PACKING DIMENSION AND AHLFORS REGULARITY OF POROUS SETS IN METRIC SPACES

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ABSTRACT. Let X be a metric measure space with an s-regular measure μ. We prove that if $A \subset X$ is ϱ-porous, then $\dim_{\mathrm{p}}(A) \le s - c\varrho^s$ where $\dim_{\mathrm{p}}$ is the packing dimension and c is a positive constant which depends on s and the structure constants of μ. This is an analogue of a well known asymptotically sharp result in Euclidean spaces. We illustrate by an example that the corresponding result is not valid if μ is a doubling measure. However, in the doubling case we find a fixed $N \subset X$ with $\mu(N) = 0$ such that $\dim_{\mathrm{p}}(A) \le \dim_{\mathrm{p}}(X) - c(\log \frac{1}{\varrho})^{-1}\varrho^t$ for all ϱ-porous sets $A \subset X \setminus N$. Here c and t are constants which depend on the structure constant of μ. Finally, we characterize uniformly porous sets in complete s-regular metric spaces in terms of regular sets by verifying that A is uniformly porous if and only if there is $t < s$ and a t-regular set F such that $A \subset F$.

1. Introduction

The purpose of this paper is twofold: we study dimensional properties of porous sets in s-regular and doubling metric measure spaces and characterize uniformly porous sets in terms of regularity. For definitions we refer to Sections 2 and 3.

In Euclidean spaces dimensional properties of porous sets have been studied extensively, see for example [BS], [JJKS], [KS], [KR], [L], [MV], [M1], [N], [PR], [S], [T] and references therein. It is well known that if $A \subset \mathbb{R}^n$ is ϱ-porous, meaning that A contains holes of relative size ϱ in all small balls, then

$$\dim_{\mathrm{p}}(A) \le n - c\varrho^n \tag{1.1}$$

where $\dim_{\mathrm{p}}$ is the packing dimension and c is a positive constant depending on n only (see [MV, T]). Furthermore, (1.1) is asymptotically sharp as ϱ tends to zero ([KR], [KS, Remark 4.2]). In [DS] and [BHR] it is shown that the dimension of a porous measure in a (globally) s-regular space is smaller than s. In this paper

Date: May 13, 2009.

2000 *Mathematics Subject Classification.* Primary 28A80; Secondary 51F99, 54E35.

Key words and phrases. Metric measure space, porosity, regularity, doubling property, packing dimension, Minkowski dimension.

The authors acknowledge the support of the Academy of Finland, projects #211229, #114821, #212753, and #214595. TR was also partially supported by the Vilho, Yrjö and Kalle Väisälä fund.

we address the question to what extend the quantitative estimate (1.1) is valid in metric measure spaces X. It turns out that the following analogue of (1.1) holds provided that X is equipped with a (locally) s-regular measure μ: if $A \subset X$ is ϱ-porous, then

$$\dim_{\mathrm{p}}(A) \leq s - c\varrho^s \tag{1.2}$$

where c is a positive constant which depends on s and the structure constants a_μ and b_μ of μ (see Theorem 4.7). Note that by [Cu, Theorem 3.16] $\dim_{\mathrm{p}}(X) = s$ provided that X is s-regular. We also show that the dependence on a_μ and b_μ is necessary, that is, unlike in $\mathbb{R}^n$, it is not possible to find c which depends on s only (see Remark 4.12.(3)).

In (1.2) it is not sufficient to assume that μ is doubling: in Example 4.8 we construct a geodesic doubling metric space X having a subset with maximal dimension and porosity. However, in general the failure of the dimension drop is due to a fixed set with μ-measure zero provided that μ is doubling. More precisely, in Theorem 4.9 we show that there exists $N \subset X$ with $\mu(N) = 0$ such that $\dim_{\mathrm{p}}(A) \leq \dim_{\mathrm{p}}(X) - c(\log \frac{1}{\varrho})^{-1}\varrho^t$ for all ϱ-porous $A \subset X \setminus N$. Here t and c are constants which depend on the structure constant c_μ of μ.

As in Euclidean spaces, in complete s-regular metric measure space X uniform porosity is closely related to regularity. We prove that $A \subset X$ is uniformly porous if and only if there are $t < s$ and a t-regular set $F \subset X$ such that $A \subset F$ (see Theorem 5.3). The easier if-part was proven in [BHR], but we give some quantitative estimates on the relations between porosity, t and s.

The paper is organized as follows: In Section 2 we discuss the concept of porosity we are using in metric measure spaces whilst Section 3 is dedicated to measure theoretic preliminaries. Dimension estimates for porous sets are dealt in Section 4. In last section we focus on connections between uniform porosity and regularity.

2. Notation

Let $X = (X, d)$ be a separable metric space and $A \subset X$. For $x \in X$ and $r > 0$, we set

$$\begin{aligned}\mathrm{por}(A, x, r) = \sup\{\varrho \geq 0 : &\text{there is } y \in X \text{ such that } B(y, \varrho r) \cap A = \emptyset \\ &\text{and } \varrho r + d(x, y) \leq r\}.\end{aligned} \tag{2.1}$$

Let $B(x, r) = \{y \in X : d(y, x) \leq r\}$ be the closed ball centred at x with radius r. Since in metric spaces the centre and the radius of a ball is not uniquely defined, we will always assume that a centre and a radius is fixed when we use the word ball. The *porosity of A at a point x* is defined to be

$$\mathrm{por}(A, x) = \liminf_{r \downarrow 0} \mathrm{por}(A, x, r) \tag{2.2}$$

and the *porosity of* A is given by

$$\operatorname{por}(A) = \inf_{x \in A} \operatorname{por}(A, x). \tag{2.3}$$

We call $A \subset X$ *porous* if $\operatorname{por}(A) > 0$, and more precisely, *$\varrho$-porous* provided that $\operatorname{por}(A) \geq \varrho$. Furthermore, $A \subset X$ is *uniformly ϱ-porous* if there exist constants $\varrho > 0$ and $r_p > 0$ such that $\operatorname{por}(A, x, r) \geq \varrho$ for all $x \in A$ and $0 < r < r_p$.

Remarks 2.1. (1) Even though it would be more accurate to use the term lower porosity for $\operatorname{por}(A, x)$ and $\operatorname{por}(A)$ to distinguish them from upper porosities defined by replacing $\liminf$ by $\limsup$ in (2.2), we keep the terminology shorter. Upper porosities are irrelevant for our purposes; there is no nontrivial upper bound for dimensions of upper porous sets. In fact, there exist sets in $\mathbb{R}^n$ with maximum upper porosity and with Hausdorff dimension n, see [M2, §4.12].

(2) We follow the convention introduced in [MMPZ] to use $\operatorname{por}(A, x, r)$ and $\operatorname{por}(A, x)$ instead of

$$\operatorname{por}^*(A, x, r) = \sup\{\varrho \geq 0 : B(y, \varrho r) \subset B(x, r) \setminus A \text{ for some } y \in X\}$$

and

$$\operatorname{por}^*(A, x) = \liminf_{r \downarrow 0} \operatorname{por}^*(A, x, r)$$

to guarantee that $0 \leq \operatorname{por}(A, x, r) \leq \frac{1}{2}$ for all $A \subset X$, $x \in A$ and $r > 0$. From the point of view of our results, however, there is no difference between por and por^* since we always have $\operatorname{por}(A, x, r) \leq \operatorname{por}^*(A, x, r) \leq 2\operatorname{por}(A, x, 2r)$, and therefore, $\operatorname{por}(A, x) \leq \operatorname{por}^*(A, x) \leq 2\operatorname{por}(A, x)$.

(3) To emphasize the underlying metric space, we write $\operatorname{por}^*_{(X,d)}$ instead of por^* in what follows. Observe that $0 \leq \operatorname{por}^*_{(\mathbb{R}^n, |\cdot|)}(A) \leq \frac{1}{2}$ for all $A \subset \mathbb{R}^n$, where $|\cdot|$ denotes the usual Euclidean metric. This is not necessarily true in general metric spaces. Indeed, choosing $0 < \varepsilon < 1$, we have $\operatorname{por}^*_{(\mathbb{R}^n, |\cdot|^\varepsilon)}(A) = \operatorname{por}^*_{(\mathbb{R}^n, |\cdot|)}(A)^\varepsilon$ for every $A \subset \mathbb{R}^n$. Hence, for example, $\operatorname{por}^*_{(\mathbb{R}^n, |\cdot|^\varepsilon)}(\{x\}) = (\frac{1}{2})^\varepsilon$ for every $x \in \mathbb{R}^n$. In the following remark, we show that $*$-porosity may be exactly one.

(4) We work in $\mathbb{R}^2$ with the polar coordinates. Define

$$X = \{(lq, 2\pi q) : 0 \leq l \leq 1 \text{ and } q \in \mathbb{Q} \cap [0, 1)\}$$

and equip X with the path metric. We claim that $\operatorname{por}^*\big(\{(0,0)\}\big) = 1$. Let $0 < r < 1$. For each $i \in \mathbb{N}$ choose $q_i \in \mathbb{Q} \cap [0, r]$ such that $\sup\{q_i : i \in \mathbb{N}\} = r$. It follows immediately that for every $i \in \mathbb{N}$ and $\varepsilon > 0$

$$B\big((q_i, 2\pi q_i), q_i - \varepsilon\big) \subset B\big((0,0), r\big) \setminus \{(0,0)\},$$

that is, $\operatorname{por}^*\big(\{(0,0)\}, (0,0), r\big) \geq (q_i - \varepsilon)/r$. Hence $\operatorname{por}^*\big(\{(0,0)\}, (0,0), r\big) = 1$ for every $0 < r < 1$ and the claim is proved.

(5) The following simple but extremely useful fact will be frequently needed: If $\mathrm{por}(A) > \varrho$, then $A = \cup_{k\in\mathbb{N}} A_k$ where

$$A_k = \{x \in A \,:\, \mathrm{por}(A,x,r) > \varrho \text{ for all } 0 < r < \frac{1}{k}\}.$$

Furthermore, given any $\varepsilon > 0$, we may, using the separability, write A_k as a union of small pieces A_{kj} such that $A_k = \cup_{j\in\mathbb{N}} A_{kj}$ and $\mathrm{diam}(A_{kj}) < \varepsilon$ for all j. (Here diam is the diameter of a set.)

