

Packing odd circuits

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PACKING ODD CIRCUITS*

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Abstract. We determine the structure of a class of graphs that do not contain the complete graph on five vertices as a “signed minor.” The result says that each graph in this class can be decomposed into elementary building blocks in which maximum packings by odd circuits can be found by flow or matching techniques. This allows us to actually find a largest collection of pairwise edge disjoint odd circuits in polynomial time (for general graphs this is NP-hard). Furthermore it provides an algorithm to test membership of our class of graphs.

Key words. odd circuits, packing, excluded minors, decomposition, signed graphs

AMS subject classifications. 05C22, 05C70, 05C75, 05C83, 90C27

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1. Introduction. The *odd circuit packing problem*, finding in a graph a largest collection of pairwise edge disjoint odd circuits, is NP-hard. In this paper we will present a class of graphs in which this problem can be solved in polynomial time. We prove that each graph in this class can be decomposed into planar graphs, graphs with a vertex meeting all odd circuits, and graphs containing at most six vertices. In such building blocks a maximum packing by odd circuits can be found by flow or matching techniques. Given a graph G in our class, our decomposition theorem allows us to combine such packings for the building blocks of G to a maximum packing by odd circuits in G . With some extra work our decomposition theorem gives an algorithm to test membership of our class.

We present everything in terms of signed graphs. The results can be stated and proved in terms of ordinary graphs without any loss of generality, but in those terms the proofs require extra maneuvering that can be avoided when speaking the language of signed graphs. A *signed graph* is a pair (G, Σ) consisting of an undirected graph G and a collection Σ of its edges. A collection F of edges in G is called *odd* in (G, Σ) if $|F \cap \Sigma|$ is odd; otherwise, F is called *even*. In particular, we speak of odd and even edges, paths, and circuits. We call (G, Σ) *Eulerian* if G is Eulerian, so if each vertex has even degree.

THEOREM 1. *The odd circuit packing problem is polynomially solvable for Eulerian signed graphs with no \tilde{K}_5 -, $K_{3,3}^{1,1}$ -, $K_{3,3}^{1,2}$ -, or $K_{3,3}^2$ -minor.*

We explain the notions used in this result. A *minor* of (G, Σ) is the result of a series of the following three operations: deletion of an edge or an isolated vertex, contraction of an even edge, and resigning. *Resigning* (on $U \subseteq V(G)$) means replacing Σ by the symmetric difference $\Sigma \Delta \delta_G(U)$ of Σ with the cut $\delta_G(U) := \{uv \in E(G) | u \in U, v \notin U\}$. Clearly, the collection $\Omega(G, \Sigma)$ of odd circuits in (G, Σ) is invariant under

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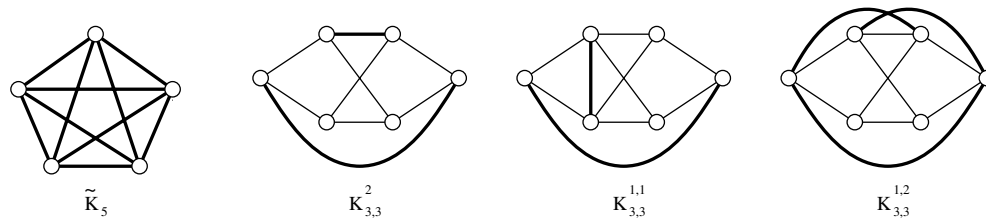


FIG. 1. Bold edges are odd; thin edges are even.

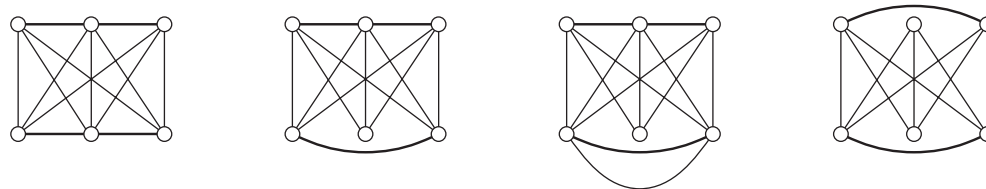


FIG. 2. Bold edges are odd; thin edges are even.

resigning. Two signed graphs are *isomorphic* if they are related through resigning and graph-isomorphism. We say that (G, Σ) has a (H, Θ) -minor or *contains* (H, Θ) if it has a minor isomorphic to (H, Θ) .

The definition of the four signed graphs “excluded” in Theorem 1 can be understood from the following (see Figure 1). If G is a graph, then $\tilde{G} := (G, E(G))$, so \tilde{K}_5 consists of the complete graph on five vertices with all edges odd. $K_{3,3}^i := (K_{3,3}, M)$, where M is a matching of size i . Finally, $K_{3,3}^{1,1}$ and $K_{3,3}^{1,2}$ are the two extensions of $K_{3,3}^1$ given in Figure 1.

In addition to Theorem 1 we prove that the signed graph property described there can be recognized in polynomial time.

THEOREM 2. *There exists a polynomial time algorithm that decides whether or not a given signed graph has a \tilde{K}_5 -, $K_{3,3}^{1,1}$ -, $K_{3,3}^{1,2}$ -, or $K_{3,3}^2$ -minor.*

As we shall see in sections 3 and 5 both Theorems 1 and 2 are a consequence of the following decomposition theorem. It is the main result of this paper.

THEOREM 3. *Let (G, Σ) be a 3-connected signed graph with no improper 3-vertex cutset and no $K_{3,3}^2$ -minor.*

- (i) *If (G, Σ) has no $K_{3,3}^1$ -minor and no \tilde{K}_5 -minor, then $|V(G)| = 5$ or G is planar or (G, Σ) is isomorphic to one of the signed graphs in Figure 2 or (G, Σ) has a blockvertex.*
- (ii) *If (G, Σ) has a $K_{3,3}^1$ minor, but no $K_{3,3}^{1,1}$ or $K_{3,3}^{1,2}$ minor, then (G, Σ) has a blockvertex.*

Here are the notions used in this result: A *blockvertex* of (G, Σ) is a vertex that is contained in every odd circuit. We call (G, Σ) *3-connected* if any two vertices in G are connected by two internally vertex disjoint paths; this allows parallel edges. (G, Σ) has an *improper 3-vertex cutset* means that it contains signed graphs (G_1, Σ_1) and (G_2, Σ_2) such that $E(G_1)$ and $E(G_2)$ are nonempty and partition $E(G)$, $|V(G_1) \cap V(G_2)| = 3$ and (G_2, Σ_2) has no odd circuits and at least four edges. The proof of (i) is in section 6, and the proof of (ii) is in sections 7–11.

We obtain not only an algorithm for the odd circuit packing problem but also a min-max relation.

THEOREM 4. *Let (G, Σ) be a signed graph with no \tilde{K}_5^- , $K_{3,3}^{1,1}$, $K_{3,3}^{1,2}$, or $K_{3,3}^2$ -minor. If G is Eulerian, then the maximum number of pairwise edge disjoint odd circuits in (G, Σ) is equal to the minimum number of edges needed to cover all odd circuits in (G, Σ) .*

This result has been generalized extensively by Geelen and Guenin [2], who proved the min-max relation for all Eulerian signed graphs with no \tilde{K}_5 -minor. This was stated as a conjecture in an earlier version of the present article. Geelen and Guenin do not use decompositions, and their methods do not seem to provide a polynomial time algorithm for finding maximum odd circuits packings. However, it does follow from their result and in fact also from the earlier characterization of “weakly bipartite graphs” by Guenin [5] that by linear programming techniques one can find in polynomial time a smallest collection of edges that cover all odd circuits in a signed graph with no \tilde{K}_5 -minor. Note that in \tilde{K}_5 itself, which is Eulerian, the min-max relation in Theorem 4 does not hold, so the Geelen–Guenin theorem is in a certain sense as strong as possible.

The min-max relation stated in Theorem 4 may fail to be true if we drop the condition that the graph is Eulerian; \tilde{K}_4 is an example. Actually it follows from a general result of Seymour [10] that the min-max relation does hold for signed graphs with no \tilde{K}_4 -minor, even if they are not Eulerian.

Theorem 3 also has consequences for the chromatic number of the graphs involved. In combination with the 4-color theorem it can be used to prove that if G has none of the forbidden minors of Theorem 1, then G is 4-colorable. (It has been conjectured by one of the authors that G is 4-colorable if \tilde{G} has no \tilde{K}_5 -minor, see Jensen and Toft [8]. Recently Guenin [6] announced a proof of this conjecture.)

Theorem 3 can be regarded as a first step towards a constructive characterization of graphs with no \tilde{K}_5 -minor, a small step though; there are quite a few other infinite families of “highly connected” graphs with no \tilde{K}_5 -minor known that are not covered by Theorem 3 (see Gerards [4]). The exclusion of $K_{3,3}^2$, $K_{3,3}^{1,1}$, and $K_{3,3}^{1,2}$ is quite restrictive. For each $\Sigma \subseteq E(K_{3,3})$, the signed graph $(K_{3,3}, \Sigma)$ is isomorphic to exactly one of $K_{3,3}^0$, $K_{3,3}^1$, and $K_{3,3}^2$. For instance, $\tilde{K}_{3,3}$ is isomorphic to $K_{3,3}^0$ and $K_{3,3}^3$ to $K_{3,3}^2$. So up to isomorphism $K_{3,3}^2$ is the only signed $K_{3,3}$ with a \tilde{K}_4 minor. $K_{3,3}^{1,1}$ and $K_{3,3}^{1,2}$ are the smallest 3-connected signed graphs that contain both $K_{3,3}^1$ and \tilde{K}_4 as minors.

2. Odd circuits in signed graphs. We mention some elementary facts on signed graphs that are good to keep in mind while reading this paper. Note that they are all known and not just for odd circuits in graphs but for general binary clutter, which are just collections of odd circuits in signed binary matroids.

A signed graph (G, Σ) is *bipartite* if $\Sigma = \delta_G(U)$ for some $U \subseteq V(G)$. So clearly, (G, Σ) is bipartite if and only if it is isomorphic to (G, \emptyset) . Hence, if (G, Σ) is bipartite it has no odd circuits. Actually the converse is also true. To see this, we may assume that G is connected and that we have resigned (G, Σ) such that Σ is as small as possible. That means that Σ does not contain a nonempty cut $\delta_G(U)$ (otherwise resigning on U replaces Σ by $\Sigma \setminus \delta_G(U)$, which then is smaller). Therefore the even edges in (G, Σ) form a connected spanning subgraph of G . Now, if (G, Σ) is nonbipartite there is an odd edge uv in Σ and, as u and v are connected by a path with all edges even, that edge is in an odd circuit. So a signed graph is bipartite if and only if it has no odd circuit.

A subset S of $E(G)$ is a *signature* of (G, Σ) if (G, S) has exactly the same odd circuits as (G, Σ) . Clearly, S is a signature if and only if all circuits are even in

$(G, S \triangle \Sigma)$. In other words, the signatures are exactly the sets $\Sigma \triangle \delta_G(U)$ for some $U \subseteq V(G)$. Each signature meets all odd circuits. Conversely, if $F \subseteq E(G)$ meets all odd circuits it contains a signature. Indeed, let H be obtained from G by deleting all edges in F . Then $(H, \Sigma \setminus F)$ has no odd circuits and so is bipartite. Thus there exists a set $U \subseteq V(H) = V(G)$ with $\Sigma \setminus F = \delta_H(U)$. In other words $\Sigma \triangle \delta_G(U) \subseteq F$, so F contains a signature, as claimed. In other words the signatures are exactly the inclusionwise minimal edge sets that meet all odd circuits, and the smallest signatures are exactly the the sets attaining the minimum in Theorem 4.

3. Packing odd circuits—algorithm and min-max relation. We actually consider a “capacitated version” of packing odd circuits, because it is slightly more convenient to work with. If G is a graph and $w \in \mathbb{Z}_+^{E(G)}$, then a w -packing is a collection of subsets of $E(G)$, repetition allowed, such that each edge e is in at most $w(e)$ members of the collection. So the maximum size of a w -packing of odd circuits in (G, Σ) is equal to

$$\nu_w(G, \Sigma) := \max \left\{ \sum_{C \in \Omega(G, \Sigma)} \lambda_C \mid \lambda \in \mathbb{Z}_+^{\Omega(G, \Sigma)} \right. \\ \left. \text{and } \sum_{C \in \Omega(G, \Sigma), C \ni e} \lambda_C \leq w(e) \text{ for each } e \in E \right\}.$$

Clearly, $\nu_w(G, \Sigma)$ is bounded from above by

$$\tau_w(G, \Sigma) := \min\{w(S) \mid S \text{ is a signature of } (G, \Sigma)\},$$

where $w(S)$ is short for $\sum_{e \in S} w(e)$.

We call a function $w \in \mathbb{Z}_+^{E(G)}$ *Eulerian* if $w(\delta_G(v))$ is even for each vertex $v \in V(G)$. Theorem 4 is equivalent with the following result:

- (1) *If (G, Σ) is a signed graph with no \tilde{K}_5^- , $K_{3,3}^{1,1}$, $K_{3,3}^{1,2}$, or $K_{3,3}^2$ -minor, then $\nu_w(G, \Sigma) = \tau_w(G, \Sigma)$ for each Eulerian $w \in \mathbb{Z}_+^{E(G)}$.*

Indeed, as the excluded minor condition is invariant under addition of even edges parallel to even edges and of odd edges parallel to odd edges and under deleting edges, (1) follows from Theorem 4, which in turn is the special case of (1) when w is the all-one function.

