Geophys. J. Int. (1994) **116**, 799–805

P-wave propagation in weakly anisotropic media

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Accepted 1993 September 7. Received 1993 September 6; in original form 1993 May 28

SUMMARY

The variation of *P*-wave speed with direction in a weakly anisotropic homogeneous elastic medium of arbitrary symmetry is expanded as a series of spherical harmonics. The results given may be used to assess the information content of measurements of the *P*-wave speed acquired over a restricted angular range, and to design experiments with the objective of imaging a limited number of anisotropy parameters. Simplifications which occur for various symmetries typical of sedimentary rocks with and without fractures are discussed. Application is made to the determination of the azimuthal anisotropy of sedimentary rocks. For propagation direction $\mathbf{n}$ with polar angle χ and azimuthal angle η defined with respect to some convenient choice of reference axes $Ox_1x_2x_3$, the variation of *P*-wave speed $v_P(\chi, \eta)$ with η for fixed χ is described by 12 parameters in the absence of symmetry. If the material contains a plane of mirror symmetry, and the axes are chosen with Ox_3 perpendicular to the symmetry plane, the variation of $v_P(\chi, \eta)$ with η for fixed χ depends on six anisotropy parameters. For orthotropic symmetry the number of required parameters is reduced to three. For small polar angles χ, $v_P(\chi, \eta)$ then varies with azimuth as $\cos 2\eta$, with amplitude determined by a single anisotropy parameter $(c_{13} - c_{23} - 2c_{44} + 2c_{55})$.

Key words: anisotropy, elastic-wave theory , *P* waves.

1 INTRODUCTION

The elastic wave speeds in isotropic materials are independent of the propagation direction and, in the case of shear waves, the direction of polarization. However, many sedimentary rocks possess an anisotropic structure resulting, for example, from fine-scale layering, the presence of oriented microcracks or fractures, or the preferred orientation of non-spherical grains or anisotropic minerals. The resulting anisotropy in the elastic wavespeeds is, therefore, of interest as an indicator of lithology (Winterstein 1986). In the absence of any symmetry, a material that can be described by a strain-energy function requires 21 independent parameters to completely specify its elastic behaviour (Nye 1957). The elastic-stiffness tensor may, therefore, be written in terms of two elastic constants of an isotropic comparison medium and 19 parameters which fully characterize the anisotropy of the medium. Material symmetry reduces the number of independent parameters, but this reduction is only applicable if the elastic stiffnesses are specified with respect to a coordinate system with axes aligned with the symmetry directions.

Thomsen (1986) has pointed out that in most cases of interest to geophysicists the anisotropy is weak. Červený (1982) and Červený & Jech (1982) have developed a theory for linearized perturbations to the traveltime in weakly anisotropic media. This has been extended to the case of degenerate *qS* waves by Jech & Pšenčík (1989). Chapman & Pratt (1992) have considered rays which, in an isotropic medium, are confined to the x_1x_3 plane. In an anisotropic medium the perturbed rays may deviate from the plane, but if the anisotropy is weak the deviation will be small. For this example, Chapman & Pratt (1992) show that the *qP* traveltimes are only sensitive to the five parameters (c_{11}, c_{33}, c_{15}, c_{35}, and $c_{13} + 2c_{55}$). Every & Sachse (1992) show that for an arbitrary propagation direction in a weakly anisotropic medium, the *P*-wave speed is most sensitive to c_{11}, c_{22}, c_{33}, c_{16}, c_{15}, c_{26}, c_{24}, c_{35} and c_{34} and the combinations $c_{12} + 2c_{66}$, $c_{23} + 2c_{44}$, $c_{13} + 2c_{55}$, $c_{14} + 2c_{56}$, $c_{25} + 2c_{46}$ and $c_{36} + 2c_{45}$. This conclusion also follows from the expressions given by Chapman & Pratt (1992).

Because of the large number of parameters necessary to describe the elastic behaviour of a material in the absence of symmetry, the inversion of seismic data suffers from non-uniqueness (MacBeth *et al.* 1993). In order to quantify the information content of *P*-wave measurements it is convenient to expand the angle-dependent *P*-wave speed in orthogonal functions. For anisotropic media the *P*-wave speed is a function of direction $\mathbf{n}$. The functions which are orthogonal over all directions of $\mathbf{n}$ are the spherical harmonics. In Section 3 the angle-dependent P-wave speed in a weakly anisotropic homogeneous elastic medium of arbitrary symmetry is expanded in a series of spherical harmonics. The results given may be used to assess the information content of measurements of the P-wave speed acquired over a restricted angular range and to design experiments with the objective of imaging a limited number of anisotropy parameters.

Many sedimentary rocks may be described, to a good approximation, as being transversely isotropic (Thomsen 1986). For such rocks the elastic stiffness tensor is invariant with respect to rotation about a direction which is often oriented perpendicular to the bedding plane. However, many sedimentary rocks contain microcracks or fractures with orientations determined by the stress history of the rock rather than by the orientation of the bedding plane. Any cracks open at depth will tend to be oriented normal to the direction of the least compressive *in situ* stress. For such rocks, observations of the seismic anisotropy have the potential of providing the orientation of the least compressive *in situ* stress direction. For example, modelling of shear waveforms in three-component shear-wave vertical seismic profiles in the Paris Basin (Crampin *et al.* 1986; Bush & Crampin 1991) is consistent with a distribution of vertical fluid-filled cracks aligned with strikes along N30°W, corresponding to the direction of maximum horizontal *in situ* stress in this area.

