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PACHPATTE INEQUALITIES ON TIME SCALES

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ABSTRACT. In the study of dynamic equations on time scales we deal with certain dynamic inequalities which provide explicit bounds on the unknown functions and their derivatives. Most of the inequalities presented are of comparison or Gronwall type and, more specifically, of Pachpatte type.

Key words and phrases: Time scales, Pachpatte inequalities, Dynamic equations and inequalities.

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1. INTRODUCTION

In this paper we present a number of dynamic inequalities that are essentially based on Gronwall's inequality. Most of these inequalities are also known as being of Pachpatte type. For a summary of related continuous inequalities, the monograph [4] by Pachpatte is an authoritative source. For the corresponding discrete inequalities, we refer the interested reader to the excellent monograph [5] by Pachpatte.

Our dynamic inequalities unify and extend the (linear) inequalities presented in the first chapters of [4, 5]. The setup of this paper is as follows: In Section 2 we give some preliminary results with respect to the calculus on time scales, which can also be found in the books by Bohner and Peterson [2, 3]. Some basic dynamic inequalities are given as established in the paper by Agarwal, Bohner, and Peterson [1]. The remaining sections deal with our dynamic inequalities. Note that they contain differential and difference inequalities as special cases, and they also contain all other dynamic inequalities, such as, for example, q-difference inequalities, as special cases.

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2. CALCULUS ON TIME SCALES

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We define the *forward jump operator* σ on $\mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\} \in \mathbb{T} \quad \text{for all} \quad t \in \mathbb{T}.$$

In this definition we put $\sigma(\emptyset) = \sup \mathbb{T}$, where $\emptyset$ is the empty set. If $\sigma(t) > t$, then we say that t is *right-scattered*. If $\sigma(t) = t$ and $t < \sup \mathbb{T}$, then we say that t is *right-dense*. The backward jump operator and left-scattered and left-dense points are defined in a similar way. The *graininess* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$ to be the number (provided it exists) such that given any $\varepsilon > 0$, there is a neighorhood U of t with

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \le \varepsilon \left|\sigma(t) - s\right| \quad \text{for all} \quad s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t, and f^Δ is the usual derivative f' if $\mathbb{T} = \mathbb{R}$ and the usual forward difference Δf (defined by $\Delta f(t) = f(t+1) - f(t)$) if $\mathbb{T} = \mathbb{Z}$.

Theorem 2.1. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If f is differentiable at t and t is right-dense, then*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is differentiable at t, then*

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \quad \textit{where} \quad f^\sigma := f \circ \sigma.$$

(v) *If f and g are differentiable at t, then so is fg with*

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f^\sigma(t) g^\Delta(t).$$

We say that $f : \mathbb{T} \to \mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left-sided limit at each left-dense point of $\mathbb{T}$. The set of rd-continuous functions will be denoted in this paper by C_{rd}, and the set of functions that are differentiable and whose derivative is rd-continuous is denoted by $\mathrm{C}^1_{\mathrm{rd}}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. In this case we define the integral of f by

$$\int_s^t f(\tau) \Delta\tau = F(t) - F(s) \quad \text{for} \quad s, t \in \mathbb{T}.$$

We say that $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}$. We denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions. We define the set of all positively regressive functions by $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t) p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. If $p, q \in \mathcal{R}$, then we define

$$p \oplus q = p + q + \mu p q, \quad \ominus q = -\frac{q}{1 + \mu q}, \quad \text{and} \quad p \ominus q = p \oplus (\ominus q).$$

If $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous and regressive, then the *exponential function* $e_p(\cdot, t_0)$ is for each fixed $t_0 \in \mathbb{T}$ the unique solution of the initial value problem

$$x^\Delta = p(t) x, \quad x(t_0) = 1 \quad \text{on} \quad \mathbb{T}.$$

We use the following four theorems which are proved in Bohner and Peterson [2].

Theorem 2.2. *If $p, q \in \mathcal{R}$, then*

(i) $e_p(t,t) \equiv 1$ *and* $e_0(t,s) \equiv 1$*;*

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s) = e_p(s,t)$*;*

(iv) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$*;*

(v) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$*;*

(vi) *if $p \in \mathcal{R}^+$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.*

Example 2.1. In order to allow for a comparison with the continuous Pachpatte inequalities given in [4], we note that, if $\mathbb{T} = \mathbb{R}$, the exponential function is given by

$$e_p(t,s) = e^{\int_s^t p(\tau)\mathrm{d}\tau}, \quad e_\alpha(t,s) = e^{\alpha(t-s)}, \quad e_\alpha(t,0) = e^{\alpha t}$$

for $s, t \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a constant and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function. To compare with the discrete Pachpatte inequalities given in [5], we also give the exponential function for $\mathbb{T} = \mathbb{Z}$ as

$$e_p(t,s) = \prod_{\tau=s}^{t-1}[1+p(\tau)], \quad e_\alpha(t,s) = (1+\alpha)^{t-s}, \quad e_\alpha(t,0) = (1+\alpha)^t$$

for $s, t \in \mathbb{Z}$ with $s < t$, where $\alpha \neq -1$ is a constant and $p : \mathbb{Z} \to \mathbb{R}$ is a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbb{Z}$. Further examples of exponential functions can be found in [2, Section 2.3].

Theorem 2.3. *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

Theorem 2.4. *If $a, b, c \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}$ such that $f(t) \geq 0$ for all $a \leq t < b$, then*

$$\int_a^b f(t)\Delta t \geq 0.$$

Theorem 2.5. *Let $t_0 \in \mathbb{T}^\kappa$ and assume $k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^\kappa$ with $t > t_0$. Also assume that $k(t,\cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\left|k(\sigma(t),\tau) - k(s,\tau) - k^\Delta(t,\tau)(\sigma(t)-s)\right| \leq \varepsilon\,|\sigma(t)-s| \quad \textit{for all} \quad s \in U,$$

where k^Δ denotes the derivative of k with respect to the first variable. Then

$$g(t) := \int_{t_0}^t k(t,\tau)\Delta\tau \quad \textit{implies} \quad g^\Delta(t) = \int_{t_0}^t k^\Delta(t,\tau)\Delta\tau + k(\sigma(t),t).$$

The next four results are proved by Agarwal, Bohner and Peterson [1]. For convenience of notation we let throughout

$$t_0 \in \mathbb{T}, \quad \mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}, \quad \text{and} \quad \mathbb{T}_0^- = (-\infty, t_0] \cap \mathbb{T}.$$

Also, for a function $b \in \mathrm{C_{rd}}$ we write

$$b \geq 0 \quad \text{if} \quad b(t) \geq 0 \text{ for all } t \in \mathbb{T}.$$

Theorem 2.6 (Comparison Theorem)**.** *Suppose $u, b \in \mathrm{C_{rd}}$ and $a \in \mathcal{R}^+$. Then*

$$u^\Delta(t) \leq a(t)u(t) + b(t) \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq u(t_0)e_a(t, t_0) + \int_{t_0}^t e_a(t, \sigma(\tau))b(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

Theorem 2.7 (Gronwall's Inequality)**.** *Suppose $u, a, b \in \mathrm{C_{rd}}$ and $b \geq 0$. Then*

$$u(t) \leq a(t) + \int_{t_0}^t b(\tau)u(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq a(t) + \int_{t_0}^t a(\tau)b(\tau)e_b(t, \sigma(\tau))\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

Remark 2.8. In the next section we show that Gronwall's inequality can be stated in different forms (see Theorem 3.1, Theorem 3.6, Theorem 3.10, and Theorem 3.12).

The next two results follow from Theorem 2.7 with $a = 0$ and $a = u_0$, respectively.

Corollary 2.9. *Suppose $u, b \in \mathrm{C_{rd}}$ and $b \geq 0$. Then*

$$u(t) \leq \int_{t_0}^t u(\tau)b(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq 0 \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

Corollary 2.10. *Suppose $u, b \in \mathrm{C_{rd}}$, $u_0 \in \mathbb{R}$, and $b \geq 0$. Then*

$$u(t) \leq u_0 + \int_{t_0}^t b(\tau)u(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq u_0 e_b(t, t_0) \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

The continuous version [4, Th. 1.2.2] of Corollary 2.10 was first proved by Bellman, while the corresponding discrete version [5, Th. 1.2.2] is due to Sugiyama.

The remaining results in this section will be needed later on in this paper.

Corollary 2.11. *If $p \in \mathcal{R}^+$ and $p(t) \leq q(t)$ for all $t \in \mathbb{T}$, then*

$$e_p(t, t_0) \leq e_q(t, t_0) \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

Proof. Let $u(t) = e_p(t, t_0)$. Then

$$u^\Delta(t) = p(t)u(t) \leq q(t)u(t).$$

Now note that $q \in \mathcal{R}^+$, so using Theorem 2.6 with $a = q$ and $b = 0$, we obtain

$$e_p(t, t_0) = u(t) \leq u(t_0)e_q(t, t_0) = e_q(t, t_0)$$

for all $t \in \mathbb{T}_0$. □

Remark 2.12. The following statements hold:

(i) If $p \geq 0$, then $e_p(t,t_0) \geq e_0(t,t_0) = 1$ by Corollary 2.11 and Theorem 2.2. Therefore $e_{\ominus p}(t,t_0) \leq 1$.

(ii) If $p \geq 0$, then $e_p(\cdot,t_0)$ is nondecreasing since $e_p^{\Delta}(t,t_0) = p(t)e_p(t,t_0) \geq 0$.

