

Packing and Covering with Non-Piercing Regions

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Abstract

In this paper, we design the first polynomial time approximation schemes for the Set Cover and Dominating Set problems when the underlying sets are non-piercing regions (which include pseudodisks). We show that the local search algorithm that yields PTASs when the regions are disks [5, 19, 28] can be extended to work for non-piercing regions. While such an extension is intuitive and natural, attempts to settle this question have failed even for pseudodisks. The techniques used for analysis when the regions are disks rely heavily on the underlying geometry, and do not extend to topologically defined settings such as pseudodisks. In order to prove our results, we introduce novel techniques that we believe will find applications in other problems.

We then consider the Capacitated Region Packing problem. Here, the input consists of a set of points with capacities, and a set of regions. The objective is to pick a maximum cardinality subset of regions so that no point is covered by more regions than its capacity. We show that this problem admits a PTAS when the regions are k -admissible regions (pseudodisks are 2-admissible), and the capacities are bounded. Our result settles a conjecture of Har-Peled (see Conclusion of [20]) in the affirmative. The conjecture was for a weaker version of the problem, namely when the regions are pseudodisks, the capacities are uniform, and the point set consists of all points in the plane.

Finally, we consider the Capacitated Point Packing problem. In this setting, the regions have capacities, and our objective is to find a maximum cardinality subset of points such that no region has more points than its capacity. We show that this problem admits a PTAS when the capacity is unity, extending one of the results of Ene et al. [16].

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1 Introduction

Geometric packing and covering problems have received wide attention in the last decade, especially in the context of approximation algorithms. Besides the inherent aesthetic appeal, the interest in the geometric setting arises from the fact that in many applications, the packing and covering problems involve geometric objects. For example, see [3, 4, 14, 24, 30]. Several tools and techniques have been developed for this purpose, but for many fundamental



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problems there are still large gaps between the known approximation factors, and the existing hardness results.

Classic techniques for solving packing and covering problems rely on grid-shifting techniques introduced by Hochbaum and Maass [22], and extensions by Erlebach et al. [17] and Chan [10]. All these algorithms are restricted to the setting where the regions are *fat*. Recent progress has been based mainly on two paradigms. The first is algorithms that use LP rounding [7, 9, 12, 18], and the other is *local search* (albeit only in the unweighted setting). Local Search has been used to obtain PTASs¹ for several problems besides packing and covering. For example, see [5, 12, 19, 23, 28] and [8, 13] for more recent work.

Har-Peled and Chan [12], and Mustafa and Ray [28] obtained PTASs for the Independent Set and Hitting Set problems respectively, via local search when the underlying regions are non-piercing². Non-piercing regions constitute a general setting studied widely, examples of which include disks, homothets of convex objects, unit height rectangles, arbitrary sized squares, etc. In contrast, for the Set Cover and Dominating Set problems, PTASs exist only when the underlying regions are disks [5, 19, 28]. Since these are natural and important problems, there have been attempts to extend these results to more general settings. The main difficulty is that the analysis for the case of disks relies heavily on the geometry. Durocher and Fraser [15] showed the existence of a PTAS for the Set Cover problem when the regions are pseudodisks satisfying a *cover-free* condition by *dualizing* and converting the problem to a Hitting Set problem. Further, they show that the approach of dualizing the problem can not be extended to work for a general family of pseudodisks.

In this paper, we develop new techniques to analyze the local search algorithm for problems when the underlying regions are non-piercing. These techniques lead to the first PTAS for the unweighted Set Cover and Dominating Set problems when the underlying regions are non-piercing. In the weighted setting, Chan et al. [11] building on the work of Varadarajan [31] obtained $O(1)$ -approximation algorithms for the Set Cover and Dominating Set problems for non-piercing regions with low union complexity. For the Set Cover problem, the current best result is a QPTAS [27] that extends the technique of Adamaszek and Wiese [1, 2] which obtains QPTAS for the Independent Set problem for polygons.

We also develop new techniques for obtaining a PTAS for the Capacitated Region packing problem when the capacities are bounded by a constant and the regions are k -admissible³. This result proves a conjecture of Har-Peled [20]. We also consider the dual problem, namely Capacitated Point Packing for non-piercing regions. We show that it admits a PTAS using local search for the special case when the capacity is unity, extending a result of Ene et al. [16], who obtained a PTAS for Capacitated Point Packing for disks with unit capacity in the plane.

2 Preliminaries

Two compact, simply connected regions A, B are said to be *non-piercing* if both $A \setminus B$ and $B \setminus A$ are connected. A set \mathcal{X} of compact, simply connected regions is non-piercing if the regions in \mathcal{X} are pairwise non-piercing. For a region A , let $\partial(A)$ denote the boundary of A . We assume $\partial(A)$ is oriented counter-clockwise. The boundary divides the plane into two

¹ A polynomial time $(1 + \epsilon)$ -approximation algorithm for any $\epsilon > 0$.

² A set of simply connected regions is said to be non-piercing if for any pair A, B of regions, the sets $A \setminus B$ and $B \setminus A$ are connected.

³ k -admissible regions are non-piercing regions whose boundaries intersect at most k times.

regions the interior of A , denoted $\text{int}(A)$, and the exterior of A , denoted $\text{ext}(A)$. By the orientation of $\partial(A)$, $\text{int}(A)$ lies to the left of $\partial(A)$. We further assume that any pair of regions in \mathcal{X} intersect *properly* by which we mean that for any two regions A, B , $\partial(A) \cap \partial(B)$, i.e., the points of intersection of their boundaries is a finite set, and at each point of intersection, their boundaries cross. In this paper, when we use the term *region*, we implicitly mean that the region is compact and simply connected, and a set of regions is assumed to be properly intersecting.

In some applications considered in this paper, we are given a set \mathcal{R} of regions as well as a set P of points in \mathbb{R}^2 . In this case, we assume that the regions in \mathcal{R} intersect properly, and each point $p \in P$ is at least at a distance $\epsilon > 0$ away from the boundary of any region in \mathcal{R} .

We will use Lemma 6 from [29], which we state here for completeness.

► **Lemma 1** ([29]). *Given a set of non-piercing regions \mathcal{X} and a set P of points in the plane, we can construct a plane graph⁴ $H = (P, E)$ in polynomial time such that for any region $X \in \mathcal{X}$, the induced sub-graph on the set of points in $P \cap X$ is connected. Furthermore, the sub-graph formed by edges of H lying within X also form a connected sub-graph on $P \cap X$.*

The Lemma as stated is slightly more general than the statement in [29]. However, this follows from their proof.

3 Local Search Framework

The local search algorithm we use in this paper is the following:

Local Search Algorithm: For a parameter k , start with a feasible solution. At each iteration, attempt to find a better feasible solution by swaps of a bounded size k of objects in the current solution with objects not in the solution. Stop and return the current solution when no such swap is possible.

In the rest of the paper, when we say the “local search algorithm”, we implicitly refer to the algorithm above. The running time of this algorithm is $n^{O(k)}$. We choose the local search parameter k to be an appropriate polynomial in $1/\epsilon$ to achieve the approximation factor of $(1 + \epsilon)$ [5, 12, 28].