In the general case of a rock possessing both an anisotropic fabric and a preferred orientation of cracks, the rock will not be transversely isotropic but will display an azimuthal anisotropy. It is shown in Section 5 that for a propagation direction **n** with polar angle χ and azimuthal angle η defined with respect to some convenient choice of reference axes $Ox_1x_2x_3$, the variation of P-wave speed $v_P(\chi, \eta)$ with η for fixed χ is described by 12 anisotropy parameters in the absence of symmetry. If the material contains a plane of mirror symmetry it is convenient to choose $Ox_1x_2x_3$ such that Ox_3 is perpendicular to the symmetry plane. The variation of $v_P(\chi, \eta)$ with η for fixed χ is then given by six anisotropy parameters. For orthotropic symmetry the number of required parameters is reduced to 3. For small angles from the vertical a single anisotropy parameter is found to dominate the azimuthal variation.

2 THE ELASTIC STIFFNESS TENSOR

It is convenient to write the fourth-order elastic stiffness tensor of the material, C_{ijkl}, in the form

$$C_{ijkl} = C^0_{ijkl} + \gamma_{ijkl}, \tag{1}$$

where C^0_{ijkl} is the elastic stiffness tensor of an isotropic comparison medium and γ_{ijkl} is the difference between C_{ijkl} and C^0_{ijkl}.

Backus (1970) has shown that an arbitrary fourth-rank tensor satisfying the 60 symmetry relations

$$\gamma_{ijkl} = \gamma_{jikl} = \gamma_{ijlk} = \gamma_{klij} \tag{2}$$

may be uniquely represented as a linear combination of 21 canonical harmonic tensors of degree 0, 2 and 4 with components denoted by γ^{lmv}_σ in the notation of Smith & Dahlen (1973). Here, l may take the values 0, 2 and 4, $0 \le m \le l$ and v is either 'c' for cosine or 's' for sine. The subscript σ may be either 'S' for symmetric or 'A' for anti-symmetric. Expressions for the γ^{lmv}_σ are given in terms of the γ_{ijkl} in the appendix of Smith & Dahlen (1973).

In the following analysis the conventional matrix notation will be used in which pairs of subscripts ij and kl are converted to single subscripts using the convention $11 \to 1$, $22 \to 2$, $33 \to 3$, 23 and $32 \to 4$, 13 and $31 \to 5$ and 12 and $21 \to 6$ (Nye 1957). In this notation, the first two γ^{lmv}_σ are given by:

$$45\gamma^{00c}_S = 3\gamma_{11} + 3\gamma_{22} + 3\gamma_{33} + 2\gamma_{12} + 2\gamma_{23} + 2\gamma_{13} + 4\gamma_{44} + 4\gamma_{55} + 4\gamma_{66}, \tag{3}$$

$$9\gamma^{00c}_A = 2\gamma_{12} + 2\gamma_{23} + 2\gamma_{13} - 2\gamma_{44} - 2\gamma_{55} - 2\gamma_{66}. \tag{4}$$

Since C^0_{ij} is isotropic it has the following non-vanishing components: $C^0_{11} = C^0_{22} = C^0_{33} = \lambda + 2\mu$, $C^0_{12} = C^0_{21} = C^0_{23} = C^0_{32} = C^0_{13} = C^0_{31} = \lambda$, $C^0_{44} = C^0_{55} = C^0_{66} = \mu$, where λ and μ are the second-order Lamé constants of the isotropic-comparison medium. It is convenient to choose the C^0_{ij} such that γ^{00c}_S and γ^{00c}_A vanish for arbitrary C_{ij}. This choice corresponds to

$$15\lambda = C_{11} + C_{22} + C_{33} + 4C_{12} + 4C_{23} + 4C_{13} - 2C_{44} - 2C_{55} - 2C_{66}, \tag{5}$$

$$15\mu = C_{11} + C_{22} + C_{33} - C_{12} - C_{23} - C_{13} + 3C_{44} + 3C_{55} + 3C_{66}. \tag{6}$$

The 19 remaining γ^{lmv}_σ are linearly independent combinations of the γ_{ij} which vanish if γ_{ij} is isotropic and fully characterize the elastic anisotropy of the material. The γ^{lmv}_σ are listed in the Appendix for completeness.