3. DYNAMIC INEQUALITIES

Note that when $p = 1$ and $q = 0$ in Theorem 3.1 below, then we obtain Theorem 2.7. For $\mathbb{T} = \mathbb{R}$, see [4, Th. 1.3.4]. For $\mathbb{T} = \mathbb{Z}$, we refer to [5, Th. 1.3.1 and Th. 1.2.3]. The proof of Theorem 3.1 below is similar to the proof of Theorem 2.7 and hence is omitted.

Theorem 3.1. *Suppose* $u, a, b, p, q \in \mathrm{C_{rd}}$ *and* $b, p \geq 0$. *Then*

$$u(t) \leq a(t) + p(t)\int_{t_0}^{t}[b(\tau)u(\tau) + q(\tau)]\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq a(t) + p(t)\int_{t_0}^{t}[a(\tau)b(\tau) + q(\tau)]e_{bp}(t,\sigma(\tau))\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

The next result follows from Theorem 3.1 with $a = q = 0$.

Corollary 3.2. *Suppose* $u, b, p \in \mathrm{C_{rd}}$ *and* $b, p \geq 0$. *Then*

$$u(t) \leq p(t)\int_{t_0}^{t} u(\tau)b(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq 0 \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

Remark 3.3. The following statements hold:

(i) If $p = 1$ in Corollary 3.2, then we get Corollary 2.9.

(ii) If $q = 0$ in Theorem 3.1 and a is nondecreasing on $\mathbb{T}$, then

$$u(t) \leq a(t) + p(t)\int_{t_0}^{t} b(\tau)u(\tau)\Delta\tau \quad \text{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq a(t)\left[1 + p(t)\int_{t_0}^{t} b(\tau)e_{bp}(t,\sigma(\tau))\Delta\tau\right] \quad \text{for all} \quad t \in \mathbb{T}_0.$$

For the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, see [4, Th. 1.3.3] and [5, Th. 1.2.4], respectively.

The next result follows from Theorem 3.1. While the continuous version [4, Th. 1.5.1] of Theorem 3.4 below is due to Gamidov, its discrete version [5, Th. 1.3.2] has been established by Pachpatte.

Theorem 3.4. *Suppose* $u, a, b_i, p_i \in \mathrm{C_{rd}}$ *and* $u, b_i, p := \max_{1\leq j\leq n} p_j \geq 0$ *for* $1 \leq i \leq n$. *Then*

$$u(t) \leq a(t) + \sum_{i=1}^{n} p_i(t)\int_{t_0}^{t} b_i(\tau)u(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies with $b := \sum_{i=1}^{n} b_i$

$$u(t) \leq a(t) + p(t)\int_{t_0}^{t} a(\tau)b(\tau)e_{bp}(t,\sigma(\tau))\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

The comparison theorem motivates us to consider the following result whose proof is similar to that of Theorem 2.6.

Theorem 3.5 (Comparison Theorem)**.** *Let* $u, b \in \mathrm{C_{rd}}$ *and* $a \in \mathcal{R}^+$. *Then*

$$u^\Delta(t) \le -a(t)u^\sigma(t) + b(t) \quad \text{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \le u(t_0)e_{\ominus a}(t, t_0) + \int_{t_0}^t b(\tau)e_{\ominus a}(t, \tau)\Delta\tau \quad \text{for all} \quad t \in \mathbb{T}_0,$$

and

$$u^\Delta(t) \le -a(t)u^\sigma(t) + b(t) \quad \text{for all} \quad t \in \mathbb{T}_0^-$$

implies

$$u(t) \ge u(t_0)e_{\ominus a}(t, t_0) + \int_{t_0}^t b(\tau)e_{\ominus a}(t, \tau)\Delta\tau \quad \text{for all} \quad t \in \mathbb{T}_0^-.$$

Proof. We calculate

$$\begin{aligned}[ue_a(\cdot, t_0)]^\Delta(t) &= u^\Delta(t)e_a(t, t_0) + u^\sigma(t)a(t)e_a(t, t_0) \\ &= \left[u^\Delta(t) + a(t)u^\sigma(t)\right] e_a(t, t_0) \\ &\le b(t)e_a(t, t_0)\end{aligned}$$

for all $t \in \mathbb{T}_0$ so that

$$u(t)e_a(t, t_0) - u(t_0)e_a(t_0, t_0) \le \int_{t_0}^t e_a(\tau, t_0)b(\tau)\Delta\tau$$

for all $t \in \mathbb{T}_0$, and hence the first claim follows. For the second claim, note that the latter inequality is reversed if $t \in \mathbb{T}_0^-$. ☐

For the continuous and discrete versions of the following three theorems, we refer the reader to [4, Th. 1.3.4, Th. 1.3.3, and Th. 1.3.5] and [5, Th. 1.2.5, Th. 1.2.6, and Th. 1.2.8], respectively.

Theorem 3.6. *Suppose* $u, b, p, q \in \mathrm{C_{rd}}$ *and* $b, p \ge 0$. *Then*

$$u(t) \le a(t) + p(t)\int_t^{t_0} [b(\tau)u^\sigma(\tau) + q(\tau)]\,\Delta\tau \quad \text{for all} \quad t \in \mathbb{T}_0^-$$

implies

$$u(t) \le a(t) + p(t)\int_t^{t_0} [b(\tau)a^\sigma(\tau) + q(\tau)]\, e_{\ominus(bp^\sigma)}(t, \tau)\Delta\tau \quad \text{for all} \quad t \in \mathbb{T}_0^-.$$

Proof. Define $z(t) := -\int_t^{t_0} [b(\tau)u^\sigma(\tau) + q(\tau)]\,\Delta\tau$. Then for all $t \in \mathbb{T}_0^-$

$$\begin{aligned}z^\Delta(t) &= b(t)u^\sigma(t) + q(t) \\ &\le b(t)\left[a^\sigma(t) - p^\sigma(t)z^\sigma(t)\right] + q(t) \\ &= -b(t)p^\sigma(t)z^\sigma(t) + b(t)a^\sigma(t) + q(t).\end{aligned}$$

Since $b, p \ge 0$, we have $bp^\sigma \in \mathcal{R}^+$, and we may apply Theorem 3.5 to obtain

$$\begin{aligned}z(t) &\ge z(t_0)e_{\ominus(bp^\sigma)}(t, t_0) + \int_{t_0}^t e_{\ominus(bp^\sigma)}(t, \tau)\left[b(\tau)a^\sigma(\tau) + q(\tau)\right]\Delta\tau \\ &= -\int_t^{t_0} e_{\ominus(bp^\sigma)}(t, \tau)\left[b(\tau)a^\sigma(\tau) + q(\tau)\right]\Delta\tau\end{aligned}$$

for all $t \in \mathbb{T}_0^-$, and therefore

$$\begin{aligned} u(t) &\leq a(t) - p(t)z(t) \\ &\leq a(t) + p(t)\int_t^{t_0} e_{\ominus(bp^\sigma)}(t,\tau)\left[b(\tau)a^\sigma(\tau) + q(\tau)\right]\Delta\tau \end{aligned}$$

for all $t \in \mathbb{T}_0^-$. ☐

Theorem 3.7. *Suppose* $u, b \in \mathrm{C_{rd}}$, $b \geq 0$, *and* $a \in \mathrm{C^1_{rd}}$. *Then*

$$u(t) \leq a(t) + \int_{t_0}^t b(\tau)u(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq a(t_0)e_b(t,t_0) + \int_{t_0}^t a^\Delta(\tau)e_b(t,\sigma(\tau))\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

Proof. Define $z(t) := a(t) + \int_{t_0}^t b(\tau)u(\tau)\Delta\tau$. Then we obtain $z^\Delta(t) \leq a^\Delta(t) + b(t)z(t)$. Applying Theorem 2.6 completes the proof. ☐

Theorem 3.8. *Suppose* $\phi, u, b, p \in \mathrm{C_{rd}}$ *and* $b, p \geq 0$. *Then*

$$u(t) \geq \phi(s) - p(t)\int_s^t b(\tau)\phi^\sigma(\tau)\Delta\tau \quad \textit{for all} \quad s, t \in \mathbb{T},\ s \leq t$$

implies

$$u(t) \geq \phi(s)e_{\ominus(p(t)b)}(t,s) \quad \textit{for all} \quad s, t \in \mathbb{T},\ s \leq t.$$

Proof. Fix $t_0 \in \mathbb{T}$. Then

$$\phi(t) \leq u(t_0) + p(t_0)\int_t^{t_0} b(\tau)\phi^\sigma(\tau)\Delta\tau \quad \text{for all} \quad t \in \mathbb{T}_0^-.$$

By Theorem 3.6, we find

$$\begin{aligned} \phi(t) &\leq u(t_0) + p(t_0)\int_t^{t_0} b(\tau)u(t_0)e_{\ominus(bp(t_0))}(t,\tau)\Delta\tau \\ &= u(t_0) + u(t_0)\int_t^{t_0} b(\tau)p(t_0)e_{bp(t_0)}(\tau,t)\Delta\tau \\ &= u(t_0) + u(t_0)\left[e_{bp(t_0)}(t_0,t) - 1\right] \\ &= u(t_0)e_{bp(t_0)}(t_0,t) \end{aligned}$$

for all $t \in \mathbb{T}_0^-$ and thus

$$u(t_0) \geq \phi(t)e_{\ominus(bp(t_0))}(t_0,t) \quad \text{for all} \quad t \in \mathbb{T}_0^-.$$

Since $t_0 \in \mathbb{T}$ was arbitrary, the claim follows. ☐

Remark 3.9. The following statements hold:

(i) The continuous version of Theorem 3.8 is due to Gollwitzer.