Now we show that Theorem 3 implies (1) hence also Theorem 4. We first consider the basic building blocks of our decomposition. For these there exist standard constructions, by Barahona and Seymour, to reduce the odd circuit packing problem to flow problems and odd cut packing problems.

- (2) *If (G, Σ) has a blockvertex then $\nu_w(G, \Sigma) = \tau_w(G, \Sigma)$ for each $w \in \mathbb{Z}_+^{E(G)}$. Moreover then we can find a maximum w -packing of odd circuits in polynomial time.*

To see this let s be a blockvertex. As the signed graph obtained by deleting s from (G, Σ) is bipartite, we may resign such that $\Sigma \subseteq \delta_G(s)$. Now construct a new graph H by adding a new vertex t and replacing each odd edge us of G with an edge ut in H . Then there is a one-to-one correspondence between odd circuits in (G, Σ) and st -paths in H . Thus (2) follows from network flow theory.

Next we discuss how to deal with the signed graphs in Figure 2 and with signed graphs (K_5, Σ) that are not isomorphic to \tilde{K}_5 . In either of these cases (G, Σ) contains a *blocking pair*. This is a pair of vertices such that each odd circuit contains at least one of these two vertices. So we can then apply the following fact:

- (3) *If (G, Σ) has a blocking pair, then $\nu_w(G, \Sigma) = \tau_w(G, \Sigma)$ for each Eulerian $w \in \mathbb{Z}_+^{E(G)}$. Moreover then we can find a maximum w -packing of odd circuits in polynomial time.*

To see this we use the same approach, due to Barahona, as in the blockvertex case. Let $\{s_1, s_2\}$ be a blocking pair. By resigning we may assume that each odd edge is incident with at least one of s_1 and s_2 . Now construct a new graph H by adding new vertices t_1 and t_2 and by replacing each odd edge us_1 of G with $u \neq s_2$ with an edge ut_1 in H ; by replacing each odd edge us_2 of G with $u \neq s_1$ with an edge ut_2 in H ; and by replacing an odd edge between s_1 and s_2 (if such edge exists) with an edge t_1s_2 in H . Then there is a one-to-one correspondence between the odd circuits in (G, Σ) and the s_1t_1 -paths and s_2t_2 -paths in H . Thus we translate the maximum w -packing of odd circuits problem into the integer 2-commodity flow problem. Note that the latter problem does not really change if we would add an edge s_1t_1 with $w(s_1t_1) = 1$ or an edge s_2t_2 with $w(s_2t_2) = 1$ or both. Hence we may assume that w is Eulerian on H . Thus (3) follows from the integer 2-commodity flow theorem of Rothschild and Whinston [9].

- (4) *If G is planar, then $\nu_w(G, \Sigma) = \tau_w(G, \Sigma)$ for each Eulerian $w \in \mathbb{Z}_+^{E(G)}$. Moreover then we can find a maximum w -packing of odd circuits in polynomial time.*

We use a construction by Seymour [12], and for ease of exposition we restrict ourselves to the case that w is the all-one function, so G is Eulerian. Hence the planar dual G^* of some embedding of G in the plane is bipartite in the ordinary graph sense. Let Σ^* be the edges of G^* corresponding to the edges in Σ . Let T denote the set of vertices of G^* that meet an even number of edges in Σ^* . We call a collection F of odd edges in G^* a T -join if and only if every vertex in T meets an odd number of edges in F and every vertex outside T meets an even number of edges in F . A cut $\delta_{G^*}(U)$ in G^* is a T -cut if $|T \cap U|$ is odd. By the relation between circuits in a plane graph and cuts in its plane dual, we see that there is a one-to-one correspondence between T -joins in G^* and signatures in (G, Σ) and between inclusionwise minimal T -cuts in G^* and odd circuits in (G, Σ) . Hence the min-max relation in (4) follows from a min-max relation by Seymour [12] that says that in any ordinary (not signed) bipartite graph the minimum size of a T -join is equal to the maximum size of a collection of pairwise disjoint T -cuts. See Barahona [1] for a polynomial algorithm for finding such a maximum collection of disjoint T -cuts; it also allows general Eulerian functions $w \in \mathbb{Z}^{E(G)}$, other than the all-one function. Thus (4) follows.

The following two results, Lemmas 5 and 6, say that all signed graphs that do not satisfy the min-max relation in (1) and are minor-minimal in this respect are 3-connected and have no improper 3-vertex cutsets.

LEMMA 5. *If (G, Σ) does not satisfy $\nu_w(G, \Sigma) = \tau_w(G, \Sigma)$ for each Eulerian $w \in \mathbb{Z}_+^{E(G)}$ and is minor-minimal in this respect, then (G, Σ) is 3-connected and has no parallel edges.*

Proof. Let (G, Σ) be a counterexample. We clearly may assume G to be 2-connected, so there exist two vertices u_1 and u_2 in G and two connected graphs G_1

and G_2 with $V(G_1) \cap V(G_2) = \{u_1, u_2\}$ such that $E(G_1)$ and $E(G_2)$ both have at least two elements and partition $E(G)$. For $i = 1, 2$, we define $\Sigma_i := \Sigma \cap E(G_i)$. Let $w \in \mathbb{Z}_+^{E(G)}$ be Eulerian with $\tau_w(G, \Sigma) > \nu_w(G, \Sigma)$.

For each signed graph (H, Θ) containing u_1 and u_2 and for $i = 0, 1$, we define

$$(5) \quad \tau_w(H, \Theta)_i := \min\{w(\Theta \Delta \delta_H(U)) \mid |U \cap \{u_1, u_2\}| = i\}.$$

Then,

$$(6) \quad \tau_w(H, \Theta) = \min\{\tau_w(H, \Theta)_0, \tau_w(H, \Theta)_1\},$$

and

$$(7) \quad \tau_w(G, \Sigma)_i = \tau_w(G_1, \Sigma_1)_i + \tau_w(G_2, \Sigma_2)_i \quad \text{for } i = 0, 1.$$

Also note that if $U \subseteq V(H)$ with $u_1 \in U$ and $u_2 \notin U$, then

$$(8) \quad \tau_w(H, \Theta)_i = \tau_w(H, \Theta \Delta \delta_H(U))_{1-i} \quad \text{for } i = 0, 1.$$

So by resigning (G, Σ) if necessary we may assume that

$$(9) \quad \tau_w(G_1, \Sigma_1)_1 \geq \tau_w(G_1, \Sigma_1)_0.$$

Let $\omega := \tau_w(G_1, \Sigma_1)_1 - \tau_w(G_1, \Sigma_1)_0$. If $\omega = 0$, let $\widehat{G}_2 := G_2$; if $\omega > 0$, let \widehat{G}_2 be obtained from G_2 by adding a new even edge e_2 between u_1 and u_2 with weight $w(e_2) := \omega$.

$$(10) \quad \tau_w(\widehat{G}_2, \Sigma_2) = \tau_w(G, \Sigma) - \tau_w(G_1, \Sigma_1)_0.$$

To see this, note that it follows from (7) that $\tau_w(\widehat{G}_2, \Sigma_2)_0 = \tau_w(G_2, \Sigma_2)_0 = \tau_w(G, \Sigma)_0 - \tau_w(G_1, \Sigma_1)_0$ and $\tau_w(\widehat{G}_2, \Sigma_2)_1 = \tau_w(G_2, \Sigma_2)_1 + \omega = \tau_w(G_2, \Sigma_2)_1 + \tau_w(G_1, \Sigma_1)_1 - \tau_w(G_1, \Sigma_1)_0 = \tau_w(G, \Sigma)_1 - \tau_w(G_1, \Sigma_1)_0$. By (6), this implies (10).

$$(11) \quad (\widehat{G}_2, \Sigma_2) \text{ is a proper minor of } (G, \Sigma).$$

Suppose this is not true. Then G_1 has no even u_1u_2 -path, and $\omega > 0$. We first prove that (G_1, Σ_1) is bipartite. Let C be a circuit in G_1 . As G is 2-connected there exist two disjoint paths from $V(C)$ to $\{u_1, u_2\}$. As the union of these paths and C does not contain an even u_1u_2 -path, C has to be even. So (G_1, Σ_1) is bipartite indeed. Hence $\Sigma_1 = \delta_{G_1}(U)$ for some $U \subseteq V(G_1)$. We may assume $u_1 \in U$. Then, as there is no even u_1u_2 -path, $u_2 \notin U$. Hence as $w(\Sigma_1 \Delta \delta_{G_1}(U)) = w(\emptyset) = 0$, we have that $\tau_w(G_1, \Sigma_1)_1 = 0$. So $\omega = 0$, which is a contradiction. This proves (11).

$$(12) \quad w(\delta_{\widehat{G}_2}(v)) \text{ is even for each } v \in V(\widehat{G}_2).$$

Indeed, as $w(\delta_G(v))$ is even for each $v \in V(G)$, (12) holds for all $v \notin \{u_1, u_2\}$. So, as there is an even number of vertices v with $w(\delta_{\widehat{G}_2}(v))$ odd, we may restrict ourselves to proving that $w(\delta_{\widehat{G}_2}(u_1))$ is even. Let $U_1 \subseteq V(G_1)$ with $U_1 \cap \{u_1, u_2\} = \{u_1\}$ such that $w(\Sigma_1 \Delta \delta_{G_1}(U_1)) = \tau_w(G_1, \Sigma_1)_1$, and let $U_0 \subseteq V(G_1)$ with $U_0 \cap \{u_1, u_2\} = \emptyset$ such that $w(\Sigma_1 \Delta \delta_{G_1}(U_0)) = \tau_w(G_1, \Sigma_1)_0$. Then we get the following (“ \equiv ” denotes

equivalence modulo 2):

$$\begin{aligned}
 w(\delta_{\widehat{G}_2}(u_1)) &= w(\delta_{G_2}(u_1)) + w(e_2) \\
 &= w(\delta_{G_2}(u_1)) + \tau_w(G_1, \Sigma_1)_1 - \tau_w(G_1, \Sigma_1)_0 \\
 &= w(\delta_{G_2}(u_1)) + w(\Sigma_1 \Delta \delta_{G_1}(U_1)) - w(\Sigma_1 \Delta \delta_{G_1}(U_0)) \\
 &\equiv w(\delta_{G_2}(u_1)) + w(\Sigma_1) + w(\delta_{G_1}(U_1)) + w(\Sigma_1) + w(\delta_{G_1}(U_0)) \\
 &\equiv w(\delta_{G_2}(u_1)) + w(\delta_{G_1}(U_1) \Delta \delta_{G_1}(U_0)) \\
 &\equiv w(\delta_{G_2}(u_1)) + w(\delta_{G_1}(U_1 \Delta U_0)) \\
 &= w(\delta_G(U_1 \Delta U_0)) \equiv 0.
 \end{aligned}$$

So (12) follows.

By (11) and (12) there exists a w -packing $\mathcal{C}^2 = \{C_1^2, \dots, C_{\tau_w(\widehat{G}_2, \Sigma_2)}^2\}$ of odd circuits in $(\widehat{G}_2, \Sigma_2)$. For each $e \in E(\widehat{G}_2)$ let $c(e)$ denote the number of members of \mathcal{C}^2 that use edge e ; abbreviate $\gamma := w(e_2)$. Assume that C_1^2, \dots, C_γ^2 are the members of \mathcal{C}^2 containing e_2 . The function $w - c$ is Eulerian on \widehat{G}_2 , and as \mathcal{C}^2 is a maximum w -packing of odd circuits, the set of edges $e \in E(\widehat{G}_2)$ with $w(e) - c(e) > 0$ contains no odd circuits. Hence, by Euler's theorem on Euler tours and since $(w - c)(e_2) = \omega - \gamma$, there exists a $(w - c)$ -packing $\mathcal{D} = \{D_1^2, \dots, D_{\omega - \gamma}^2\}$ of even circuits in $(\widehat{G}_2, \Sigma_2)$ that all contain e_2 .

$$(13) \quad \text{We may assume that } \gamma = 0 \text{ or } \omega - \gamma = 0.$$

If both are positive, then C_1^2 contains e_2 and D_1^2 exists; by definition D_1^2 also contains e_2 . The set $C_1^2 \Delta D_1^2$ contains an odd circuit, C say. As $C_1^2 \Delta D_1^2$ does not contain e_2 , neither does C . Replacing in \mathcal{C}^2 the odd circuit C_1^2 with C yields a w -packing of the same size as \mathcal{C}^2 that has only $c(e_2) - 1$ members using e_2 . This proves (13).

If $\omega = 0$, let $\widehat{G}_1 := G_1$. If $\gamma = \omega > 0$, let \widehat{G}_1 be obtained from G_1 by adding an odd edge e_1 between u_1 and u_2 with $w(e_1) := \omega$. If $\omega > 0 = \gamma$, let \widehat{G}_1 be obtained from G_1 by adding an even edge f_1 between u_1 and u_2 with $w(f_1) := \omega$. If e_1 is included in \widehat{G}_1 , we define $\widehat{\Sigma}_1 := \Sigma_1 \cup \{e_1\}$; otherwise, $\widehat{\Sigma}_1 := \Sigma_1$.

$$(14) \quad (\widehat{G}_1, \widehat{\Sigma}_1) \text{ is a proper minor of } (G, \Sigma).$$

If e_1 exists in $(\widehat{G}_1, \widehat{\Sigma}_1)$, then $\gamma > 0$, so there exists an odd circuit using e_2 in (G_2, Σ_2) , for instance, C_1^2 . So in that case there is an odd u_1u_2 -path in (G_2, Σ_2) . If f_1 exists in $(\widehat{G}_1, \widehat{\Sigma}_1)$, then $\omega - \gamma > 0$, so there exists an even circuit using e_2 in (G_2, Σ_2) , for instance, D_1^2 . Hence, in that case there is an even u_1u_2 -path in (G_2, Σ_2) . This proves (14).

$$(15) \quad w(\delta_{\widehat{G}_1}(v)) \text{ is even for each } v \in V(\widehat{G}_1).$$

This is obvious as the weight of the added edge is $w(e_2)$ and as w is Eulerian on G and on \widehat{G}_2 .