3 THE VELOCITY OF P WAVES

Consider the propagation of P waves in the direction **n** shown in Fig. 1. In this figure χ and η are the polar and azimuthal angles of the vector **n** with respect to an arbitrary set of reference axes $Ox_1x_2x_3$. It is convenient to introduce a normalized velocity $r(\zeta, \eta)$ (Sayers 1988) defined by:

$$r(\zeta, \eta) = v_P(\zeta, \eta)/4\pi\bar{v}_P, \tag{7}$$

where $v_P(\zeta, \eta)$ is the P-wave speed in the direction **n**,

$$4\pi\bar{v}_P = \int_0^{2\pi}\int_{-1}^{1} v_P(\zeta, \eta)\, d\zeta\, d\eta \tag{8}$$

and $\zeta = \cos\chi$. For weak anisotropy $r(\zeta, \eta)$ may be expanded as a linear combination of spherical harmonics with expansion coefficients R_{lm} (Sayers 1988):

$$r(\zeta, \eta) = \sum_{l=0}^{4}\sum_{m=-l}^{l} R_{lm}P^m_l(\zeta)\exp(-im\eta). \tag{9}$$

Here $P^m_l(\zeta)$ is the *normalized* associated Legendre function and $P^m_l(\zeta) = (-1)^m P^{\bar{m}}_l(\zeta)$ with $\bar{m} = -m$ (Roe 1965). The coefficients R_{lm} are, in general, complex and can be obtained from eq. (9) as follows:

$$R_{lm} = (2\pi)^{-1}\int_0^{2\pi}\int_{-1}^{1} r(\zeta, \eta)P^m_l(\zeta)\exp(im\eta)\, d\zeta\, d\eta. \tag{10}$$

Since the P-wave velocities are centro-symmetric,

$$r(\zeta, \eta) = r(-\zeta, \eta + \pi).$$

Expanding both sides of this equation in spherical harmonics and utilizing the relation $P_l^m(-\zeta) = (-1)^{l+m} P_l^m(\zeta)$ gives $R_{lm} = (-1)^l R_{lm}$. This requires R_{lm} to be identically zero when l is odd.

It is convenient to write R_{lm} in the form:

$$R_{lm} = \alpha_{lm} + i\beta_{lm}. \tag{11}$$

Since the velocity $v_P(\zeta, \eta)$ is a real quantity, the following relationships hold: $\alpha_{l\bar{m}} = \alpha_{lm}$, $\beta_{l\bar{m}} = -\beta_{lm}$ where $\bar{m} = -m$. hence $\beta_{lm} = 0$ for $m = 0$. Eq. (9) may therefore be written in the form:

$$r(\zeta, \eta) = \sum_{l=0}^{4} \sum_{m=-l}^{l} P_l^m(\zeta)[\alpha_{lm} \cos m\eta + \beta_{lm} \sin m\eta]. \tag{12}$$

The R_{lm} may be obtained using the variational method (Jeffreys 1961; Aki & Richards 1980). Defining quantities A_{lm} and B_{lm} by $A_{lm} = 8\pi\rho v_P^{02} N_{lm}\alpha_{lm}$, $B_{lm} = 8\pi\rho v_P^{02} N_{lm}\beta_{lm}$ where

$$N_{lm} = \sqrt{\frac{2l+1}{2} \frac{(l-|m|)!}{(l+|m|)!}}, \tag{13}$$

the non-zero A_{lm} and B_{lm} are found to be given in terms of the γ_σ^{lmv} by the following equations:

$$A_{00} = 2\rho v_P^{02}, \qquad B_{00} = 0,$$
$$A_{20} = -12\gamma_S^{20c}, \qquad B_{20} = 0,$$
$$A_{21} = -2\gamma_S^{21c}, \qquad B_{21} = -2\gamma_S^{21s},$$
$$A_{22} = \gamma_S^{22c}, \qquad B_{22} = \gamma_S^{22s},$$
$$A_{40} = 8\gamma_S^{40c}, \qquad B_{40} = 0,$$
$$A_{41} = 4\gamma_S^{41c}/5, \qquad B_{41} = 4\gamma_S^{41s}/5,$$
$$A_{42} = -\gamma_S^{42c}/15, \qquad B_{42} = -2\gamma_S^{42s}/15,$$
$$A_{43} = -2\gamma_S^{43c}/105, \qquad B_{43} = 2\gamma_S^{43s}/105,$$
$$A_{44} = \gamma_S^{44c}/210, \qquad B_{44} = \gamma_S^{44s}/210.$$

It follows from eq. (12) that the azimuthal anisotropy depends on 12 parameters (γ_S^{21c}, γ_S^{21s}, γ_S^{22c}, γ_S^{22s}, γ_S^{41c}, γ_S^{41s}, γ_S^{42c}, γ_S^{42s}, γ_S^{43c}, γ_S^{43s}, γ_S^{44c} and γ_S^{44s}).

The R_{lm} are seen to be independent of the quantities γ_A^{20c}, γ_A^{21c}, γ_A^{21s}, γ_A^{22c} and γ_A^{22s} which cannot, therefore, be obtained using P-wave velocities alone. In order to determine these quantities, a measurement of the shear-wave anisotropy is required.

In the laboratory, ultrasonic P-wave velocity measurements can be made in a sufficient number of directions so that eq. (10) may be used directly to obtain the γ_σ^{lmv} (Sayers 1988). An example is the data of Thill, Willard & Bur (1969) who measured the ultrasonic P-wave velocity in a large number of directions in a spherical sample of Salisbury granite. In seismic studies it is not possible to obtain a complete angular coverage. Eq. (10) may then be used to assess the information content of measurements of the P-wave velocity over a restricted angular range and to optimize the acquisition geometry for further measurements. For example, velocity measurements in directions for which $P_l^m(\zeta)$ in eq. (10) is small contain little information relevant for determining R_{lm} whilst measurements in directions for which $P_l^m(\zeta)$ is large are essential for its determination.