3. Measure theory in metric spaces

This section contains some basic facts of measure and dimension theory in metric spaces that will be needed later. Recall that X is a separable metric space. By a measure we always mean a Borel regular outer measure defined on all subsets of X, see [M2, Definition 1.1]. We say that μ is σ-finite if $X = \cup_{k\in\mathbb{N}} A_k$ where $\mu(A_k) < \infty$ for each k.

The separability assumption is natural given our interest in dimension estimates since the Hausdorff dimension of a non-separable metric space X is infinite and usually one can find porous sets $A \subset X$ that are non-separable. Moreover, no σ-finite doubling measures exist in non-separable spaces.

We denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure defined on X. As in [M2, §5.3], we define for a bounded set $A \subset X$, $\lambda \geq 0$ and $r > 0$

$$M^\lambda(A,r) = \inf\{kr^\lambda : A \subset \bigcup_{i=1}^{k} B(x_i,r) \text{ for some } x_i \in X, k \in \mathbb{N}\}$$

with the interpretation $\inf\emptyset = \infty$. The (upper) Minkowski dimension of a bounded set A is

$$\dim_{\mathrm{M}}(A) = \inf\{\lambda : \limsup_{r\downarrow 0} M^\lambda(A,r) < \infty\}.$$

The packing dimension of $A \subset X$ is given by

$$\dim_{\mathrm{p}}(A) = \inf\bigl\{\sup_i \dim_{\mathrm{M}}(A_i) \,:\, A_i \text{ is bounded and } A \subset \bigcup_{i=1}^{\infty} A_i\bigr\}.$$

Alternatively, the packing dimension may be defined in terms of the (radius based) packing measures $\mathcal{P}^s$ (see Cutler [Cu, §3.1] for the definition) by the identity (here $\sup\emptyset = 0$)

$$\dim_{\mathrm{p}}(A) = \sup\{s \geq 0 \,:\, \mathcal{P}^s(A) > 0\},$$

see [Cu, Theorem 3.11]. Since $\mathcal{H}^\lambda(A) \leq \liminf_{r\downarrow 0} M^\lambda(A,r)$ for all bounded sets $A \subset X$, we immediately get $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{p}}(A) \leq \dim_{\mathrm{M}}(A)$, where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. It is also easy to see that $\dim_{\mathrm{p}}(X) < \infty$ whenever X carries a doubling measure, consult [Cu, Theorem 3.16].

Let $s > 0$. A measure μ on X is *s-regular on a set $A \subset X$* if there are constants $0 < a_\mu \le b_\mu$ and $r_\mu > 0$ such that

$$a_\mu r^s \le \mu\big(B(x,r)\big) \le b_\mu r^s \tag{3.1}$$

for all $x \in A$ and $0 < r < r_\mu$. A set $A \subset X$ is *s-regular* if there is a measure μ which is s-regular on A and $\mu(X \setminus A) = 0$. In particular, a metric space X is *s-regular* if there is a measure μ which is s-regular on X.

A measure μ on X is called *doubling* if there are constants $c_\mu \ge 1$ and $r_\mu > 0$ such that

$$0 < \mu\big(B(x,2r)\big) \le c_\mu \mu\big(B(x,r)\big) < \infty \tag{3.2}$$

for every $x \in X$ and $0 < r < r_\mu$. A metric space is *doubling* if there exists a constant $N \in \mathbb{N}$ such that for each $r > 0$, every closed ball with radius $2r$ can be covered by a family of at most N closed balls of radius r. Notice that an s-regular measure on X is doubling, and moreover, by [LS], every complete doubling metric space carries a doubling measure.

Often in the literature it is assumed that (3.1) and (3.2) are valid for all $0 < r \le \operatorname{diam}(X)$, that is, μ is globally s-regular or doubling (see for example [BHR]). However, for our purposes this is not needed by Remark 2.1.(5). The following example shows that it is not always possible to choose $r_\mu = \operatorname{diam}(X)$.

Example 3.1. Equip $X = [0,1] \times \mathbb{N} \subset \mathbb{R}^2$ with the metric d defined by

$$d\big((x_1,y_1),(x_2,y_2)\big) = \begin{cases} |x_1 - x_2|, & y_1 = y_2, \\ 1, & y_1 \ne y_2. \end{cases}$$

Let $\mu = \mathcal{H}^1$ be the length measure on X. Now $r \le \mu(B(x,r)) \le 2r$ whenever $0 < r < 1$, but $\mu(B(x,1)) = \infty$ for all $x \in X$.

It is straightforward to see that in separable metric spaces all doubling measures are σ-finite, in particular, this is true for s-regular measures.

An easy exercise leads to the following lemma:

Lemma 3.2. *Suppose that μ is a doubling measure on X. For all $x \in X$, $0 < r < r_\mu$ and $\alpha > 1$ we have*

$$\mu\big(B(x,\alpha r)\big) \le c_\mu^{\frac{\log \alpha}{\log 2}+1} \mu\big(B(x,r)\big). \tag{3.3}$$

Moreover, if μ is s-regular, then

$$\mu\big(B(x,\alpha r)\big) \le \frac{b_\mu}{a_\mu} \alpha^s \mu\big(B(x,r)\big) \tag{3.4}$$

for all $x \in X$, $0 < r < r_\mu$ and $\alpha > 1$.

Let μ be an s-regular measure on X and $A \subset X$. For all $\lambda \ge 0$ and $r > 0$ we define

$$M_\mu^\lambda(A,r) = \frac{\mu\big(A(r)\big)}{r^{s-\lambda}},$$

where

$$A(r) = \{x \in X : \operatorname{dist}(x, A) < r\}$$

is the open r-neighbourhood of A and $\operatorname{dist}(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance of x from A. The following easy lemma shows how $M_\mu^\lambda(A, r)$ can be utilized to calculate $\dim_{\mathrm{M}}(A)$. We give a detailed proof for the convenience of the reader.

Lemma 3.3. *Suppose that μ is an s-regular measure on X. Let $A \subset X$ and $\lambda \geq 0$. Then*

$$2^{-s} b_\mu^{-1} M_\mu^\lambda(A, r) \leq M^\lambda(A, r) \leq 2^s a_\mu^{-1} M_\mu^\lambda(A, r)$$

whenever $0 < r < \frac{r_\mu}{2}$.

Proof. Fix $0 < r < \frac{r_\mu}{2}$. For the right-hand side inequality, we may assume that $\mu\big(A(r)\big) < \infty$. Since X is separable, there exists a maximal collection of mutually disjoint balls $\{B(x_i, \frac{r}{2})\}_{i \in I}$, where I is a countable index set and $x_i \in A$ for all $i \in I$. Observe that the maximality trivially implies

$$A \subset \bigcup_{i \in I} B(x_i, r).$$

Let $\#I$ be the number of elements in I. Since

$$\mu\big(A(r)\big) \geq \mu\Big(\bigcup_{i \in I} B(x_i, \frac{r}{2})\Big) = \sum_{i \in I} \mu\big(B(x_i, \frac{r}{2})\big) \geq \#I a_\mu 2^{-s} r^s,$$

it follows that $\#I < \infty$. This in turn implies that

$$M_\mu^\lambda(A, r) = \frac{\mu\big(A(r)\big)}{r^{s-\lambda}} \geq a_\mu 2^{-s} \#I r^\lambda \geq a_\mu 2^{-s} M^\lambda(A, r).$$

For the left-hand side inequality, we may assume that $M^\lambda(A, r) < \infty$. Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in X$ such that

$$A \subset \bigcup_{i=1}^k B(x_i, r) \quad \text{and} \quad M^\lambda(A, r) \geq k r^\lambda - \varepsilon.$$

Since now

$$\mu\big(A(r)\big) \leq \mu\Big(\bigcup_{i=1}^k B(x_i, 2r)\Big) \leq \sum_{i=1}^k \mu\big(B(x_i, 2r)\big) \leq k b_\mu (2r)^s,$$

we get

$$M_\mu^\lambda(A, r) = \frac{\mu\big(A(r)\big)}{r^{s-\lambda}} \leq 2^s b_\mu k r^\lambda \leq 2^s b_\mu \big(M^\lambda(A, r) + \varepsilon\big).$$

The proof is finished by letting $\varepsilon \downarrow 0$. □

The next observation shows that any porous set on a space carrying a doubling measure must have zero measure. Note that the proposition (with its simple proof) is easily seen to hold for upper porous sets as well.

Proposition 3.4. *Suppose that μ is a doubling measure on X. If $A \subset X$ is porous then $\mu(A) = 0$.*

Proof. By Remark 2.1.(5), we may assume that A is uniformly ϱ-porous for some $\varrho > 0$. Furthermore, we may assume that A is closed since it is clear from the definition that the closure of a uniformly ϱ-porous set is uniformly ϱ-porous. Assume on the contrary that $\mu(A) > 0$. By Remark 2.1.(5), we may assume that μ is globally doubling. Thus using the density point theorem [H, Theorem 1.8], we choose $x \in A$ for which

$$\lim_{r \downarrow 0} \frac{\mu\big(A \cap B(x,r)\big)}{\mu\big(B(x,r)\big)} = 1. \tag{3.5}$$

Since A is uniformly ϱ-porous we find $0 < r_p < r_\mu$ such that for all $0 < r < r_p$ and $\varrho' < \varrho$ there exists $y \in X$ for which

$$B(y, \varrho' r) \subset B(x,r) \setminus A.$$

Hence by Lemma 3.2, we get for any $0 < r < r_p$

$$\frac{\mu\big(B(x,r) \setminus A\big)}{\mu\big(B(x,r)\big)} \geq \frac{\mu\big(B(y,\varrho' r)\big)}{\mu\big(B(x,r)\big)} \geq c_\mu^{\frac{\log \varrho'}{\log 2} - 1} > 0,$$

contrary to (3.5). □

4. Dimension estimates for porous sets

It is well known that in $\mathbb{R}^n$

$$\dim_{\mathrm{p}}(A) \leq n - c\varrho^n \tag{4.1}$$

for any ϱ-porous set $A \subset \mathbb{R}^n$. Here c is a positive constant depending on n. In particular, $\dim_{\mathrm{p}}(A) < n$ for all porous sets $A \subset \mathbb{R}^n$. In this section we discuss whether these estimates are valid in s-regular metric measure spaces and, more generally, on spaces that carry a doubling measure. In [DS, Lemma 5.8] it is stated that in the s-regular case $\dim_{\mathrm{p}}(A) \leq s - \eta$, where η depends on porosity and the constants of the s-regular measure. The proof is based on generalized dyadic cubes whose side lengths are powers of ϱ. Thus (as in $\mathbb{R}^n$) this argument will give that $\eta = c(\log \frac{1}{\varrho})^{-1} \varrho^s$. In [BHR, Lemma 3.12] a different method is used to show that even the Assouad dimension of a porous subset of a globally s-regular space is less than s. (Recall that the Assouad dimension is always at least the packing dimension.) Pushing this argument further we will show that the factor $(\log \frac{1}{\varrho})^{-1}$ is not needed in the s-regular case. As a tool we need the following generalization of (2.1)-(2.3) which is a modification of the mean ε-porosity from [KR].

