in \mathbf{Z}_{\geq 0}[\Sigma]$, let ν_κ be the p-adic character of Γ' such that $\nu_\kappa(\mathrm{rec}_{\Sigma_p}(y)) = y^\kappa$, for $y \in 1 + pO_p$. Let d^κ be the p-adic differential operator corresponding to κ (*cf.* §5.1). Equation (6.1) follows from the fact that

$$(d^\kappa \sum_i a_i \circ (f_{\lambda,i}(t)))|_{t=1} = \int_{\Gamma^-} \nu_\kappa \, d\mathcal{L}^-_{\Sigma,\lambda}. \tag{6.2}$$

In view of the linear independence of $(a_i \circ (f_{\lambda,i}))_i$ (*cf.* [19, Thm. 3.20]), it follows that

$$\mu(\mathcal{L}^-_{\Sigma,\lambda}) = \min_i \mu(f_{\lambda,i}(t)) = \min_i \mu(f_{\lambda,i}). \tag{6.3}$$

For a p-adic Hilbert modular form f, let $f(t_{\mathfrak{p}})$ be obtained from $f(t)$ by substituting $t_{\mathfrak{p}'} = 1$, for all $\mathfrak{p}' \neq \mathfrak{p}$.

Let

$$f^-_{\Sigma,\lambda,\mathfrak{p}} = \sum_i a_{i,\mathfrak{p}} \circ (f_{\lambda,i}(t_{\mathfrak{p}})). \tag{6.4}$$

Recall that $\Sigma_{\mathfrak{p}} \subset \Sigma$ denotes the subset of infinite places of F corresponding to $\mathfrak{p}$, via ι_p. For $\sigma \in \Sigma_{\mathfrak{p}}$, let $d^{\sigma'}$ be the formal differential operator given by $f(t_{\mathfrak{p}}) \mapsto t^\sigma_{\mathfrak{p}} \frac{\partial f}{\partial t^\sigma_{\mathfrak{p}}}$. In view of Proposition 5.2, it follows that

$$d^{\sigma'}(f(t_{\mathfrak{p}})) = (d_\sigma f)(t_{\mathfrak{p}}). \tag{6.5}$$

From now, let $\kappa \in \mathbf{Z}_{\geq 0}[\Sigma_{\mathfrak{p}}]$. Let $d^{\kappa'}$ be the corresponding formal differential operator.

We now have

$$d^{\kappa'}(f^-_{\Sigma,\lambda,\mathfrak{p}})|_{t_{\mathfrak{p}}=1} = (d^{\kappa}L^-_{\Sigma,\lambda})|_{t=1} = \int_{\Gamma^-} \nu_\kappa \, d\mathcal{L}^-_{\Sigma,\lambda} = \int_{\Gamma^-_{\mathfrak{p}}} \nu_\kappa \, d\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}. \tag{6.6}$$

The first two equalities follow from (6.5) and (6.2), respectively. The last equality follows from the fact that for $\kappa \in \mathbf{Z}_{\geq 0}[\Sigma_{\mathfrak{p}}]$, the character ν_κ factors through $\Gamma^-_{\mathfrak{p}}$.

In other words, $d^{\kappa'}(f^-_{\Sigma,\lambda,\mathfrak{p}})|_{t_{\mathfrak{p}}=1}$ interpolates the κ-th moment of the measure $\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}$. Thus, $f^-_{\Sigma,\lambda,\mathfrak{p}} = L^-_{\Sigma,\lambda,\mathfrak{p}}$, up to an automorphism of $\overline{\mathbf{Z}}_p[\![\Gamma^-_{\mathfrak{p}}]\!]$.

In view of the linear independence of $(a_{i,\mathfrak{p}} \circ (f_{\lambda,i}|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}))_i$ (*cf.* Theorem 4.1), it follows that

$$\mu(L^-_{\Sigma,\lambda,\mathfrak{p}}) = \min_i \mu(f_{\lambda,i}(t_{\mathfrak{p}})) = \min_i \mu(f_{\lambda,i}|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}})). \tag{6.7}$$

In view of Corollary 3.8, this finishes the proof of the Theorem.