(ii) When $\mathbb{T} = \mathbb{R}$,

$$e_{\ominus(p(t)b)}(t,s) = e^{-p(t)\int_s^t b(\tau)d\tau},$$

and when $\mathbb{T} = \mathbb{Z}$,

$$e_{\ominus(p(t)b)}(t,s) = \prod_{\tau=s}^{t-1}[1 + p(t)b(\tau)]^{-1}.$$

(iii) If $p = 1$ in Theorem 3.8, then we obtain $u(t) \geq \phi(s) e_{\ominus b}(t, s)$.

The following Volterra type inequality reduces to Theorem 2.7 if $k = p = 1$ and $q = 0$. For $\mathbb{T} = \mathbb{R}$, it is due to Norbury and Stuart and can be found in [4, Th. 1.4.3]. For $\mathbb{T} = \mathbb{Z}$, see [5, Th. 1.3.4 and Th. 1.3.3].

Theorem 3.10. *Suppose $u, a, b, p, q \in \mathrm{C}_{\mathrm{rd}}$ and $u, b, p, q \geq 0$. Let $k(t,s)$ be defined as in Theorem 2.5 such that $k(\sigma(t), t) \geq 0$ and $k^{\Delta}(t,s) \geq 0$ for $s, t \in \mathbb{T}$ with $s \leq t$. Then*

$$u(t) \leq a(t) + p(t) \int_{t_0}^{t} k(t,\tau) \left[b(\tau) u(\tau) + q(\tau) \right] \Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq a(t) + p(t) \int_{t_0}^{t} \bar{b}(\tau) e_{\bar{a}}(t, \sigma(\tau)) \Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0,$$

where

$$\bar{a}(t) = k(\sigma(t), t) b(t) p(t) + \int_{t_0}^{t} k^{\Delta}(t,\tau) b(\tau) p(\tau) \Delta\tau$$

and

$$\bar{b}(t) = k(\sigma(t), t) \left[a(t) b(t) + q(t) \right] + \int_{t_0}^{t} k^{\Delta}(t,\tau) \left[a(\tau) b(\tau) + q(\tau) \right] \Delta\tau.$$

Proof. Define $z(t) := \int_{t_0}^{t} k(t,\tau) \left[b(\tau) u(\tau) + q(\tau) \right] \Delta\tau$. Then for all $t \in \mathbb{T}_0$

$$\begin{aligned} z^{\Delta}(t) &= k(\sigma(t), t) \left[b(t) u(t) + q(t) \right] + \int_{t_0}^{t} k^{\Delta}(t,\tau) \left[b(\tau) u(\tau) + q(\tau) \right] \Delta\tau \\ &\leq \left\{ k(\sigma(t), t) b(t) p(t) + \int_{t_0}^{t} k^{\Delta}(t,\tau) b(\tau) p(\tau) \Delta\tau \right\} z(t) \\ &\qquad + k(\sigma(t), t) \left[a(t) b(t) + q(t) \right] + \int_{t_0}^{t} k^{\Delta}(t,\tau) \left[a(\tau) b(\tau) + q(\tau) \right] \Delta\tau \\ &= \bar{a}(t) z(t) + \bar{b}(t). \end{aligned}$$

In view of $\bar{a} \in \mathcal{R}^{+}$, we may apply Theorem 2.6 to obtain

$$z(t) \leq z(t_0) e_{\bar{a}}(t, t_0) + \int_{t_0}^{t} e_{\bar{a}}(t, \sigma(\tau)) \bar{b}(\tau) \Delta\tau = \int_{t_0}^{t} e_{\bar{a}}(t, \sigma(\tau)) \bar{b}(\tau) \Delta\tau$$

for all $t \in \mathbb{T}_0$. Since $u(t) \leq a(t) + p(t) z(t)$ holds for all $t \in \mathbb{T}_0$, the claim follows. □

Corollary 3.11. *In addition to the assumptions of Theorem 3.10 with $p = b = 1$ and $q = 0$, suppose that a is nondecreasing. Then*

$$u(t) \leq a(t) + \int_{t_0}^{t} k(t,\tau) u(\tau) \Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq a(t) e_{\bar{a}}(t, t_0) \quad \textit{for all} \quad t \in \mathbb{T}_0,$$

where

$$\bar{a}(t) = k(\sigma(t), t) + \int_{t_0}^{t} k^{\Delta}(t,\tau) \Delta\tau.$$

Proof. By Theorem 3.10 with

$$\begin{aligned}\bar{b}(t) &= k(\sigma(t),t)a(t) + \int_{t_0}^{t} k^{\Delta}(t,\tau)a(\tau)\Delta\tau \\ &\leq \left\{k(\sigma(t),t) + \int_{t_0}^{t} k^{\Delta}(t,\tau)\Delta\tau\right\} a(t) \\ &= \bar{a}(t)a(t),\end{aligned}$$

we obtain for all $t \in \mathbb{T}_0$

$$\begin{aligned}u(t) &\leq a(t) + \int_{t_0}^{t} \bar{b}(\tau)e_{\bar{a}}(t,\sigma(\tau))\Delta\tau \\ &\leq a(t)\left\{1 + \int_{t_0}^{t} \bar{a}(\tau)e_{\bar{a}}(t,\sigma(\tau))\Delta\tau\right\} \\ &= a(t)\left\{1 + e_{\bar{a}}(t,t_0) - e_{\bar{a}}(t,t)\right\} \\ &= a(t)e_{\bar{a}}(t,t_0),\end{aligned}$$

where we have also used Theorem 2.2 and Theorem 2.3. □

The following theorem with $k = 1$ reduces to Theorem 3.6.

Theorem 3.12. *Suppose* $u, a, b, p, q \in \mathrm{C_{rd}}$ *and* $u, b, p, q \geq 0$. *Let* $k(t,s)$ *be defined as in Theorem 2.5 such that* $k(\sigma(t),t) \geq 0$ *for all* $t \in \mathbb{T}_0^-$ *and* $k^{\Delta}(t,s) \leq 0$ *for* $s, t \in \mathbb{T}_0^-$ *with* $s \geq t$. *Then*

$$u(t) \leq a(t) + p(t)\int_{t}^{t_0} k(t,\tau)\left[b(\tau)u^{\sigma}(\tau) + q(\tau)\right]\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0^-$$

implies

$$u(t) \leq a(t) + p(t)\int_{t}^{t_0} \bar{b}(\tau)e_{\ominus\bar{a}}(t,\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0^-,$$

where

$$\bar{a}(t) = k(\sigma(t),t)b(t)p(\sigma(t)) - \int_{t}^{t_0} k^{\Delta}(t,\tau)b(\tau)p^{\sigma}(\tau)\Delta\tau$$

and

$$\bar{b}(t) = k(\sigma(\tau),t)\left[b(t)a^{\sigma}(t) + q(t)\right] - \int_{t}^{t_0} k^{\Delta}(t,\tau)\left[b(\tau)a^{\sigma}(\tau) + q(\tau)\right]\Delta\tau.$$

Proof. Define $z(t) := -\int_t^{t_0} k(t,\tau)\left[b(\tau)u^{\sigma}(\tau) + q(\tau)\right]\Delta\tau$. Then for all $t \in \mathbb{T}_0^- \setminus \{t_0\}$

$$\begin{aligned}z^{\Delta}(t) &= k(\sigma(t),t)\left[b(t)u^{\sigma}(t) + q(t)\right] - \int_{t}^{t_0} k^{\Delta}(t,\tau)\left[b(\tau)u^{\sigma}(\tau) + q(\tau)\right]\Delta\tau \\ &\leq -\left\{k(\sigma(t),t)b(t)p^{\sigma}(t) - \int_{t}^{t_0} k^{\Delta}(t,\tau)b(\tau)p^{\sigma}(\tau)\Delta\tau\right\} z^{\sigma}(t) \\ &\qquad + k(\sigma(t),t)\left[b(t)a^{\sigma}(t) + q(t)\right] - \int_{t}^{t_0} k^{\Delta}(t,\tau)\left[b(\tau)a^{\sigma}(\tau) + q(\tau)\right]\Delta\tau \\ &= -\bar{a}(t)z^{\sigma}(t) + \bar{b}(t).\end{aligned}$$

In view of $\bar{a} \in \mathcal{R}^+$, we may apply Theorem 3.5 to obtain for all $t \in \mathbb{T}_0^-$

$$z(t) \geq z(t_0)e_{\ominus\bar{a}}(t,t_0) - \int_{t}^{t_0} e_{\ominus\bar{a}}(t,\tau)\bar{b}(\tau)\Delta\tau = -\int_{t}^{t_0} e_{\ominus\bar{a}}(t,\tau)\bar{b}(\tau)\Delta\tau.$$

Since $u(t) \leq a(t) - p(t)z(t)$ for all $t \in \mathbb{T}_0^-$, the claim follows. □

Corollary 3.13. *In addition to the assumptions of Theorem 3.12 with $p = b = 1$ and $q = 0$, suppose that a is nondecreasing. Then*

$$u(t) \le a(t) + \int_t^{t_0} k(t,\tau)u^\sigma(\tau)\Delta\tau \quad \text{for all} \quad t \in \mathbb{T}_0^-$$

implies

$$u(t) \le a(t)e_{\bar a}(t_0,t) \quad \text{for all} \quad t \in \mathbb{T}_0^-,$$

where

$$\bar a(t) = k(\sigma(t),t) - \int_t^{t_0} k^\Delta(t,\tau)\Delta\tau.$$

Proof. The proof is similar to the proof of Corollary 3.11, this time using Theorem 3.12 instead of Theorem 3.10. Note also that this time we have $\bar b(t) \le \bar a(t)a^\sigma(t)$. ☐

The continuous versions of our next two results are essentially due to Greene and can be found in [4, Th. 1.6.2 and Th. 1.6.1]. Their discrete versions [5, Th. 1.3.8 and Th. 1.3.7] are proved by Pachpatte. Note that for the discrete versions, "normal" exponential functions are used, while we employ time scales exponential functions below.