Definition 4.1. Let $0 < \varrho \leq 1$, $D > 1$, $0 < p \leq 1$ and $n_0, k_0 \in \mathbb{N}$. For all $k \in \mathbb{N}$ and $x \in X$, we denote by $A_k(x)$ the annulus

$$A_k(x) = \{y \in X \,:\, D^{-k} < d(x,y) \leq D^{-k+1}\}.$$

Furthermore, for $A \subset X$ define

$$\psi_k(x) = \begin{cases} 1 & \text{if } A_k(x) \text{ contains } y \text{ with } \operatorname{dist}(y,A) \geq \varrho d(y,x) \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$S_{k_0,n}(x) = \sum_{k=k_0+1}^{k_0+n} \psi_k(x).$$

The set $A \subset X$ is $(\varrho, D, p, n_0, k_0)$-mean porous if $S_{k_0,n}(x) \geq pn$ for all $x \in A$ and $n \geq n_0$.

Lemma 4.3 generalizes the arguments of [MV, Lemma 2.8], [KR, Theorem 2.1] and [BHR, Lemma 3.12] to our setting. For the purpose of proving it, we state an auxiliary result that can be found from [B, Lemma 4.2] and for the convenience of the reader we prove it. Moreover, we use the notation χ_B for the characteristic function of a set $B \subset X$.

Lemma 4.2. *Suppose that μ is a globally doubling measure on X, that is, $r_\mu = \operatorname{diam}(X)$. Let $\{x_k\}_{k\in\mathbb{N}}$ be a collection of points in X and let $\{r_k\}_{k\in\mathbb{N}}$ and $\{a_k\}_{k\in\mathbb{N}}$ be sequences of positive real numbers. Then for all $R \geq 1$ and $1 \leq q < \infty$*

$$\| \sum_{k\in\mathbb{N}} a_k \chi_{B(x_k, Rr_k)} \|_{L^q_\mu(X)} \leq C_B R^t q \| \sum_{k\in\mathbb{N}} a_k \chi_{B(x_k, r_k)} \|_{L^q_\mu(X)},$$

where $t = \frac{\log c_\mu}{\log 2}$ and C_B depends on c_μ only. Moreover, if μ is s-regular, then we may choose $t = s$ and C_B depends only on a_μ, b_μ and s.

Proof. A straightforward calculation gives the claim in the case $q = 1$. For the case $q > 1$, let $\phi \in L^p(X)$, where $p = q/(q-1)$. Then we have

$$\begin{aligned}
&\int_X |\phi(x) \sum_{k\in\mathbb{N}} a_k \chi_{B(x_k,Rr_k)}(x)|\, d\mu(x) \\
&\qquad \le \sum_{k\in\mathbb{N}} a_k \int_{B(x_k,Rr_k)} |\phi(x)|\, d\mu(x) \\
&\qquad \le \sum_{k\in\mathbb{N}} a_k \mu(B(x_k,Rr_k)) \inf_{x\in B(x_k,r_k)} M\phi(x) \\
&\qquad \le c_\mu R^t \int_X M\phi(x) \sum_{k\in\mathbb{N}} a_k \chi_{B(x_k,r_k)}\, d\mu(x) \\
&\qquad \le c_\mu R^t \Big(\int_X (M\phi)^p\, d\mu\Big)^{\frac{1}{p}} \Big(\int_X \big(\sum_{k\in\mathbb{N}} a_k \chi_{B(x_k,r_k)}\big)^q\, d\mu\Big)^{\frac{1}{q}} \\
&\qquad \le c_\mu R^t c_1 \Big(\frac{p}{p-1}\Big)^{\frac{1}{p}} \Big(\int_X |\phi|^p\, d\mu\Big)^{\frac{1}{p}} \Big(\int_X \big(\sum_{k\in\mathbb{N}} a_k \chi_{B(x_k,r_k)}\big)^q\, d\mu\Big)^{\frac{1}{q}},
\end{aligned}$$

where we have used the non-centred Hardy-Littlewood maximal function

$$M\phi(x) = \sup_{B(y,r)\ni x} \mu(B(y,r))^{-1} \int_{B(y,r)} |\phi|\, d\mu,$$

Lemma 3.2, Hölder's inequality and the fact that when $p > 1$

$$\int_X (M\phi)^p\, d\mu \le c_1^p \frac{p}{p-1} \int_X |\phi|^p\, d\mu,$$

where c_1 depends only on c_μ (see [H, Theorem 2.2, Remark 2.5]). Thus the claim follows from the duality of L^q-spaces.

In the case μ is s-regular replace in the proof c_μ with b_μ/a_μ and t with s and notice that c_1 depends in this case only on a_μ, b_μ and s. □

Lemma 4.3. *Suppose that μ is a doubling measure on X. Let $x_0 \in X$ and $0 < R < \frac{r_\mu}{2}$. If $A \subset B(x_0,R)$ is $(\varrho, D, p, n_0, k_0)$-mean porous, then there is an absolute constant C_1 and a constant C_2 that depends only on c_μ so that*

$$\mu(A(r)) \le C_1 D^{k_0\delta} \mu(A(2D^{-k_0})) r^\delta \text{ for all } r < D^{-n_0-k_0}$$

where $\delta = C_2(\log D)^{t-1} D^{-3t} p \varrho^t$ and $t = \frac{\log c_\mu}{\log 2}$. Moreover, if μ is s-regular then we may choose $t = s$ and C_2 depends only on a_μ, b_μ and s.

Proof. By restricting our measure μ to the ball $B(x_0,R)$ we may assume that it is globally doubling (or globally s-regular). Define

$$\widetilde{\mathcal{B}} = \big\{B(z,r_z) : z \in A(D^{-k_0}) \setminus \overline{A} \text{ and } r_z = \frac{\log D}{20D^2} \operatorname{dist}(z,A)\big\}.$$

By the $5r$-covering theorem (see [H, Theorem 1.2]) we find a countable pairwise disjoint subfamily $\mathcal{B}$ of $\widetilde{\mathcal{B}}$ such that

$$A(D^{-k_0}) \setminus \overline{A} \subset \bigcup_{B(z,r_z)\in\mathcal{B}} B(z,5r_z). \tag{4.2}$$

Letting $j \in \mathbb{N}$ and $x \in A(2^{-j})$, choose $x' \in A$ such that $d(x,x') < 2^{-j}$. Assume that there is $k \geq k_0+1$ with $\psi_k(x') = 1$. Take $y \in A_k(x')$ such that $\operatorname{dist}(y,A) \geq \varrho d(y,x')$. Using (4.2) we find $B(z,r_z) \in \mathcal{B}$ such that $y \in B(z,5r_z)$. We proceed by showing that

$$B(z,5r_z) \subset A_{k+1}(x') \cup A_k(x') \cup A_{k-1}(x'). \tag{4.3}$$

Since

$$\operatorname{dist}(z,A) \leq d(z,y) + \operatorname{dist}(y,A) \leq 5r_z + D^{-k+1} = \frac{\log D}{4D^2}\operatorname{dist}(z,A) + D^{-k+1},$$

we obtain

$$\operatorname{dist}(z,A) \leq D^{-k+1}\big(1 - \frac{\log D}{4D^2}\big)^{-1},$$

giving

$$10r_z \leq \frac{\log D}{2D^2}\big(1 - \frac{\log D}{4D^2}\big)^{-1} D^{-k+1} \leq \frac{2}{3}\log D D^{-k-1}.$$

The claim (4.3) now follows as $\operatorname{dist}(X \setminus (A_{k-1}(x') \cup A_k(x') \cup A_{k+1}(x')), A_k(x')) \geq D^{-k-1}(D-1) > 10r_z$, and $B(z,5r_z) \cap A_k(x') \neq \emptyset$.

Next we conclude that

$$x \in B\big(z, \frac{75D^3}{\varrho \log D} r_z\big) \tag{4.4}$$

under the assumption that $D^{-k} \geq 2^{-j}$. Indeed, since

$$\operatorname{dist}(z,A) \geq \operatorname{dist}(y,A) - d(z,y) \geq \varrho d(y,x') - 5r_z = \varrho d(y,x') - \frac{\log D}{4D^2}\operatorname{dist}(z,A),$$

we get

$$\operatorname{dist}(z,A) \geq \varrho d(y,x')\big(1 + \frac{\log D}{4D^2}\big)^{-1}.$$

This in turn implies that

$$r_z \geq \frac{\log D}{20D^2}\varrho d(y,x')\big(1 + \frac{\log D}{4D^2}\big)^{-1} \geq \frac{\varrho \log D}{25} D^{-k-2}.$$

Hence

$$D^{-k} \leq \frac{25D^2}{\varrho \log D} r_z,$$

and therefore,

$$\begin{aligned} d(x,z) &\leq d(x,x') + d(x',y) + d(y,z) \leq 2^{-j} + D^{-k+1} + 5r_z \\ &\leq D^{-k} + D^{-k+1} + 5r_z \leq \frac{25D^2}{\varrho \log D} r_z + \frac{25D^3}{\varrho \log D} r_z + \frac{25D^3}{\varrho \log D} r_z \leq \frac{75D^3}{\varrho \log D} r_z. \end{aligned}$$

This gives (4.4).