When $p|h^-_K$, the power series $L^-_{\Sigma,\lambda}$ restricted to an explicit finite open cover of Γ^- is still of the form (5.1) (*cf.* [23, §5.2]). Thus, a similar argument proves the Theorem. □

In most of the cases, $\mu(L^-_{\Sigma,\lambda})$ has been explicitly determined (*cf.* [19] and [23]). Thus, we obtain a formula for $\mu(L^-_{\Sigma,\lambda,\mathfrak{p}})$.

Remark. A class of Rankin-Selberg anticyclotomic p-adic L-functions for Hilbert modular forms is constructed in [24]. It also satisfies a property analogoues to (6.1) ([24, §6.2]). Thus, by an argument similar to the proof of Theorem 6.1, we get an analogue of Theorem 6.1. In the near future, the first named author hopes to consider Rankin-Selberg anticyclotomic p-adic L-functions for quaternionic modular forms.

6.2. **μ-invariant of the cyclotomic derivative of Katz $\mathfrak{p}$-adic L-function.** In this subsection, we determine the μ-invariant of the cyclotomic derivative of Katz $\mathfrak{p}$-adic L-function, when the branch character λ is self-dual with the root number -1 (*cf.* Theorem C).

Let K^+_∞ be the cyclotomic $\mathbf{Z}_p$-extension of K and $K_{\mathfrak{p},\infty} = K^-_{\mathfrak{p},\infty}K^+_\infty$. Let $\Gamma = \mathrm{Gal}(K^-_\infty K^+_\infty/K)$ and $\Gamma_{\mathfrak{p}} = \mathrm{Gal}(K_{\mathfrak{p},\infty}/K)$. Let $\mathcal{L}_{\Sigma,\lambda}$ (resp. $\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}$) be the p-adic measure on Γ (resp. $\Gamma_{\mathfrak{p}}$) obtained by the pull-back of $\mathcal{L}_{\mathfrak{C},\Sigma}$ (*cf.* §6.1) along λ. We call $\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}$ as the Katz $\mathfrak{p}$-adic L-function with branch character λ. Let $L_{\Sigma,\lambda}(T_1, T_2, ..., T_d, S) \in \overline{\mathbf{Z}}_p[\![\Gamma]\!]$ (resp. $L_{\Sigma,\lambda,\mathfrak{p}}(-, S) \in \overline{\mathbf{Z}}_p[\![\Gamma_{\mathfrak{p}}]\!]$) be the power series of $\mathcal{L}_{\Sigma,\lambda}$ (resp. $\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}$). Here, $T_1, ..., T_d$ are the anticyclotomic variables and S is the cyclotomic variable.

From now on, suppose that λ is self-dual i.e. $\lambda|_{\mathbf{A}_F^\times} = \tau_{K/F}|.|_{\mathbf{A}_F}$, where $\tau_{K/F}$ is the quadratic character associated to K/F and $|.|_{\mathbf{A}_F}$ is the adelic norm. In particular, the global root number of λ is ± 1. Now, suppose that the global root number is -1. In view of the functional equation of Hecke L-function, this root number condition forces all the Hecke L-values appearing in the interpolation property of $\mathcal{L}^-_{\Sigma,\lambda}$ to vanish. Accordingly, we have $\mathcal{L}^-_{\Sigma,\lambda} = 0$. This also follows from the functional equation of $\mathcal{L}_{\Sigma,\lambda}$ (*cf.* [14, §5]). In particular, we have $\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}} = 0$.

We can consider the cyclotomic derivatives

$$L'_{\Sigma,\lambda} = (\frac{\partial}{\partial S} L_{\Sigma,\lambda}(T_1, ..., T_d, S))|_{S=0} \tag{6.8}$$

and $L'_{\Sigma,\lambda,\mathfrak{p}}$ (defined analogously).

The μ-invariants of these functions are related by the following theorem.

Theorem 6.2. *Suppose that $p \nmid h_K^-$, where h_K^- is the relative class number given by $h_K^- = h_K/h_F$. Then,*

$$\mu(L'_{\Sigma,\lambda}) = \mu(L'_{\Sigma,\lambda,\mathfrak{p}}).$$

PROOF. We follow the notation in the proof of Theorem 6.1.