By (14) and (15) there exists a w -packing $\mathcal{C}^1 = \{C_1^1, \dots, C_{\tau_w(\widehat{G}_1, \widehat{\Sigma}_1)}^1\}$ of odd circuits in $(\widehat{G}_1, \widehat{\Sigma}_1)$.

$$(16) \quad \tau_w(\widehat{G}_1, \widehat{\Sigma}_1) = \tau_w(G_1, \Sigma_1)_0 + \gamma = \tau_w(\widehat{G}_1, \widehat{\Sigma}_1)_0.$$

Note that $\gamma = \omega = w(e_1)$ if e_1 exists and $\gamma = 0$ if e_1 does not exist. Hence, $\tau_w(\widehat{G}_1, \widehat{\Sigma}_1)_0 = \tau_w(G_1, \Sigma_1)_0 + \gamma$. Similarly, $\omega - \gamma = \omega = w(f_1)$ if f_1 exists, and

$\omega - \gamma = 0$ if f_1 does not exist. Hence, by the definition of ω we get $\tau_w(\widehat{G}_1, \widehat{\Sigma}_1)_1 = \tau_w(G_1, \Sigma_1)_1 + \omega - \gamma = \tau_w(G_1, \Sigma_1)_0 + 2\omega - \gamma \geq \tau_w(G_1, \Sigma_1)_0 + \gamma$. By (6), this proves (16).

As $\tau_w(\widehat{G}_1, \widehat{\Sigma}_1) = \tau_w(\widehat{G}_1, \widehat{\Sigma}_1)_0$ there exists a minimum weight signature containing e_1 as soon as e_1 exists, that is as soon as $\gamma > 0$. Hence, by ‘‘complementary slackness’’ there are exactly γ odd circuits in \mathcal{C}^1 that contain e_1 . Assume that C_1^1, \dots, C_γ^1 contain e_1 and that $C_{\gamma+1}^1, \dots, C_{\gamma+k}^1$ are the members of \mathcal{C}^1 containing f_1 . Note that $k \leq \omega - \gamma$.

Now let \mathcal{C} be the collection of the following odd circuits:

$$\begin{aligned} & (C_i^1 \setminus \{e_1\}) \cup (C_i^2 \setminus \{e_2\}) \text{ for } i = 1, \dots, \gamma, \\ & (C_i^1 \setminus \{f_1\}) \cup (D_{i-\gamma}^2 \setminus \{e_2\}) \text{ for } i = \gamma + 1, \dots, \gamma + k, \\ & C_i^1 \text{ for } i = \gamma + k + 1, \dots, \tau_w(\widehat{G}_1, \widehat{\Sigma}_1), \\ & C_i^2 \text{ for } i = \gamma + 1, \dots, \tau_w(\widehat{G}_2, \Sigma_2). \end{aligned}$$

Clearly, \mathcal{C} is a w -packing in G . Its size is $\tau_w(\widehat{G}_1, \widehat{\Sigma}_1) + \tau_w(\widehat{G}_2, \Sigma_2) - \gamma$. By (10) and (16) this is equal to $\tau_w(G, \Sigma)$. Hence, $\nu_w(G, \Sigma) \geq \tau_w(G, \Sigma)$, contrary to our assumption. This proves the lemma. \square

LEMMA 6. *If (G, Σ) does not satisfy $\nu_w(G, \Sigma) = \tau_w(G, \Sigma)$ for each Eulerian $w \in \mathbb{Z}_+^{E(G)}$ and is minor-minimal in this respect, then (G, Σ) has no improper 3-vertex cutset.*

Proof. Let (G, Σ) be a counterexample; by Lemma 5 it is 3-connected. Then (G, Σ) contains a signed graph (G_1, Σ_1) and a bipartite signed graph (G_2, Σ_2) such that $E(G_1)$ and $E(G_2)$ partition $E(G)$, $V(G_1) \cap V(G_2) = \{u_1, u_2, u_3\}$, and $|E(G_2)| \geq 4$. By resigning, we may assume that $\Sigma_2 = \emptyset$. Let $w \in \mathbb{Z}_+^{E(G)}$ be Eulerian with $\tau_w(G, \Sigma) > \nu_w(G, \Sigma)$.

For each signed graph (H, Θ) containing $\{u_1, u_2, u_3\}$, we define

$$(17) \quad \tau_w(H, \Theta)_0 := \min\{w(\Theta \triangle \delta_H(U)) \mid U \cap \{u_1, u_2, u_3\} = \emptyset\},$$

and, for each $i = 1, 2, 3$,

$$(18) \quad \tau_w(H, \Theta)_i := \min\{w(\Theta \triangle \delta_H(U)) \mid U \cap \{u_1, u_2, u_3\} = \{u_i\}\}.$$

Then,

$$(19) \quad \tau_w(H, \Theta) = \min\{\tau_w(H, \Theta)_0, \tau_w(H, \Theta)_1, \tau_w(H, \Theta)_2, \tau_w(H, \Theta)_3\}.$$

Moreover, we define

$$(20) \quad \begin{aligned} \omega_1 &:= \frac{1}{2}[\tau_w(G_2, \emptyset)_2 + \tau_w(G_2, \emptyset)_3 - \tau_w(G_2, \emptyset)_1], \\ \omega_2 &:= \frac{1}{2}[\tau_w(G_2, \emptyset)_1 + \tau_w(G_2, \emptyset)_3 - \tau_w(G_2, \emptyset)_2], \\ \omega_3 &:= \frac{1}{2}[\tau_w(G_2, \emptyset)_1 + \tau_w(G_2, \emptyset)_2 - \tau_w(G_2, \emptyset)_3]. \end{aligned}$$

Then,

$$(21) \quad \omega_1, \omega_2, \text{ and } \omega_3 \text{ are nonnegative.}$$

To prove that, for ω_1 , choose for $i = 2, 3$ a set $U_i \subseteq V(G_2)$ with $U_i \cap \{u_1, u_2, u_3\} = \{u_i\}$ and $w(\delta_{G_2}(U_i)) = \tau_w(G_2, \emptyset)_i$. Then, as $(V(G_2) \setminus (U_2 \cup U_3)) \cap \{u_1, u_2, u_3\} = \{u_1\}$, we get that $\tau_w(G_2, \emptyset)_1 \leq w(\delta_{G_2}(V(G_2) \setminus (U_2 \cup U_3))) = w(\delta_{G_2}(U_2 \cup U_3)) \leq w(\delta_{G_2}(U_2)) + w(\delta_{G_2}(U_3)) = \tau_w(G_2, \emptyset)_2 + \tau_w(G_2, \emptyset)_3$. So indeed, $\omega_1 \geq 0$ and (21) follows.

Moreover,

$$(22) \quad \omega_1, \omega_2, \text{ and } \omega_3 \text{ are integers.}$$

To see that note that the fact that $w(\delta_{G_2}(v))$ is even for each $v \in V(G_2) \setminus \{u_1, u_2, u_3\}$ has the following two consequences: $w(\delta_{G_2}(u_1)) + w(\delta_{G_2}(u_2)) + w(\delta_{G_2}(u_3))$ is even and, for $i = 1, 2, 3$, $w(\delta_{G_2}(U_i)) - w(\delta_{G_2}(u_i))$ is even if $U_i \cap \{u_1, u_2, u_3\} = \{u_i\}$. Hence, by the definition of $\tau_w(G_2, \emptyset)_i$, the number $\tau_w(G_2, \emptyset)_1 + \tau_w(G_2, \emptyset)_2 + \tau_w(G_2, \emptyset)_3$ is even. So (22) follows.

We define both \widehat{G}_1 and \widehat{G}_2 by adding to G_1 and to G_2 the edges $e_1 := u_2u_3, e_2 := u_1u_3$, and $e_3 := u_1u_2$. Moreover, we define $w(e_i) = \omega_i$ for $i = 1, 2, 3$. Similar calculations as in the proof of Lemma 5 show that

$$(23) \quad w(\delta_{\widehat{G}_j}(v)) \text{ is even for each } v \in V(\widehat{G}_j) \text{ and } j = 1, 2.$$

Next we define $\widehat{\Sigma}_2 := \{e_1, e_2, e_3\}$. Straightforward calculations show that

$$(24) \quad \tau_w(\widehat{G}_1, \Sigma_1)_i = \tau_w(G, \Sigma)_i \text{ and } \tau_w(\widehat{G}_2, \widehat{\Sigma}_2)_i = \tau_w(G_2, \Sigma_2)_i + \omega_i = \omega_1 + \omega_2 + \omega_3$$

for each $i = 0, 1, 2, 3$ and thus that

$$(25) \quad \tau_w(\widehat{G}_1, \Sigma_1) = \tau_w(G, \Sigma) \text{ and } \tau_w(\widehat{G}_2, \widehat{\Sigma}_2) = \omega_1 + \omega_2 + \omega_3.$$

From the facts that $|E(G_2)| \geq 4$ and that G is 3-connected, it easily follows that $(\widehat{G}_1, \Sigma_1)$ is a proper minor of (G, Σ) . Hence, $\nu_w(\widehat{G}_1, \Sigma_1) = \tau_w(\widehat{G}_1, \Sigma_1)$. So by (25), there exists a w -packing \mathcal{C}^1 in $(\widehat{G}_1, \Sigma_1)$ consisting of $\tau_w(G, \Sigma)$ odd circuits.

As $\{u_1, u_2\}$ is a blocking pair of $(\widehat{G}_2, \widehat{\Sigma}_2)$, it follows from (3) and (23) that $\nu_w(\widehat{G}_2, \widehat{\Sigma}_2) = \tau_w(\widehat{G}_2, \widehat{\Sigma}_2)$. Thus by (25) there exists a w -packing \mathcal{C}^2 in $(\widehat{G}_2, \widehat{\Sigma}_2)$ consisting of $\omega_1 + \omega_2 + \omega_3$ odd circuits.

As $\{e_1, e_2, e_3\}$ is a minimum weight signature of $(\widehat{G}_2, \widehat{\Sigma}_2)$, there are by complementary slackness for each i exactly ω_i members of \mathcal{C}^2 that intersect $\{e_1, e_2, e_3\}$ in exactly e_i . So there exists a w -packing $\mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{P}^3$ in (G_2, Σ_2) such that each \mathcal{P}^i is a collection of ω_i even paths connecting the ends of e_i . Using the paths in \mathcal{P}^i to replace occurrences of e_i in the members of \mathcal{C}^1 , we can turn \mathcal{C}^1 into a w -packing consisting of $\tau_w(G, \Sigma)$ odd circuits in (G, Σ) , contradicting our assumption that $\tau_w(G, \Sigma) > \nu_w(G, \Sigma)$. This proves the lemma. \square

Proof of Theorem 4 (from Theorem 3). We prove (1), which implies Theorem 4. From Lemmas 5 and 6 and from (2) and (4), we see that we may assume that $|V(G)| = 5$ or that (G, Σ) is one of the signed graphs in Figure 2. In the latter case (G, Σ) has a blocking pair; thus, (3) applies. So we may assume $|V(G)| = 5$. By Lemma 5 we may assume that G has no parallel edges. This means that G is isomorphic to a subgraph of K_5 . As (G, Σ) is not isomorphic to \widetilde{K}_5 , (G, Σ) has a blocking pair. So again (3) applies. This proves Theorem 4. \square

Proof of Theorem 1 (from Theorem 3). Clearly, if (G, Σ) has a blockvertex or a blocking pair or if G is planar, we can find a maximum w -packing of odd circuits by (2), (3), and (4). So it remains to explain how we can algorithmically deal with 2-separations and improper 3-separations.

First consider an improper 3-separation $(G_1, \Sigma_1), (G_2, \Sigma_2)$ of (G, Σ) as in the proof of Lemma 6. We follow that proof. So we assume that $\Sigma_2 = \emptyset$. Finding $\omega_1, \omega_2, \omega_3$ amounts to calculating $\tau_w(G_2, \Sigma_2)_i$ for $i = 1, 2, 3$, which is just the minimum weight

of a cut in G_2 separating u_i from $\{u_1, u_2, u_3\} \setminus \{u_i\}$, so that can be solved by flow techniques. As $\{u_1, u_2\}$ is a blocking pair in $(\widehat{G}_2, \widehat{\Sigma}_2)$ finding a maximum w -packing of odd circuits in $(\widehat{G}_2, \widehat{\Sigma}_2)$ can be done by solving an integer 2-commodity flow problem. As explained in the proof of Lemma 6 the solution of that gives a collection of paths in G_2 that can be used to transform a maximum w -packing of odd circuits in $(\widehat{G}_1, \Sigma_1)$ to a maximum w -packing of odd circuits in (G, Σ) . As all this can be done in polynomial time, we have a polynomial time reduction from the odd circuit packing problem in (G, Σ) to the odd circuit packing problem in $(\widehat{G}_1, \Sigma_1)$, which is a proper minor of (G, Σ) .