Let \mathcal{R} and \mathcal{B} denote an optimal solution and the solution returned by the local search algorithm, respectively. To analyze the approximation factor of the algorithm, we need to construct a suitable bipartite graph on the elements in \mathcal{R} and \mathcal{B} . We refer to the exposition by Aschner et al. [5] as well as [12, 28] for a more complete description of this framework and its analysis. The following Theorem then follows.

► **Theorem 2** ([5, 12, 28]). *Consider a problem Π .*

1. *Suppose Π is a minimization problem. If there exists a bipartite graph $H = (\mathcal{R} \cup \mathcal{B}, E)$, that belongs to a family of graphs having a balanced vertex separator of sub-linear size, and it satisfies the local-exchange property: For any subset $\mathcal{B}' \subseteq \mathcal{B}$, $(\mathcal{B} \setminus \mathcal{B}') \cup N(\mathcal{B}')$ is a feasible solution. Then, the Local Search algorithm is a PTAS for Π . Here, $N(\mathcal{B}')$ denotes the set of neighbors of \mathcal{B}' in H .*
2. *Suppose Π is a maximization problem. If there exists a bipartite graph $H = (\mathcal{R} \cup \mathcal{B}, E)$ that belongs to a family of graphs having a balanced vertex separator of sub-linear size,*

⁴ An embedding of a planar graph in the plane such that the vertices are points and edges are continuous curves between the end-points.

and such that it satisfies the local-exchange property: For any $\mathcal{R}' \subseteq \mathcal{R}$, $(\mathcal{B} \cup \mathcal{R}') \setminus N(\mathcal{R}')$ is a feasible solution. Then, the Local Search algorithm is a PTAS for Π . Here, as above $N(\mathcal{R}')$ denotes the set of neighbors of \mathcal{R}' in H .

Note that it can be assumed that $\mathcal{R} \cap \mathcal{B} = \emptyset$. This is because, otherwise, the common elements can be removed from both the sets and then the analysis can be restricted to the modified sets.

In Sections 6, 7, and 8, we construct graphs satisfying the conditions of Theorem 2 above, and thereby obtain a PTAS for the Dominating Set, Set Cover, Capacitated Region Packing, and Capacitated Point Packing problems. *Note that we only need to show the existence of these graphs and we do not require their construction to be algorithmic as they are only used in the analysis and not in the algorithm.*

4 Our Results

In this paper, we study the following problems.

Set Cover: Given a finite set of non-piercing regions \mathcal{X} , and a set of points P covered by the union of the regions in \mathcal{X} , compute $\mathcal{Y} \subseteq \mathcal{X}$ of smallest cardinality such that for each $p \in P$, there exists $Y \in \mathcal{Y}$ s.t., $p \in Y$.

Note that when the regions are disks, a PTAS for this problem follows from a PTAS for the Hitting-Set problem of halfspaces in \mathbb{R}^3 [28] via lifting. However, this technique does not generalize even for pseudodisks. An appealing approach is to try to *dualize* the problem, and use results for the Hitting Set problem. However, as Durocher and Frazer [15] observed, such a dual does not exist. Currently, the best approximation algorithm is a QPTAS given by Mustafa et al., [27] that works even in the weighed setting. In the unweighted setting, the work of Har-Peled and Quanrud [21] implies a PTAS for the above problem under the assumption that the regions are fat, and that no point is contained in more than a constant number of regions. We obtain a PTAS without these assumptions. The only requirement is that the regions are non-piercing, and in this sense, our work complements the work of Har-Peled and Quanrud [21].

► **Theorem 3.** *The Local Search algorithm yields a PTAS for the geometric Set Cover problem when the regions are non-piercing.*

Dominating Set: Given a finite set of non-piercing regions \mathcal{X} , find a subset $\mathcal{Y} \subseteq \mathcal{X}$ of smallest cardinality such that for each $X \in \mathcal{X}$, there exists $Y \in \mathcal{Y}$ so that $Y \cap X \neq \emptyset$.

Gibson and Pirwani [19] gave a PTAS for this problem, via local search when restricted to disks. However, their construction of the planar graph uses power diagrams [6] which strongly relies on the fact that the regions are circular disks. It is not clear how one could generalize their result to the setting of non-piercing regions, or even pseudodisks. Har-Peled and Quanrud [21] prove that local search yields a PTAS for *low-density graphs*. However, it is not clear how we could apply these techniques even in the setting where the regions are fat. We obtain a PTAS for this problem when the regions are non-piercing. In Section 6, we prove the following:

► **Theorem 4.** *The Local Search algorithm yields a PTAS for the Dominating Set problem in the intersection graph of non-piercing regions.*

Capacitated Region Packing: Given a finite set of non-piercing regions \mathcal{X} , a set of points P each having a constant capacity ℓ , compute the largest cardinality set $\mathcal{Y} \subseteq \mathcal{X}$ such that each point of P is contained in at most ℓ regions of \mathcal{Y} .

Currently, the best algorithm for this problem is by Ene et al., [16] that is an $O(1)$ -approximation. This algorithm works even in the weighted setting as long as the regions have linear union complexity. Aschner et al. [5] gave PTAS for fat objects, Har-Peled [20] gave a QPTAS for family of pseudodisks. The results in [5, 20] is for the special case where $P = \mathbb{R}^2$. Har-Peled [20] conjectured that a PTAS must exist for this problem. Indeed, we obtain a PTAS for the more general problem. In Section 7 we prove the following theorem.

► **Theorem 5.** *The Local search algorithm yields a PTAS for the Capacitated Region Packing problem when the regions are k -admissible for $k = O(1)$, and the capacities are bounded above by a constant.*

Capacitated Point Packing: Given a finite set of non-piercing regions \mathcal{X} , with a constant capacity ℓ and a set of points P , compute the largest cardinality set $Q \subseteq P$ such that each region in \mathcal{X} contains at most ℓ points of Q .

Ene et al., [16] gave $O(1)$ -approximation algorithms for disks in plane with arbitrary capacities. For unit capacities, they show a PTAS for halfspaces in \mathbb{R}^3 and disks in the plane. To the best of our knowledge, there is no known $O(1)$ -approximation algorithm for pseudodisks, even for unit capacity (standard pack points). We show that the problem admits a PTAS when the regions have unit capacity. In Section 8 we prove the following theorem.

► **Theorem 6.** *The Local search algorithm yields a PTAS for the Capacitated Point Packing problem for non-piercing regions when the regions have unit capacity.*

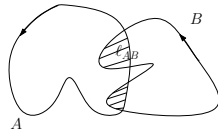
We define the notion of *lens bypassing*, and use this to build graphs with a small separator for the Set Cover and the Dominating Set problems thus obtaining a PTAS for these problems. Lens-bypassing is a finer tool than *lens-cutting* used in [27], and allows us to simplify one intersection at a time, instead of all intersections with one region at a time. This technique may find applications elsewhere. For the Capacitated Region Packing problem, we argue that the natural intersection graph relevant to the problem has a small separator by comparing it to a planar graph on the points.