Theorem 3.14. *Suppose $u, v, f, g, p, q, b_i \in \mathrm{C_{rd}}$ and $u, v, f, p, q, b_i \ge 0$, $i \in \{1,2,3,4\}$. Then*

$$u(t) \le f(t) + p(t)\left[\int_{t_0}^t b_1(\tau)u(\tau)\Delta\tau + \int_{t_0}^t e_q(\tau,t_0)b_2(\tau)v(\tau)\Delta\tau\right] \quad \text{for all} \quad t \in \mathbb{T}_0$$

and

$$v(t) \le g(t) + p(t)\left[\int_{t_0}^t e_{\ominus q}(\tau,t_0)b_3(\tau)u(\tau)\Delta\tau + \int_{t_0}^t b_4(\tau)v(\tau)\Delta\tau\right] \quad \text{for all} \quad t \in \mathbb{T}_0$$

imply

$$u(t) \le e_q(t,t_0)Q(t) \quad \text{and} \quad v(t) \le Q(t) \quad \text{for} \quad t \in \mathbb{T}_0,$$

where

$$Q(t) = f(t) + g(t) + p(t)\int_{t_0}^t [f(\tau) + g(\tau)]\, b(\tau)e_{bp}(t,\sigma(\tau))\Delta\tau$$

with

$$b(t) = \max\{b_1(t) + b_3(t),\ b_2(t) + b_4(t)\}.$$

Proof. We define $w(t) = e_{\ominus q}(t,t_0)u(t) + v(t)$. By Remark 2.12 we obtain for all $t \in \mathbb{T}_0$

$$\begin{aligned} w(t) &\le e_{\ominus q}(t,t_0)f(t) + g(t) \\ &\qquad + p(t)\int_{t_0}^t \Big\{[e_{\ominus q}(t,t_0)b_1(\tau) + e_{\ominus q}(\tau,t_0)b_3(\tau)]\,u(\tau) \\ &\qquad\qquad + [e_{\ominus q}(t,\tau)b_2(\tau) + b_4(\tau)]\,v(\tau)\Big\}\Delta\tau \\ &\le e_{\ominus q}(t,t_0)f(t) + g(t) \\ &\qquad + p(t)\int_{t_0}^t \Big\{e_{\ominus q}(\tau,t_0)\,[b_1(\tau) + b_3(\tau)]\,u(\tau) + [b_2(\tau) + b_4(\tau)]\,v(\tau)\Big\}\Delta\tau \\ &\le e_{\ominus q}(t,t_0)f(t) + g(t) + p(t)\int_{t_0}^t b(\tau)w(\tau)\Delta\tau \\ &\le f(t) + g(t) + p(t)\int_{t_0}^t b(\tau)w(\tau)\Delta\tau. \end{aligned}$$

Now $b, p \geq 0$ so that Theorem 3.1 yields for all $t \in \mathbb{T}_0$

$$w(t) \leq f(t) + g(t) + p(t) \int_{t_0}^{t} [f(\tau) + g(\tau)]\, b(\tau) e_{bp}(t, \sigma(\tau)) \Delta\tau = Q(t).$$

Hence

$$u(t) = e_q(t, t_0) w(t) - e_q(t, t_0) v(t) \leq e_q(t, t_0) Q(t)$$

and

$$v(t) = w(t) - e_{\ominus q}(t, t_0) u(t) \leq Q(t)$$

for all $t \in \mathbb{T}_0$. $\square$

Corollary 3.15. *In addition to the assumptions of Theorem 3.14 with* $f(t) \equiv c_1$, $g(t) \equiv c_2$, *and* $p(t) \equiv 1$, *suppose* $c_1, c_2 \in \mathbb{R}$. *Then*

$$u(t) \leq c_1 + \int_{t_0}^{t} [b_1(\tau) u(\tau) + e_q(\tau, t_0) b_2(\tau) v(\tau)]\, \Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

and

$$v(t) \leq c_2 + \int_{t_0}^{t} [e_{\ominus q}(\tau, t_0) b_3(\tau) u(\tau) + b_4(\tau) v(\tau)]\, \Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0$$

imply with $c = c_1 + c_2$

$$u(t) \leq c e_{b \oplus q}(t, t_0) \quad \textit{and} \quad v(t) \leq c e_b(t, t_0) \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

Proof. In this case we find, using Theorem 2.2 and Theorem 2.3, that

$$Q(t) = c + \int_{t_0}^{t} c b(\tau) e_b(t, \sigma(\tau)) \Delta\tau = c e_b(t, t_0).$$

Hence $u(t) \leq e_q(t, t_0) c e_b(t, t_0) = c e_{b \oplus q}(t, t_0)$ and $v(t) \leq c e_b(t, t_0)$ for all $t \in \mathbb{T}_0$ by Theorem 3.14. $\square$

4. FURTHER DYNAMIC INEQUALITIES

Our first few results are, even for the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, more general than any result given in [4, 5].

Theorem 4.1. *Suppose* $u, a, b, c, d, p, w \in \mathrm{C_{rd}}$ *such that* $u, a, b, c, p, w \geq 0$. *Then*

$$u(t) \leq w(t) + p(t) \int_{t_0}^{t} \left\{ [a(\tau) + b(\tau)] u(\tau) + b(\tau) p(\tau) \int_{t_0}^{\tau} [c(s) u(s) + d(s)] \Delta s \right\} \Delta\tau$$

for all $t \in \mathbb{T}_0$ *implies*

$$\begin{aligned} u(t) \leq w(t) + p(t) \int_{t_0}^{t} &[a(\tau) + b(\tau)] \\ &\times \left\{ w(\tau) + p(\tau) \int_{t_0}^{\tau} e_{p(a+b+c)}(\tau, \sigma(s)) [(a + b + c) w + d](s) \Delta s \right\} \Delta\tau \end{aligned}$$

for all $t \in \mathbb{T}_0$.

Proof. Define

$$z(t) := \int_{t_0}^{t} \left\{ [a(\tau) + b(\tau)] u(\tau) + b(\tau) p(\tau) \int_{t_0}^{\tau} [c(s) u(s) + d(s)] \Delta s \right\} \Delta\tau$$

and

$$r(t) := z(t) + \int_{t_0}^{t} \left\{ c(\tau) \left[w(\tau) + p(\tau) z(\tau) \right] + d(\tau) \right\} \Delta\tau.$$

Then we have $u(t) \le w(t) + p(t)z(t)$, $z(t) \le r(t)$, and

$$\begin{aligned} z^{\Delta}(t) &= [a(t)+b(t)]u(t) + b(t)p(t)\int_{t_0}^{t}[c(\tau)u(\tau)+d(\tau)]\Delta\tau \\ &\le [a(t)+b(t)]w(t) + a(t)p(t)z(t) + b(t)p(t)r(t) \\ &\le [a(t)+b(t)][w(t)+p(t)r(t)] \end{aligned}$$

and therefore

$$\begin{aligned} r^{\Delta}(t) &= z^{\Delta}(t) + c(t)[w(t)+p(t)z(t)] + d(t) \\ &\le [a(t)+b(t)][w(t)+p(t)r(t)] + c(t)[w(t)+p(t)r(t)] + d(t) \\ &= [(a+b+c)p](t)r(t) + [(a+b+c)w+d](t). \end{aligned}$$

By Theorem 2.6 we find

$$r(t) \le \int_{t_0}^{t} e_{(a+b+c)p}(t,\sigma(\tau))[(a+b+c)w+d](\tau)\Delta\tau$$

since $r(t_0)=0$. Using this in $z^{\Delta}(t) \le [a(t)+b(t)][w(t)+p(t)r(t)]$ and integrating the resulting inequality completes the proof. □

In certain cases it will be possible to further evaluate the integral occurring in Theorem 4.1. To this end we present the following useful auxiliary result, which is an extension of Theorem 2.3.

Theorem 4.2. *Suppose $f : \mathbb{T} \to \mathbb{R}$ is differentiable. If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$\int_a^b f(t)p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a)f(a) - e_p(c,b)f(b) + \int_a^b e_p(c,\sigma(t))f^{\Delta}(t)\Delta t.$$

Proof. We use Theorem 2.2 and integration by parts:

$$\begin{aligned} \int_a^b e_p(c,\sigma(t))p(t)f(t)\Delta t &= \int_a^b e_{\ominus p}(\sigma(t),c)p(t)f(t)\Delta t \\ &= \int_a^b \frac{1}{1+\mu(t)p(t)} e_{\ominus p}(t,c)p(t)f(t)\Delta t \\ &= -\int_a^b (\ominus p)(t)e_{\ominus p}(t,c)f(t)\Delta t \\ &= -\int_a^b e_{\ominus p}^{\Delta}(t,c)f(t)\Delta t \\ &= -\left\{ e_{\ominus p}(b,c)f(b) - e_{\ominus p}(a,c)f(a) - \int_a^b e_{\ominus p}(\sigma(t),c)f^{\Delta}(t)\Delta t \right\} \\ &= e_p(c,a)f(a) - e_p(c,b)f(b) + \int_a^b e_p(c,\sigma(t))f^{\Delta}(t)\Delta t, \end{aligned}$$

which completes the proof. □

Using Theorem 4.2, we now present the following result.