Clearly, $D^{-k} \geq 2^{-j}$ provided that $k \leq \frac{\log 2}{\log D} j$. Thus if $\psi_k(x') = 1$ for $k_0 + 1 \leq k \leq \frac{\log 2}{\log D} j$ we find, by (4.4), a ball $B(x_k, r_{x_k}) \in \mathcal{B}$ such that $x \in B(x_k, \frac{75D^3}{\varrho \log D} r_{x_k})$. The fact that A is $(\varrho, D, p, n_0, k_0)$-mean porous gives

$$S_{k_0,n}(x') = \sum_{k=k_0+1}^{k_0+n} \psi_k(x') \geq pn \text{ whenever } n \geq n_0.$$

Letting $j_0 > \frac{\log D}{\log 2}(n_0 + k_0)$, we have for all $j \geq j_0$

$$\#\{k : k_0 + 1 \leq k \leq \frac{\log 2}{\log D} j \text{ and } \psi_k(x') = 1\} \geq \frac{p}{2}\Big(\frac{\log 2}{\log D} j - k_0\Big).$$

Combining this with (4.3) implies that for all $j \geq j_0$ and $x \in A(2^{-j})$

$$\sum_{B(z,r_z) \in \mathcal{B}} \chi_{B(z, \frac{75D^3}{\varrho \log D} r_z)}(x) \geq \frac{p}{6}\Big(\frac{\log 2}{\log D} j - k_0\Big). \tag{4.5}$$

Indeed, for at least $\frac{p}{2}(\frac{\log 2}{\log D} j - k_0)$ different k's we find a ball $B(x_k, r_{x_k}) \in \mathcal{B}$ such that $x \in B(x_k, \frac{75D^3}{\varrho \log D} r_{x_k})$. However, because of (4.3) each $B(x_k, r_{x_k})$ can be taken into account at most three times.

We finish the proof by verifying that for all $j \geq j_0$

$$\mu(A(2^{-j})) \leq 11 D^{k_0 \delta} \mu(A(2D^{-k_0})) 2^{-j\delta} \tag{4.6}$$

where

$$\delta = \frac{\log 2}{18 C_B 75^t} (\log D)^{t-1} D^{-3t} p \varrho^t \text{ and } t = \frac{\log c_\mu}{\log 2},$$

and C_B is the constant from Lemma 4.2. (If μ is s-regular replace here and at the rest of the proof t with s.) Notice that $\delta < 1$. Our claim easily follows from this. For (4.6) it suffices to show that for all $j \geq j_0$

$$\int_{A(2^{-j})} 2^{\gamma(\varrho \log D)^t (\frac{p}{6} k_0 + \sum_{B(z,r_z) \in \mathcal{B}} \chi_{B(z, \frac{75D^3}{\varrho \log D} r_z)}(x))} \, d\mu(x) \leq 11 D^{k_0 \delta} \mu(A(2D^{-k_0})) \tag{4.7}$$

with $\gamma = (3C_B(75D^3)^t)^{-1}$. This is so because from (4.5) we obtain for all $x \in A(2^{-j})$ that

$$2^{\gamma(\varrho \log D)^t \sum_{B(z,r_z) \in \mathcal{B}} \chi_{B(z, \frac{75D^3}{\varrho \log D} r_z)}(x))} \geq 2^{\gamma(\varrho \log D)^t \frac{p}{6}(\frac{\log 2}{\log D} j - k_0)}.$$

Moreover, combining this with (4.7) gives

$$\begin{aligned}
\mu(A(2^{-j})) &= \mu(A(2^{-j})) 2^{-j\gamma(\varrho \log D)^t \frac{p}{6} \frac{\log 2}{\log D}} 2^{j\gamma(\varrho \log D)^t \frac{p}{6} \frac{\log 2}{\log D}} \\
&= 2^{-j\gamma(\varrho \log D)^t \frac{p}{6} \frac{\log 2}{\log D}} \int_{A(2^{-j})} 2^{\gamma(\varrho \log D)^t \frac{p}{6} \frac{\log 2}{\log D} j} \, d\mu \\
&\leq 11 D^{k_0 \delta} \mu(A(2D^{-k_0})) 2^{-j\gamma(\varrho \log D)^t \frac{p}{6} \frac{\log 2}{\log D}},
\end{aligned}$$

and therefore (4.6) is valid.

To prove (4.7), write

$$u(x)=\gamma(\varrho\log D)^t\sum_{B(z,r_z)\in\mathcal{B}}\chi_{B(z,\frac{75D^3}{\varrho\log D}r_z)}(x).$$

Now we obtain from Lemma 4.2 (used with $q=k$) and from the fact that for $B\in\mathcal{B}$ we have $B\subset A(2D^{-k_0})$ that

$$\begin{aligned}\int_{A(2^{-j})}2^{u(x)}\,d\mu(x)&\le\int_{A(D^{-k_0})}\exp(u(x))\,d\mu(x)\\&\le\mu(A(D^{-k_0}))+\sum_{k=1}^{\infty}\frac{(\gamma(\varrho\log D)^t)^k}{k!}\int_X\Big(\sum_{B(z,r_z)\in\mathcal{B}}\chi_{B(z,\frac{75D^3}{\varrho\log D}r_z)}(x)\Big)^k\,d\mu(x)\\&\le\mu(A(D^{-k_0}))+\sum_{k=1}^{\infty}\frac{(\gamma(\varrho\log D)^tC_B(75D^3)^tk)^k}{k!(\varrho\log D)^{tk}}\int_X\Big(\sum_{B\in\mathcal{B}}\chi_B(x)\Big)^k\,d\mu(x)\\&\le\mu(A(2D^{-k_0}))\Big(1+\sum_{k=1}^{\infty}\frac{(\gamma C_B(75D^3)^tk)^k}{k!}\Big)\\&=\mu(A(2D^{-k_0}))\Big(1+\sum_{k=1}^{\infty}\frac{1}{k!}\Big(\frac{k}{3}\Big)^k\Big)\\&\le 11\mu(A(2D^{-k_0})),\end{aligned}$$

where the last inequality follows since

$$\frac{\frac{1}{(k+1)!}\big(\frac{k+1}{3}\big)^{k+1}}{\frac{1}{k!}\big(\frac{k}{3}\big)^k}=\frac{1}{3}\Big(1+\frac{1}{k}\Big)^k\uparrow\frac{e}{3}$$

as $k\to\infty$. Finally,

$$\begin{aligned}\int_{A(2^{-j})}&2^{\gamma(\varrho\log D)^t(\frac{p}{6}k_0+\sum_{B(z,r_z)\in\mathcal{B}}\chi_{B(z,\frac{75D^3}{\varrho\log D}r_z)}(x))}\,d\mu(x)\\&=2^{\gamma(\varrho\log D)^t\frac{p}{6}k_0}\int_{A(2^{-j})}2^{u(x)}\,d\mu(x)\\&\le 11\cdot2^{\gamma(\varrho\log D)^t\frac{p}{6}k_0}\mu(A(2D^{-k_0}))\\&=11D^{k_0\delta}\mu(A(2D^{-k_0}))\end{aligned}$$

finishing the proof. □

Remark 4.4. (1) Assume that $\operatorname{por}(A,x,r)>\varrho$ for all $x\in A$ and $0<r<r_p$. Let $D>1$. Choose $k_0\in\mathbb{N}$ such that $D^{-k_0}<r_p$. Then there is $y\in X$ such that $B(y,\varrho D^{-k_0})\subset B(x,D^{-k_0})\setminus A$ which in turn implies that $y\in A_k(x)$ for some $k_0+1\le k\le k_0-\lfloor\frac{\log\varrho}{\log D}\rfloor$, where $\lfloor a\rfloor$ is the greatest integer i satisfying $i\le a$. Repeating this argument for $k_0-m\lfloor\frac{\log\varrho}{\log D}\rfloor$, $m=1,2,\dots$, we see that

A is $(\varrho, D, -\frac{1}{2}\lfloor\frac{\log\varrho}{\log D}\rfloor^{-1}, -\lfloor\frac{\log\varrho}{\log D}\rfloor, k_0)$-mean porous. Note that it is not possible to obtain mean porosity with $p=1$ or p close to 1 unless one takes $D=\frac{1}{\varrho}$. The reason for this is that in general metric spaces for fixed D the annuli $A_k(x)$ may be empty for many k's. However, in s-regular spaces one may find a D independent of ϱ such that all annuli are non-empty as explained in the next remark.

(2) Assume that μ is an s-regular measure on X and $\operatorname{por}(A,x,r)>\varrho$ for all $x\in A$ and $0<r<r_p$. Let $l=\frac{1}{2}(\frac{a_\mu}{b_\mu})^{\frac{1}{s}}$ and $D=2(1-l)^{-1}l^{-2}$. Choose $k_0\in\mathbb{N}$ such that $D^{-k_0}<\min\{r_p,r_\mu\}$. We verify that A is $(\frac{1}{3}l^2\varrho, D, 1, 1, k_0)$-mean porous. Consider $0<r\le D^{-k_0}$. Since μ is s-regular and $r<r_\mu$ there exists $y\in B(x,lr)\setminus B(x,l^2r)$. Assuming that there is $z\in B(y,\frac{1}{2}l^2r)\cap A$, we find, using uniform porosity of A and the fact $r<r_p$, $w\in X$ such that $B(w,\frac{1}{2}\varrho l^3r)\cap A=\emptyset$ and $\frac{1}{2}\varrho l^3r+d(w,z)\le\frac{1}{2}l^3r$. This gives

$$d(w,x)\ge d(x,y)-d(y,z)-d(z,w)\ge\frac{1}{2}l^2(1-l)r=\frac{r}{D}$$

and

$$d(w,x)\le d(x,y)+d(y,z)+d(z,w)<\frac{3}{2}lr<r.$$

Letting $k\ge k_0+1$ and choosing $r=D^{-(k-1)}$ in the above inequalities gives $w\in A_k(x)$ for all $k\ge k_0+1$. This combined with the fact that

$$\operatorname{dist}(w,A)\ge\frac{1}{2}\varrho l^3r>\frac{1}{3}l^2\varrho d(w,x)$$

gives the claim. If $B(y,\frac{1}{2}l^2r)\cap A=\emptyset$ we may take $w=y$.

In the following two corollaries we verify scaling properties of measures of r-neighbourhoods of bounded uniformly porous sets for small scales r.

Corollary 4.5. *Suppose that μ is a doubling measure on X. Let $x_0\in X$ and $0<r_0<\frac{\varrho}{12}\min\{r_p,r_\mu\}$. Then there exists a constant $C_3>0$ which depends only on c_μ such that for any uniformly ϱ-porous set $A\subset X$ we have*

$$\mu\big((A\cap B(x_0,r_0))(r)\big)\le C_3c_\mu^{-\frac{\log\varrho}{\log 2}}\mu(B(x_0,r_0))\Big(\frac{r}{r_0}\Big)^\delta \text{ for all } 0<r<r_0,$$

where $t=\frac{\log c_\mu}{\log 2}$ and $\delta=c(\log\frac{1}{\varrho})^{-1}\varrho^t$ with c depending only on c_μ.