5 Lens Bypassing

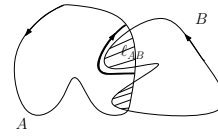
For two regions A and B , each connected component of $A \cap B$ bounded by two arcs, one from the boundary of A and the other from that of B is called a *lens*. Since the boundary of any region is oriented counter-clockwise, observe that the arcs from A and B forming the boundary of a lens are oriented in opposite directions.

Let L_{AB} denote the set of lenses formed by the intersection of regions A and B . We define a *lens bypassing* for a lens $\ell_{AB} \in L_{AB}$ formed by A and B as follows: Leaving B as is, we modify the boundary of A to follow the boundary of B along the arc of B bounding ℓ_{AB} , at an arbitrarily small distance $\beta > 0$ away from this arc. In this case, we say that we do lens bypassing in favor of B . If we did the reverse, we would call this lens bypassing in favor of A . More formally, let D_β be a ball of radius β . Then, bypassing lens ℓ_{AB} in favor of B is the operation of replacing A by $A' = A \setminus (\ell_{AB} \oplus D_\beta)$, where \oplus denotes a Minkowski-sum. Figures 1 and 2 shows the operation of lens-bypassing for non-piercing regions.

► **Remark.** Our goal in lens-bypassing is to preserve the union of the regions while simplifying the arrangement. For this, we need $\beta = 0$ in the definition of lens-bypassing. But, this would



■ **Figure 1** The figure shows lenses in the arrangement of non-piercing regions.



■ **Figure 2** The figure shows the operation of bypassing lens ℓ_{AB} in favor of B

imply that A' and B share a portion of their boundaries and they are no longer properly intersecting. To avoid this technical complication, we take β to be an arbitrarily small positive quantity. However, for the rest of the paper, we do not make this distinction and state our results as if $\beta = 0$ for better readability. For instance, we say that lens-bypassing does not change the union of the regions. This is to be understood as: $A' \cup B$ contains all points in $A \cup B$ that are at least a distance $\delta > 0$ away from $\partial(A)$ and $\partial(B)$. Here, we assume that $\beta < \delta$ for an arbitrarily small positive quantity δ .

► **Proposition 6.1.** *Let A and B be two regions in \mathbb{R}^2 . Let ℓ_{AB} be a lens formed by A and B . Let A', B be the regions obtained by bypassing the lens ℓ_{AB} in favor of B . Then, A' and B are non-piercing regions if and only if A, B are non-piercing.*

For two regions A and B , let $X(A, B)$ denote the intersection points of $\partial(A)$ and $\partial(B)$. Let σ_{AB} denote the cyclic sequence of $X(A, B)$ along $\partial(A)$, i.e., walking in counter-clockwise order along $\partial(A)$. Similarly, let σ_{BA} denote the cyclic sequence of the intersection points $X(A, B)$ along $\partial(B)$. For two points x, y on ∂A , we let $\gamma_{xy}(A)$ denote the arc on $\partial(A)$ from x to y in counter-clockwise direction along ∂A . We use γ_{xy} , when the region A is clear from context. Two cyclic sequences σ and σ' on the same set of elements are said to be reverse-cyclic if σ can be obtained from σ' by reversing the order. For example x_1, x_2, x_3, x_4, x_1 and x_4, x_3, x_2, x_1, x_4 are reverse-cyclic. For a cyclic sequence σ , we say x precedes y in σ , or $x \prec_\sigma y$ if x immediately precedes y in the cyclic sequence σ .

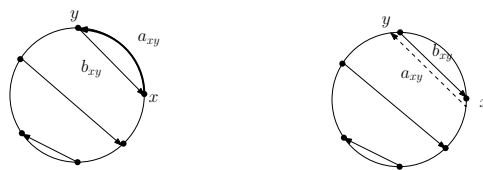
For a pair of reverse-cyclic sequences σ, σ' on the same set of elements, we define a *lens* in the sequences as pair of elements x, y that appear consecutively in σ and appear consecutively in σ' , but in reverse-cyclic order; i.e., $x \prec_\sigma y$ and $y \prec_{\sigma'} x$. Bypassing a lens xy in a pair σ, σ' of reverse-cyclic sequences is the operation of removing x and y from σ and σ' , i.e., $\pi = \sigma \setminus \{x, y\}$ and $\pi' = \sigma' \setminus \{x, y\}$.

For regions A, B consider the cyclic sequences σ_{AB} and σ_{BA} . If x, y form a lens in the sequences σ_{AB} and σ_{BA} , it is easy to see that the arcs of $\partial(A)$ and $\partial(B)$ between points x and y in $X(A, B)$ form a lens of the regions A and B . If $x \prec_{\sigma_{AB}} y$ and $x \prec_{\sigma_{BA}} y$, then the region bounded by the arcs of $\partial(A)$ and $\partial(B)$ between points x and y forms a region that is contained in either $A \setminus B$ or $B \setminus A$.

We will see that the pair of cyclic sequences σ_{AB} and σ_{BA} of the intersection points of regions A and B are reverse-cyclic if and only if the regions A, B are non-piercing. Further, a lens in the sequences σ_{AB} and σ_{BA} corresponds to a lens in the intersection of A and B , and bypassing a lens in σ_{AB} and σ_{BA} is the operation of lens bypassing in A and B . The following proposition is intuitively clear, and we skip the easy proof.

► **Proposition 6.2.** *If σ, σ' are two reverse-cyclic sequences on the same set X of elements, then*

1. *For a lens in the sequences σ, σ' formed by elements x, y , bypassing the lens leaves the sequences reverse-cyclic, i.e., the sequences $\pi = \sigma \setminus \{x, y\}$ and $\pi' = \sigma' \setminus \{x, y\}$, are reverse-cyclic.*



■ **Figure 3** The figure on the left shows the segments of the boundary of B in A . The figure on the right shows the lens bypassed to obtain the modified region for A . This is shown as a dotted line.

2. π and π' are reverse-cyclic sequences, where π and π' are obtained from σ and σ' respectively, by adding elements $x, y \notin X$ between the same pair of consecutive elements in σ and σ' such that $x \prec_{\pi} y$ and $y \prec_{\pi'} x$.

We now give a combinatorial characterization of non-piercing regions that will be useful for the rest of the paper.

► **Theorem 7.** *Two regions A, B in \mathbb{R}^2 are non-piercing if and only if σ_{AB} and σ_{BA} are reverse-cyclic.*

Proof. We prove both directions by induction on $|X(A, B)|$. If A and B are non-piercing, then we show that σ_{AB} and σ_{BA} are reverse-cyclic. The base case is when $|X(A, B)| = 0$; sequences σ_{AB} and σ_{BA} are empty and are therefore reverse-cyclic. Suppose the theorem holds for $|X(A, B)| < k$. Given two regions A, B with $|X(A, B)| = k$. Consider $S = \partial(B) \cap A$. Since $\partial(B)$ is not self-intersecting, S is a set of non-intersecting chords connecting disjoint pairs of points in $X(A, B)$.