So there exists a polynomial time algorithm for the odd circuit packing problem in 3-connected signed graphs with no K_{5^-} , $K_{3,3}^{1,1}$ -, $K_{3,3}^{1,2}$ -, or $K_{3,3}^2$ -minor. Next we consider the case that the signed graph is not 3-connected. Here there are certain issues involved that need extra care. Consider a 2-separation $(G_1, \Sigma_1), (G_2, \Sigma_2)$ of (G, Σ) as in the proof of Lemma 5. If we can find such separation with (G_1, Σ_1) and (G_2, Σ_2) both bipartite, then u_1 is a blockvertex of (G, Σ) , and we can solve the odd circuit packing problem by flow techniques. So we assume that no such 2-separations exist. Therefore as of now we assume that we selected (G_1, Σ_1) and (G_2, Σ_2) such that (G_2, Σ_2) is nonbipartite and under that condition $E(G_1)$ is inclusionwise minimal.

Let (G_1^1, Σ_1^1) be obtained from (G_1, Σ_1) by adding an odd edge e_1 connecting u_1 and u_2 , and let (G_1^0, Σ_1^0) be obtained from (G_1, Σ_1) by adding an even edge f_1 connecting u_1 and u_2 . Then as (G_2, Σ_2) is nonbipartite both (G_1^1, Σ_1^1) and (G_1^0, Σ_1^0) are proper minors of (G, Σ) . Moreover, by minimality of $E(G_1)$ these graphs are 3-connected so we do have a polynomial time algorithm for solving any odd circuit packing problem in (G_1^1, Σ_1^1) or (G_1^0, Σ_1^0) . This is important since as we will see we need to solve three such problems in these signed graphs.

For both $i = 0$ and $i = 1$, we can find $\tau_w(G_1, \Sigma_1)_i$ in polynomial time as it amounts to finding a minimum weight signature in (G_1^i, Σ_1^i) where the extra edge between u_1 and u_2 gets a very high weight. Thus we can calculate ω in polynomial time. Now solve the odd circuit packing problem in the signed graph $(\widehat{G}_2, \Sigma_2)$ constructed in the proof of Lemma 5. We do this recursively, so we may use 2-separations again. We also find the collection of even circuits \mathcal{D}_2 (which is just a flow problem) and adjust the solution such that γ is either 0 or ω , as in (13). Now we solve the odd circuit packing problem on $(\widehat{G}_1, \widehat{\Sigma}_1)$. Since \widehat{G}_1 is 3-connected, we can do this without recursively using 2-separations. Now we combine the optimal packing of odd circuits in $(\widehat{G}_1, \widehat{\Sigma}_1)$ with the optimal packing of odd circuits in $(\widehat{G}_2, \Sigma_2)$ and with the collection \mathcal{D}^2 of even circuits to a solution for the odd circuit packing problem in (G, Σ) .

This recursive method using 2-separations calls itself only in $(\widehat{G}_2, \Sigma_2)$ and for just a single function w . Hence, it runs in polynomial time. \square

4. Subdivisions, homeomorphs, and minors; links and bridges. If P is a path containing vertices u and v , then P_{uv} denotes the uv -subpath of P .

Subdividing an edge uv of (G, Σ) is replacing it with a uv -path P that is internally vertex disjoint with G and replacing Σ with $(\Sigma \setminus \{uv\}) \cup \Sigma_P$, where Σ_P is any subset of $E(P)$ with the same parity as $\Sigma \cap \{uv\}$. A (G, Σ) -*subdivision* is the result of a series of subdivisions of edges in (G, Σ) . If G is just a graph, so with no signing, *subdividing an edge* and G -*subdivision* are defined similarly.

A (G, Σ) -*homeomorph* is a signed graph that is isomorphic to a (G, Σ) -subdivision. Clearly, if a signed graph has a (G, Σ) -homeomorph it has a (G, Σ) -minor. If G has maximum degree 3, the converse is true as well. In particular, for $i = 0, 1, 2$, (G, Σ) has a $K_{3,3}^i$ -minor if and only if it has a $K_{3,3}^i$ -homeomorph.

Let G be a graph; a *leg* of G is a path such that all of its internal vertices have degree 2 in G and its ends have degree at least 3. Let H be a subgraph of G , and let u and v be two of its vertices. A *uv-link* of H , or just *link* of H , is a uv -path that intersects H exactly in $\{u, v\}$.

If G is a graph and X is a set of vertices, then $G - X$ is the graph obtained from G by deleting the vertices in X and the edges incident to them; if X is a set of edges (or a subgraph with edges), then $G - X$ is obtained by deleting only the edges in X .

A subgraph B of G is called a *bridge* of H if either B consists of a single edge not in $E(H)$ that has both ends in $V(H)$ or B consists of a component of $G - V(H)$ together with the edges from this component to H and their ends in H .

5. Recognizing if a graph has a \widetilde{K}_{5^-} , $K_{3,3}^{1,1}$ -, $K_{3,3}^{1,2}$ -, or $K_{3,3}^2$ -minor. We describe how to decide in polynomial time if a graph has a \widetilde{K}_{5^-} , $K_{3,3}^{1,1}$ -, $K_{3,3}^{1,2}$ -, or $K_{3,3}^2$ -minor or not. The algorithm is based on the decomposition in Theorem 3. The idea is standard: we can check in polynomial time if G is planar or if (G, Σ) has a blockvertex or is one of the signed graphs in Figure 2, so we need only recursive procedures for the cases that (G, Σ) is not 3-connected or has improper 3-vertex cutsets. In case (G, Σ) is not 3-connected such a procedure is straightforward, but dealing with decompositions along improper 3-vertex cutsets needs some extra care. So we describe that in detail.

Assume (G, Σ) is 3-connected and contains an improper 3-vertex cutset $\{u_1, u_2, u_3\}$. So, after resigning if necessary, we may assume that G contains graphs G_1 and G_2 with $\Sigma \cap E(G_2) = \emptyset$ such that $E(G_1)$ and $E(G_2)$ partition $E(G)$, $V(G_1) \cap V(G_2) = \{u_1, u_2, u_3\}$, and $|E(G_2)| \geq 4$. Let G^+ be defined by adding to G_1 a new vertex u^+ and three new even edges u^+u_1 , u^+u_2 , and u^+u_3 . Then (G^+, Σ) is a minor of (G, Σ) . So if it has a \widetilde{K}_{5^-} , $K_{3,3}^{1,1}$ -, $K_{3,3}^{1,2}$ -, or $K_{3,3}^2$ -minor, then so does (G, Σ) . Also if (G, Σ) has a \widetilde{K}_{5^-} , $K_{3,3}^{1,2}$ -, or $K_{3,3}^2$ -minor, (G^+, Σ) will have such a minor. But, as $K_{3,3}^{1,1}$ has improper 3-vertex cutsets, (G, Σ) may have a $K_{3,3}^{1,1}$ -minor whereas (G^+, Σ) does not. Fortunately, it can be checked in polynomial time if this happens, as we will explain now. Let G^- be obtained from G_2 by adding a new vertex u^- and three new edges u^-u_1 , u^-u_2 , and u^-u_3 . The following observation is straightforward.

(26) (G, Σ) has a $K_{3,3}^{1,1}$ -minor if and only if one of the following holds:

- (i) G^- has a $K_{3,3}$ -subdivision in which u^- has degree 3 and (G^+, Σ) has a \widetilde{K}_4 -homeomorph in which u^+ has degree 3 and at least one of u_1, u_2 , and u_3 has degree 2.
- (ii) G^- has a $K_{3,3}$ -subdivision in which u^- has degree 3 and at least one of u_1, u_2 , and u_3 has degree 2 and (G^+, Σ) has a \widetilde{K}_4 -homeomorph in which u^+ has degree 3.
- (iii) (G^+, Σ) has a $K_{3,3}^{1,1}$ -minor.

So when we encounter an improper 3-separation, we first check if (26i) or (26ii) applies. If so we decide that our signed graph has a $K_{3,3}^{1,1}$ -minor. If not we just replace (G, Σ) with (G^+, Σ) and search for the existence of a \widetilde{K}_{5^-} , $K_{3,3}^{1,1}$ -, $K_{3,3}^{1,2}$ -, or $K_{3,3}^2$ -minor in (G^+, Σ) recursively. To check if (26i) or (26ii) applies we use the following two results:

(27) *If v is a degree 3 vertex in a simple 3-connected graph H , then v is a degree 3 vertex in some $K_{3,3}$ -subdivision in H if and only if H is nonplanar (Seymour [11]).*

(28) *If v is a degree vertex in a simple 3-connected signed graph (H, Θ) , then v is a degree 3 vertex in some \tilde{K}_4 -homeomorph in (H, Θ) if and only if (H, Θ) has a \tilde{K}_4 -homeomorph.*

We will prove (28) below; (27) is immediate from (11.2) in Seymour [11]. By (27), we can check the condition on G^- in (26i) by checking if G^- is nonplanar. For checking the condition on G^- in (26ii), we construct for each $i = 1, 2, 3$ and each neighbor of $x \neq u^-$ of u_i the graph $G_{i,x}^-$ by deleting from G^- all edges incident with u_i except u^-u_i and u_ix . If $G_{i,x}^-$ is nonplanar for some i and some x , the condition on G^- in (26ii) is satisfied; otherwise, it is not.

By (28), we can check the condition on (G^+, Σ) in (26ii) by checking if (G^+, Σ) contains a \tilde{K}_4 -homeomorph. This can be done in polynomial time by an algorithm by Gerards, Lovász, Schrijver, Seymour, Shih, and Truemper based on decomposing signed graphs with no \tilde{K}_4 -homeomorph (see Gerards [3]; actually the algorithm amounts to applying Truemper’s algorithm [13] for recognizing if a binary clutter has a Q_6 -minor to the clutter of odd circuits in (G^+, Σ)). Finally to check if (G^+, Σ) satisfies the condition in (26i), we construct for each $i = 1, 2, 3$ and each neighbor of $x \neq u^+$ of u_i the graph $G_{i,x}^+$ by deleting from G^+ all edges incident with u_i except u^-u_i and u_ix . If $G_{i,x}^+$ contains a \tilde{K}_4 -homeomorph for some i and some x , the condition on G^- in (26ii) is satisfied; otherwise, it is not.

So to see that we can decide in polynomial time if a signed graph has a \tilde{K}_{5^-} , $K_{3,3}^{1,1}$, $K_{3,3}^{1,2}$, or $K_{3,3}^2$ -minor, it remains only to prove (28).

Proof of (28). Suppose it is false; let (H, Θ) be a minimal counterexample.

(29) *Each \tilde{K}_4 -homeomorph K satisfies $V(K) \supseteq V(H) \setminus \{u\}$.*

Suppose it is not true; let K be a \tilde{K}_4 -homeomorph and x be a vertex not in $V(K) \cup \{u\}$. As H is 3-connected, x has a neighbor y such that $\{x, y\} \not\subseteq \{u, u_1, u_2, u_3\}$. Then $H \setminus xy$ contains K . So if $H \setminus xy$ is a subdivision of a simple 3-connected graph H' , it follows, as (H, Θ) is a minimal counterexample, that H' contains a \tilde{K}_4 -homeomorph containing u . As H itself does not contain such a homeomorph, this is impossible. So $H \setminus xy$ is not a subdivision of a simple 3-connected graph. Then, as $|V(H)| \geq |V(K) \cup \{x\}| \geq 5$, (11.1) in Seymour [11] says that H/xy is 3-connected. H/xy may have parallel edges though. Let H'' be a subgraph of H/xy consisting of one edge from each parallel class of H/xy . We may choose H'' such that it contains K . Note that u has also degree 3 in H'' . Hence, as (H, Θ) is a minimal counterexample, H'' contains a \tilde{K}_4 -homeomorph containing u . But then also H contains such a \tilde{K}_4 -homeomorph; this contradiction proves (29).

(30) *(H, Θ) contains a \tilde{K}_4 -homeomorph \bar{K} with $V(\bar{K}) = V(H)$.*

Indeed, let K be a \tilde{K}_4 -homeomorph in H . If $u \notin V(K)$, then, by (29), u has all three neighbors on K . From this it is straightforward to check that the union of K and the three edges incident with u contains a \tilde{K}_4 -homeomorph \bar{K} using u . By (29), $V(\bar{K}) = V(H)$. So (30) follows.

Take \bar{K} as in (30). Then as u does not have degree 3 in K , we may assume that uu_1 and uu_2 are edges of the same leg, say, P , of \bar{K} . By (28), u_3 lies on \bar{K} . If u_3 does not lie on P , then it is straightforward to find in $K \cup \{uu_3\}$ a \tilde{K}_4 -homeomorph in which u has degree 3. So u_3 lies on P as well, see Figure 3 (left). As indicated there,

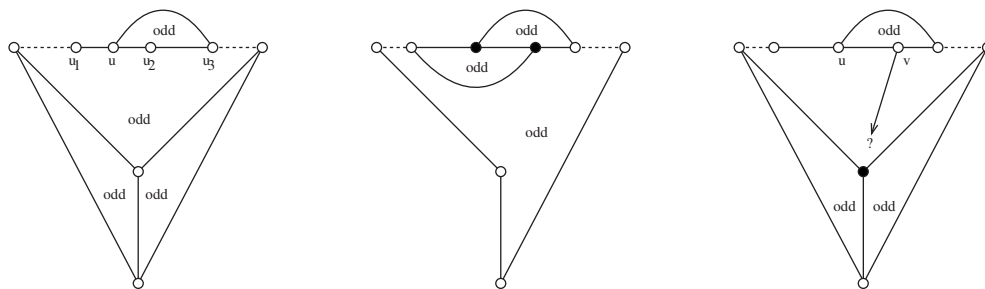


FIG. 3. The word “odd” indicates that the corresponding face is bounded by an odd circuit. Dashed edges may have length zero.

the circuit $P_{u_3u} \cup \{uu_3\}$ is odd as otherwise $(\bar{K} - P_{u_3u}) \cup \{uu_3\}$ is a \tilde{K}_4 -homeomorph that misses u_2 , contradicting (29). As H is 3-connected, $P_{u_1u_3} - u_1 - u_3$ contains a vertex v that is adjacent to a vertex $w \in V(\bar{K}) \setminus V(P_{u_1u_3})$. As u had degree 3 in H , $v \neq u$.