For general anisotropy, the 14 independent γ_S^{lmv} for $l = 2$ and $l = 4$ may be obtained from the R_{lm}. Inverting the equations for the γ_σ^{lmv} given in the Appendix to obtain the γ_{ij} and eliminating the γ_A^{lmv} which cannot be obtained from P-wave measurements allows the following elastic constants to be obtained:

$$\gamma_{11} = 6\gamma_S^{20c} + 6\gamma_S^{22c} + 3\gamma_S^{40c} + \gamma_S^{42c} + \gamma_S^{44c},$$
$$\gamma_{22} = 6\gamma_S^{20c} - 6\gamma_S^{22c} + 3\gamma_S^{40c} - \gamma_S^{42c} + \gamma_S^{44c},$$
$$\gamma_{33} = -12\gamma_S^{20c} + 8\gamma_S^{40c},$$
$$\gamma_{16} = 3\gamma_S^{22s} + \gamma_S^{42s} + \gamma_S^{44s},$$
$$\gamma_{15} = 3\gamma_S^{21c} + 3\gamma_S^{41c} + \gamma_S^{43c},$$
$$\gamma_{26} = 3\gamma_S^{22s} + \gamma_S^{42s} - \gamma_S^{44s},$$
$$\gamma_{24} = 3\gamma_S^{21s} + 3\gamma_S^{41s} + \gamma_S^{43s},$$
$$\gamma_{35} = 3\gamma_S^{21c} - 4\gamma_S^{41c},$$
$$\gamma_{34} = 3\gamma_S^{21s} - 4\gamma_S^{41s}.$$

In addition, the following combinations of elastic constants can also be determined using only P waves:

$$\gamma_{12} + 2\gamma_{66} = 6\gamma_S^{20c} + 3\gamma_S^{40c} - 3\gamma_S^{44c},$$
$$\gamma_{23} + 2\gamma_{44} = -3\gamma_S^{20c} - 3\gamma_S^{22c} - 12\gamma_S^{40c} + 3\gamma_S^{42c},$$
$$\gamma_{13} + 2\gamma_{55} = -3\gamma_S^{20c} + 3\gamma_S^{22c} - 12\gamma_S^{40c} - 3\gamma_S^{42c},$$
$$\gamma_{14} + 2\gamma_{56} = 3\gamma_S^{21s} + 3\gamma_S^{41s} - 3\gamma_S^{43s},$$
$$\gamma_{25} + 2\gamma_{46} = 3\gamma_S^{21c} + 3\gamma_S^{41c} - 3\gamma_S^{43c},$$
$$\gamma_{36} + 2\gamma_{45} = 3\gamma_S^{22s} - 6\gamma_S^{42s}.$$

Thus, for a generally anisotropic medium γ_{11}, γ_{22}, γ_{33}, γ_{16}, γ_{15}, γ_{26}, γ_{24}, γ_{35} and γ_{34} and the combinations $\gamma_{12} + 2\gamma_{66}$, $\gamma_{23} + 2\gamma_{44}$, $\gamma_{13} + 2\gamma_{55}$, $\gamma_{14} + 2\gamma_{56}$, $\gamma_{25} + 2\gamma_{46}$ and $\gamma_{36} + 2\gamma_{45}$ can be obtained using P waves alone. This conclusion is in agreement with that of Every & Sachse (1992) and also follows from the expressions given by Chapman & Pratt (1992).

4 LONGITUDINAL DIRECTIONS

A longitudinal direction is one in which three pure modes can propagate (Helbig 1993). Koloder (1966) has shown that three or more longitudinal directions exist in every anisotropic solid. All symmetry axes are longitudinal directions but not all longitudinal axes are symmetry directions. If Ox_3 is a longitudinal direction, the elastic stiffnesses c_{34} and c_{35} are zero even if the material lacks any symmetry. Furthermore, a rotation of the coordinate system about Ox_3 exists such that the elastic stiffness component c_{45} is also zero. Norris (1989) has shown that even in the absence of any symmetry there are at least three coordinate systems with respect to which there are only 18 non-zero elastic constants, the elastic constants c_{34}, c_{35} and c_{45} being zero when expressed in any of these coordinate systems. The 21 independent parameters describing the elastic behaviour may therefore be expressed in terms of the 18 non-vanishing elastic constants and three Euler angles which define the orientation of the measurement coordinate system with respect to one of these coordinate systems.