Let $\gamma_{yx}(B)$ be a chord from y to x in S of smallest length, where the length of a chord in S joining x to y is the number of points of $X(A, B)$ encountered when going from y to x in σ_{AB} . We claim that x and y are adjacent in σ_{AB} since points between y and x along σ_{AB} can not be connected by a chord of S to a point outside without intersecting $\gamma_{yx}(B)$, and all such chords have smaller length than $\gamma_{yx}(B)$, contradicting the fact that $\gamma_{yx}(B)$ is the chord of smallest length.

Suppose $x \prec_{\sigma_{AB}} y$. Let $\gamma_{xy}(A)$ be the arc on $\partial(A)$ joining x to y . Since $\gamma_{yx}(B)$ is a chord joining y to x , it follows that $y \prec_{\sigma_{BA}} x$. Then, the region bounded by $\gamma_{yx}(B)$ and $\gamma_{xy}(A)$ forms a lens ℓ_{AB} . Suppose we bypass ℓ_{AB} in favor of B then, by Proposition 6.1 regions A and B remain non-piercing and $|X(A, B)|$ decreases by 2. Figure 3 shows this operation. By the inductive hypothesis, the two sequences are reverse-cyclic. Adding this pair x, y of adjacent points in both sequences σ_{AB} and σ_{BA} (in opposite order), the sequences remain reverse-cyclic by Proposition 6.2.

If $y \prec_{\sigma_{AB}} x$. Let a_{yx} be the arc joining y to x on $\partial(A)$. Since $\gamma_{yx}(B)$ lies in A , the region R bounded by a_{yx} and $\gamma_{yx}(B)$ lies in $A \setminus B$. In this case, we show $|X(A, B)| = 2$. Suppose not. Then, there are points u, v different from x, y in $X(A, B)$. If b_{uv} is a chord of S joining u to v . Then, around this region, we again obtain a region of $A \setminus B$ disconnected from R , contradicting the fact that $A \setminus B$ is connected. If $|X(A, B)| = 2$, then the sequences are reverse-cyclic.

We show the reverse direction again by induction on $|X(A, B)|$. If $|X(A, B)| = 0$, the regions are disjoint and are therefore non-piercing. Suppose that the theorem is true for $|X(A, B)| < k$. Let A, B be two regions with $|X(A, B)| = k$ and such that σ_{AB} and σ_{BA} are reverse-cyclic. By applying the lens-bypassing as before between two points x, y such that $x \prec_{\sigma_{AB}} y$ (in σ_{BA} , $y \prec_{\sigma_{BA}} x$, by virtue of the sequences being reverse-cyclic) the sequences remain reverse-cyclic by Proposition 6.2 and $|X(A, B)|$ has reduced by 2. By the inductive

hypothesis, A and B are non-piercing. Adding the two intersection points x, y results in adding the lens between x and y , and this does not change $A \setminus B$ or $B \setminus A$, and the regions are non-piercing. ◀

► **Corollary 8.** *For two non-piercing regions A, B if there exist $x \prec_{\sigma_{AB}} y$ and $x \prec_{\sigma_{BA}} y$, then $|X(A, B)| = 2$.*

► **Corollary 9.** *For two non-piercing regions A, B , such that one is not contained in the other, $A \cap B$ is a collection of disjoint lenses.*

The lenses formed in an arrangement of a set \mathcal{X} of regions can be ordered as a partial order by inclusion. That is, lenses $\ell_{AB} \prec \ell_{CD}$ if $\ell_{AB} \subseteq \ell_{CD}$. Note that either C or D could be equal to A or B . Now we prove the key lemma about lenses.

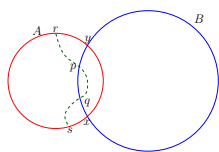
► **Lemma 10.** *Let \mathcal{X} be a set of non-piercing regions. Let ℓ_{AB} be a minimal lens, defined by regions $A, B \in \mathcal{X}$. Let A' be the region obtained from A by bypassing the lens ℓ_{AB} in favor of B . Then, the regions $\mathcal{X}' = (\mathcal{X} \setminus \{A\}) \cup \{A'\}$ is a set of non-piercing regions.*

Proof. Let x, y be the two vertices of the lens ℓ_{AB} so that the arcs bounding ℓ_{AB} are $\gamma_{xy}(A)$, and $\gamma_{yx}(B)$. Let C be any region intersecting A . We will show that after bypassing the lens ℓ_{AB} in favor of B , the modified region A' remains non-piercing with respect to C . In particular, we will show that $\sigma_{A'C}$ and $\sigma_{CA'}$ remain reverse-cyclic. Let S be the set of chords in ℓ_{AB} formed by $\partial(C)$.

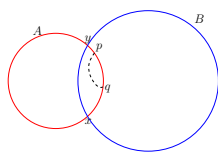
Suppose there is a chord $\gamma_{pq}(C)$ in S , such that p and q both lie on the boundary of the lens ℓ_{AB} defined by B , i.e., on $\gamma_{yx}(B)$. If q precedes p along $\partial(B)$, i.e., on σ_{BC} then $\gamma_{pq}(C)$ and $\gamma_{qp}(B)$ form a lens contained in ℓ_{AB} , contradicting the minimality of ℓ_{AB} . Therefore, it follows that p must precede q along $\partial(B)$, i.e., on σ_{BC} . But, this implies that p precedes q in both reverse-cyclic sequences σ_{BC} and σ_{CB} . By Corollary 8 therefore, $\partial(B)$ and $\partial(C)$ do not have any other points of intersection. In particular, this implies that $\partial(C)$ does not intersect the boundary of the lens ℓ_{AB} at any point other than p or q and so there are no other chords in S . After lens-bypassing therefore, $\sigma_{A'C}$ and $\sigma_{CA'}$ are obtained by inserting consecutive points p, q (in fact by points p', q' arbitrarily close to p and q respectively, but we do not make this distinction) between the same two consecutive points in opposite order into σ_{AC} and σ_{CA} . By Proposition 6.2 the sequences remain reverse-cyclic. This case is shown in Figure 4.

Now, suppose there is a chord $\gamma_{pq}(C)$ in S that joins two points on the boundary of the ℓ_{AB} defined by $\partial(A)$, namely $\gamma_{xy}(A)$. If q precedes p on $\gamma_{xy}(A)$, then this forms a lens contained in ℓ_{AB} contradicting the minimality of ℓ_{AB} . Otherwise, p precedes q in σ_{AC} . But this implies that p precedes q in both reverse-cyclic sequences σ_{AC} and σ_{CA} . Hence, by Corollary 8 $\partial(A)$ and $\partial(C)$ have no more points of intersection. In this case, after lens-bypassing, $\partial(A')$ and $\partial(C)$ do not intersect. Hence, $\sigma_{A'C}$ and $\sigma_{CA'}$ are empty and therefore the regions remain non-piercing. In fact, in this case we will have $A' \subseteq C$. This case is shown in Figure 5.