Theorem 4.3. *Suppose $u, a, b, c, d, p, w \in \mathrm{C_{rd}}$ such that $u, a, b, c, p, w \ge 0$. Furthermore assume that w is differentiable and that p is nonincreasing. Then*

$$u(t) \le w(t) + p(t)\int_{t_0}^{t}\left\{[a(\tau)+b(\tau)]u(\tau) + b(\tau)p(\tau)\int_{t_0}^{\tau}[c(s)u(s)+d(s)]\Delta s\right\}\Delta\tau$$

for all $t \in \mathbb{T}_0$ *implies*

$$u(t) \le w(t) + p(t) \int_{t_0}^{t} [a(\tau) + b(\tau)] \times \left\{ e_{(a+b+c)p}(\tau, t_0) w(t_0) + \int_{t_0}^{\tau} e_{p(a+b+c)}(\tau, \sigma(s))[w^\Delta(s) + p(\tau) d(s)] \Delta s \right\} \Delta\tau$$

for all $t \in \mathbb{T}_0$.

Proof. Using Theorem 4.2 and the fact that p is nonincreasing, we employ Theorem 4.1 to find

$$u(t) \le w(t) + p(t) \int_{t_0}^{t} [a(\tau) + b(\tau)] z(\tau) \Delta\tau,$$

where

$$\begin{aligned} z(t) &:= w(t) + p(t) \int_{t_0}^{t} e_{(a+b+c)p}(t, \sigma(\tau))[(a+b+c)w + d](\tau) \Delta\tau \\ &\le w(t) + p(t) \int_{t_0}^{t} e_{(a+b+c)p}(t, \sigma(\tau)) d(\tau) \Delta\tau \\ &\quad + \int_{t_0}^{t} e_{(a+b+c)p}(t, \sigma(\tau))(a+b+c)(\tau) p(\tau) w(\tau) \Delta\tau \\ &= p(t) \int_{t_0}^{t} e_{(a+b+c)p}(t, \sigma(\tau)) d(\tau) \Delta\tau + e_{(a+b+c)p}(t, t_0) w(t_0) \\ &\quad + \int_{t_0}^{t} e_{(a+b+c)p}(t, \sigma(\tau)) w^\Delta(\tau) \Delta\tau, \end{aligned}$$

and this completes the proof. □

Corollary 4.4. *Under the same assumptions of Theorem 4.3 we can conclude*

$$u(t) \le e_{(a+b+c)p}(t, t_0) w(t_0) + \int_{t_0}^{t} \left\{ w^\Delta(\tau) + [a(\tau) + b(\tau)] \int_{t_0}^{\tau} e_{(a+b+c)p}(\tau, \sigma(s))[w^\Delta(s) + p(\tau) d(s)] \Delta s \right\} \Delta\tau$$

for all $t \in \mathbb{T}_0$.

Proof. The estimate

$$p(t) \int_{t_0}^{t} [a(\tau) + b(\tau)] e_{(a+b+c)p}(\tau, t_0) \Delta\tau \le \int_{t_0}^{t} [a(\tau) + b(\tau) + c(\tau)] p(\tau) e_{(a+b+c)p}(\tau, t_0) \Delta\tau$$

completes the proof as the latter integral may be evaluated directly. □

The following two results (for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, see [4, Th. 1.7.2 (iv) and Th. 1.7.4] and [5, Th. 1.4.4 and Th. 1.4.2], respectively) are immediate consequences of Theorem 4.1.

Corollary 4.5. *Suppose* $u, b, c, p, w \in \mathrm{C_{rd}}$ *such that* $u, b, c, p, w \ge 0$. *Then*

$$u(t) \le w(t) + p(t) \int_{t_0}^{t} b(\tau) \left\{ u(\tau) + p(\tau) \int_{t_0}^{\tau} c(s) u(s) \Delta s \right\} \Delta\tau$$

for all $t \in \mathbb{T}_0$ *implies*

$$u(t) \le w(t) + p(t) \int_{t_0}^{t} b(\tau) \left\{ w(\tau) + p(\tau) \int_{t_0}^{\tau} e_{p(b+c)}(\tau, \sigma(s))(b+c)(s) w(s) \Delta s \right\} \Delta\tau$$

for all $t \in \mathbb{T}_0$.

Proof. Put $a = d = 0$ in Theorem 4.1. □

Corollary 4.6. *If we suppose in addition to the assumptions of Corollary 4.5 that p is nonincreasing and w is nondecreasing, then*

$$u(t) \le w(t)\left[1 + p(t)\int_{t_0}^{t} b(\tau)e_{(b+c)p}(\tau, t_0)\Delta\tau\right] \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

Proof. We have

$$p(t)\int_{t_0}^{t} e_{(b+c)p}(t, \sigma(\tau))w(\tau)(b+c)(\tau)\Delta\tau \le w(t)\int_{t_0}^{t} e_{(b+c)p}(t, \sigma(\tau))p(\tau)(b+c)(\tau)\Delta\tau,$$

and the latter integral can be directly evaluated using Theorem 2.3, hence yielding the result. □

Remark 4.7. The right-hand side of the inequality in Corollary 4.6 can be further estimated and then evaluated by Theorem 2.3 so that the statement of Corollary 4.6 can be replaced by

$$u(t) \le w(t)e_{(b+c)p}(t, t_0) \quad \text{for all} \quad t \in \mathbb{T}_0.$$

In the following theorem we state some easy consequences of Theorem 4.3. See [4, Th. 1.7.2] for $\mathbb{T} = \mathbb{R}$ and [5, Th. 1.4.6] for $\mathbb{T} = \mathbb{Z}$.

Theorem 4.8. *Suppose $u, a, b, c, d, q \in \mathrm{C}_{\mathrm{rd}}$ and $u, a, b, c, q \ge 0$. Let u_0 be a nonnegative constant. Then*

(i) $$u(t) \le u_0 + \int_{t_0}^{t} b(\tau)\left[u(\tau) + q(\tau) + \int_{t_0}^{\tau} c(s)u(s)\Delta s\right]\Delta\tau, \quad t \in \mathbb{T}_0$$

implies

$$u(t) \le u_0 + \int_{t_0}^{t} Q(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0,$$

where

$$Q(t) = b(t)\left\{u_0 e_{b+c}(t, t_0) + \int_{t_0}^{t} b(\tau)q(\tau)e_{b+c}(t, \sigma(\tau))\Delta\tau + p(t)\right\};$$

(ii) $$u(t) \le u_0 + \int_{t_0}^{t} b(\tau)\left\{u(\tau) + \int_{t_0}^{\tau}[c(s)u(s) + d(s)]\Delta s\right\}\Delta\tau, \quad t \in \mathbb{T}_0$$

implies

$$u(t) \le u_0 + \int_{t_0}^{t} b(\tau)\left[u_0 e_{b+c}(\tau, t_0) + \int_{t_0}^{\tau} e_{b+c}(\tau, \sigma(s))d(s)\Delta s\right]\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0;$$

(iii) $$u(t) \le u_0 + \int_{t_0}^{t} a(s)u(s)\Delta s + \int_{t_0}^{t} b(s)\left[u(s) + \int_{t_0}^{s} c(\tau)u(\tau)\Delta\tau\right]\Delta s, \quad t \in \mathbb{T}_0$$

implies

$$u(t) \le u_0 e_{a+b+c}(t, t_0) \quad \textit{for all} \quad t \in \mathbb{T}_0.$$

Proof. In each case we use Theorem 4.3, for (i) with

$$a = d = 0, \quad p = 1, \quad \text{and} \quad w(t) = u_0 + \int_{t_0}^{t} b(\tau)q(\tau)\Delta\tau,$$

for (ii) with

$$a = 0, \quad p = 1, \quad \text{and} \quad w = u_0,$$

and for (iii) with

$$d = 0, \quad p = 1, \quad \text{and} \quad w = u_0.$$

In (i) and (ii), the claim follows directly, while the calculation

$$\begin{aligned} u(t) &\leq u_0 \left\{ 1 + \int_{t_0}^{t} [a(\tau) + b(\tau)] e_{a+b+c}(\tau, t_0) \Delta\tau \right\} \\ &\leq u_0 \left\{ 1 + \int_{t_0}^{t} [a(\tau) + b(\tau) + c(\tau)] e_{a+b+c}(\tau, t_0) \Delta\tau \right\} \\ &= u_0 e_{a+b+c}(t, t_0) \end{aligned}$$

completes the proof of statement (iii). □

For further reference we state the following corollary, whose continuous and discrete versions can be found in [4, Th. 1.7.1] and [5, Th. 1.4.1], respectively.