Consider a set of axes $Ox_1'x_2'x_3'$ specified with respect to

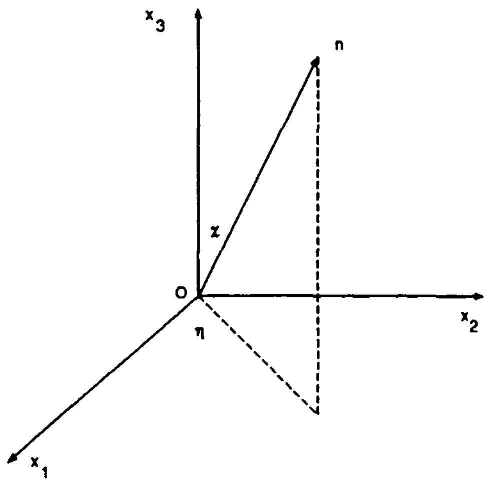

Figure 1. Orientation of the vector **n** with respect to a Cartesian set of axes $Ox_1x_2x_3$.

the axes $Ox_1x_2x_3$ by three Euler angles α, β and γ. The Euler angles are defined such that $Ox_1'x_2'x_3'$ is obtained by (i) a rotation of β about Ox_3, (ii) a rotation of α about the new z axis that results, and (iii) a rotation of γ about Ox_3'. Consider the measurement direction **n** defined with respect to $Ox_1x_2x_3$ by polar angle χ and azimuthal angle η (see Fig. 1). If the polar and azimuthal angles of **n** in the coordinate system $Ox_2'x_2'x_3'$ are denoted by θ and ϕ then the normalized velocity introduced in Section 3 may be expanded in the coordinate system $Ox_1'x_2'x_3'$ as a linear combination of spherical harmonics with expansion coefficients R'_{lm}:

$$r(\xi, \phi) = \sum_{l=0}^{4} \sum_{m=-l}^{l} R'_{lm} P_l^m(\xi) \exp(-im\phi), \tag{14}$$

where $\xi = \cos\theta$. If Ox_3' is a longitudinal direction then $c'_{34} = c'_{35} = 0$ and therefore

$$3\gamma_S'^{21c} = 4\gamma_S'^{41c} \tag{15}$$

$$3\gamma_S'^{21s} = 4\gamma_S'^{41s} \tag{16}$$

in the coordinate frame $Ox_1'x_2'x_3'$. Using the addition theorem for spherical harmonics allows the expansion coefficients R'_{lm} to be written in terms of the R_{lm} appearing in eq. (9). This allows the polar angle β and azimuthal angle α of the longitudinal direction with respect to the coordinate axes $Ox_1x_2x_3$ to be determined by searching for values of α and β such that eqs (15) and (16) are satisfied. The third Euler angle γ requires shear-wave measurements for its determination.

5 SIMPLIFICATIONS FOR VARIOUS MATERIAL SYMMETRIES

The equations given above are valid for arbitrary material symmetry. If the material displays symmetry the equations simplify as follows.

5.1 Material with a single plane of mirror symmetry

A sedimentary rock containing several sets of fractures with normals lying in the bedding plane is an example of a medium with a single plane of mirror symmetry if, in the absence of fractures, the rock is transversely isotropic with symmetry axis perpendicular to the bedding plane. For a material with a single plane of mirror symmetry it is convenient to choose the coordinate axes $Ox_1x_2x_3$ with the normal to the mirror plane along Ox_3. For an arbitrary choice of Ox_1 and Ox_2 within the mirror plane, the elastic behaviour of the material may then be described by 13 independent parameters. It follows from Section 4 that a choice of Ox_1 and Ox_2 exists such that $c_{45} = 0$ and therefore the number of independent elastic constants in this coordinate frame is reduced to 12. It is not possible, however, to obtain the appropriate orientation of Ox_1 and Ox_2 from P-wave measurements alone, since P-wave measurements are only sensitive to the combination $c_{36} + 2c_{45}$ and cannot be used to obtain c_{36} and c_{45} separately.

With Ox_3 chosen to lie along the normal to the mirror plane, it follows that $v_P(\zeta, \eta) = v_P(-\zeta, \eta)$ and that $R_{lm} = 0$ if m is odd. γ_S^{21c}, γ_S^{21s}, γ_S^{41c}, γ_S^{41s}, γ_S^{43c} and γ_S^{43s} are therefore identically zero for a material with a single plane of mirror symmetry with normal along Ox_3 and the azimuthal anisotropy is seen from eq. (12) to depend on six anisotropy parameters (γ_S^{22c}, γ_S^{22s}, γ_S^{42c}, γ_S^{42s}, γ_S^{44c} and γ_S^{44s}). For an arbitrary choice of Ox_1 and Ox_2 within the mirror plane, the 13 non-zero γ_{ij} are γ_{11}, γ_{22}, γ_{33}, γ_{12}, γ_{23}, γ_{13}, γ_{44}, γ_{55}, γ_{66}, γ_{16}, γ_{26}, γ_{36} and γ_{45}. Thus γ_A^{21c} and γ_A^{21s} also vanish.

The elastic anisotropy of the medium is therefore determined by 11 independent parameters. Eight of these parameters (γ_S^{20c}, γ_S^{22c}, γ_S^{22s}, γ_S^{40c}, γ_S^{42c}, γ_S^{42s}, γ_S^{44c} and γ_S^{44s}) may be determined using measured P-wave speeds. The remaining three parameters (γ_A^{20c}, γ_A^{22c} and γ_A^{22s}) require a measurement of the shear-wave anisotropy for their determination. Expressions for these parameters are given in the Appendix.