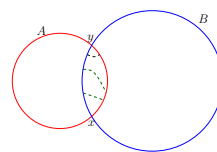
If none of the above hold, then all chords in S have one end-point on $\gamma_{xy}(A)$ and the other end-point on $\gamma_{yx}(B)$. In this case, after lens-bypassing $\sigma_{A'C}$ and $\sigma_{CA'}$ are obtained by replacing for each chord of C , the end-point of the chord on $\gamma_{xy}(A)$ by its other end-point on $\gamma_{yx}(B)$. The sequences σ_{AC} and $\sigma_{A'C}$ are identical except for renaming of the points in $\gamma_{xy}(A)$ by corresponding points in $\gamma_{yx}(B)$. The same holds for the sequence σ_{CA} and $\sigma_{CA'}$. Therefore the sequences $\sigma_{A'C}$ and $\sigma_{CA'}$ remain reverse-cyclic. This case is shown in Figure 6. ◀



■ **Figure 4** There is a chord of C between two points on $\gamma_{yx}(B)$.



■ **Figure 5** There is a chord of C between two points on $\gamma_{xy}(A)$.



■ **Figure 6** All chords of C connect a point on $\gamma_{xy}(A)$ and a point $\gamma_{yx}(B)$.

► **Corollary 11.** Let ℓ_{AB} be a minimal lens. Let A' be the region obtained after bypassing ℓ_{AB} in favor of B . Then, for any region $C \neq B$, $A \cap C \neq \emptyset$ implies $A' \cap C \neq \emptyset$.

6 Dominating Set and Set Cover

In this section, we first describe the construction of a graph over a set of non-piercing regions. The graph we construct satisfies the conditions required for Theorem 2, and thereby obtaining a PTAS for the Dominating Set and Set Cover problems.

For the Set Cover problem, in order to satisfy the local-exchange property, we require a bipartite graph $H = (\mathcal{R} \cup \mathcal{B}, E)$ with a small separator so that for any set $\mathcal{S} \subseteq \mathcal{B}$, the set $\mathcal{B}' = (\mathcal{B} \setminus \mathcal{S}) \cup N(\mathcal{S})$ is a Set Cover, where $N(\mathcal{S})$ is the set of neighbors of the vertices in H corresponding to regions in \mathcal{S} . In other words, the required graph $H = (\mathcal{R} \cup \mathcal{B}, E)$ must satisfy the following property: For each point $p \in P$, there is an $R \in \text{RED}(p)$ and $B \in \text{BLUE}(p)$ that are adjacent in H , where $\text{RED}(p)$ is the set of regions in \mathcal{R} containing p , and $\text{BLUE}(p)$ is the set of regions in \mathcal{B} containing p .

For the Dominating Set problem, the property that the bipartite graph H is required to satisfy is that for any region $X \in \mathcal{X}$, there is a region $R \in \text{RED}(X)$ and a region $B \in \text{BLUE}(X)$ such that R and B are adjacent in H , where $\text{RED}(X)$ is the set of regions in \mathcal{R} intersecting X , and $\text{BLUE}(X)$ is the set of regions in \mathcal{B} intersecting X . We let \mathcal{G} denote the regions of \mathcal{X} that are not in \mathcal{R} or \mathcal{B} . Since we assumed that $\mathcal{R} \cap \mathcal{B} = \emptyset$, the three sets form a partition of the input \mathcal{X} .

We give a graph construction of a planar graph for any set $\mathcal{X} = \mathcal{R} \sqcup \mathcal{B} \sqcup \mathcal{G}$ of non-piercing regions from which the construction of the graphs for the Dominating Set and Set Cover problems follow.

Henceforth we refer to the regions in \mathcal{R}, \mathcal{B} and \mathcal{G} as RED, BLUE, and GREEN regions respectively. We call a region $X \in \mathcal{X}$ *bi-chromatic* if and only if X intersects a RED as well as a BLUE region. We assume that a region intersects itself. Therefore a RED region is bi-chromatic if it intersects a BLUE region. Similarly, a point in the plane is called bi-chromatic if and only if it is contained in both a RED as well as a BLUE region. As before, we let $\text{RED}(X)$ and $\text{BLUE}(X)$ denote the RED and BLUE regions intersecting X , respectively. Similarly, for any point p , we let $\text{RED}(p)$ and $\text{BLUE}(p)$ denote the RED and BLUE regions containing p respectively. We now define the properties that our constructed graph satisfies.

► **Definition 12.** Let $\mathcal{X} = \mathcal{R} \sqcup \mathcal{G} \sqcup \mathcal{B}$ be a set of non-piercing regions. A bipartite planar graph $H = (\mathcal{R} \cup \mathcal{B}, E)$ is called a *locality-preserving* graph for \mathcal{X} if it satisfies the following properties.

- P1. For each bi-chromatic region $X \in \mathcal{X}$, there exists an $R \in \text{RED}(X)$ and a $B \in \text{BLUE}(X)$ such that R and B are adjacent in H .
- P2. For each bi-chromatic point p in the plane, there exists an $R \in \text{RED}(p)$ and a $B \in \text{BLUE}(p)$ such that R and B are adjacent in H .

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We can assume that in any instance, all GREEN regions are bi-chromatic since GREEN regions that are not bi-chromatic can be removed without changing the problem. We say that a graph H on $\mathcal{R} \cup \mathcal{B}$ *satisfies* a region $X \in \mathcal{X}$ if Property P1 holds for X . We also state this as “region X is *satisfied* by H ”. We use a similar terminology for the points. If in a given instance \mathcal{X} , there are regions $R \in \mathcal{R}$, $B \in \mathcal{B}$ and $G \in \mathcal{G}$ such that $R \cap B \cap G \neq \emptyset$, we say that \mathcal{X} has a RED-BLUE-GREEN intersection. The main theorem we prove in this section is the following:

► **Theorem 13.** *For any set $\mathcal{X} = \mathcal{R} \sqcup \mathcal{G} \sqcup \mathcal{B}$ of non-piercing regions, there is a locality-preserving graph H .*

The broad approach to construct a locality-preserving graph H for \mathcal{X} is as follows. If the instance \mathcal{X} satisfies certain additional conditions, then we can directly describe the construction of such a graph. If the instance \mathcal{X} does not satisfy these additional conditions, we show that can *reduce* the instance to one that does.

In order to do this, we describe a sequence of reduction steps that either remove a region, or bypasses a minimal lens in the arrangement of the regions, thus getting us closer to an arrangement enjoying the additional conditions alluded to above. These reduction steps have the crucial property that if we are given a locality-preserving graph for the reduced instance, we can obtain a locality-preserving graph for the original instance. We start with a construction of a locality-preserving graph for an instance \mathcal{X} satisfying the additional conditions. Then, in a sequence of lemmas, namely Lemma 15, 16 and 17 we describe the reduction steps for an instance not enjoying the additional properties. Finally, we can prove Theorem 13.