Corollary 4.9. *Suppose* $u, b, c \in \mathrm{C_{rd}}$ *and* $u, b, c \geq 0$. *Let* u_0 *be a nonnegative constant. Then*

$$u(t) \leq u_0 + \int_{t_0}^{t} b(\tau) \left[u(\tau) + \int_{t_0}^{\tau} c(s) u(s) \Delta s \right] \Delta\tau \quad \text{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq u_0 \left\{ 1 + \int_{t_0}^{t} b(\tau) e_{b+c}(\tau, t_0) \Delta\tau \right\} \quad \text{for all} \quad t \in \mathbb{T}_0.$$

Proof. This follows from Theorem 4.8 (i) with $q = 0$ or from Theorem 4.8 (iii) with $a = 0$. □

Remark 4.10. As in Remark 4.7, we can replace the conclusion in Corollary 4.9 by

$$u(t) \leq u_0 e_{b+c}(t, t_0) \quad \text{for all} \quad t \in \mathbb{T}_0.$$

For $\mathbb{T} = \mathbb{Z}$ in the following result, we refer to [5, Th. 1.4.3].

Theorem 4.11. *Suppose* $u, a, b, c \in \mathrm{C_{rd}}$, $a > 0$, *and* $u, b, c \geq 0$. *Let* u_0 *be a nonnegative constant. Then*

$$u(t) \leq a(t) \left\{ u_0 + \int_{t_0}^{t} b(\tau) \left[u(\tau) + \int_{t_0}^{\tau} c(s) u(s) \Delta s \right] \Delta\tau \right\} \quad \text{for all} \quad t \in \mathbb{T}_0$$

implies

(i) $$u(t) \leq u_0 a(t) \left\{ 1 + \int_{t_0}^{t} b(\tau) e_{b+c}(\tau, t_0) \Delta\tau \right\} \quad \text{for all} \quad t \in \mathbb{T}_0$$

if $0 < a(t) \leq 1$ *holds for all* $t \in \mathbb{T}$, *and*

(ii) $$u(t) \leq a(t) u_0 \left\{ 1 + \int_{t_0}^{t} a(\tau) b(\tau) e_{a(b+c)}(\tau, t_0) \Delta\tau \right\} \quad \text{for all} \quad t \in \mathbb{T}_0$$

if $a(t) \geq 1$ *holds for all* $t \in \mathbb{T}$.

Proof. Since $a(t) > 0$, we have

$$\frac{u(t)}{a(t)} \leq u_0 + \int_{t_0}^{t} b(\tau) \left[u(\tau) + \int_{t_0}^{\tau} c(s) u(s) \Delta s \right] \Delta\tau.$$

First we assume that $0 < a(t) \leq 1$ holds for all $t \in \mathbb{T}$. Then

$$\frac{u(t)}{a(t)} \leq u_0 + \int_{t_0}^{t} b(\tau) \left[\frac{u(\tau)}{a(\tau)} + \int_{t_0}^{\tau} c(s) \frac{u(s)}{a(s)} \Delta s \right] \Delta\tau.$$

We apply Corollary 4.9 to obtain

$$\frac{u(t)}{a(t)} \leq u_0 \left\{ 1 + \int_{t_0}^{t} b(\tau) e_{b+c}(\tau, t_0) \Delta\tau \right\} \quad \text{for all} \quad t \in \mathbb{T}_0$$

so that (i) follows. Next we assume that $a(t) \geq 1$ holds for all $t \in \mathbb{T}$. Then

$$\begin{aligned} \frac{u(t)}{a(t)} &\leq u_0 + \int_{t_0}^{t} b(\tau) \left[u(\tau) + \int_{t_0}^{\tau} c(s) u(s) \Delta s \right] \Delta\tau \\ &= u_0 + \int_{t_0}^{t} b(\tau) \left[\frac{u(\tau)}{a(\tau)} a(\tau) + \int_{t_0}^{\tau} c(s) \frac{u(s)}{a(s)} a(s) \Delta s \right] \Delta\tau \\ &\leq u_0 + \int_{t_0}^{t} b(\tau) \left[\frac{u(\tau)}{a(\tau)} a(\tau) + a(\tau) \int_{t_0}^{\tau} c(s) \frac{u(s)}{a(s)} a(s) \Delta s \right] \Delta\tau \\ &= u_0 + \int_{t_0}^{t} b(\tau) a(\tau) \left[\frac{u(\tau)}{a(\tau)} + \int_{t_0}^{\tau} c(s) \frac{u(s)}{a(s)} a(s) \Delta s \right] \Delta\tau. \end{aligned}$$

We again apply Corollary 4.9 to obtain

$$\frac{u(t)}{a(t)} \leq u_0 \left\{ 1 + \int_{t_0}^{t} a(\tau) b(\tau) e_{a(b+c)}(\tau, t_0) \Delta\tau \right\} \quad \text{for all} \quad t \in \mathbb{T}_0$$

so that (ii) follows. Hence the proof is complete. □

Remark 4.12. If $c = 0$ in the above theorem with $a \geq 0$ and $u_0 \in \mathbb{R}$, then we can use Theorem 3.1, Theorem 2.3, and Theorem 2.2 to conclude

$$u(t) \leq u_0 a(t) e_{ab}(t, t_0) \quad \text{for all} \quad t \in \mathbb{T}_0.$$

This improves [5, Th. 1.2.7] (Ma's inequality) for the case $\mathbb{T} = \mathbb{Z}$, where under the assumptions $a > 0$ and $u_0 \geq 0$ a similar result as in Theorem 4.11 is shown.

Remark 4.13. If $a = 1$ in Theorem 4.11, then we get Corollary 2.10.

In [4, Th. 1.7.5] for $\mathbb{T} = \mathbb{R}$ and in [5, Th. 1.4.8] for $\mathbb{T} = \mathbb{Z}$, a and b are assumed to be positive to get the result which we give next.

Theorem 4.14. *Suppose* $u, a, b, c, p \in \mathrm{C_{rd}}$ *and* $u, a, b, c, p \geq 0$. *Let* u_0 *be a nonnegative constant. Then*

(i) $$u(t) \leq u_0 + \int_{t_0}^{t} a(s) \left[p(s) + \int_{t_0}^{s} c(\tau) u(\tau) \Delta\tau \right] \Delta s, \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq \left\{ u_0 + \int_{t_0}^{t} a(s) p(s) \Delta s \right\} e_{aC}(t, t_0) \quad \textit{for all} \quad t \in \mathbb{T}_0,$$

where

$$C(t) = \int_{t_0}^{t} c(\tau) \Delta\tau;$$

(ii) $$u(t) \leq u_0 + \int_{t_0}^{t} a(s) \left[p(s) + \int_{t_0}^{s} b(\tau) \left(\int_{t_0}^{\tau} c(\gamma) u(\gamma) \Delta\gamma \right) \Delta\tau \right] \Delta s, \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq \left\{ u_0 + \int_{t_0}^{t} a(\tau) p(\tau) \Delta\tau \right\} e_{a\xi}(t, t_0) \quad \textit{for all} \quad t \in \mathbb{T}_0,$$

where

$$\xi(t) = \int_{t_0}^{t} b(\tau) \left(\int_{t_0}^{\tau} c(\gamma)\Delta\gamma \right) \Delta\tau.$$

Proof. We only prove (i) here since the proof of (ii) can be completed by following the same ideas as in the proof of (i) given below with suitable changes. First we define

$$z(t) := u_0 + \int_{t_0}^{t} a(s) \left[p(s) + \int_{t_0}^{s} c(\tau)u(\tau)\Delta\tau \right] \Delta s.$$

Then $z(t_0) = u_0$, $u(t) \le z(t)$, and

$$z^{\Delta}(t) = a(t) \left\{ p(t) + \int_{t_0}^{t} c(\tau)u(\tau)\Delta\tau \right\} \ge 0.$$

This implies that z is nondecreasing. Therefore

$$z^{\Delta}(t) \le a(t)p(t) + a(t) \int_{t_0}^{t} c(\tau)z(\tau)\Delta\tau \le a(t)p(t) + a(t)C(t)z(t).$$

By Theorem 2.6 we obtain

$$z(t) \le z(t_0)e_{aC}(t,t_0) + \int_{t_0}^{t} e_{aC}(t,\sigma(\tau))a(\tau)p(\tau)\Delta\tau.$$

Since $u(t) \le z(t)$, we get

$$u(t) \le u_0 e_{aC}(t,t_0) + \int_{t_0}^{t} e_{aC}(t,\sigma(\tau))a(\tau)p(\tau)\Delta\tau.$$

By Theorem 2.2 and Remark 2.12 we get the desired result. □

Our next result slightly differs from the corresponding results for $\mathbb{T} = \mathbb{R}$ as given in [4, Th. 1.7.3] and for $\mathbb{T} = \mathbb{Z}$ as given in [5, Th. 1.4.7].

Theorem 4.15. *Suppose* $u, b, c, q \in \mathrm{C_{rd}}$ *and* $u, b, c, q \ge 0$. *Let* u_0 *be a nonnegative constant. Then*

$$u(t) \le u_0 + \int_{t_0}^{t} b(s) \left\{ u(s) + \int_{t_0}^{s} c(\tau) \left[u(\tau) + \int_{t_0}^{\tau} q(\gamma)u(\gamma)\Delta\gamma \right] \Delta\tau \right\} \Delta s, \quad t \in \mathbb{T}_0$$

implies

$$u(t) \le u_0 e_{\phi}(t,t_0) \quad \textit{for all} \quad t \in \mathbb{T}_0,$$

where

$$\phi(t) = b(t) + c(t) \left\{ 1 + \int_{t_0}^{t} q(\gamma)\Delta\gamma \right\}.$$

Proof. We define

$$z(t) := u_0 + \int_{t_0}^{t} b(s) \left\{ u(s) + \int_{t_0}^{s} c(\tau) \left[u(\tau) + \int_{t_0}^{\tau} q(\gamma)u(\gamma)\Delta\gamma \right] \Delta\tau \right\} \Delta s$$

and

$$r(t) := z(t) + \int_{t_0}^{t} c(\tau) \left[z(\tau) + \int_{t_0}^{\tau} q(\gamma)z(\gamma)\Delta\gamma \right] \Delta\tau.$$

We observe that z is nondecreasing and use Theorem 2.6 to get the desired result. □

The final result in this section is more general than Theorem 3.8. For $\mathbb{T} = \mathbb{R}$, see [4, Th. 1.7.6]. For $\mathbb{T} = \mathbb{Z}$, see [5, Th. 1.4.5].