First consider the case that w lies on P . Then the circuit $P_{vw} \cup \{vw\}$ is odd as otherwise $(\bar{K} - P_{vw}) \cup \{vw\}$ is a \tilde{K}_4 -homeomorph that misses either u_1 or u_3 , contradicting (29). So $\bar{K} \cup \{uu_3, vw\}$ contains a subgraph as indicated in the middle picture in Figure 3, where u is one of the two black vertices. That subgraph is a \tilde{K}_4 -homeomorph, and u is a degree 3 vertex of it. This contradicts our assumption that no such homeomorph exists. So we may assume that w is not on P .

Then upto symmetry w lies on a leg of \bar{K} that has the black vertex as an end, as indicated in Figure 3 (right). From the fact that \bar{K} is a \tilde{K}_4 -homeomorph, it is again a straightforward case check that $\bar{K} \cup \{uu_3, vw\}$ contains a \tilde{K}_4 -homeomorph in which u has degree 3. This concludes the proof of (28). \square

6. Nonbipartite subdivisions of $K_{3,3}$: Proof of Theorem 3(i). We now prove Theorem 3(i). We denote the six degree-3 vertices of a $K_{3,3}$ -subdivision K by $r_1^K, r_2^K, r_3^K, r_4^K, r_5^K$, and r_6^K , where the numbering is such that there is a leg between r_i^K and r_j^K if and only if $i = 1, 3, 5$ and $j = 2, 4, 6$. We denote such a leg by P_{ij}^K .

Proof of Theorem 3(i). Suppose the theorem is false. Let (G, Σ) be a minimal counterexample. As G is 3-connected, has no parallel edges, and is not planar and not isomorphic to K_5 , it follows from Kuratowski’s theorem and a well-known and easy result of Hall [7] that G contains a $K_{3,3}$ -subdivision. No $K_{3,3}$ -subdivision in G contains odd circuits, as otherwise there would be a $K_{3,3}^1$ - or a $K_{3,3}^2$ -homeomorph.

Let K be any $K_{3,3}$ -subdivision. By resigning, we may assume that all edges in K are even.

$$(31) \text{ Each odd link of } K \text{ has both ends in } \{r_1^K, r_3^K, r_5^K\} \text{ or both ends in } \{r_2^K, r_4^K, r_6^K\}.$$

Suppose there is a link P contradicting (31). Then $K \cup P$ contains a $K_{3,3}$ -subdivision using P as part of one of its legs. As P is odd and all edges in K are even, this is a $K_{3,3}^1$ -subdivision; this contradiction proves (31).

$$(32) \text{ Each odd link of } K \text{ is an edge.}$$

Suppose this is not true; let P be a link of K contradicting (32). By (31), we may assume that the ends of P are r_1^K and r_3^K . As P is not an edge and G is 3-connected, there exists a link Q of $K \cup P$ with one end in $V(P) \setminus \{r_1^K, r_3^K\}$ and one end, say, r

in $V(K) \setminus \{r_1^K, r_3^K\}$. Clearly, $P \cup Q$ contains an odd link of K with end r . So, by (31), r has to be r_5^K . Now $(K \cup P \cup Q) - P_{21}^K - P_{23}^K - P_{25}^K$ is a $K_{3,3}^1$ -homeomorph; this contradiction proves (32).

G has at least seven vertices, as otherwise Theorem 3(i) is easily verified. It is straightforward to derive from that and the fact that G is 3-connected that (G, Σ) has a $K_{3,3}$ -subdivision with at least seven vertices. Fix such a $K_{3,3}$ -subdivision, and call it K . Let F be the edges of G that form the odd links of K . So each edge in F has both ends in $\{r_1^K, r_3^K, r_5^K\}$ or both ends in $\{r_2^K, r_4^K, r_6^K\}$. For each edge uv of F , there are three internally vertex disjoint uv paths in K . Hence, $G - F$ is 3-connected. Moreover, $G - F$ has no odd circuits because if it had, then by the 3-connectivity of $G - F$ there would exist an odd link of K that is not an edge of F , contradicting (32). So we may resign (G, Σ) such that the edges in F are odd and the edges in $G - F$ are even.

(33) *If $i = 1, 3, 5$ and $j = 2, 4, 6$ and if r_i^K and r_j^K are both ends of some edge in F , then P_{ij}^K consists of a single edge.*

Suppose this is false. Then, as $G - F - r_i^K - r_j^K$ is connected, K has an even link Q with one end in $P_{ij}^K - r_i^K - r_j^K$ and one end not in P_{ij}^K . Then Q is contained in a $K_{3,3}$ -subdivision in $K \cup Q$. This $K_{3,3}$ -subdivision has an odd link contradicting (32). So (33) follows.

(34) *We may assume that $r_1^K r_3^K$ and $r_2^K r_4^K$ are in F and that $r_1^K r_5^K, r_2^K r_6^K$ and $r_4^K r_6^K$ are not in F .*

If no edge in F has its end in $\{r_1^K, r_3^K, r_5^K\}$, then $\{r_2^K, r_4^K, r_6^K\}$ is an improper 3-vertex cutset. Hence, by symmetry, we may assume that $r_1^K r_3^K$ and $r_2^K r_4^K$ are in F . As K has at least seven vertices, it follows from (33) that at least one of $r_1^K, r_2^K, \dots, r_6^K$ is not an end of an edge in F . So, again by symmetry, we may assume that $r_2^K r_6^K$ and $r_4^K r_6^K$ are not in F . Now if both $r_1^K r_5^K$ and $r_3^K r_5^K$ are in F , then $r_1^K r_3^K, r_1^K r_5^K, r_3^K r_5^K, r_2^K r_4^K$, and K contains a \tilde{K}_5 -homeomorph. Thus (34) follows.

$$(35) \quad F = \{r_1^K r_3^K, r_2^K r_4^K\}.$$

If not, then by (34), $F = \{r_1^K r_3^K, r_3^K r_5^K, r_2^K r_4^K\}$. Now as F has at least seven vertices it follows from (33) that $P_{61}^K \cup P_{63}^K \cup P_{65}^K$ has at least four edges. Since $\{r_1^K, r_3^K, r_5^K\}$ is not an improper 3-vertex cutset this means that (G, Σ) has the signed graph in Figure 4 as a minor (possibly with r_2^K and r_4^K interchanged). That signed graph has a $K_{3,3}^1$ -subdivision, so (35) follows.

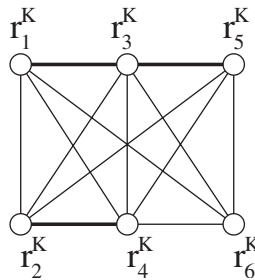


FIG. 4. Bold edges are odd; thin edges are even.

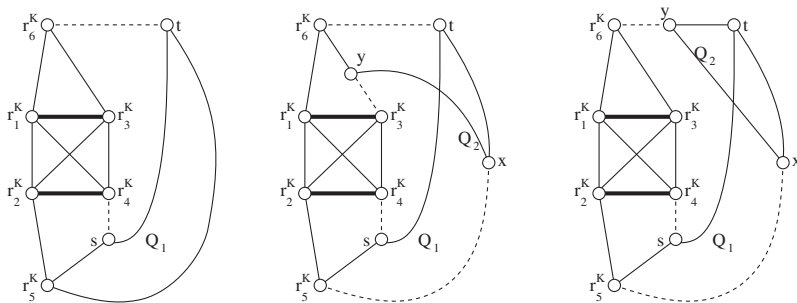


FIG. 5. Bold edges are odd paths; thin edges are even paths; and dashed edges may have length zero.

By (33) and (35), each of $P_{12}^K, P_{14}^K, P_{32}^K,$ and P_{34}^K is a single edge. Hence, by symmetry, we may assume that $P_{61}^K \cup P_{63}^K \cup P_{65}^K$ has at least four edges. Since $\{r_1^K, r_3^K, r_5^K\}$ is not an improper 3-vertex cutset, that means that K has an st -link Q_1 with s on $(P_{52}^K \cup P_{54}^K) - r_5^K$ and t on $(P_{61}^K \cup P_{63}^K \cup P_{65}^K) - r_1^K - r_3^K - r_5^K$. Choose K and Q_1 such that t is as close as possible to $P_{61}^K \cup P_{63}^K$ in $P_{61}^K \cup P_{63}^K \cup P_{65}^K$. We may assume that s lies on P_{54}^K .

$$(36) \quad t \text{ lies on } P_{65}^K.$$

If not, $K \cup Q_1$ contains a $K_{3,3}$ -subdivision that has an odd link contradicting (31).

So we have a situation as depicted in Figure 5 (left). Since $\{r_2^K, r_4^K, t\}$ is not an improper 3-vertex cutset, $K \cup Q_1$ has an xy -link Q_2 with x on $(P_{52}^K \cup P_{54}^K \cup Q_1 \cup (P_{56}^K)_{r_5^K t}) - r_2^K - r_4^K - t$ and y on $(P_{61}^K \cup P_{63}^K \cup (P_{65}^K)_{r_6^K t}) - t$. As K and Q_1 are chosen such that t is as close as possible to $P_{61}^K \cup P_{63}^K$ the end x of Q_2 has to lie on $(P_{56}^K)_{r_5^K t} - t$. If y lies on $P_{63}^K - r_6^K$ (see Figure 5 (middle)) then $(K \cup Q_1 \cup Q_2) - r_2^K r_3^K - r_1^K r_4^K - (P_{63}^K)_{r_6^K y} - (P_{52}^K)_{r_5^K s}$ is a $K_{3,3}$ -subdivision. Hence, y does not lie on $P_{63}^K - r_6^K$ and, by symmetry, also does not lie on $P_{61}^K - r_6^K$. So y lies on $(P_{65}^K)_{r_6^K t} - t$ (see Figure 5 (right)). Now replacing K with $(K \cup Q_2) - (P_{65}^K)_{xy}$ and Q_1 with $Q_1 \cup (P_{65}^K)_{ty}$ yields a contradiction against the fact that K and Q_1 are chosen such that t is as close as possible to $P_{61}^K \cup P_{63}^K$. This proves Theorem 3(i). \square

7. $K_{3,3}^1$ -subdivisions and $K_{3,3}^1$ -extensions. As of now, if K is a $K_{3,3}^1$ -subdivision in (G, Σ) , we will assume that the unique odd leg is P_{12}^K . In that case, we can always resign (G, Σ) such that the only odd edge in K is the edge in P_{12}^K with end r_1^K ; unless stated otherwise, we will assume that if we call a $K_{3,3}^1$ -subdivision K , it has such a canonical signing. Under these assumptions we define $T_1^K := P_{14}^K \cup P_{16}^K$, $T_2^K := P_{23}^K \cup P_{25}^K$, $\text{CAGE}(K) := P_{34}^K \cup P_{36}^K \cup P_{45}^K \cup P_{56}^K$, and $\text{CORE}(K) := V(\text{CAGE}(K)) \setminus \{r_3^K, r_4^K, r_5^K, r_6^K\}$ (see Figure 6).

Clearly, these labelings of vertices and legs of a $K_{3,3}^1$ -subdivision and the indicated canonical signing are not unique. For instance if we interchange index 1 with index 2, interchange index pair $\{4, 6\}$ with index pair $\{3, 5\}$, and resign (G, Σ) on the internal vertices of P_{12}^K , we obtain another labeling and canonical signing as indicated above. When we use this symmetry, we refer to it as *left-right symmetry*. Simpler symmetries are *35-symmetry*, that is interchanging index 3 with index 5, and *46-symmetry*.

Our strategy in proving Theorem 3(ii) is to start with a $K_{3,3}^1$ -subdivision in (G, Σ) . Such a $K_{3,3}^1$ -subdivision has blockvertices and improper 3-vertex cutsets. So, assuming

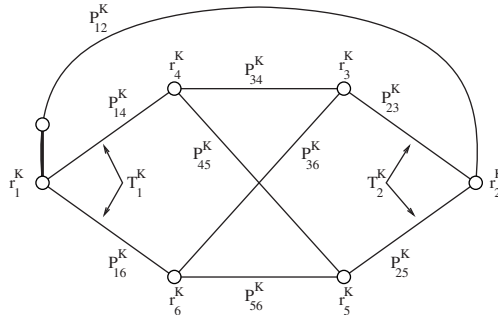


FIG. 6. A $K_{3,3}^1$ -subdivision F .

(G, Σ) does not have these features, more structure should be available. We try to grasp that structure by studying the links of the $K_{3,3}^1$ -subdivision. Ideally such a link, or a combination of a few of them, provides a contradiction by establishing one of the forbidden minors in Theorem 3(ii). There are other links, however, that do not provide any extra structure other than some extra $K_{3,3}^1$ -subdivisions, for instance, even links with no end on the unique odd leg of the $K_{3,3}^1$ -subdivision. To avoid chasing such useless links, we include many of them in our initial structure; that is, we start with a “ $K_{3,3}^1$ -extension” rather than with just a $K_{3,3}^1$ -subdivision.

Consider a signed graph F consisting of

- six special vertices, $r_1^F, r_2^F, r_3^F, r_4^F, r_5^F$, and r_6^F ,
- five internally vertex disjoint paths, $P_{12}^F, P_{14}^F, P_{16}^F, P_{23}^F$, and P_{25}^F , where P_{ij}^F is an $r_i^F r_j^F$ -path whose edges are all even, except for the edge of P_{12}^F adjacent to r_1^F which is odd,
- a 2-connected subgraph $\text{CAGE}(F)$ with even edges only that shares with these paths exactly the vertices r_3^F, r_4^F, r_5^F , and r_6^F .