5.2 Orthotropic symmetry

A material with orthotropic symmetry has three orthogonal planes of mirror symmetry. A sedimentary rock containing a set of fractures with normals lying in the same direction within the bedding plane is an example of such a medium if, in the absence of fractures, the rock is transversely isotropic with a symmetry axis perpendicular to the bedding plane. If the normals to the three orthogonal mirror planes are chosen to lie along the coordinate axes, the elastic behaviour of the medium may be described by nine independent elastic constants. There are, of course, three different such choices of coordinate axes.

With the coordinate axes chosen to lie along the normals to the three orthogonal mirror planes, it follows that $R_{lm} = 0$ if m is odd and that $R_{lm} = R_{l\bar{m}}$ if m is even. It follows that the β_{lm} are all zero and $\alpha_{21} = \alpha_{41} = \alpha_{43} = 0$. Eq. (12) therefore reduces to the following simple form:

$$\begin{aligned} 4\pi r(\zeta, \eta) = 1 + 4\pi[&\alpha_{20}P_2^0(\zeta) + 2\alpha_{22}P_2^2(\zeta)\cos 2\eta \\ &+ \alpha_{40}P_4^0(\zeta) + 2\alpha_{42}P_4^2(\zeta)\cos 2\eta \\ &+ 2\alpha_{44}P_4^4(\zeta)\cos 4\eta]. \end{aligned} \tag{17}$$

For orthotropic symmetry γ_S^{21c}, γ_S^{21s}, γ_S^{22s}, γ_S^{41c}, γ_S^{41s}, γ_S^{42s}, γ_S^{43c}, γ_S^{43s} and γ_S^{44s} are identically zero and the azimuthal anisotropy is seen from eq. (17) to depend on three anisotropy parameters (γ_S^{22c}, γ_S^{42c} and γ_S^{44c}). The nine

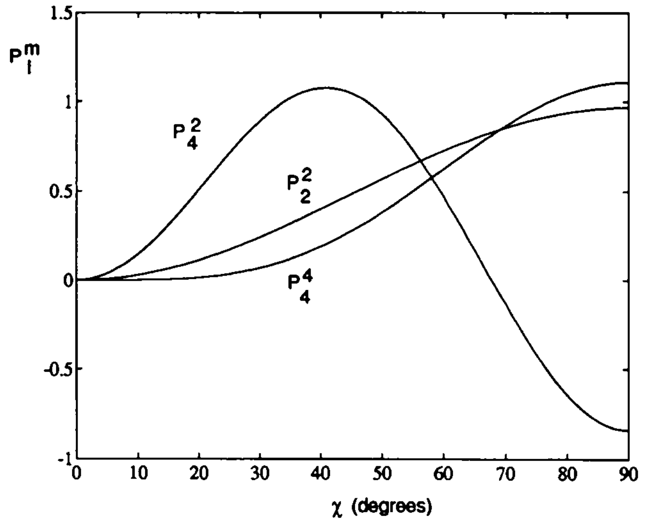

Figure 2. Variation of the three *normalized* associated Legendre functions $P_2^2(\zeta)$, $P_4^2(\zeta)$ and $P_4^4(\zeta)$ with polar angle χ. $\zeta = \cos\chi$.

non-zero γ_{ij} are γ_{11}, γ_{22}, γ_{33}, γ_{12}, γ_{23}, γ_{13}, γ_{44}, γ_{55} and γ_{66}. γ_A^{21c}, γ_A^{21s} and γ_A^{22s} are therefore also identically zero. The elastic anisotropy of the medium is therefore determined by seven independent anisotropy parameters which are given in the Appendix. Five of these parameters (γ_S^{20c}, γ_S^{22c}, γ_S^{40c}, γ_S^{42c} and γ_S^{44c}) may be determined using measured *P*-wave speeds. The remaining two parameters (γ_A^{20c} and γ_A^{22c}) require a measurement of the shear-wave anisotropy for their determination.

Figure 2 plots the three *normalized* associated Legendre functions $P_2^2(\zeta)$, $P_4^2(\zeta)$ and $P_4^4(\zeta)$ which occur in the azimuthally dependent terms in eq. (17). For polar angles less than about 30°, $P_4^4(\zeta)$ is seen to be much smaller than $P_2^2(\zeta)$ and $P_4^2(\zeta)$. For polar angles less than about 30°, $v_P(\zeta, \eta)$ will therefore vary with η as $\cos 2\eta$, the term in $\cos 4\eta$ being negligible. Thus, for polar angles less than about 30°, the azimuthal anisotropy will be insensitive to the anisotropy parameter $c_{11} + c_{22} - 2c_{12} - 4c_{66}$. For small polar angle χ, $P_2^2(\zeta)$ and $P_4^2(\zeta)$ vary as χ^2 whilst $P_4^4(\zeta)$ varies as χ^4. It follows that for small χ the azimuthal anisotropy is given by a single anisotropy parameter $(c_{13} - c_{23} - 2c_{44} + 2c_{55})$ and eq. (17) becomes:

$$v_P(\chi, \eta)/v_{33} = 1 - \frac{\chi^2}{2c_{33}}[(2c_{33} - c_{13} - c_{23} - 2c_{44} - 2c_{55}) - (c_{13} - c_{23} - 2c_{44} + 2c_{55})\cos 2\eta]. \quad (18)$$

Here v_{33} is the *P*-wave speed for propagation along Ox_3.