Proof. Let $0<\varrho'<\varrho$ be such that $r_0<\frac{\varrho'}{12}\min\{r_p,r_\mu\}$. Choose k_0 to be the largest integer so that $\frac{6}{\varrho'}r_0\le 2^{-k_0}<\min\{r_p,r_\mu\}$. By Remark 4.4.(1), A is $(\varrho',2,p,n_0,k_0)$-mean porous for $p=c_0\log(\frac{1}{\varrho'})^{-1}$ and $n_0=-\lfloor\frac{\log\varrho'}{\log 2}\rfloor$, where $c_0>0$ is an absolute constant. Thus by Lemma 4.3 we obtain for all $0<r<2^{-n_0-k_0}$ that

$$\mu\big((A\cap B(x_0,r_0))(r)\big)\le C_12^{k_0\delta}\mu\big((A\cap B(x_0,r_0))(2^{-k_0+1})\big)r^\delta.$$

Notice that $(A\cap B(x_0,r_0))(2^{-k_0+1})\subset B(x_0,\frac{25}{\varrho'}r_0)$. The claim follows by applying the doubling condition and using Lemma 3.2, by noting that $r_0<2^{-n_0-k_0}$, $2^{k_0\delta}<r_0^{-\delta}$ and by letting ϱ' tend to ϱ. □

Corollary 4.6. *Suppose that μ is an s-regular measure on X. There exist positive constants D_1 and C_4 which depend only on a_μ, b_μ and s so that for any $x_0\in X$ and $0<r_0<D_1\min\{r_p,r_\mu\}$ we have for any uniformly ϱ-porous $A\subset X$ that*

$$\mu\big((A\cap B(x_0,r_0))(r)\big)\le C_4\mu(B(x_0,r_0))\Big(\frac{r}{r_0}\Big)^\delta \text{ for all } 0<r<r_0,$$

where $\delta=c\varrho^s$ and c depends only on a_μ,b_μ and s.

Proof. Let $l=\frac12(\frac{a_\mu}{b_\mu})^{\frac1s}$ and $D=2(1-l)^{-1}l^{-2}$. Set $D_1=\frac12 D^{-2}$. Let $0<\varrho'<\varrho$. Choose k_0 to be the largest integer so that $Dr_0\le D^{-k_0}<\min\{r_p,r_\mu\}$. By Remark 4.4.(2), A is $(\frac13 l^2\varrho',D,1,1,k_0)$-mean porous. Thus by Lemma 4.3 we obtain for all $0<r<D^{-1-k_0}$ that

$$\mu\big((A\cap B(x_0,r_0))(r)\big)\le C_1D^{k_0\delta}\mu\big((A\cap B(x_0,r_0))(2D^{-k_0})\big)r^\delta,$$

where $\delta=c(\varrho')^s$ for a constant c that depends only on a_μ,b_μ and s.

Notice that $(A\cap B(x_0,r_0))(2D^{-k_0})\subset B(x_0,(1+2D/D_1)r_0)$. The claim follows by applying the s-regularity, by noting that $r_0<D^{-1-k_0}$ and $D^{k_0\delta}<r_0^{-\delta}$ and by letting ϱ' tend to ϱ.

□

Next we prove the analogue of (4.1) for s-regular metric spaces. As mentioned in the Introduction this is asymptotically sharp as ϱ tends to zero in $\mathbb{R}^n$ and thus also in metric spaces.

Theorem 4.7. *Suppose μ is s-regular on X. If $A\subset X$ is ϱ-porous, then*

$$\dim_{\mathrm{p}}(A)\le s-c\varrho^s.$$

Moreover, if A is uniformly ϱ-porous and $\operatorname{diam}(A)<r_\mu$, *then*

$$\dim_{\mathrm{M}}(A)\le s-c\varrho^s.$$

Here c is as in Corollary 4.6.

Proof. By Remark 2.1.(5), for any $\varrho'<\varrho$ we may represent A as a countable union of sets A_{ij} with $\operatorname{diam}(A_{ij})<D_1\min\{\frac1i,r_\mu\}$ such that $\operatorname{por}(A,x,r)>\varrho'$ for all $x\in A_{ij}$ and $0<r<\frac1i$, where D_1 is as in Corollary 4.6. Moreover, if A is uniformly ϱ-porous with $\operatorname{diam}(A)<r_\mu$, then it is a finite union of such sets. Thus it is enough to show that $\dim_{\mathrm{M}}(A_{ij})\le s-\delta$ for all i and j where $\delta=c(\varrho')^s$ as in Corollary 4.6. Letting $x\in A_{ij}$ and using Lemma 3.3 and Corollary 4.6 we

get for large i

$$\begin{aligned}\limsup_{r\downarrow 0} M^\lambda(A_{ij},r) &\le 2^s a_\mu^{-1}\limsup_{r\downarrow 0}\frac{\mu\big(A_{ij}(r)\big)}{r^{s-\lambda}}\\ &\le 2^s a_\mu^{-1} C\limsup_{r\downarrow 0}\mu\big(B(x,1/i)\big)i^\delta r^{\delta+\lambda-s}=0\end{aligned}$$

if $\lambda > s-\delta$. Here C is a constant which is independent of r. This gives the claim. □

Theorem 4.7 is not true if we only assume that μ is doubling. An easy example is given by defining $X = (\{0\}\cup_{j\in\mathbb{N}}\{2^{-j}\})\times[0,1]$ with the metric inherited from $\mathbb{R}^2$ and letting μ be any doubling measure on X. If $N=\{0\}\times[0,1]$, then $\dim_{\mathrm{M}}(N)=\dim_{\mathrm{p}}(N)=1=\dim_{\mathrm{p}}(X)$. However, it is easy to see that N is uniformly $\frac{1}{3}$-porous. The following example shows that one can perceive similar behaviour in geodesic metric spaces as well, even for maximally porous sets. Recall that a metric space (X,d) is geodesic if for each pair of points $x,y\in X$ there exists a path $\gamma\colon[0,1]\to X$ such that $\gamma(0)=x$, $\gamma(1)=y$, and the length of γ is equal to $d(x,y)$.

Example 4.8. We give an example of a complete geodesic doubling metric space having a subset with maximal dimension and porosity. The construction is an infinite tree with branches getting smaller and smaller as we go deeper into the tree. The metric is the natural path metric induced by the branches (see Figure 1).

Letting

$$\mathcal{N}_0=\{\overline{0}\}\ ,\ \mathcal{N}_n=\{1,2\}^n\times]0,2^{-n}]\text{ for all }\mathbb{N}\setminus\{0\}\text{ and }\mathcal{N}_\infty=\{1,2\}^{\mathbb{N}},$$

define

$$\mathcal{N}=\mathcal{N}_\infty\cup\bigcup_{n=0}^{\infty}\mathcal{N}_n.$$

The metric is given as follows: Put $n(\overline{0})=0$ and for $\overline{0}\ne x\in\mathcal{N}$, let $n(x)\in\mathbb{N}\cup\{\infty\}$ such that $x=(x_1,\ldots,x_{n(x)},x_{n(x)+1})\in\mathcal{N}_{n(x)}$. We denote the distance of x from the root $\overline{0}$ by

$$l(x)=\begin{cases}1-2^{-(n(x)-1)}+x_{n(x)+1} & \text{, if } 0<n(x)<\infty\\ 1 & \text{, if } n(x)=\infty\\ 0 & \text{, if } n(x)=0.\end{cases}$$

Given $n\in\mathbb{N}$, let

$$x|_n=\begin{cases}(x_1,x_2,\ldots,x_n) & \text{, if } n(x)\ge n\\ \emptyset & \text{, if } n(x)<n\end{cases}$$

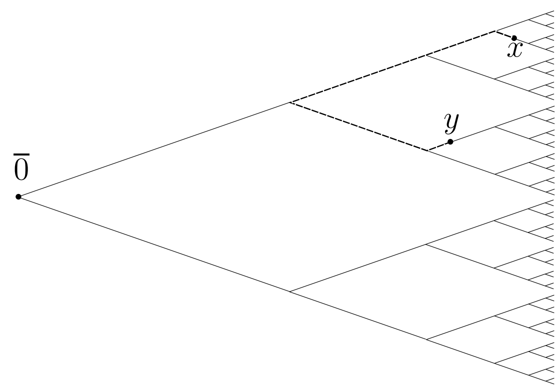

Figure 1: An illustration of space $\mathcal{N}$. The distance from x to y is measured along the dashed line.

be the restriction of x. Define the longest common route $\delta(x, y)$ of a point $x = (x_1, x_2, \ldots) \in \mathcal{N}$ and $y = (y_1, y_2, \ldots) \in \mathcal{N}$ from $\overline{0}$ by

$$\delta(x, y) = \sup\{m : x|_m = y|_m \neq \emptyset\}$$

with the convention $\sup \emptyset = 0$. The longest common part of x and y, $x \wedge y \in \mathcal{N}_{\delta(x,y)}$ is defined as

$$x \wedge y = \begin{cases} (x|_{\delta(x,y)}, \min\{2^{-\delta(x,y)}, x_{\delta(x,y)+1}, y_{\delta(x,y)+1}\}) & \text{, if } 0 < \delta(x, y) < \infty \\ \overline{0} & \text{, if } \delta(x, y) = 0 \\ x & \text{, if } \delta(x, y) = \infty. \end{cases}$$

With these notations we define the metric $d : \mathcal{N} \times \mathcal{N} \to [0, \infty[$ by

$$d(x, y) = |l(x) - l(x \wedge y)| + |l(y) - l(x \wedge y)|.$$

Geometrically, the distance $d(x, y)$ is the length of the shortest path joining x and y in the tree $\cup_{n=0}^{\infty} \mathcal{N}_n$, see Figure 1. Also, $\mathcal{N}_\infty$ is the metric boundary of $\cup_{n=0}^{\infty} \mathcal{N}_n$ and $\operatorname{dist}(\overline{0}, \mathcal{N}_\infty) = 1$.