We define $T_1^F := P_{14}^F \cup P_{16}^F$, $T_2^F := P_{23}^F \cup P_{25}^F$, and $\text{CORE}(F) := V(\text{CAGE}(F)) \setminus \{r_3^F, r_4^F, r_5^F, r_6^F\}$.

The set of $K_{3,3}^1$ -subdivisions K in F with $P_{12}^K = P_{12}^F$ and $\text{CAGE}(K) \subseteq \text{CAGE}(F)$ is denoted by $\mathcal{K}(F)$. Note that for each $K_{3,3}^1$ -subdivision K in $\mathcal{K}(F)$ we can choose the numbering such that: $r_1^K = r_1^F, r_2^K = r_2^F, P_{14}^K \supseteq P_{14}^F, P_{16}^K \supseteq P_{16}^F, P_{23}^K \supseteq P_{23}^F$, and $P_{25}^K \supseteq P_{25}^F$.

For $u \in V(F)$ we define the following

- If $u \notin \text{CORE}(F)$, then $\mathcal{K}_u(F) := \mathcal{K}(F)$.
- If $u \in \text{CORE}(F)$, then $\mathcal{K}_u(F)$ consists of those $K_{3,3}^1$ -subdivisions $K \in \mathcal{K}(F)$ with $u \in \text{CORE}(K)$.

We call F a $K_{3,3}^1$ -extension if $\mathcal{K}(F) \neq \emptyset$ and for each $u \in \text{CORE}(F)$ there exists a $K_{3,3}^1$ -subdivision K in F with $u \in \text{CORE}(K)$ and (after resigning) $P_{12}^K = P_{12}^F$ (see Figure 7).

Note that each $K_{3,3}^1$ -subdivision is a $K_{3,3}^1$ -extension. A $K_{3,3}^1$ -extension F is called *extreme* in (G, Σ) if, even after resigning, there is no $K_{3,3}^1$ -extension F' with $P_{12}^{F'} \subsetneq P_{12}^F$ or with $P_{12}^{F'} = P_{12}^F$ and $\text{CAGE}(F') \supsetneq \text{CAGE}(F)$.

8. Links of $K_{3,3}^1$ -extensions. As of now we call signed graphs with no $K_{3,3}^2$ -, $K_{3,3}^{1,1}$ -, or $K_{3,3}^{1,2}$ -minor *clean*. In this section we characterize the type of links an extreme $K_{3,3}^1$ -extension in a clean signed graph can have (see Figure 8).

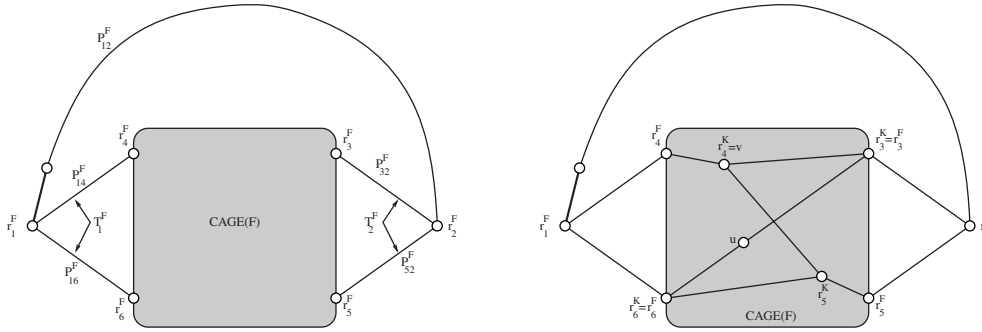


FIG. 7. Left: a $K_{3,3}^1$ -extension F . Right: a $K_{3,3}^1$ -extension F with a $K_{3,3}^1$ -subdivision K that lies in $\mathcal{K}(F)$ and in $\mathcal{K}_u(F)$ but not in $\mathcal{K}_v(F)$.

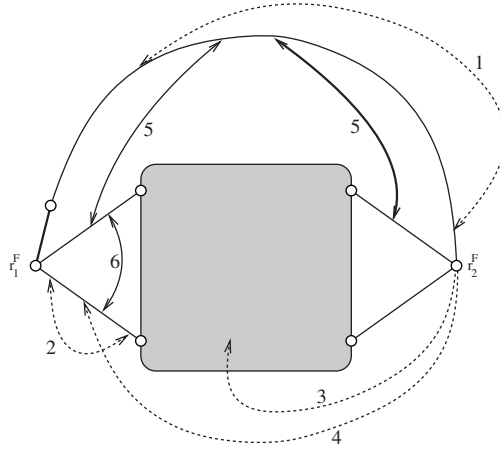


FIG. 8. A $K_{3,3}^1$ -extension F with all possible links (upto symmetry, numbers indicate types, thin lines are even links, bold lines are odd links, and dotted lines have either parity).

LEMMA 7. Let F be an extreme $K_{3,3}^1$ -extension in a clean signed graph, and let P be a link of F . Then P is exactly one of the following types:

- Type 1. Both ends of P lie on P_{12}^F .
- Type 2. Both ends of P lie on P_{ij}^F , where (i, j) is $(1, 4), (1, 6), (2, 3),$ or $(2, 5)$.
- Type 3. P connects r_i^F with a vertex in $\text{CORE}(F)$, where $i = 1$ or $i = 2$.
- Type 4. P connects r_i^F with a vertex in $T_{3-i}^F - r_{3-i}^F$, where $i = 1$ or $i = 2$.
- Type 5. P connects a vertex of $P_{12}^F - r_1^F - r_2^F$ with a vertex on $T_i^F - r_i^F$, where $i = 1$ or $i = 2$.
- Type 6. P connects the two components of $T_i^F - r_i^F$, where $i = 1$ or $i = 2$.

Moreover, a link P of Type 5 is even when $i = 1$ and odd when $i = 2$; all links of Type 6 are even.

We denote the collection of type t links of F by \mathcal{L}_t^F . If $t = 2, 5, 6$, $\mathcal{L}_{t,i}^F$ denotes the collection of links in \mathcal{L}_t^F with an end in T_i^F . If $t = 1, 3, 4$, $\mathcal{L}_{t,i}^F$ denotes the collection of links in \mathcal{L}_t^F with r_i^F as an end. So if $t \neq 1$, $\mathcal{L}_{t,1}^F$ and $\mathcal{L}_{t,2}^F$ partition \mathcal{L}_t^F . The set of even links in \mathcal{L}_t^F is denoted by \mathcal{E}_t^F , and the set of odd links is denoted by \mathcal{O}_t^F . Similarly, we define $\mathcal{E}_{t,i}^F$ and $\mathcal{O}_{t,i}^F$.

It is the statement of Lemma 7 that the collection of links of an extreme $K_{3,3}^1$ -extension F in a clean signed graph is equal to

$$\mathcal{L}_1^F \cup \mathcal{L}_2^F \cup \mathcal{L}_3^F \cup \mathcal{L}_4^F \cup \mathcal{E}_{5,1}^F \cup \mathcal{O}_{5,2}^F \cup \mathcal{E}_6^F.$$

Mind that $\mathcal{E}_{5,1}^F$ corresponds to $\mathcal{O}_{5,2}^F$ under left-right symmetry, and $\mathcal{O}_{5,1}^F$ corresponds to $\mathcal{E}_{5,2}^F$.

Proof of Lemma 7. Suppose the theorem is false; let F and P form a counterexample. Note that as (G, Σ) has no $K_{3,3}^{1,1}$ -minor, $\mathcal{O}_6^F = \emptyset$. So

$$(37) \quad P \notin \mathcal{L}_1^F \cup \mathcal{L}_2^F \cup \mathcal{L}_3^F \cup \mathcal{L}_4^F \cup \mathcal{E}_{5,1}^F \cup \mathcal{O}_{5,2}^F \cup \mathcal{L}_6^F.$$

We first prove

$$(38) \quad P \text{ has no end on } P_{12}^F.$$

If not, then as $P \notin \mathcal{L}_1^F \cup \mathcal{L}_2^F \cup \mathcal{L}_3^F \cup \mathcal{L}_4^F$, one end of P , say, u , lies on $P_{12}^F - r_1^F - r_2^F$ and the other end, say, v , does not lie on P_{12}^F . With u and v in those positions we may assume, by left-right symmetry, that P is even. So as $P \notin \mathcal{E}_{5,1}^F$, v does not lie on T_1^F . Let $K \in \mathcal{K}_v(F)$. Then v is not on T_1^K . By 35-symmetry and 46-symmetry we may assume that v lies on $P_{43}^F \cup P_{32}^F - r_4^F - r_2^F$. If v lies on P_{43}^F , let $S := P_{32}^F$; if v lies on P_{32}^F , let S be the vr_2^F -subpath of P_{32}^F . Then $K' = (K \cup P) - S$ is a $K_{3,3}^1$ -extension with $P_{12}^{K'}$ strictly contained in P_{12}^F ; this contradicts that F is extreme. So (38) follows.

$$(39) \quad \text{Both ends of } P \text{ lie in the core of } F.$$

Suppose this is not true; then by symmetry we may assume that P has an end u in $P_{14}^F - r_1^F$. Then by (37) and (38) the other end, say, v of P lies on $T_2^F - r_2^F$ or in the core of F . Let $K \in \mathcal{K}_v(F)$. Then by 35-symmetry, we may assume that v lies on $(P_{43}^K \cup P_{63}^K \cup P_{32}^K) - r_2^K - r_4^K - r_6^K$. If v lies on P_{43}^K , let S be the $r_4^K v$ -subpath of P_{43}^K ; otherwise, $S := P_{43}^K$. If v lies on P_{23}^K , let R be the $r_3^K v$ -subpath of P_{23}^K ; otherwise, $R := \{v\}$. Let Q be the ur_4^F -subpath of P_{14}^F . Then $K' := (K \cup P) - S$ is a $K_{3,3}$ -subdivision with odd leg P_{12}^F . Moreover, the leg of K' containing P shares no end with P_{12}^F . Hence, as (G, Σ) has no $K_{3,3}^2$ -minor, that leg is even. So K' is a $K_{3,3}^1$ -subdivision. The vertices of $(P \cup Q \cup R) - u - v$ lie in $\text{CORE}(K')$. Hence, $F \cup P$ is a $K_{3,3}^1$ -extension that has a larger core than F has, a contradiction. So (39) follows.

Let u and v be the two ends of P . Let $K \in \mathcal{K}_u(F)$. As $\text{CAGE}(F) - u$ is connected, it contains a path from v to K . Let P' be the union of this path with P , then P' is a leg of K with one end in $\text{CORE}(K)$ and the other end not in P_{12}^K . Hence, as (G, Σ) has no $K_{3,3}^2$ -minor, P' is even. So P' is contained in the cage of a (unique) $K_{3,3}^1$ -subdivision in $K \cup P$. Hence, $F \cup P$ is a $K_{3,3}^1$ -extension with a larger core than F , a contradiction. \square

9. Pairs of links of $K_{3,3}^1$ -extensions. We study the occurrence of pairs of links of $K_{3,3}^1$ -extensions of different types, but first we give an easy fact.

LEMMA 8. *Let a, b_1, b_2 be vertices in a 3-connected signed graph. Each nonbipartite bridge of a, b_1, b_2 contains an odd ab_1 -path disjoint from b_2 or an odd ab_2 -path disjoint from b_1 .*

Proof. Let C be an odd circuit in the bridge. As the graph is 3-connected, there exist three vertex disjoint paths from C to $\{a, b_1, b_2\}$. So the bridge contains an odd path P with ends in $\{a, b_1, b_2\}$. Assume P is not as claimed. Then it is a $b_1 b_2$ -path.

As $\{b_1, b_2\}$ is not a 2-vertex cutset, there exists a path Q from a to P that is disjoint from $\{b_1, b_2\}$. Clearly $P \cup Q$ contains an odd ab_1 -path or an odd ab_2 -path; it obviously misses one of b_1 and b_2 . \square

If F is an $K_{3,3}^1$ -extension, then $\Lambda_i^F := \mathcal{O}_{2,i}^F \cup \mathcal{O}_{3,i}^F \cup \mathcal{O}_{4,i}^F \cup \mathcal{L}_{5,i}^F$ for $i = 1, 2$.