5.3 Transverse isotropy

The elastic stiffness tensor of a transversely isotropic material is invariant with respect to rotations about a symmetry axis and may be described by five independent elastic constants. An example is a sedimentary rock for which the bedding plane is a plane of isotropy.

If the axis of rotational symmetry is chosen to lie along Ox_3, it follows that $v_P(\zeta, \eta)$ is independent of η and that $R_{lm} = 0$ if $m \neq 0$. The non-vanishing γ_{ij} are $\gamma_{11} = \gamma_{22}$, γ_{33}, γ_{12}, $\gamma_{23} = \gamma_{13}$, $\gamma_{44} = \gamma_{55}$ and $\gamma_{66} = (\gamma_{11} - \gamma_{12})/2$ and eq. (12) becomes:

$$4\pi r(\zeta) = 1 + 4\pi[\alpha_{20}P_2^0(\zeta) + \alpha_{40}P_4^0(\zeta)]. \quad (19)$$

The elastic anisotropy of the medium is therefore determined by three independent anisotropy parameters (γ_S^{20c}, γ_A^{20c} and γ_S^{40c}). Two of these parameters may be determined using measured *P*-wave speeds and are given by the following equations:

$$63\gamma_S^{20c} = 4c_{11} - 3c_{33} - c_{13} - 2c_{44}, \quad (20)$$

$$35\gamma_S^{40c} = c_{11} + c_{33} - 2c_{13} - 4c_{44}. \quad (21)$$

The remaining parameter requires a measurement of the shear-wave anisotropy and is given by the following equation:

$$9\gamma_A^{20c} = -c_{11} + 3c_{12} - 2c_{13} + 2c_{44}. \quad (22)$$

For small polar angle χ, eq. (19) may be written in the form:

$$v_P(\chi)/v_{33} = 1 + \delta_{\text{weak}}\chi^2, \quad (23)$$

where v_{33} is the *P*-wave speed for propagation along Ox_3 and

$$\delta_{\text{weak}} = -(c_{33} - c_{13} - 2c_{44})/c_{33}. \quad (24)$$

It is interesting to compare eq. (23) with Thomsen's result (Thomsen 1986) for small χ:

$$v_P(\chi)/v_{33} = 1 + \delta_{\text{Thomsen}}\chi^2, \quad (25)$$

where δ_{Thomsen} is Thomsen's anisotropy factor δ (Thomsen 1986):

$$\delta_{\text{Thomsen}} = \frac{(c_{13} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}. \quad (26)$$

Fig. 3 compares δ_{weak} with δ_{Thomsen} using the elastic constants for the sedimentary rocks listed by Thomsen (1986). δ_{Thomsen} may be written in the form:

$$\delta_{\text{Thomsen}} = \delta_{\text{weak}} + \delta_{\text{weak}}^2/2(1 - c_{44}/c_{33}). \quad (27)$$

This equation is plotted in Fig. 3 for the average value

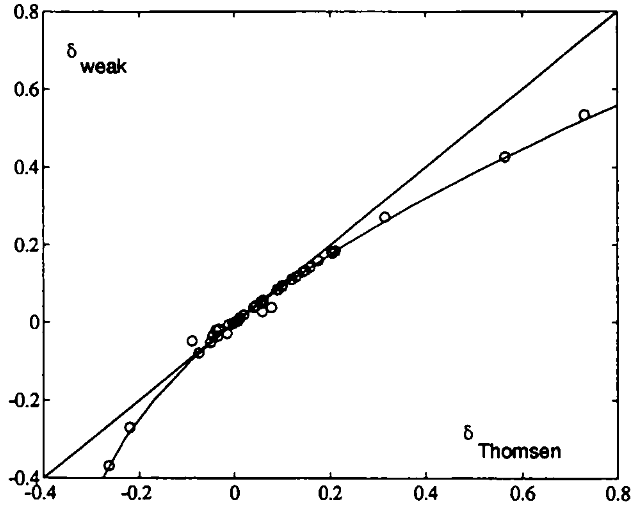

Figure 3. Comparison of δ_{weak} defined by eq. (24) with δ_{Thomsen} defined by eq. (26) for the sedimentary rocks listed by Thomsen (1986). The curve is the prediction of eq. (27) for the average value $c_{44}/c_{33} = 0.3514$.

$c_{44}/c_{33} = 0.3514$ calculated for the sedimentary rocks listed by Thomsen (1986).

6 CONCLUSION

Because of the large number of parameters necessary to describe the elastic behaviour of a material in the absence of symmetry, the inversion of seismic data suffers from non-uniqueness (MacBeth *et al.* 1993). In order to quantify the information content of *P*-wave measurements it is convenient to expand the angle-dependent *P*-wave speed in orthogonal functions. For anisotropic media, the *P*-wave speed is a function of direction $\mathbf{n}$. The functions which are orthogonal over all directions of $\mathbf{n}$ are the spherical harmonics. The angle-dependent *P*-wave speed in a weakly anisotropic homogeneous elastic medium of arbitrary symmetry is expanded in a series of spherical harmonics. This allows 14 of the parameters controlling the seismic anisotropy to be determined by numerical integration given *P*-wave speed measurements in a sufficient number of directions. In seismic studies it is not usually possible to obtain a complete angular coverage. In this case the results given may be used to assess the information content of measurements of the *P*-wave speed acquired over a restricted angular range and to design experiments with the objective of imaging a limited number of anisotropy parameters.