► **Lemma 14.** *Suppose $\mathcal{X} = \mathcal{R} \sqcup \mathcal{G} \sqcup \mathcal{B}$ is a set of non-piercing regions satisfying the following properties:*

1. $R \cap R' = \emptyset$, for all $R, R' \in \mathcal{R}$ and $B \cap B' = \emptyset$, for all $B, B' \in \mathcal{B}$, i.e., the RED regions are pairwise disjoint, and the BLUE regions are pairwise disjoint.
2. For each $R \in \mathcal{R}, B \in \mathcal{B}$ and $G \in \mathcal{G}$, $R \cap B \cap G = \emptyset$, i.e., there is no RED-BLUE-GREEN intersection.

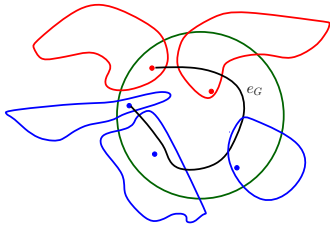
Then, there is a locality-preserving graph for \mathcal{X} .

Proof. In order to construct the graph, we temporarily add RED and BLUE points to the arrangement of the regions in the following way: For each intersection $R \cap G$ of a RED region R and a GREEN region G , we place a RED point in $R \cap G$. Similarly, we place a BLUE point for each BLUE-GREEN intersection. Since there are no RED-BLUE-GREEN intersections, observe that in the interior of any GREEN region the RED and BLUE regions are disjoint. Therefore, for any GREEN region, the point we place corresponding to a RED region does not lie in a BLUE region, and vice-versa.

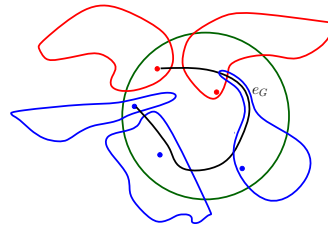
Now, by Lemma 1, applied to the GREEN regions and the RED and BLUE points we place, there is a plane graph K such that for each GREEN region $G \in \mathcal{G}$, there is an edge in K between a RED point contained in G and a BLUE point contained in G lying entirely in G .

For each $G \in \mathcal{G}$, we pick one such edge e_G arbitrarily. Let r and b be the RED and BLUE end-points of e_G , respectively. As observed earlier, r lies in a RED region, and b lies in a BLUE region. So, walking from b to r along e_G we encounter a RED region R and a BLUE region B that are consecutive along e_G . We now extend B along e_G so that R and B now intersect. Note that this can be done in a way that they remain non-piercing. An example is shown in Figures 7 and 8.

We remove any GREEN region with a RED-BLUE-GREEN intersection, and repeat the operation above on the remaining GREEN regions. Since the graph K is planar, the extended



■ **Figure 7** The edge e_G .



■ **Figure 8** Extending the BLUE region to intersect the adjacent RED region along e_G .

BLUE regions remain disjoint. Extending a BLUE region along an edge of K chosen for a GREEN region may intersect other GREEN regions in an arbitrary fashion. However, this does not matter as the GREEN regions will not play any role henceforth. It is possible that the same BLUE region is extended multiple times to intersect the same RED region. However, the regions remain non-piercing as each extension of the BLUE ensures this property.

By extending the BLUE regions, we have the property that for each $G \in \mathcal{G}$, there is now an $R \in \text{RED}(G)$ and $B \in \text{BLUE}(G)$ such that $R \cap B \neq \emptyset$. Therefore, the intersection graph of the RED and BLUE regions gives us the desired locality-preserving graph.

Since we ensure that the RED regions are pairwise disjoint, the BLUE regions are pairwise disjoint, and $\mathcal{R} \cup \mathcal{B}$ is non-piercing, the intersection graph of $\mathcal{R} \cup \mathcal{B}$ is planar as observed by Chan and Har-Peled [12]. ◀

We now describe the reductions for an instance \mathcal{X} that does not satisfy the conditions of Lemma 14. The reduction steps are the following: We first show that we can remove RED-BLUE-GREEN intersections if any in our instance. Then, we show that if our instance has two regions such that one is contained in another, then we can remove one of them (this statement is not entirely accurate; we do not get rid of containments where a RED or BLUE region is contained in a GREEN region, but such containments do not affect our construction). Then, we show that we can decrease the number vertices in the arrangement by bypassing minimal lenses. The latter two reductions are applied repeatedly until none apply. At that point, we can show that the instance satisfies the conditions of Lemma 14 and a locality-preserving graph can thus be constructed.

► **Lemma 15.** *Let $\mathcal{X} = \mathcal{R} \sqcup \mathcal{B} \sqcup \mathcal{G}$ be a set of non-piercing regions. Suppose there exists $R \in \mathcal{R}, B \in \mathcal{B}$ and $G \in \mathcal{G}$ such that $R \cap B \cap G \neq \emptyset$, i.e., the instance contains a RED-BLUE-GREEN intersection. Then, a locality-preserving graph for the reduced instance $\mathcal{X}' = \mathcal{X} \setminus G$ is a locality-preserving graph for \mathcal{X} .*

Proof. Let H' be a locality-preserving graph for \mathcal{X}' . By Property P2, there is an edge between a region $R \in \text{RED}(p)$ and $B \in \text{BLUE}(p)$ for $p \in R \cap B \cap G$. This implies that H' is locality-preserving for \mathcal{X} since $\text{RED}(p) \subseteq \text{RED}(G)$ and $\text{BLUE}(p) \subseteq \text{BLUE}(G)$. ◀

► **Lemma 16.** *Let $\mathcal{X} = \mathcal{R} \sqcup \mathcal{B} \sqcup \mathcal{G}$ be a set of non-piercing regions with no RED-BLUE-GREEN intersections. If there are two regions P and Q such that $P \subseteq Q$, and either both P and Q are GREEN, or both are not GREEN, a locality-preserving graph for a suitable reduced instance \mathcal{X}' with one less region than \mathcal{X} implies a locality-preserving graph for \mathcal{X} .*

Proof. Note that, as mentioned in the conditions of the lemma, we do not deal with the case where Q is GREEN and P is either RED or BLUE. Thus, we have the following cases.

Case 1: Q is either RED or BLUE, and P is GREEN. This case does not arise, since we assume that there are no RED-BLUE-GREEN intersections and all GREEN regions are bi-chromatic.

Case 2: Q is either RED or BLUE, and P is either RED or BLUE. The reduced instance in this case is obtained by removing P , i.e., $\mathcal{X}' = \mathcal{X} \setminus P$. If H' is a locality-preserving graph for \mathcal{X}' , then we obtain the locality-preserving graph H for \mathcal{X} by adding an edge between P and Q if they have distinct colors. Otherwise, we add P to any RED or BLUE region intersecting P and having a color different from P , if such a region exists. If such a region does not exist, then we set $H = H'$.

Now we argue that the graph H so constructed is locality-preserving for \mathcal{X} . In obtaining the graph H , we added at most one vertex of degree 1 to H' , and hence H is planar. Since the only regions affected by the removal of P are those that intersect P , we only argue about such regions. Similarly the only points affected are those lying in P and we argue only about these points.