LEMMA 9. *Let F be an extreme $K_{3,3}^1$ -extension in a 3-connected clean signed graph with no blockvertex and no improper 3-vertex cutset. If Λ_1^F and Λ_2^F are nonempty, then either $\Lambda_1^F = \mathcal{O}_{2,1}^F \cup \mathcal{L}_{5,1}^F$ and $\Lambda_2^F = \mathcal{O}_{4,2}^F$ or $\Lambda_1^F = \mathcal{O}_{4,1}^F$ and $\Lambda_2^F = \mathcal{O}_{2,2}^F \cup \mathcal{L}_{5,2}^F$.*

Proof. First we prove some easy facts. In items (40)–(45), K is a $K_{3,3}^1$ -subdivision in a clean signed graph.

$$(40) \quad \text{If } Q_1 \in \mathcal{O}_{2,1}^K \text{ and } Q_2 \in \mathcal{O}_{2,2}^K, \text{ then they intersect.}$$

Indeed, if Q_1 and Q_2 did not intersect, then the unique $K_{3,3}$ -subdivision in $K \cup Q_1 \cup Q_2$ that contains both Q_1 and Q_2 would be a $K_{3,3}^2$ -subdivision.

$$(41) \quad \text{If } Q_1 \in \mathcal{O}_{2,1}^K \text{ and } Q_2 \in \mathcal{O}_{3,2}^K \cup \mathcal{L}_{5,2}^K, \text{ then they intersect.}$$

By contracting edges in the cage of F and along P_{12}^F , we can turn K into a $K_{3,3}^1$ -subdivision K' so that $Q_2 \in \mathcal{O}_{2,2}^{K'}$. As Q_1 is also in $\mathcal{O}_{2,1}^{K'}$ it follows from (40) that Q_1 and Q_2 intersect after these contractions. As these intersections cannot lie on K' , the paths also intersected before the contractions were carried out. So (41) holds indeed.

$$(42) \quad \text{If } Q_1 \in \mathcal{O}_{4,1}^K \text{ and } Q_2 \in \mathcal{O}_{4,2}^K, \text{ then they intersect.}$$

If not, $K \cup Q_1 \cup Q_2$ contains a $K_{3,3}^{1,2}$ -minor.

$$(43) \quad \text{If } Q_1 \in \mathcal{O}_{3,1}^K \text{ and } Q_2 \in \mathcal{O}_{4,2}^K, \text{ then they intersect.}$$

If not, we can contract edges in the cage of K such that Q_1 and Q_2 stay disjoint and K turns into a $K_{3,3}^1$ -subdivision K' with $Q_1 \in \mathcal{O}_{4,1}^{K'}$ and $Q_2 \in \mathcal{O}_{4,2}^{K'}$, contradicting (42).

By a similar contraction argument we derive the following from (41):

$$(44) \quad \text{If } Q_1 \in \mathcal{O}_{3,1}^K \text{ and } Q_2 \in \mathcal{L}_{5,2}^K, \text{ then they intersect.}$$

Note that (41), (43), and (44) have “left-right symmetrical” versions obtained by swapping the second subscripts 1 and 2. We will not list all such versions but just refer to them by mentioning left-right symmetry.

$$(45) \quad \text{If } Q_1 \in \mathcal{O}_{3,1}^K \text{ and } Q_2 \in \mathcal{O}_{3,2}^K, \text{ then they intersect outside } K.$$

If Q_1 and Q_2 do not intersect at all, it is possible to contract edges in the cage of K such that K turns into a $K_{3,3}^1$ -subdivision K' with $Q_1 \in \mathcal{O}_{2,1}^{K'} \cup \mathcal{O}_{4,1}^{K'}$ and Q_2 still in $\mathcal{O}_{3,2}^{K'}$. If $Q_1 \in \mathcal{O}_{2,1}^{K'}$ this contradicts (41); if $Q_1 \in \mathcal{O}_{4,1}^{K'}$ this contradicts (43), by left-right symmetry. If Q_1 and Q_2 meet only in the cage of K , so at their ends, we can contract edges in $\text{CAGE}(K)$ such that we obtain the signed graph in Figure 9(a) as a minor. As is illustrated in that figure, that signed graph has a $K_{3,3}^{1,1}$ -homeomorph, a contradiction. So (45) follows indeed.

Now let F be an extreme $K_{3,3}^1$ -extension in a clean signed graph (G, Σ) with no blockvertex and no improper 3-vertex cutset.

$$(46) \quad \text{At least one of } \mathcal{O}_{2,1}^F \cup \mathcal{O}_{3,1}^F \text{ and } \mathcal{O}_{2,2}^F \cup \mathcal{L}_{5,2}^F \text{ is empty.}$$

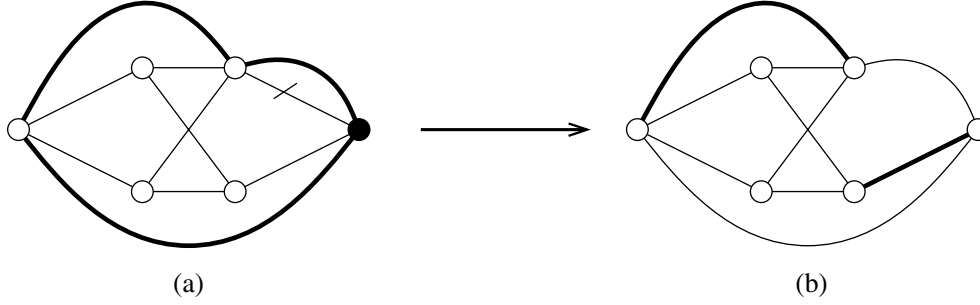


FIG. 9. Bold edges are odd; thin edges are even. To obtain (b) from (a), resign on the black vertex and delete the “crossed” edge.

Suppose this is false; let $P_1 \in \mathcal{O}_{2,1}^F \cup \mathcal{O}_{3,1}^F$ and $P_2 \in \mathcal{O}_{2,2}^F \cup \mathcal{L}_{5,2}^F$. If $P_1 \in \mathcal{O}_{3,1}^F$, let u be its end in the core of F ; otherwise, let u be any vertex of F . Choose $K \in \mathcal{K}_u(F)$. Then $P_1 \in \mathcal{O}_{2,1}^K \cup \mathcal{O}_{3,1}^K$ and $P_2 \in \mathcal{O}_{2,2}^K \cup \mathcal{L}_{5,2}^K$. Hence, it follows from (40), (41), (44), and left-right symmetry that P_1 and P_2 intersect. Clearly this intersection lies outside F . Hence, $P_1 \cup P_2$ contains a link of F that has one end in $(T_1^F \cup \text{core}(F)) - r_1^F$ and one end in $T_2^F - r_2^F$. As this contradicts Lemma 7, (46) follows.

$$(47) \quad \text{At least one of } \mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F \text{ and } \mathcal{O}_{4,2}^F \text{ is empty.}$$

Suppose this is false; let $P_1 \in \mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F$ and $P_2 \in \mathcal{O}_{4,2}^F$. If $P_1 \in \mathcal{O}_{3,1}^F$, let u be its end in the core of K ; otherwise, let u be any vertex of F . Choose $K \in \mathcal{K}_u(F)$. Then $P_1 \in \mathcal{O}_{3,1}^K \cup \mathcal{O}_{4,1}^K$ and $P_2 \in \mathcal{O}_{4,2}^K$. Hence, it follows from (42) and (43) that P_1 and P_2 intersect. Clearly this intersection lies outside F . Hence, $P_1 \cup P_2$ contains a link of F that has one end in $T_1^F - r_1^F$ and one end in $(T_2^F \cup \text{core}(F)) - r_2^F$. As this contradicts Lemma 7, (47) follows.

Now assume that the lemma is false and that F is a counterexample. Hence, Λ_1^F and Λ_2^F are both nonempty.

$$(48) \quad \mathcal{O}_2^F \text{ is empty.}$$

Suppose this is false; assume $\mathcal{O}_{2,1}^F \neq \emptyset$. Then by (46) and left-right symmetry $\Lambda_2^F = \mathcal{O}_{4,2}^F$. So $\mathcal{O}_{4,2}^F \neq \emptyset$. Hence, (47) implies that $\Lambda_1^F = \mathcal{O}_{2,1}^F \cup \mathcal{L}_{5,1}^F$. This contradicts that F is a counterexample, so (48) follows.

We consider two cases.

Case 1. \mathcal{O}_3^F is empty.

$$(49) \quad \mathcal{L}_{5,1}^F \text{ and } \mathcal{L}_{5,2}^F \text{ are not empty.}$$

If $\mathcal{L}_{5,1}^F = \emptyset$, then, by (48) and as \mathcal{O}_3^F is empty, $\Lambda_1^F = \mathcal{O}_{4,1}^F$ and $\Lambda_2^F = \mathcal{O}_{4,2}^F \cup \mathcal{L}_{5,2}^F$. Hence, as F falsifies the lemma, both $\mathcal{O}_{4,1}^F$ and $\mathcal{O}_{4,2}^F$ are nonempty, contradicting (47).

$$(50) \quad \mathcal{O}_4^F \text{ is empty.}$$

Suppose this is false; assume $Q \in \mathcal{O}_{4,1}^F$. Let $P_1 \in \mathcal{L}_{5,1}^F$ and $P_2 \in \mathcal{L}_{5,2}^F$. By Lemma 7, Q and P_1 are vertex disjoint and P_1 and P_2 are internally vertex disjoint. Let P'_2 be the link of $F \cup Q$ that is contained in P_2 and has one end on P_{12}^K . Let P''_2 be the link of F in $\mathcal{L}_{5,2}^F$ contained in $P'_2 \cup Q$. By symmetry, we may assume that P_1 has an end on P_{14}^F

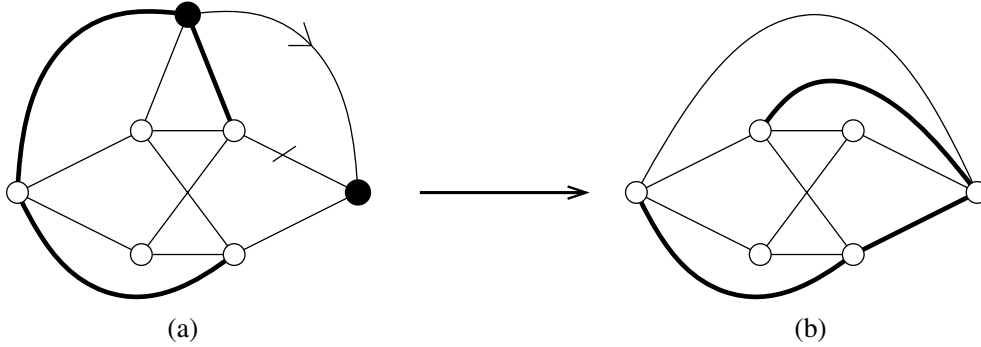


FIG. 10. Bold edges are odd; thin edges are even. To obtain (b) from (a), resign on the black vertices, delete the “crossed” edge, and contract the “directed” edge.

and that P_2'' has an end on P_{23}^F . Note that, by Lemma 7, $P_1 \in \mathcal{E}_{5,1}^F$ and $P_2'' \in \mathcal{O}_{5,2}^F$. If Q has an end in P_{25}^F , then by construction of P_2'' links Q and P_2'' are disjoint. In that case, $K \cup Q \cup P_1 \cup P_2''$ contains the signed graph in Figure 10(a) as a minor, and as illustrated in Figure 10 that signed graph has a $K_{3,3}^{1,2}$ -minor. So Q has an end in P_{23}^F . If Q and P_2'' share edges, resign (if necessary) to make them even, and contract them. Now it is easy to see that $K \cup Q \cup P_2''$ has the signed graph in Figure 9(a) as a minor, hence also a $K_{3,3}^{1,1}$ -minor. That contradicts the cleanliness of (G, Σ) , so (50) follows indeed.

(51) *There exists a vertex $v \in P_{12}^F$ such that each path in \mathcal{L}_5^F has v as one of its ends.*

By (49), it suffices to prove that if $P_1 \in \mathcal{L}_{5,1}^F$ has end p_1 on P_{12}^F and $P_2 \in \mathcal{L}_{5,2}^F$ has end p_2 on P_{12}^F , then $p_1 = p_2$. Suppose this is not the case. Choose $K \in \mathcal{K}(F)$. By Lemma 7, P_1 and P_2 are vertex disjoint. If p_1 lies between r_1^K and p_2 along P_{12}^K , then the unique $K_{3,3}$ -subdivision in $K \cup P_1 \cup P_2$ that contains P_{12}^K , P_1 , and P_2 is a $K_{3,3}^2$ -subdivision. So p_1 lies between p_2 and r_2^K along P_{12}^K . Then $K \cup P_1 \cup P_2$ is a subdivision of the signed graph in Figure 11(a). Hence, as illustrated in Figure 11(b), it contains a $K_{3,3}^1$ -extension F' with $P_{12}^{F'} = (P_{12}^F)_{p_1 p_2}$. That contradicts the extremeness of F , so (51) follows.

As G is 3-connected, $\{r_1^F, r_2^F\}$ is not a 2-vertex cutset of $G - v$. Hence, it follows from (51) that P_{12}^F consists of only two edges: $r_1^F v$ and vr_2^F . Fix $P_1 \in \mathcal{E}_{5,1}^F$ and $P_2 \in \mathcal{O}_{5,2}^F$. Resign on the internal vertices of P_1 and P_2 so that all edges on P_1 and on $P_2 - v$ are even. As (G, Σ) has no blockvertex, $(G, \Sigma) - v$ contains an odd circuit. Hence, as $G - v$ is 2-connected, $(F \cup P_1 \cup P_2) - v$ has an odd link Q contained in $G - v$. By Lemma 7, (48), (50), and (51), and as \mathcal{O}_3^F is empty, Q is disjoint with P_1 and P_2 , and $Q \in \mathcal{O}_1^F$. So the ends of Q are r_1^F and r_2^F . Consider the $K_{3,3}^1$ -subdivision $(F - P_{12}^F) \cup Q$; it is extreme in $F \cup P_1 \cup P_2 \cup Q$. The union of P_1 and P_2 is a link of that $K_{3,3}^1$ -subdivision that contradicts Lemma 7. So Case 1 cannot apply.

Case 2. \mathcal{O}_3^F is not empty.

If $\mathcal{O}_{3,1}^F$ is not empty, then by (46) and (47), $\Lambda_2^F = \mathcal{O}_{3,2}^F$, so $\mathcal{O}_{3,2}^F$ is nonempty as well. Hence, by left-right symmetry it follows from $\mathcal{O}_3^F \neq \emptyset$ that $\mathcal{O}_{3,1}^F = \Lambda_1^F \neq \emptyset$ and $\mathcal{O}_{3,2}^F = \Lambda_2^F \neq \emptyset$.