Application is made to the determination of the azimuthal anisotropy of sedimentary rocks. Since the *P*-wave velocity is centro-symmetric, the azimuthal anisotropy for arbitrary anisotropy is given by eq. (12). For propagation direction $\mathbf{n}$ with polar angle χ and azimuthal angle η defined with respect to some convenient choice of reference axes $Ox_1x_2x_3$, the variation of *P*-wave speed $v_P(\chi, \eta)$ with η for fixed χ is therefore described by 12 parameters in the absence of any symmetry. If the material contains a plane of mirror symmetry, it is convenient to choose $Ox_1x_2x_3$ such that Ox_3 is perpendicular to the symmetry plane. The variation of $v_P(\chi, \eta)$ with η for fixed χ is then determined by six parameters. For orthotropic symmetry the number of required parameters is reduced to three. Many seismic experiments are performed with polar angles less than 30°. Such experiments are found to be insensitive to one of the three parameters $(c_{11} + c_{22} - 2c_{12} - 4c_{66})$ controlling the azimuthal variation. For small polar angles χ, $v_P(\chi, \eta)$ is found to vary with azimuth as $\cos 2\eta$, with a single anisotropy parameter $(c_{13} - c_{23} - 2c_{44} + 2c_{55})$ dominating the azimuthal variation.

ACKNOWLEDGMENTS

I wish to thank C.H. Chapman and D.E. Miller for discussions, and the referees for helpful comments and suggestions.

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APPENDIX

With the choice of isotropic comparison medium given by eqs (5) and (6), γ_S^{00c} and γ_A^{00c} vanish. The remaining anisotropy factors $\gamma_\sigma^{lm\nu}$ have been given by Smith & Dahlen (1973) and are listed below for completeness.

The $\gamma_\sigma^{lm\nu}$ which can be obtained from a measurement of the angular variation of the *P*-wave speed are given by:

$$126\gamma_S^{20c} = 3\gamma_{11} + 3\gamma_{22} - 6\gamma_{33} + 2\gamma_{12} - \gamma_{23} - \gamma_{13} - 2\gamma_{44} - 2\gamma_{55} + 4\gamma_{66},$$

$$21\gamma_S^{21c} = 3\gamma_{15} + 3\gamma_{35} + \gamma_{25} + 2\gamma_{64},$$

$$21\gamma_S^{21s} = 3\gamma_{24} + 3\gamma_{34} + \gamma_{14} + 2\gamma_{65},$$

$$42\gamma_S^{22c}=3\gamma_{11}-3\gamma_{22}+\gamma_{13}-\gamma_{23}+2\gamma_{55}-2\gamma_{44},$$
$$21\gamma_S^{22s}=3\gamma_{16}+3\gamma_{26}+\gamma_{36}+2\gamma_{54},$$
$$280\gamma_S^{40c}=3\gamma_{11}+3\gamma_{22}+8\gamma_{33}+2\gamma_{12}-8\gamma_{13}-8\gamma_{23}+4\gamma_{66}-16\gamma_{55}-16\gamma_{44},$$
$$28\gamma_S^{41c}=3\gamma_{15}-4\gamma_{35}+\gamma_{25}+2\gamma_{64},$$
$$28\gamma_S^{41s}=3\gamma_{24}-4\gamma_{34}+\gamma_{14}+2\gamma_{65},$$
$$14\gamma_S^{42c}=\gamma_{11}-\gamma_{22}-2\gamma_{13}+2\gamma_{23}-4\gamma_{55}+4\gamma_{44},$$
$$14\gamma_S^{42s}=\gamma_{16}+\gamma_{26}-2\gamma_{36}-4\gamma_{54},$$
$$4\gamma_S^{43c}=\gamma_{15}-\gamma_{25}-2\gamma_{64},$$
$$4\gamma_S^{43s}=\gamma_{24}-\gamma_{14}-2\gamma_{65},$$
$$8\gamma_S^{44c}=\gamma_{11}+\gamma_{22}-2\gamma_{12}-4\gamma_{66},$$
$$2\gamma_S^{44s}=\gamma_{16}-\gamma_{26}.$$

In the above a misprint occuring in the equation for γ_S^{41s} in Smith & Dahlen (1973) has been corrected.

The γ_σ^{lmv} which cannot be obtained from P waves are given by:

$$9\gamma_A^{20c}=2\gamma_{12}-\gamma_{13}-\gamma_{23}+\gamma_{44}+\gamma_{55}-2\gamma_{66},$$
$$3\gamma_A^{21c}=2\gamma_{25}-2\gamma_{64},$$
$$3\gamma_A^{21s}=2\gamma_{14}-2\gamma_{65},$$
$$3\gamma_A^{22c}=\gamma_{13}-\gamma_{23}+\gamma_{44}-\gamma_{55},$$
$$3\gamma_A^{22s}=2\gamma_{36}-2\gamma_{54}.$$