Theorem 4.16. *Suppose* $\phi, u, p, b, c \in \mathrm{C_{rd}}$ *and* $\phi, u, b, c, p \geq 0$. *Then*

$$u(t) \geq \phi(s) - p(t) \int_s^t b(\tau) \left[\phi^\sigma(\tau) + \int_{\sigma(\tau)}^t c(\gamma)\phi^\sigma(\gamma)\Delta\gamma \right] \Delta\tau \quad \textit{for all} \quad s, t \in \mathbb{T},\ s \leq t$$

implies

$$u(t) \geq \left\{ \phi(s) + \int_s^t c(\gamma)\phi^\sigma(\gamma)\Delta\gamma \right\} e_{\ominus(p(t)b+c)}(t, s) \quad \textit{for all} \quad s, t \in \mathbb{T},\ s \leq t.$$

Proof. By assumption we have

$$\phi(s) \leq u(t) + p(t) \int_s^t b(\tau) \left[\phi^\sigma(\tau) + \int_{\sigma(\tau)}^t c(\gamma)\phi^\sigma(\gamma)\Delta\gamma \right] \Delta\tau.$$

Define

$$z(s) := -u(t) - p(t) \int_s^t b(\tau) \left[\phi^\sigma(\tau) + \int_{\sigma(\tau)}^t c(\gamma)\phi^\sigma(\gamma)\Delta\gamma \right] \Delta\tau.$$

This implies that $\phi(s) \leq -z(s)$, $z(t) = -u(t)$, and

$$\begin{aligned} z^\Delta(s) &= p(t)b(s) \left\{ \phi^\sigma(s) + \int_{\sigma(s)}^t c(\gamma)\phi^\sigma(\gamma)\Delta\gamma \right\} \\ &\leq -p(t)b(s) \left\{ z^\sigma(s) + \int_{\sigma(s)}^t c(\gamma)z^\sigma(\gamma)\Delta\gamma \right\}. \end{aligned}$$

Define

$$r(s) := z(s) + \int_s^t c(\gamma)z^\sigma(\gamma)\Delta\gamma.$$

Then we get $r(t) = z(t)$, $z^\Delta(s) \leq -p(t)b(s)r^\sigma(s)$, and $r(s) \leq z(s)$. Thus

$$r^\Delta(s) = z^\Delta(s) - c(s)z^\sigma(s) \leq -\left[p(t)b(s) + c(s)\right] r^\sigma(s).$$

By Theorem 3.5 we obtain

$$r(s) \geq r(t)e_{\ominus(p(t)b+c)}(s, t) = -u(t)e_{\ominus(p(t)b+c)}(s, t),$$

and therefore

$$z(s) + \int_s^t c(\gamma)z^\sigma(\gamma)\Delta\gamma \geq -u(t)e_{\ominus(p(t)b+c)}(s, t).$$

Since $z(s) \leq -\phi(s)$, we get

$$-\phi(s) - \int_s^t c(\gamma)\phi^\sigma(\gamma)\Delta\gamma \geq -u(t)e_{\ominus(p(t)b+c)}(s, t).$$

This gives the desired result. □

Remark 4.17. The following statements hold:

(i) If $c = 0$ in Theorem 4.16, then we obtain Theorem 3.8.

(ii) In [4, 5],

$$u(t) \geq \phi(s) \left\{ 1 + p(t) \int_s^t b(\tau)e_{p(t)b+c}(t, \sigma(\tau))\Delta\tau \right\}^{-1}$$

is given as a result of Theorem 4.16 instead. With only minor alterations of our proof presented above, the corresponding claim can be verified, too.

5. INEQUALITIES INVOLVING DELTA DERIVATIVES

In this section we establish some inequalities involving functions and their delta derivatives. We give the estimates on the delta derivative of functions and consequently on the functions themselves. Continuous and discrete versions (all due to Pachpatte) of the four theorems presented in this section may be found in [4, Th. 1.8.1, Th. 1.8.2, and Th. 1.8.3] and [5, Th. 1.5.1, Th. 1.5.2, Th. 1.5.3, and Th. 1.5.4], respectively.

Theorem 5.1. *Suppose* $u, u^\Delta, a, b, c \in \mathrm{C_{rd}}$ *and* $u, u^\Delta, a, b, c \geq 0$. *Then*

(i) $$u^\Delta(t) \leq a(t) + b(t)\int_{t_0}^t c(s)\left[u(s)+u^\Delta(s)\right]\Delta s \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies, provided that $b(t) \geq 1$ *holds for all* $t \in \mathbb{T}$,

$$u^\Delta(t) \leq a(t) + b(t)\int_{t_0}^t c(s)\left[\bar{a}(s)+b(s)\bar{b}(s)\right]\Delta s \quad \textit{for all} \quad t \in \mathbb{T}_0,$$

where

$$\bar{a}(t) = u(t_0) + a(t) + \int_{t_0}^t a(s)\Delta s \quad \textit{and} \quad \bar{b}(t) = \int_{t_0}^t c(s)e_{b(c+1)}(t,\sigma(s))\bar{a}(s)\Delta s;$$

(ii) $$u^\Delta(t) \leq a(t) + b(t)\left\{u(t) + \int_{t_0}^t c(s)\left[u(s)+u^\Delta(s)\right]\Delta s\right\} \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u^\Delta(t) \leq a(t) + b(t)\left\{u(t_0)e_{b+c+bc}(t,t_0) + \int_{t_0}^t e_{b+c+bc}(t,\sigma(\tau))a(\tau)[c(\tau)+1]\Delta\tau\right\}$$

for all $t \in \mathbb{T}_0$.

Proof. In order to prove (i) we define $z(t) := \int_{t_0}^t c(s)\left[u(s)+u^\Delta(s)\right]\Delta s$. Then we have

$$u^\Delta(t) \leq a(t) + b(t)z(t).$$

Integrating both sides of this inequality from t_0 to t provides

$$u(t) \leq u(t_0) + \int_{t_0}^t [a(s)+b(s)z(s)]\Delta s.$$

This implies that

$$z^\Delta(t) \leq c(t)\left\{\bar{a}(t) + b(t)\left[z(t) + \int_{t_0}^t b(s)z(s)\Delta s\right]\right\}.$$

Now we define $r(t) := z(t) + \int_{t_0}^t b(s)z(s)\Delta s$ to obtain

$$r^\Delta(t) \leq b(t)\left[c(t)+1\right]r(t) + c(t)\bar{a}(t).$$

By Theorem 2.6, we get

$$r(t) \leq \int_{t_0}^t e_{b(c+1)}(t,\sigma(\tau))c(\tau)\bar{a}(\tau)\Delta\tau = \bar{b}(t)$$

since $r(t_0) = z(t_0) = 0$. This implies that

$$z^\Delta(t) \leq c(t)\left[\bar{a}(t) + b(t)\bar{b}(t)\right].$$

Upon integrating both sides of the latter inequality, we arrive at

$$z(t) \le \int_{t_0}^{t} c(\tau) \left[\bar{a}(\tau) + b(\tau)\bar{b}(\tau)\right] \Delta\tau.$$

Since $u^\Delta \le a + bz$ holds, we get the desired result. The proof of (ii) is shorter than the first part: First we define $z(t) := u(t) + \int_{t_0}^{t} c(s) \left[u(s) + u^\Delta(s)\right] \Delta s$. Then one can get easily that

$$z^\Delta(t) \le [b(t) + c(t) + c(t)b(t)]\, z(t) + a(t)\, [c(t) + 1]\,.$$

Applying Theorem 2.6 and $u^\Delta \le a + bz$ completes the proof. ☐

Theorem 5.2. *Suppose* $u, u^\Delta, a, b, c, p \in \mathrm{C_{rd}}$ *and* $u, u^\Delta, a, b, c, p \ge 0$. *Then*

(i) $$u^\Delta(t) \le a(t)u(t) + b(t)\left\{p(t) + u(t) + \int_{t_0}^{t} c(s)u(s)\Delta s\right\} \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \le u(t_0)e_a(t, t_0) + \int_{t_0}^{t} e_a(t, \sigma(\tau))b(\tau)\, [p(\tau) + \bar{a}(\tau)]\, \Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0,$$

where

$$\bar{a}(t) = u(t_0)e_{a+b+c}(t, t_0) + \int_{t_0}^{t} e_{a+b+c}(t, \sigma(\tau))b(\tau)p(\tau)\Delta\tau;$$