The space $(\mathcal{N}, d)$ is obviously geodesic. We verify that

$$\dim_{\mathrm{M}}(\mathcal{N}) = \dim_{\mathrm{H}}(\mathcal{N}_\infty) = 1 \tag{4.8}$$

and

$$\operatorname{por}(\mathcal{N}_\infty) = \frac{1}{2}. \tag{4.9}$$

Indeed, since $\operatorname{dist}(\mathcal{N}_n, \mathcal{N}_\infty) = 2^{-n}$, the set $\mathcal{N} \setminus \cup_{k=1}^{n} \mathcal{N}_k$ can be covered by 2^n closed balls centred in $\mathcal{N}_n$ with radii 2^{-n}. On the other hand, $\cup_{k=1}^{n} \mathcal{N}_k$ can be covered by $n2^n$ balls with radii 2^{-n}, and therefore, $\mathcal{N}$ can be covered by $(n+1)2^n$ such balls. Hence $\dim_{\mathrm{H}}(\mathcal{N}_\infty) \leq \dim_{\mathrm{M}}(\mathcal{N}) \leq 1$. Clearly, $\dim_{\mathrm{H}}(\mathcal{N}_\infty) = 1$ which gives (4.8). For (4.9), take any $x \in \mathcal{N}_\infty$ and $0 < r < 1$. Choose $n \in \mathbb{N}$ such that $2^{-n} \le r < 2^{-n+1}$. Since $\operatorname{dist}((x|_n, 2^{-n+1} - r), \mathcal{N}_\infty) = r$, we have for all $\varepsilon > 0$

$$B((x|_n, 2^{-n+1} - r), (1-\varepsilon)r) \subset B(x, 2r) \setminus \mathcal{N}_\infty.$$

This implies (4.9).

Note that the space $(\mathcal{N}, d)$ is doubling with a doubling constant 3, that is, every closed ball with radius $2r$ can be covered with 3 closed balls with radius r. In particular, it carries a doubling measure by [LS]. Moreover, while $\cup_{k=0}^n \mathcal{N}_k$ is 1-regular for all n and $\mathcal{N}_\infty$ is also 1-regular, the set $\mathcal{N} \setminus \mathcal{N}_\infty$ is not.

By taking a closer look at the above examples one recognizes that porous sets having maximal dimension are exceptional: there is a set N with $\mu(N) = 0$ so that $\dim_{\mathrm{p}}(A) < \dim_{\mathrm{p}}(X)$ for all porous sets $A \subset X \setminus N$. This is not a coincidence as indicated by the following result.

Theorem 4.9. *Suppose that μ is a doubling measure on X. Then there is $N \subset X$ such that $\mu(N) = 0$ and*

$$\dim_{\mathrm{p}}(A) \le \dim_{\mathrm{p}}(X) - c(\log \tfrac{1}{\varrho})^{-1} \varrho^t$$

for all ϱ-porous sets $A \subset X \setminus N$ where $t = \frac{\log c_\mu}{\log 2}$. Here c is as in Corollary 4.5.

We prove Theorem 4.9 using Corollary 4.5 and the following two lemmas.

Lemma 4.10. *Suppose that μ is a doubling measure on X. Then there is $N \subset X$ with $\mu(N) = 0$ such that for all $s > \dim_{\mathrm{p}}(X)$ we have*

$$\lim_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} = \infty$$

for every $x \in X \setminus N$.

Proof. Choose a decreasing sequence (s_k) such that $s_k \downarrow \dim_{\mathrm{p}}(X)$ as $k \to \infty$. We claim that for all $k \in \mathbb{N}$ there is $N_k \subset X$ with $\mu(N_k) = 0$ such that

$$\lim_{r \downarrow 0} \frac{\mu(B(x,r))}{r^{s_k}} = \infty \tag{4.10}$$

for all $x \in X \setminus N_k$. Fix $k \in \mathbb{N}$ and suppose to the contrary that there are $0 < D < \infty$ and a Borel set $A \subset X$ such that $\mu(A) > 0$ and $\liminf_{r \downarrow 0} \mu(B(x,r))/r^{s_k} < D$ for all $x \in A$. By [Cu, Theorem 3.16] there exists $C > 0$ such that $\mathcal{P}^{s_k}(A) \ge \frac{1}{CD}\mu(A) > 0$ which is impossible since $\dim_{\mathrm{p}}(A) \le \dim_{\mathrm{p}}(X) < s_k$. Thus (4.10) is proved. Defining $N = \cup_{k=0}^\infty N_k$, verifies the claim. □

The following lemma is a substitute for Lemma 3.3 in metric spaces that carry a doubling measure.

Lemma 4.11. *Suppose that μ is a doubling measure on X. Let $0 < \lambda < \dim_{\mathrm{p}}(X)$ and let $N \subset X$ be as in Lemma 4.10. If $A \subset X \setminus N$ has the property that*

$$\limsup_{r \downarrow 0} \frac{\mu(A(r))}{r^\lambda} < \infty,$$

then $\dim_{\mathrm{p}}(A) \leq \dim_{\mathrm{p}}(X) - \lambda$.

Proof. Fix $x_0 \in X$. Let $s > \dim_{\mathrm{p}}(X)$ and define

$$A_{ki} = \{x \in A \cap B(x_0, i) : \mu(B(x,r)) > r^s \text{ for all } 0 < r < \frac{1}{k}\}$$

for all k, $i \in \mathbb{N}$. Since $A \subset X \setminus N$ we see from Lemma 4.10 that $A = \cup_{k,i} A_{ki}$. It suffices to show that

$$\dim_{\mathrm{M}}(A_{ki}) \leq s - \lambda \text{ for all } k, i \in \mathbb{N}. \tag{4.11}$$

Let k, $i \in \mathbb{N}$. Choose $r_0 > 0$ and $C < \infty$ such that $\mu(A_{ki}(r)) \leq \mu(A(r)) \leq Cr^\lambda$ when $0 < r < r_0$. If $0 < r < \min\{\frac{1}{2}r_0, \frac{1}{k}\}$, we choose $\{B(x_l, r)\}_{l \in I}$ to be a maximal countable collection of mutually disjoint balls centered at A_{ki}. Note that trivially

$$A_{ki} \subset \bigcup_{l \in I} B(x_l, 2r). \tag{4.12}$$

Now

$$C 2^\lambda r^\lambda \geq \mu(A_{ki}(2r)) \geq \sum_{l \in I} \mu(B(x_l, r)) \geq \# I r^s.$$

This implies that $\# I \leq C 2^\lambda r^{\lambda - s} < \infty$ which combined with (4.12) gives

$$M^{s-\lambda}(A_{ki}, 2r) \leq \# I (2r)^{s-\lambda} \leq C 2^s.$$

Now (4.11) follows as $r \downarrow 0$. □

Proof of Theorem 4.9. Let N be as in Lemma 4.10 and let $A \subset X \setminus N$. Fix $\varrho' < \varrho$. By Remark 2.1.(5), A is a countable union of sets of the form

$$E = \{x \in A \cap B(x_0, r_0) : \mathrm{por}(A, x, r) > \varrho' \text{ for all } 0 < r < r_0\}$$

where $x_0 \in X$ and $r_0 > 0$. Since $\varrho' < \varrho$ is arbitrary, it suffices to show that $\dim_{\mathrm{p}}(E) \leq \dim_{\mathrm{p}}(X) - \delta$ where δ is as in Corollary 4.5 with ϱ replaced by ϱ'. This follows from Lemma 4.11 since $E \subset X \setminus N$ and $\limsup_{r \downarrow 0} \mu(E(r))/r^\delta < \infty$ by Corollary 4.5. □

Remark 4.12. (1) We do not know if $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}(X) - \delta$ for some $\delta > 0$ in Theorem 4.9. Of course, this question is relevant only when $\dim_{\mathrm{H}}(X) < \dim_{\mathrm{p}}(X)$.

(2) The essential qualitative difference between Theorems 4.7 and 4.9 is that in the doubling case we have an extra factor $(\log \frac{1}{\varrho})^{-1}$ in δ. This extra factor appears also in $\mathbb{R}^n$ if one makes a simple estimate for the Minkowski dimension using mesh cubes whose side lengths are powers of ϱ. To obtain the optimal upper bound in $\mathbb{R}^n$ one has to utilize the porosity at "all balls", that is, one

has to use mesh cubes whose side lengths are powers of D for some fixed D independent of ϱ. In our proofs we use annuli and mean porosity instead of mesh cubes. As pointed out in Remark 4.4.(1), the annuli $A_k(x)$ defined using some fixed D may be empty in general metric spaces for many k. Thus we obtain mean porosity with p containing the factor $(\log \frac{1}{\varrho})^{-1}$. We do not know whether this factor $(\log \frac{1}{\varrho})^{-1}$ is necessary or not in Theorem 4.9 although in Remark 4.4.(1) it is which can be seen by considering a disjoint union of Cantor sets C_{ϱ_i} where ϱ_i tends to zero.

(3) In Theorem 4.7 the constant c depends on a_μ, b_μ and s. One may ask whether it is possible that c depends on s only. However, this is not the case. If in Example 4.8 one chooses $\mathcal{N}_n = \{1,2\}\times]0,\lambda^n]$ for $\lambda < \frac{1}{2}$, then the resulting space $\mathcal{N}$ is 1-regular, $\mathrm{por}(\mathcal{N}_\infty) = \frac{1}{2}$ and $\dim_{\mathrm{p}}(\mathcal{N}_\infty) \to 1$ as $\lambda \to \frac{1}{2}$. Note that in this case $\frac{b_\mu}{a_\mu} \to \infty$ and thus $c \to 0$ as $\lambda \to \frac{1}{2}$.

5. Uniform porosity and regular sets

In this section we prove that if X is s-regular and complete, then uniformly porous and regular subsets of X are closely related in the following sense: $A \subset X$ is uniformly porous if and only if there is $0 < t < s$ and a t-regular set $F \subset X$ such that $A \subset F$. In $\mathbb{R}^n$ this is well known, see for example [S], [L, Theorem 5.2], [Ca, Proposition 4.3], [KS, (proof of) Theorem 4.1], and [KV, Example 6.8]. We begin by showing that for each $0 < t < s$ there exists a t-regular set $F \subset X$. This is a consequence of the following lemma which was proven in $\mathbb{R}^n$ in an unpublished Licentiate thesis of Pirjo Saaranen, see also [MS, Theorem 3.1]. The proof for complete s-regular spaces is almost the same but it is included here for the sake of completeness. We denote by $\mathrm{spt}(\mu)$ the support of μ, that is, $\mathrm{spt}(\mu)$ is the smallest closed set F with $\mu(X \setminus F) = 0$.