In the proof of [3, Thm. 3.2], it is shown there are p-adic Hilbert modular forms $(\mathbf{f}'_{\lambda,i})_i$ such that

$$L'_{\Sigma,\lambda} = \frac{1}{\log_p(1+p)} \sum_i a_i \circ (f'_{\lambda,i}(t)), \tag{6.9}$$

up to an automorphism of $\overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$. More precisely, $f'_{\lambda,i}$ is the p-adic derivative of $f_{\lambda N^s,i}$ at $s = 0$. Being a p-adic limit of classical Hilbert modular forms, it is a p-adic Hilbert modular form.

In view of (6.4), (6.6) and by a similar argument as in the proof of [loc. cit., Thm 3.2], it follows that

$$L'_{\Sigma,\lambda,\mathfrak{p}} = \frac{1}{\log_p(1+p)} \sum_i a_{i,\mathfrak{p}} \circ (f'_{i,\lambda}(t_{\mathfrak{p}})), \tag{6.10}$$

up to an automorphism of $\overline{\mathbf{Z}}_p[\![\Gamma_{\mathfrak{p}}^-]\!]$.

We finish the proof in the same way as in Theorem 6.1.

□

In most of the cases, $\mu(L'_{\Sigma,\lambda})$ has been explicitly determined (*cf.* [3, Thm. A]). Thus, we obtain a formula for $\mu(L'_{\Sigma,\lambda,\mathfrak{p}})$.

Remark. When $p|h_K^-$, we do not know an expression for $L_{\Sigma,\lambda}$ in terms of the t-expansion of certain Hilbert modular forms. Such an expression seems to be essential in the above proof.

6.3. **$\mathfrak{p}$-version of a conjecture of Gillard.** In this subsection, we prove a $\mathfrak{p}$-version of a conjecture Gillard regarding the vanishing of the μ-invariant of Katz p-adic L-function (*cf.* Theorem D).

Recall that we have Katz $\mathfrak{p}$-adic L-function $\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}} \in \overline{\mathbf{Z}}_p[\![\Gamma_{\mathfrak{p}}]\!]$ (*cf.* §6.2). As a consequence of Theorem 6.1 and the results of [20] and [2], we prove a $\mathfrak{p}$-version of a conjecture of Gillard regarding the vanishing of the μ-invariant of Katz p-adic L-function (*cf.* [12, Conj. (i)]). The conjecture was originally formulated for the $(d+1)$-variable Katz p-adic L-function.

Theorem 6.3. $\mu(\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}) = 0.$

Proof. Let $\mathfrak{X}^+$ be the set consisting of finite order characters $\epsilon : \Gamma^+ \to \mu_{p^\infty}$. For every $\epsilon \in \mathfrak{X}^+$, we regard ϵ as a Hecke character.

In [20] and [2], it has been shown that

$$\liminf_{\epsilon\in\mathfrak{X}^+} \mu(\mathcal{L}^-_{\Sigma,\lambda\epsilon}) = 0. \tag{6.11}$$

In view of Theorem 6.1, this finishes the proof

$$0 \le \mu(\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}) \le \liminf_{\epsilon} \mu(\mathcal{L}^-_{\Sigma,\lambda\epsilon}). \tag{6.12}$$

□

References

[1] M. Brakocevic, *Anticyclotomic p-adic L-function of central critical Rankin-Selberg L-value*, IMRN, Vol. 2011, no. 21, (2011), 4967-5018.

[2] A. Burungale and M.-L. Hsieh, *The vanishing of the μ-invariant of p-adic Hecke L-functions for CM fields*, IMRN, no. 5, (2013), 1014-1027.

[3] A. Burungale, *On the μ-invariant of the cyclotomic derivative of a Katz p-adic L-function*, J. Inst. Math. Jussieu 14 (2015), no. 1, 131-148.

[4] A. Burungale, *An $l \neq p$-interpolation of genuine p-adic L-functions*, Res. Math. Sci. 3 (2016), Paper No. 16, 26 pp.