(ii) $$u^\Delta(t) \le a(t)u(t) + b(t)\left\{p(t) + u(t) + \int_{t_0}^{t} c(s)u^\Delta(s)\Delta s\right\} \quad \textit{for all} \quad t \in \mathbb{T}_0$$

implies

$$u(t) \le u(t_0)e_a(t, t_0) + \int_{t_0}^{t} e_a(t, \sigma(\tau)) \left[p(\tau) + \bar{b}(\tau)\right] b(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}_0,$$

where

$$\bar{b}(t) = u(t_0)e_{(1+c)(a+b)}(t, t_0) + \int_{t_0}^{t} e_{(1+c)(a+b)}(t, \sigma(\tau))[1 + c(\tau)]b(\tau)p(\tau)\Delta\tau.$$

Proof. To prove (i) we define $z(t) = u(t) + \int_{t_0}^{t} c(s)u(s)\Delta s$. Then we obtain

$$z(t) \ge u(t), \quad z(t_0) = u(t_0), \quad \text{and} \quad u^\Delta(t) \le a(t)z(t) + b(t)\, [p(t) + z(t)]\,.$$

This implies that

$$z^\Delta(t) \le [a(t) + b(t) + c(t)]\, z(t) + b(t)p(t).$$

By Theorem 2.6, $z(t) \le \bar{a}(t)$. Hence $u^\Delta(t) \le a(t)u(t) + b(t)\, [p(t) + \bar{a}(t)]$. Applying again Theorem 2.6 gives us the desired result. Finally, in order to prove (ii), we define $z(t) = u(t) + \int_{t_0}^{t} c(s)u^\Delta(s)\Delta s$ and apply Theorem 2.6 twice. ☐

For $\mathbb{T} = \mathbb{Z}$, our result of the second part of the following theorem is different than in [5, Th. 1.5.3].

Theorem 5.3. *Suppose* $u, u^\Delta, a, b \in \mathrm{C_{rd}}$ *and* $u, u^\Delta, a, b \ge 0$. *Then*

$$u^\Delta(t) \le u(t_0) + \int_{t_0}^{t} a(s) \left\{u(s) + u^\Delta(s) + \int_{t_0}^{s} b(\tau)u^\Delta(\tau)\Delta\tau\right\} \Delta s$$

for all $t \in \mathbb{T}_0$ *implies*

$$u^\Delta(t) \le u(t_0) \left[1 + 2\int_{t_0}^{t} a(\tau)e_{1+a+b}(\tau, t_0)\Delta\tau\right] \quad \textit{for all} \quad t \in \mathbb{T}_0;$$

$$u^\Delta(t) \le u(t_0) + \int_{t_0}^t a(s)\left\{u(s) + u^\Delta(s) + \int_{t_0}^s b(\tau)\left[u(\tau) + u^\Delta(\tau)\right]\Delta\tau\right\}\Delta s$$

for all $t \in \mathbb{T}_0$ *implies*

$$u^\Delta(t) \le u(t_0)\left\{2e_{1+a}(t,t_0) + \int_{t_0}^t b(\tau)\left[1 + 2e_{2+a+b}(\tau,t_0) + \int_{t_0}^\tau b(s)e_{2+a+b}(\tau,\sigma(s))\Delta s\right]e_{1+a}(t,\sigma(\tau))\Delta\tau\right\}.$$

Proof. To prove (i), we define

$$z(t) := u(t_0) + \int_{t_0}^t a(s)\left[u(s) + u^\Delta(s)\right]\Delta s + \int_{t_0}^t a(s)\left[\int_{t_0}^s b(\tau)u^\Delta(\tau)\Delta\tau\right]\Delta s$$

in order to get

$$z^\Delta(t) \le a(t)\left[u(t_0) + z(t) + \int_{t_0}^t z(s)\Delta s + \int_{t_0}^t b(\tau)z(\tau)\Delta\tau\right].$$

Next define

$$r(t) := u(t_0) + z(t) + \int_{t_0}^t z(s)\Delta s + \int_{t_0}^t b(\tau)z(\tau)\Delta\tau$$

to obtain

$$r^\Delta(t) \le [a(t) + b(t) + 1]\, r(t).$$

We apply Theorem 2.6 twice to get the desired result. To prove (ii), we define

$$z(t) := u(t_0) + \int_{t_0}^t a(s)\left[u(s) + u^\Delta(s)\right]\Delta s + \int_{t_0}^t a(s)\left\{\int_{t_0}^s b(\tau)[u(\tau) + u^\Delta(\tau)]\Delta\tau\right\}\Delta s$$

to get

$$z^\Delta(t) \le a(t)\left\{u(t_0) + \int_{t_0}^t z(s)\Delta s + z(t) + \int_{t_0}^t b(\tau)\left[u(t_0) + \int_{t_0}^\tau z(s)\Delta s + z(\tau)\right]\Delta\tau\right\}.$$

Defining

$$r(t) := u(t_0) + \int_{t_0}^t z(s)\Delta s + z(t) + \int_{t_0}^\tau b(\tau)\left[u(t_0) + \int_{t_0}^\tau z(s)\Delta s + z(\tau)\right]\Delta\tau$$

provides

$$r^\Delta(t) \le [a(t) + 1]\, r(t) + b(t)\left\{u(t_0) + \int_{t_0}^t r(s)\Delta s + r(t)\right\}.$$

By Theorem 5.2 (i), we obtain

$$r(t) \le u(t_0)\left\{2e_{1+a}(t,t_0) + \int_{t_0}^t e_{1+a}(t,\sigma(\tau))b(\tau) \times \left[1 + 2e_{2+a+b}(\tau,t_0) + \int_{t_0}^\tau e_{a+b+2}(\tau,\sigma(s))b(s)\Delta s\right]\Delta\tau\right\}.$$

Since $u^\Delta(t) \le z(t) \le r(t)$, the proof is complete. □

We note that the inequalities in our final result provide estimates on $u^{\Delta\Delta}(t)$ and consequently, after solving, estimates on $u(t)$.

Theorem 5.4. *Suppose* $u, u^{\Delta}, u^{\Delta\Delta}, a, b, c \in \mathrm{C_{rd}}$ *and* $u, u^{\Delta}, u^{\Delta\Delta}, a, b, c \geq 0$. *Then*

(i) $$u^{\Delta\Delta}(t) \leq a(t) + b(t) \int_{t_0}^{t} c(s) \left[u(s) + u^{\Delta}(s)\right] \Delta s \quad \text{for all} \quad t \in \mathbb{T}_0$$

implies

$$u^{\Delta\Delta}(t) \leq a(t) + b(t) \int_{t_0}^{t} p(\tau) e_q(t, \sigma(\tau)) \Delta\tau,$$

where

$$p(t) = c(t) \left\{ u(t_0) + (t - t_0 + 1) u^{\Delta}(t_0) + \int_{t_0}^{t} \left[a(\tau) + \int_{t_0}^{\tau} a(s) \Delta s \right] \Delta\tau \right\}$$

and

$$q(t) = c(t) \int_{t_0}^{t} \left[b(\tau) + \int_{t_0}^{\tau} b(s) \Delta s \right] \Delta\tau;$$

(ii) $$u^{\Delta}(t) \leq a(t) + b(t) \left\{ u^{\Delta}(t) + \int_{t_0}^{t} c(s) \left[u(s) + u^{\Delta}(s)\right] \Delta s \right\} \quad \text{for all} \quad t \in \mathbb{T}_0$$

implies

$$u^{\Delta\Delta}(t) \leq a(t) + b(t) \left\{ u^{\Delta}(t_0) e_q(t, t_0) + \int_{t_0}^{t} p(\tau) e_q(t, \sigma(\tau)) \Delta\tau \right\},$$

where

$$p(t) = a(t) + c(t) \left\{ u(t_0) + (t - t_0 + 1) u^{\Delta}(t_0) + \int_{t_0}^{t} \left[a(\tau) + \int_{t_0}^{\tau} a(s) \Delta s \right] \Delta\tau \right\}$$

and

$$q(t) = b(t) + c(t) \int_{t_0}^{t} \left[b(\tau) + \int_{t_0}^{\tau} b(s) \Delta s \right] \Delta\tau.$$

Proof. In order to prove (i), we define $z(t) := \int_{t_0}^{t} c(\tau) \left[u(\tau) + u^{\Delta}(\tau)\right] \Delta\tau$ and obtain

$$u^{\Delta}(t) \leq u^{\Delta}(t_0) + \int_{t_0}^{t} \left[a(\tau) + b(\tau) z(\tau)\right] \Delta\tau \leq u^{\Delta}(t_0) + \int_{t_0}^{t} a(\tau) \Delta\tau + z(t) \int_{t_0}^{t} b(\tau) \Delta\tau,$$

where we have used the fact that z is nondecreasing. This implies that

$$u(t) \leq u(t_0) + u^{\Delta}(t_0)(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{\tau} a(s) \Delta s \Delta\tau + z(t) \int_{t_0}^{t} \int_{t_0}^{\tau} b(s) \Delta s \Delta\tau$$

and therefore $z^{\Delta}(t) \leq p(t) + q(t) z(t)$. Then by Theorem 2.6 we get

$$z(t) \leq \int_{t_0}^{t} e_q(t, \sigma(\tau)) p(\tau) \Delta\tau$$

since $z(t_0) = 0$. Applying the inequality $u^{\Delta\Delta} \leq a + bz$ completes the proof. To prove (ii), we define $z(t) := u^{\Delta}(t) + \int_{t_0}^{t} c(s) \left[u(s) + u^{\Delta}(s)\right] \Delta s$. Now, by following the same arguments as in the proof of (i) given above, we get the required inequality. $\square$

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