Lemma 5.1. *Assume that X is complete, μ is s-regular on X and $0 < t < s$. Let $z \in X$ and $0 < R < r_\mu$. Then there is a measure ν with $\mathrm{spt}(\nu) \subset B(z, 2R)$ such that $\nu(B(z, 2R)) = R^t$ and*

$$a_\nu r^t \leq \nu(B(x, r)) \leq b_\nu r^t$$

for all $x \in \mathrm{spt}(\nu)$ and $0 < r < R$. Here the constants $a_\nu > 0$ and $b_\nu > 0$ depend only on s, t, a_μ and b_μ.

Proof. Let $0 < r \leq 2^{-2-1/t}R$. Recalling that X is separable, we choose a maximal countable collection $\{B(x_i, 2^{1+1/t}r)\}_{i\in I}$ of mutually disjoint balls centred at $B(z, R)$. We may take $x_1 = z$. It follows that the balls $B(x_i, 2^{2+1/t}r)$ cover the set $B(z,R)$. Thus

$$\begin{aligned}\#I b_\mu (2^{1+1/t}r)^s \geq & \sum_{i=1}^{\#I} \mu(B(x_i, 2^{1+1/t}r)) \geq \sum_{i=1}^{\#I} \frac{a_\mu}{2^s b_\mu} \mu(B(x_i, 2^{2+1/t}r)) \\ \geq & \frac{a_\mu}{2^s b_\mu} \mu(B(z,R)) \geq \frac{a_\mu^2}{2^s b_\mu} R^s\end{aligned}$$

by Lemma 3.2, and hence

$$c_1 \Big(\frac{R}{r}\Big)^s \leq \#I \text{ for all } 0 < r \leq 2^{-2-1/t}R, \tag{5.1}$$

where

$$c_1 = \frac{a_\mu^2}{2^{2s+s/t} b_\mu^2}.$$

Choose $d > 0$ so small that

$$d < \frac{1}{10} 2^{-1/t},\ d^{s-t} \leq \frac{c_1}{2} \text{ and } d^t \leq \frac{1}{2}. \tag{5.2}$$

Taking $r = dR$ in the above process gives disjoint balls $B(x_i, 2^{1+1/t}dR)$ with $x_i \in B(z,R)$. Moreover, by (5.1) we get the following estimate for the number m of such balls

$$\frac{c_1}{d^s} \leq m. \tag{5.3}$$

Let $M \in \mathbb{N}$ be such that

$$d^{-t} - \frac{1}{2} \leq M < d^{-t} + \frac{1}{2}. \tag{5.4}$$

Because

$$d^{-t} + \frac{1}{2} \leq \frac{c_1}{2d^s} + \frac{1}{2} < m$$

by (5.2) and (5.3), we may take M balls from the collection of disjoint balls $\{B(x_i, 2^{1+1/t}dR) : i = 1, \ldots, m\}$. Having roughly advanced towards our t-regular measure on this scale by taking suitable balls, we proceed by adjusting the radii of the balls to get exactly the regularity we want.

Fix $d_1 < 1$ so that $d_1^t = M^{-1}$. Inequalities (5.4) and (5.2) combine to give

$$d_1^t \leq \frac{2d^t}{2 - d^t} \leq \frac{4}{3} d^t < 2d^t \tag{5.5}$$

which implies that the balls $B_i = B(x_i, 2d_1R)$, $i = 1, \ldots, M$ are disjoint. Next we repeat the process by taking $B(x_i, d_1R)$ as $B(z,R)$. In this manner we get disjoint balls $B(x_{ij}, 2^{1+1/t}dd_1R)$, $j = 1, \ldots, m_{2i}$, where $x_{ij} \in B(x_i, d_1R)$, $x_{i1} = x_i$ and

$$\frac{c_1}{d^s} \leq m_{2i}.$$

For all $i = 1, \ldots, M$ choose M balls from these collections and adjust the radii to be $d_1^2 R$. Now the balls $B_{ij} = B(x_{ij}, 2d_1^2 R)$ are disjoint and $B_{ij} \subset B_i$ for all $i, j = 1, \ldots, M$, because by (5.5)

$$d_1 R + 2d_1^2 R < d_1 R + 2^{1+1/t} d_1 d R = d_1 R(1 + 2^{1+1/t} d) < 2d_1 R.$$

We continue this process. At step k we obtain for all sequences $(i_1, \ldots, i_{k-1}) \in \{1, \ldots, M\}^{k-1}$ disjoint balls $B_{i_1 \ldots i_k}$, $i_k = 1, \ldots, M$, with centres in the ball $B(x_{i_1 \ldots i_{k-1}}, d_1^{k-1} R)$ and with radii $2d_1^k R$ such that

$$B_{i_1 \ldots i_k} \subset B_{i_1 \ldots i_{k-1}}.$$

Furthermore, $x_{i_1 \ldots i_{k-1} 1} = x_{i_1 \ldots i_{k-1}}$. Set

$$\tilde{\nu}_{(i_1 \ldots i_k)} = M^{-k} \tag{5.6}$$

for all $k \in \mathbb{N}$ and $(i_1, \ldots, i_k) \in \{1, \ldots, M\}^k$. Since X is s-regular, it follows that $B(z, 2R)$ is totally bounded and hence also compact by the completeness of X. Thus all closed balls in $B(z, 2R)$ are compact and the set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{i_1, \ldots, i_k} B_{i_1 \ldots i_k} \subset B(z, 2R)$$

is nonempty and compact. By the disjointness of the balls $B_{i_1 \ldots i_k}$, we may identify the sequence $(i_1, \ldots, i_k)$ with $E \cap B_{i_1 \ldots i_k}$. It follows from (5.6) that $\tilde{\nu}$ is a probability measure on the algebra generated by the sets $E \cap B_{i_1 \ldots i_k}$. Hence, by the Carathéodory-Hahn extension theorem [WZ, Theorem 11.19], $\tilde{\nu}$ extends to a Borel probability measure on E.

Next we prove that $\tilde{\nu}$ is t-regular. Take any $x \in \operatorname{spt}(\tilde{\nu})$ and $0 < r < (1 - 2d_1)R$. Let $l \in \mathbb{N}$ be such that

$$(1 - 2d_1) d_1^{l+1} R \le r < (1 - 2d_1) d_1^l R. \tag{5.7}$$

By (5.5) and (5.2) we have $d_1 < \frac{1}{10}$. Thus inequalities (5.7) guarantee that

$$B_{i_1 \ldots i_{l+2}} \subset B(x, r) \subset B_{i_1 \ldots i_l}$$

for some $(i_1, \ldots, i_{l+2}) \in \{1, \ldots, M\}^{l+2}$. This in turn implies with (5.6) and (5.7) that

$$\frac{d_1^{2t}}{(1 - 2d_1)^t R^t} r^t \le d_1^{(l+2)t} = M^{-(l+2)} \le \tilde{\nu}(B(x, r)) \le M^{-l} = d_1^{lt} \le \frac{1}{(1 - 2d_1)^t R^t d_1^t} r^t.$$

Finally, defining $\nu = R^t \tilde{\nu}$ gives the desired measure. □

As an immediate consequence we obtain:

Corollary 5.2. *Assume that X is complete and μ is s-regular on X. Then for all $0 < t < s$ there is $F \subset X$ which is t-regular.*

Now we are ready to state the main theorem of this section.

Theorem 5.3. *Suppose that μ is s-regular on X. If $0 < t < s$ and $A \subset X$ is t-regular, then A is uniformly ϱ-porous for some $\varrho > 0$. Conversely, if X is complete and $A \subset X$ is uniformly ϱ-porous, then for all $s - c\varrho^s < t < s$ there exists a t-regular set $F \subset X$ so that $A \subset F$. Here c is as in Corollary 4.6.*

Proof. The first part is proven in [BHR, Lemma 3.12]. We reprove it here to obtain a quantitative estimate which in a sense is optimal (see Remark 5.4). Let ν be a t-regular measure on A. Pick $x \in A$ and $0 < r < \frac{1}{2}\min\{r_\mu, r_\nu\}$, and take $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, we choose a maximal countable collection $\{B(x_i, 2^{-k-1}r)\}_{i \in I_k}$ of mutually disjoint balls centred at $B(x,r)$. It follows that the balls $B(x_i, 2^{-k}r)$ cover the set $B(x,r)$.

From the s-regularity of μ we get

$$a_\mu r^s \le \mu\big(B(x,r)\big) \le \sum_{i \in I_k} \mu\big(B(x_i, 2^{-k}r)\big) \le \# I_k b_\mu 2^{-ks} r^s,$$

and therefore,

$$\# I_k \ge \frac{a_\mu}{b_\mu} 2^{ks}. \tag{5.8}$$

Taking any set $J_k \subset I_k$ for which $B(x_j, 2^{-k-2}r) \cap A \neq \emptyset$ as $j \in J_k$ and using the fact that ν is t-regular on A and $r < r_\nu$, we have

$$\begin{aligned} b_\nu 2^t r^t \ge \nu\big(B(x,2r)\big) &\ge \sum_{j \in J_k} \nu\big(B(x_j, 2^{-k-1}r)\big) \\ &\ge \sum_{j \in J_k} \nu\big(B(z_j, 2^{-k-2}r)\big) \ge \# J_k a_\nu 2^{-(k+2)t} r^t, \end{aligned}$$

where $z_j \in B(x_j, 2^{-k-2}r) \cap A$ as $j \in J_k$. Thus

$$\# J_k \le \frac{b_\nu}{a_\nu} 2^{(k+3)t}.$$

This upper bound is strictly smaller than the lower bound in (5.8) when

$$k > \frac{\log\big(\frac{b_\mu b_\nu}{a_\mu a_\nu} 2^{3t}\big)}{(s-t)\log 2} =: K(\mu, \nu).$$

Choosing $k > K(\mu,\nu)$, gives $I_k \setminus J_k \neq \emptyset$, and we find $i_0 \in I_k$ such that $B(x_{i_0}, 2^{-k-2}r) \cap A = \emptyset$. It follows that

$$\operatorname{por}^*(A, x, 2r) \ge 2^{-k-3} \tag{5.9}$$

whenever $x \in A$ and $0 < r < \frac{1}{2}\min\{r_\mu, r_\nu\}$. The claim follows from Remark 2.1.(2).