[5] A. Burungale and H. Hida, *André-Oort conjecture and nonvanishing of central L-values over Hilbert class fields*, Forum Math. Sigma 4 (2016), e20, 26 pp.

[6] A. Burungale, *$\mathfrak{p}$-rigidity and $\mathfrak{p}$-independence of quaternionic modular forms modulo p*, preprint 2015.

[7] A. Burungale and D. Disegni, *On the non-vanishing of p-adic heights for CM abelian varieties, and the arithmetic of Katz p-adic L-functions*, preprint 2016.

[8] C.-L. Chai, *Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli*, Invent. Math. 121 (1995), 439-479.

[9] C.-L. Chai, *Families of ordinary abelian varieties: canonical coordinates, p-adic monodromy, Tate-linear subvarieties and Hecke orbits*, preprint, 2003. Available at http://www.math.upenn.edu/~chai/papers.html .

[10] C.-L. Chai, *Hecke orbits as Shimura varieties in positive characteristic*, International Congress of Mathematicians. Vol. II, 295-312, Eur. Math. Soc., Zurich, 2006.

[11] E. Eischen, M. Harris, J.-S. Li and C. Skinner, *p-adic L-functions for unitary groups*, preprint, arXiv:1602.01776.

[12] R. Gillard, *Remarques sur l'invariant mu d'Iwasawa dans le cas CM*, Sém. Théor. Nombres Bordeaux (2) 3 (1991), no. 1, 13-26.

[13] M. Harris, J.-S. Li and C. Skinner, *p-adic L-functions for unitary Shimura varieties. I. Construction of the Eisenstein measure.*, Doc. Math. 2006, Extra Vol., 393–464.

[14] H. Hida and J. Tilouine, *Anticyclotomic Katz p-adic L-functions and congruence modules*, Ann. Sci. Ecole Norm. Sup., (4) 26 (1993), no. 2, 189-259.

[15] H. Hida, *p-Adic Automorphic Forms on Shimura Varieties*, Springer Monogr. in Math., Springer-Verlag, New York, 2004.

[16] H. Hida, *Anticyclotomic Main Conjectures*, in the Coates anniversary volume from Documenta Math. Ducumenta Math. Volume Coates (2006), 465-532.

[17] H. Hida, *Irreducibility of the Igusa tower*, Acta Math. Sin. (Engl. Ser.) 25 (2009), 1-20.

[18] H. Hida, *Quadratic excercises in Iwasawa theory*, IMRN (2009), no. 5, 912-952.

[19] H. Hida, *The Iwasawa μ-invariant of p-adic Hecke L-functions*, Ann. of Math. (2) 172 (2010), 41-137.

[20] H. Hida, *Vanishing of the μ-Invariant of p-Adic Hecke L-functions*, Compositio Math. 147 (2011), 1151-1178.

[21] H. Hida, *Local indecomposability of Tate modules of non CM abelian varieties with real multiplication*, J. Amer. Math. Soc. 26 (2013), no. 3, 853–877.

[22] H. Hida, *Elliptic Curves and Arithmetic Invariants*, Springer Monographs in Mathematics, Springer, New York, 2013. xviii+449 pp.

[23] M.-L. Hsieh, *On the μ-invariant of anticyclotomic p-adic L-functions for CM fields*, J. reine angew. Math., 688 (2014), 67-100.

[24] M.-L. Hsieh, *Special values of anticyclotomic Rankin-Selbeg L-functions*, Doc. Math. 19 (2014), 709-767 .

[25] N. M. Katz, *p-adic L-functions for CM fields*, Invent. Math., 49 (1978), no. 3, 199-297.

[26] N. M. Katz, *Serre-Tate local moduli*, in Algebraic Surfaces (Orsay, 1976-78), Lecture Notes in Math. 868, Springer-Verlag, New York, 1981, pp. 138-202.

[27] G. Shimura, *On analytic families of polarized abelian varieties and automorphic functions*, Ann. of Math. (2) 78 (1963), 149-192.

[28] G. Shimura, *Abelian varieties with complex multiplication and modular functions*, Princeton Mathematical Series, 46. Princeton University Press, Princeton, NJ, 1998.

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