Case 2a: P and Q have the same color. To show that H is locality-preserving for \mathcal{X} , note that if P is bi-chromatic, then the edge we added satisfies P . Any other region intersecting P also intersects a region of the same color as P , namely Q . Therefore, such a region is satisfied by H' . Similarly, all bi-chromatic points in P remain bi-chromatic in \mathcal{X}' since they are contained in Q . Therefore, it follows that the edges in H' satisfy all bi-chromatic points with respect to \mathcal{X} .

Case 2b: P and Q have different colors. In this case, we connect P to Q . This satisfies the region P . Since $P \subset Q$, all other regions intersecting P also intersect Q and are therefore satisfied by the edge we added. By the same argument, all bi-chromatic points in P are also satisfied by H .

Case 3: P and Q are both GREEN. In this case, $\mathcal{X}' = \mathcal{X} \setminus Q$ is the desired reduced instance. If H' is a locality-preserving graph, then H' is also locality-preserving with respect to \mathcal{X} , since any region intersecting P also intersects Q . ◀

► **Lemma 17.** *Let $\mathcal{X} = \mathcal{R} \sqcup \mathcal{B} \sqcup \mathcal{G}$ be a set of non-piercing regions such that there are no RED-BLUE-GREEN intersections. Then, a locality-preserving graph H' for a reduced instance \mathcal{X}' is a locality-preserving graph for \mathcal{X} . Here, \mathcal{X}' is obtained by bypassing a minimal lens ℓ_{PQ} using the following rules:*

1. *If P and Q have the same color. Bypass ℓ_{PQ} in favor of either P or Q chosen arbitrarily.*
2. *If ℓ_{PQ} is contained in a region $R \in \mathcal{R}$, where R is distinct from P and Q , then bypass the lens in favor of the region that is not RED.*
3. *If ℓ_{PQ} is contained in a region $B \in \mathcal{B}$, where B is distinct from P and Q , then bypass the lens in favor of the region that is not BLUE.*

Proof. We assume without loss of generality that we bypass the lens ℓ_{PQ} in favor of Q , and let P' be the resulting region corresponding to P . Then, $\mathcal{X}' = (\mathcal{X} \setminus P) \cup P'$. By Lemma 10, the regions in \mathcal{X}' are non-piercing since we bypass a minimal lens ℓ_{PQ} .

Suppose P and Q have the same color. By Corollary 11, any region $X \neq Q$ intersecting P also intersects P' . This ensures that the set of RED or BLUE regions intersecting a region $X \neq Q$ remains unchanged in \mathcal{X}' . Thus, H' satisfies all regions except possibly Q . Since P and Q have the same color, Q remains bi-chromatic in \mathcal{X}' if it was bi-chromatic in \mathcal{X} . Since

$P \cup Q$ remains unchanged due to lens-bypassing and the fact that they have the same color, all bi-chromatic points in \mathcal{X} remain bi-chromatic in \mathcal{X}' . Since no region gains a new intersection and no point is contained in a new region when going from \mathcal{X} to \mathcal{X}' , a locality-preserving graph H' for \mathcal{X}' is also locality-preserving for \mathcal{X} .

Now suppose that P and Q have distinct colors, then we bypass ℓ_{PQ} only if it lies in a RED or a BLUE region. Let us assume that ℓ_{PQ} is contained in a RED region R . The other case is symmetric. By our assumption that there is no RED-BLUE-GREEN intersection, one of P or Q must be RED. Since we assume that we bypass ℓ_{PQ} in favor of Q , P is RED. We claim that H' is locality-preserving with respect to \mathcal{X} .

By Corollary 11, it follows that for every region $X \neq P, Q$, the set of regions intersecting X does not change. These regions are therefore satisfied by H' . For the region Q , while it possibly loses its intersection with P , it continues to intersect a RED region, namely R . Therefore, Q is also satisfied by H' . The region P' may not be bi-chromatic even if P was bi-chromatic. This can happen only when Q is the unique BLUE region intersecting P , which is RED by assumption. However, in that case, consider any point $p \in \ell_{PQ}$. Such a point p is bi-chromatic in \mathcal{X}' since it lies in Q and R . Therefore the point p is satisfied in H' . Now, the fact that P is satisfied in H' follows from the fact that any region containing p also intersects P . ◀

Proof of Theorem 13. Given an instance $\mathcal{X} = \mathcal{R} \sqcup \mathcal{B} \sqcup \mathcal{G}$ of non-piercing regions, we first remove all RED-BLUE-GREEN intersections by applying Lemma 15. Then, we repeatedly apply Lemma 16 followed by Lemma 17 until neither applies. At this point, we claim that the RED regions are pairwise disjoint, and the BLUE regions are pairwise disjoint.

First, note that no RED or BLUE region is contained in another RED or BLUE region by Lemma 16. In particular, this implies that any intersection of a pair P, Q of RED regions is a union of lenses formed by them. However, any such lens ℓ_{PQ} is removed by Lemma 17 so long as it is minimal. We argue that ℓ_{PQ} must be minimal. Suppose not. Then, there is a minimal lens ℓ_{WZ} contained in ℓ_{PQ} . We can check that for all possible colors of W and Z , Lemma 17 applies to ℓ_{WZ} and is thus bypassed. A symmetric argument applies for the BLUE regions.

Thus, when the conditions of Lemma 16 or Lemma 17 do not apply, the RED regions are pairwise disjoint, the BLUE regions are pairwise disjoint, and there are no RED-BLUE-GREEN intersections. Hence, the conditions of Lemma 14 apply and we can obtain a locality-preserving graph H' . This implies a locality-preserving graph H for the instance \mathcal{X} . ◀

Now, we can prove that the Dominating Set and Set Cover problem for non-piercing regions admits a PTAS.

Proof of Theorem 4. Let \mathcal{R} denote an optimal solution to the Dominating Set problem. Let \mathcal{B} denote a solution returned by local search. Let \mathcal{G} be the remaining regions. Recall that we assume $\mathcal{R} \cap \mathcal{B} = \emptyset$. Let $\mathcal{X} = \mathcal{R} \sqcup \mathcal{B} \sqcup \mathcal{G}$. Note that since every region in \mathcal{X} is bi-chromatic, Property P1 is precisely the locality condition required for the graph of the Dominating Set problem. Now, Theorem 13 gives us a planar graph satisfying Property P1, and is therefore the desired graph. Since planar graphs have separators of size $O(\sqrt{n})$ [25], where n is the number of regions in the input \mathcal{X} , by Theorem 2 a $(1 + \epsilon)$ -approximation algorithm follows. ◀

Proof of Theorem 3. Let \mathcal{R} denote an optimal solution to the Set Cover problem. Let \mathcal{B} denote a solution returned by local search. Recall that we assume $\mathcal{R} \cap \mathcal{B} = \emptyset$. Let $\mathcal{X} = \mathcal{R} \cup \mathcal{B}$,

and let $\mathcal{G} = \emptyset$. Since every point P is in $R \cap B$, for some $R \in \mathcal{R}$ and $B \in \mathcal{B}$, Property P2 is precisely the locality condition required for the Set Cover problem. Now, Theorem 13 gives us a planar graph satisfying the required locality conditions. The graph returned by Theorem 13 is planar. Since planar graphs have separators of size $O(\sqrt{n})$ [25], where n is the number of regions in the input \mathcal{X} , by Theorem 2 a $(1 + \epsilon)$ -approximation follows. ◀

7 Capacitated Region Packing

Recall that in the Capacitated Region Packing problem, we are given a family of r -admissible regions \mathcal{X} , a set of points P , and a positive integer constant ℓ . We want to find a maximum sized subset $\mathcal{X}' \subseteq \mathcal{X}$ such that every point $p \in P$ is contained in at most ℓ regions in \mathcal{X}' . Unlike the earlier results in this paper, here we require that the regions be r -admissible for a constant r , i.e., they are non-piercing and their boundaries intersect at most a constant number of times.