Now we prove the opposite direction. The idea is to use Lemma 5.1 to build a regular measure inside the voids of suitable reference balls. Let $0 < \varrho' < \varrho$ and $r_p > 0$ be such that

$$\operatorname{por}(A, x, r) > \varrho' \tag{5.10}$$

for all $x \in A$ and $0 < r < r_p$. Set $\gamma = \frac{\varrho'}{5}$ and fix $n_0 \in \mathbb{N}$ such that $\gamma^{n_0} < D_1 \min\{r_p, r_\mu\}$ (see Corollary 4.6). From Corollary 4.6 we see that for all $x \in X$ and $0 < r_0 \leq \gamma^{n_0}$

$$\mu((A \cap B(x, r_0))(r)) \leq C_4 \mu(B(x, r_0)) \Big(\frac{r}{r_0}\Big)^\delta \text{ for all } 0 < r < r_0 \tag{5.11}$$

where $\delta = c(\varrho')^s$.

Consider $s - \delta < t < s$. For all $j \in \mathbb{N}$, we choose a maximal collection of disjoint balls $\{B(x_{ji}, \gamma^{n_0+j})\}$ so that $x_{ji} \in A$ for all $i \in \mathbb{N}$. Now we clearly have

$$A \subset \bigcup_i B(x_{ji}, 2\gamma^{n_0+j}). \tag{5.12}$$

For each $i \in \mathbb{N}$ we find, using inequality (5.10) and the fact $r < r_p$, points $z_{ji} \in B(x_{ji}, \gamma^{n_0+j})$ so that $B(z_{ji}, \varrho'\gamma^{n_0+j}) \subset B(x_{ji}, \gamma^{n_0+j}) \setminus A$. Define

$$B_{ji} = B(z_{ji}, \tfrac{\varrho'}{5}\gamma^{n_0+j}) = B(z_{ji}, \gamma^{n_0+j+1}) \text{ and } 2B_{ji} = B(z_{ji}, 2\gamma^{n_0+j+1}).$$

Then obviously $2B_{ji} \subset B(z_{ji}, \varrho'\gamma^{n_0+j}) \subset B(x_{ji}, \gamma^{n_0+j}) \setminus A$. Lemma 5.1 implies that for all $i \in \mathbb{N}$ there is a t-regular measure ν_{ji} on $\operatorname{spt}(\nu_{ji}) \subset 2B_{ji}$ with $\nu_{ji}(2B_{ji}) = \gamma^{(n_0+j+1)t}$. Note that the constants $a_{\nu_{ji}}$ and $b_{\nu_{ji}}$ are the same for all j and i, say $a_{\nu_{ji}} = a$ and $b_{\nu_{ji}} = b$. Moreover, it is evident by the properties of ν_{ji} that we may choose $r_{\nu_{ji}} = 3\gamma^{n_0+j}$ by adjusting a and b. We conclude that for all $x \in \operatorname{spt}(\nu_{ji})$ and $0 < r < 3\gamma^{n_0+j}$

$$ar^t \leq \nu_{ji}(B(x, r)) \leq br^t. \tag{5.13}$$

Setting

$$F = \bigcup_{j,i} \operatorname{spt}(\nu_{ji}) \cup A$$

and

$$\nu = \sum_{j,i} \nu_{ji},$$

we clearly have $A \subset F$ and $\nu(X \setminus F) = 0$, and therefore, it suffices to prove that ν is t-regular on F.

We first verify that ν is t-regular on A by showing that there are constants C_1 and C_2 such that for all $x \in A$ and $0 < r < \frac{1}{2}\gamma^{n_0+1}$ we have

$$C_1 r^t \leq \nu(B(x, r)) \leq C_2 r^t. \tag{5.14}$$

For the purpose of proving (5.14), fix $k \in \mathbb{N}$ so that

$$\gamma^{n_0+k+1} \leq r < \gamma^{n_0+k}.$$

From (5.12) it follows easily that $2B_{k+3,i} \subset B(x, r)$ for some i giving

$$\nu(B(x, r)) \geq \nu_{k+3,i}(2B_{k+3,i}) = \gamma^{(n_0+k+4)t} \geq \gamma^{4t} r^t.$$

Thus we may choose $C_1 = \gamma^{4t}$ in (5.14).

For the remaining inequality in (5.14), denote by N_j the number of balls $2B_{ji}$ that intersect $B(x,r)$. Assuming that $j \geq k$, we obtain

$$\bigcup_{2B_{ji}\cap B(x,r)\neq\emptyset} 2B_{ji} \subset (A \cap B(x, \frac{2}{\gamma}r))(\gamma^{n_0+j}). \tag{5.15}$$

Indeed, if $2B_{ji} \cap B(x,r) \neq \emptyset$, then also $B(x_{ji}, \gamma^{n_0+j}) \cap B(x,r) \neq \emptyset$ giving $d(x_{ji}, x) \leq r + \gamma^{n_0+j} \leq r + \frac{1}{\gamma}r \leq \frac{2}{\gamma}r$. Hence (5.15) is valid since $x_{ji} \in A$. Now (5.15) implies with (5.11) that

$$\begin{aligned}
N_j a_\mu 2^s \gamma^{(n_0+j+1)s} &\leq \mu\big(\bigcup_{2B_{ji}\cap B(x,r)\neq\emptyset} 2B_{ji}\big) \leq \mu((A \cap B(x, \frac{2}{\gamma}r))(\gamma^{n_0+j})) \\
&\leq C_4 \mu(B(x, \frac{2}{\gamma}r))\big(\frac{\gamma^{n_0+j}}{\frac{2}{\gamma}r}\big)^\delta \leq C_4 b_\mu \big(\frac{2}{\gamma}r\big)^{s-\delta}\gamma^{(n_0+j)\delta} \\
&\leq C_4 b_\mu \big(\frac{2}{\gamma}\big)^{s-\delta}\gamma^{(n_0+k)s}\gamma^{(j-k)\delta},
\end{aligned}$$

and therefore,

$$N_j \leq c_3 \gamma^{(k-j)(s-\delta)} \text{ for all } j \geq k, \tag{5.16}$$

where c_3 is a constant independent of k and j. Note that $N_j = 0$ provided that $j \leq k-1$. This is true because

$$\operatorname{dist}(2B_{ji}, A) \geq 3\gamma^{n_0+j+1} \geq 3\gamma^{n_0+k} > r.$$

From (5.16) we obtain

$$\begin{aligned}
\nu(B(x,r)) &= \sum_{j,i} \nu_{ji}(B(x,r) \cap 2B_{ji}) \leq \sum_{j=k}^{\infty} N_j \gamma^{(n_0+j+1)t} \\
&\leq c_3 \gamma^{t(n_0+1)+k(s-\delta)} \sum_{j=k}^{\infty} \gamma^{j(\delta+t-s)} \\
&= \frac{c_3}{1-\gamma^{\delta+t-s}} \gamma^{t(n_0+k+1)} \leq C_2 r^t
\end{aligned}$$

where $C_2 = \frac{c_3}{1-\gamma^{\delta+t-s}}$. This completes the verification of (5.14).

We finish the proof by showing that ν is t-regular on $\operatorname{spt}(\nu_{ji})$ for all j and i. Fix j and i and let $y \in \operatorname{spt}(\nu_{ji})$. We first derive the lower bound in the definition of t-regularity. If $0 < r \leq 3\gamma^{n_0+j}$, then $\nu(B(y,r)) \geq \nu_{ji}(B(y,r)) \geq ar^t$ by (5.13). On the other hand, if $3\gamma^{n_0+j} < r < \gamma^{n_0+1}$, we get $B(x_{ji}, \frac{r}{2}) \subset B(y,r)$. Applying (5.14) implies that $\nu(B(y,r)) \geq \nu(B(x_{ji}, \frac{r}{2})) \geq C_1 2^{-t} r^t$.

For the upper regularity bound we first assume that $0 < r \leq \gamma^{n_0+j+1}$. Now $B(y,r) \cap 2B_{j'i'} = \emptyset$ for all $(j', i') \neq (j, i)$. Indeed, if $j = j'$ this follows from $B(y,r) \subset B(x_{ji}, \gamma^{n_0+j})$ and $2B_{ji'} \subset B(x_{ji'}, \gamma^{n_0+j})$. In the case $j' < j$ assume that $w \in B(y,r)$. Then $d(w, A) \leq d(w, x_{ji}) \leq 2\gamma^{n_0+j}$. Further, if $w \in 2B_{j'i'}$, then $d(w,A) \geq \frac{3\varrho'}{5}\gamma^{n_0+j'} \geq 3\gamma^{n_0+j}$. The final case $j' > j$ is similar. Thus $\nu(B(y,r)) = \nu_{ji}(B(y,r)) \leq br^t$ by (5.13). Finally, supposing that $\gamma^{n_0+j+1} < r < \frac{\varrho'}{12}\gamma^{n_0+1}$ gives $B(y,r) \subset B(x_{ji}, \frac{6r}{\varrho'})$. Hence $\nu(B(y,r)) \leq \nu(B(x_{ji}, \frac{6r}{\varrho'})) \leq C_2\big(\frac{6}{\varrho'}\big)^t r^t$ by (5.14). □

Remark 5.4. (1) Observe that in (5.9) the porosity ϱ is proportional to $\big(\frac{a_\nu}{b_\nu}\big)^{\frac{1}{s-t}}$. The following simple construction shows that this is sharp. Let $0 < t < 1$ and choose from $[0,1]$ N evenly distributed intervals of length $N^{-\frac{1}{t}}$. Repeat this construction and let ν be the natural measure on the resulting Cantor set A. Then A is t-regular, $\mathrm{por}(A) \approx \frac{1-N^{1-\frac{1}{t}}}{N}$ and $\frac{a_\nu}{b_\nu} = \frac{N^{t-1}}{(1-N^{1-\frac{1}{t}})^t}$. As N tends to infinity, $\mathrm{por}(A) \approx \frac{1}{N} \approx \big(\frac{a_\nu}{b_\nu}\big)^{\frac{1}{1-t}}$. We do not know what is the best asymptotic behaviour as $\frac{a_\nu}{b_\nu}$ is fixed and $t \to s$.

(2) We do not know whether the completeness is needed in Theorem 5.3. Our method does not work without the completeness since in that case the set E constructed in the proof of Lemma 5.1 may be empty.

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