Let $\mathcal{X} = \mathcal{R} \cup \mathcal{B}$ and $k = 2\ell$. Note that the depth of a point in P with respect to \mathcal{X} , that is the number of regions in \mathcal{X} containing it, is at most k . For this problem, the graph we construct is as follows: For each point $p \in P$, add an edge between all regions in \mathcal{R} containing p and all regions in \mathcal{B} containing p . It is easy to check that this graph satisfies the local-exchange property as stated in Theorem 2. We now show that this graph has a small balanced separator. In fact, we show the following super-graph $H(\mathcal{X}, k)$ has a small, balanced separator. For each point $p \in \mathbb{R}^2$ whose depth is at most k , add an edge between all pairs of regions in $\mathcal{R} \cup \mathcal{B}$ containing p .

► **Theorem 18.** *Given a set \mathcal{X} of r -admissible regions for a constant r , the graph $H(\mathcal{X}, k)$ on \mathcal{X} has a balanced separator of size $O(k^{3/2}\sqrt{|\mathcal{X}|})$.*

Proof. In order to prove the statement for $H(\mathcal{X}, k)$, we prove it for an isomorphic graph that is the intersection graph of a family of trees \mathcal{T} that we obtain in the following way: We put one point in every cell whose depth is at most k in the arrangement of the regions in \mathcal{X} , and call this point set P . By Lemma 1 there exists a plane graph G_P on the point set P such that the subgraph induced by $X \cap P$ is connected, for every $X \in \mathcal{X}$. For each $X \in \mathcal{X}$, consider an arbitrary spanning tree T_X of the subgraph induced by $X \cap P$. Observe that the intersection graph of the family of these spanning trees $\mathcal{T} = \{T_X \mid X \in \mathcal{X}\}$ is isomorphic to $H(\mathcal{X}, k)$. We now claim that such an intersection graph has a small and balanced separator.

► **Lemma 19.** *Given a family of trees \mathcal{T} as above, its intersection graph has a balanced separator of size $O(k^{3/2}\sqrt{|\mathcal{T}|})$.*

Proof. We prove this statement as follows. We assign appropriate weights on the points in P , and use the fact that the graph G_P on the points, constructed by applying Lemma 1 is planar, and therefore has a balanced weighted separator.

We assign weights to the points in P as follows: We start by assigning a weight of 0 to each point in P . For each region $X \in \mathcal{X}$, we add $1/|X \cap P|$ to the weight of each point in X . Thus, the weight of a point in P is given by $wt(p) = \sum_{X|p \in X} 1/|X \cap P|$.

By the Lipton-Tarjan separator theorem [25], G_P has a separator S of size $O(\sqrt{|P|})$ such that removing S separates the graph into two disjoint sets A and B , each of which has at most $2/3$ of the total weight of the points. The separator S for G_P gives a separator for the graph \mathcal{T} by taking $\mathcal{S} = \{T \mid S \cap T \neq \emptyset\}$. We claim that \mathcal{S} is a small, balanced separator. The fact that removing \mathcal{S} separates \mathcal{T} into two parts \mathcal{A}, \mathcal{B} follows since a tree containing a vertex from A and a vertex from B must contain a vertex from S .

To show that $|\mathcal{S}| \leq O(k^{3/2}\sqrt{|\mathcal{T}|})$, we proceed as follows. By our construction, every point in P has depth at most k . This implies that $|\mathcal{S}| \leq k|S|$. However, $|S| \leq O(\sqrt{|P|})$, as S is a separator in G_P . Since r -admissible regions have linear union complexity [32], the Clarkson-Shor technique [26, p. 141] implies that the number of cells in the arrangement is at most $O(k|\mathcal{T}|)$. Thus, $|P| = O(k|\mathcal{T}|)$ and hence, $|\mathcal{S}| \leq O(k\sqrt{k|\mathcal{T}|})$.

Now, we need to show that \mathcal{S} is balanced. To see this, let \mathcal{T}_A be the set of trees whose vertex set is a subset of A . \mathcal{T}_B is defined similarly. Since, the weight of all trees in \mathcal{T}_A were distributed among the points in A , $wt(A) \geq |\mathcal{T}_A|$. Also, from the planar separator theorem we know that $wt(A) \leq \frac{2}{3}|\mathcal{T}|$. Therefore, $|\mathcal{T}_A| \leq \frac{2}{3}|\mathcal{T}|$. The same holds for \mathcal{T}_B . Therefore, \mathcal{S} is a balanced separator of size $O(k^{3/2}\sqrt{|\mathcal{T}|})$ of the intersection graph of \mathcal{T} . ◀

From Lemma 19, it follows that $H(\mathcal{X}, k)$ has a balanced separator of size $O(k^{3/2}\sqrt{|\mathcal{X}|})$. ◀

Proof of Theorem 5. The graph constructed satisfies the conditions of Theorem 2. Therefore a PTAS follows. ◀

8 Capacitated Point Packing

Recall that in the Capacitated Point Packing problem, we are given a set P of n points, a set \mathcal{X} of non-piercing regions, and a positive integer constant ℓ . The goal is to obtain the maximum sized subset of points $Q \subseteq P$ such that for every region $X \in \mathcal{X}$, $|X \cap Q| \leq \ell$. We consider the Capacitated Point Packing problem when $\ell = 1$. We show that the Local search algorithm yields a PTAS for this special case.

Proof of Theorem 6. We construct a bipartite graph on the union of red and blue points in the following way. We put an edge between a red point and a blue point if they are contained in some region $X \in \mathcal{X}$. It is easy to check that this graph satisfies the local-exchange property as stated in Theorem 2. As both the local search and the optimum solutions are feasible, every region contains at most one red point and at most one blue point. Observe that the graph constructed in Lemma 1 is such a graph, because it ensures that the graph induced by the points contained in an input region is connected. This in turn means that there is always an edge between a red point and a blue point contained in the same region. Lemma 1 states that this graph is planar. Thus, the graph constructed satisfies the condition of Theorem 2. Therefore, a PTAS follows. ◀

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