(52) *Each link in $\mathcal{O}_{3,1}^F$ intersects each link in $\mathcal{O}_{3,2}^F$ outside F .*

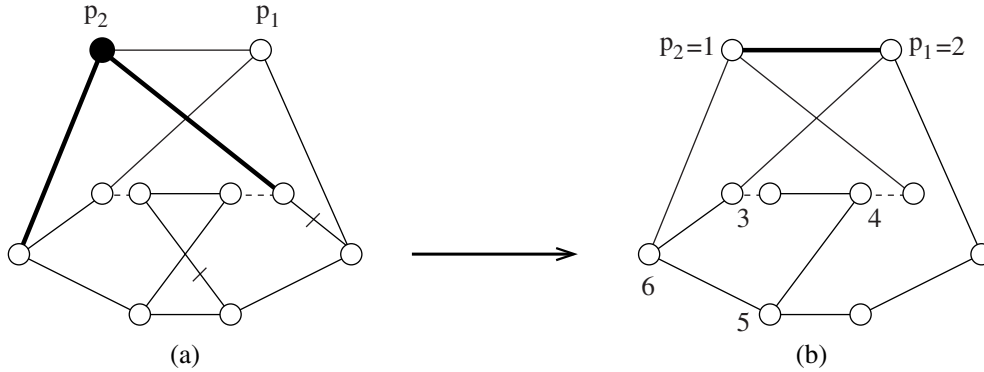


FIG. 11. Bold edges are odd paths; thin edges are even paths; and dashed edges may have length zero. To obtain (b) from (a), resign on the black vertex and delete the “crossed” edges. The numbers $i = 1, \dots, 6$ indicate the vertices $r_i^{F'}$.

Suppose this is false, and let $P_1 \in \mathcal{O}_{3,1}^F$ and $P_2 \in \mathcal{O}_{3,2}^F$ be disjoint outside F . Let p_1 be the end of P_1 in the core of F , and let p_2 be the end of P_2 in the core of F . Let $K \in \mathcal{K}_{p_1}(F)$. If $p_2 \neq p_1$, let P be a path in the cage of F that misses p_1 and connects p_2 to $\text{CAGE}(K)$ (as $\text{CAGE}(F)$ is 2-connected, such P exists); if $p_2 = p_1$, let P consist only of p_2 . Then $P_2 \cup P \in \mathcal{O}_{3,2}^K \cup \mathcal{O}_{4,2}^K$ and $P_1 \in \mathcal{O}_{3,1}^K$. Moreover, these paths are disjoint. This contradicts (43) and (45). So (52) follows.

(53) All links in \mathcal{O}_3^F have the same end in the core of F ; we call that end p .

If not, then as $\mathcal{O}_{3,1}^F$ and $\mathcal{O}_{3,2}^F$ are nonempty, there would be a link in $\mathcal{O}_{3,1}^F$ and a link, in $\mathcal{O}_{3,2}^F$ that have different ends in the core of F . By (52) the union of two such links would contain a link of F that contradicts Lemma 7. So (53) follows.

Let \mathcal{B} be the bridge of $\{r_1^F, r_2^F, p\}$ that contains $\text{CAGE}(F)$.

(54) P_{12}^F and all links in \mathcal{O}_3^F lie outside \mathcal{B} .

That P_{12}^F lies outside \mathcal{B} follows as $\mathcal{L}_5^F = \emptyset$. Suppose \mathcal{B} contains a link P in \mathcal{O}_3^F . Then as $\mathcal{B} - r_1^F - r_2^F - p$ is connected, it contains a path Q from $P - r_1^F - r_2^F - p$ to $F - r_1^F - r_2^F - p$. Now $P \cup Q$ contains a link of F with one end outside $\{r_1^F, r_2^F, p\}$. This contradicts Lemma 7. So (54) follows.

So $\{r_1^F, r_2^F, p\}$ is a 3-vertex cutset separating the core of F from the links in \mathcal{O}_3^F . As this is not an improper 3-vertex cutset, bridge \mathcal{B} contains an odd circuit. Hence, by Lemma 8, \mathcal{B} contains an odd path that connects p to one of r_1^F and r_2^F and that does not contain the other vertex in $\{r_1^F, r_2^F\}$. Clearly such a path contains an odd link of F with at most one end in $\{r_1^F, r_2^F\}$. As $\Lambda_1^F \cup \Lambda_2^F = \mathcal{O}_3^F$, this contradicts (54). So the lemma follows. \square

LEMMA 10. Let K be a $K_{3,3}^1$ -subdivision in a clean signed graph, let Q_1 be an st -link in $\mathcal{O}_{2,1}^K$ with $s \in P_{14}^K$ and $t \in (P_{14}^K)_{sr_4^F}$, and let Q_2 be an $r_2^K p$ -link of $K \cup Q_1$ with $p \in (Q_1 \cup (P_{14}^K)_{r_4^K s}) - s$. Then the unique $r_2^K r_4^K$ -path P' in $(Q_1 \cup Q_2 \cup P_{14}^K) - s$ is even.

Proof. Suppose P' is odd. If necessary resign on p such that $P' - Q_2$ is even, and contract $P' - Q_2$, $(P_{14}^K)_{r_4^K t}$ and $(P_{14}^K)_{r_1^F s}$. This yields a subdivision of the signed graph in Figure 9(a). As illustrated in Figure 9, that signed graph has a $K_{3,3}^{1,1}$ -minor, a contradiction. \square

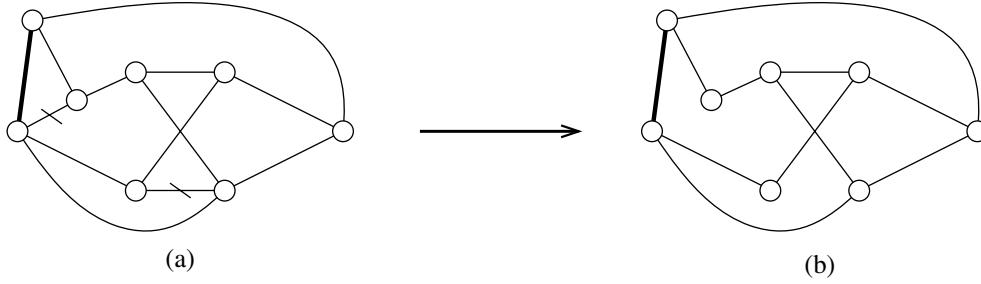


FIG. 12. Bold edges are odd paths; thin edges are even paths. To obtain (b) from (a), delete the “crossed” edges.

LEMMA 11. Let F be an extreme $K_{3,3}^1$ -extension in a clean signed graph. Then $\mathcal{L}_{5,1}^F = \emptyset$ or $\mathcal{E}_{3,1}^F \cup \mathcal{E}_{4,1}^F = \emptyset$.

Proof. Suppose this is not true. Then we may assume that there exists a p_1p_2 -link $P \in \mathcal{L}_{5,1}^F$ and an $r_1^F q$ -link $Q \in \mathcal{E}_{3,1}^F \cup \mathcal{E}_{4,1}^F$ with $p_2 \in P_{14}^F$ and that $q \in \text{CORE}(F) \cup T_2^F$. Choose $K \in \mathcal{K}_q(F)$. By 35-symmetry we may assume that $q \in P_{45}^K \cup P_{65}^K \cup P_{52}^K$. Let R be the intersection of P_{65}^K with the $r_6^K q$ -subpath of $P_{45}^K \cup P_{65}^K \cup P_{52}^K$. By Lemma 7, P is even and disjoint with Q . Now deleting R and $(P_{14}^F)_{r_1^F p_2}$ from $K \cup P \cup Q$ yields a $K_{3,3}^1$ -subdivision F' with $P_{12}^{F'} = (P_{12}^F)_{r_1^F p_1}$. As $P_{12}^{F'}$ is properly contained in P_{12}^F , this contradicts the extremeness of F . (See Figure 12 for the special case that $q = r_5^K$.) \square

The results so far say that certain combinations of links cannot occur; here is a lemma that says that certain links force other ones.

LEMMA 12. Let F be an extreme $K_{3,3}^1$ -extension in a 3-connected clean signed graph with no blockvertex and no improper 3-vertex cutset. If $\mathcal{O}_{2,1}^F \cup \mathcal{L}_{5,1}^F \neq \emptyset$, then $\mathcal{L}_{3,1}^F \cup \mathcal{L}_{4,1}^F \neq \emptyset$.

Proof. Let F be a counterexample. As $\mathcal{O}_{2,1}^F \cup \mathcal{L}_{5,1}^F \neq \emptyset$, it follows from Lemma 9 that $\mathcal{L}_{5,2}^F = \emptyset$. So, as also $\mathcal{L}_{3,1}^F \cup \mathcal{L}_{4,1}^F = \emptyset$, it follows from Lemma 7 that r_1^F does not lie in the bridge \mathcal{B} of $\{r_2^F, r_4^F, r_6^F\}$ that contains $\text{CAGE}(F) \cup T_2^F$. As $\{r_2^F, r_4^F, r_6^F\}$ is no improper 3-vertex cutset, \mathcal{B} contains an odd circuit. Hence, by Lemma 8, \mathcal{B} contains an odd path that has both ends in $\{r_2^F, r_4^F, r_6^F\}$ and that is disjoint from the third vertex in $\{r_2^F, r_4^F, r_6^F\}$. Such a path contains an odd link of F . By Lemma 7, that odd link is in $\mathcal{O}_{2,2}^F \cup \mathcal{O}_{3,2}^F$. As that contradicts Lemma 9, the lemma follows. \square

LEMMA 13. Let F be an extreme $K_{3,3}^1$ -extension in a 3-connected clean signed graph that has no blockvertex and no improper 3-vertex cutset. If $Q \in \mathcal{O}_{2,1}^F$ with ends on P_{1j}^F with $j = 4, 6$ and $P \in \mathcal{O}_{4,2}^F$, then P intersects $Q \cup P_{1j}^F$.

Proof. Let P and Q be as indicated. Assume P and $Q \cup P_{1j}^F$ do not intersect. By Lemma 9, $\mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F = \emptyset$, and thus, by Lemma 12, $\mathcal{E}_{3,1}^F \cup \mathcal{E}_{4,1}^F \neq \emptyset$. Let $R \in \mathcal{E}_{3,1}^F \cup \mathcal{E}_{4,1}^F$. Then, by Lemma 7, R is internally vertex disjoint with P and Q . Hence, we have the signed graph in Figure 13(a) as a minor. As indicated in Figure 13 that signed graph has a $K_{3,3}^{1,2}$ -minor, a contradiction. \square

10. Handles. A *handle* of a $K_{3,3}^1$ -extension F is a link in \mathcal{O}_2^F with no end in $\{r_1^F, r_2^F\}$. The following lemma says that in a counterexample to Theorem 3(ii) each extreme $K_{3,3}^1$ -extension has a handle.

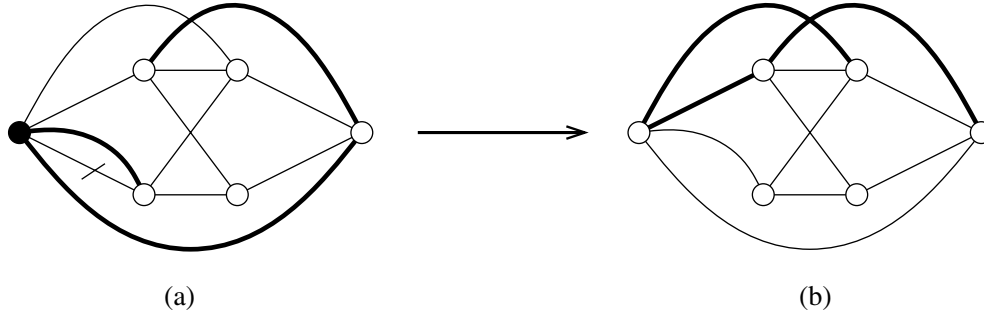


FIG. 13. Bold edges are odd; thin edges are even. To obtain (b) from (a), resign on the black vertex and delete the “crossed” edge.

LEMMA 14. *Each extreme $K_{3,3}^1$ -extension F in a 3-connected clean signed graph (G, Σ) with no blockvertex and no improper 3-vertex cutset has a handle.*

Proof. Let (G, Σ) and F form a counterexample; thus, F has no handle. Let $B := T_1^F \cup \text{CAGE}(F) \cup T_2^F$.

$$(55) \quad (G, \Sigma) - r_1^F - r_2^F \text{ contains an odd circuit, say, } C.$$

Suppose this is not true; then we may assume, by resigning, that all edges not incident with r_1^F or r_2^F are even. It is easy to see that this resigning can be done such that all edges in B are even. In other words $\Sigma \subseteq (\delta_G(r_1^F) \cup \delta_G(r_2^F)) - B$.

As (G, Σ) has no blockvertex, there exists an odd circuit disjoint from r_2^F . As $G - r_1^F - r_2^F$ is connected, F has a link Q_1 that closes with $F - r_2^F$ an odd circuit. Moreover, as (55) is false, all such odd circuits go through r_1^F . So, as $\Sigma \subseteq (\delta_G(r_1^F) \cup \delta_G(r_2^F)) - B$, we have that $Q_1 \in \mathcal{L}_{1,1}^F \cup \mathcal{O}_{2,1}^F \cup \mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F \cup \mathcal{L}_{5,1}^F$. By symmetry F also has a link $Q_2 \in \mathcal{L}_{1,2}^F \cup \mathcal{O}_{2,2}^F \cup \mathcal{O}_{3,2}^F \cup \mathcal{O}_{4,2}^F \cup \mathcal{L}_{5,2}^F$ that closes with $F - r_1^F$ an odd circuit.

First assume that P_{12}^F consists of a single edge. Then, $Q_1, Q_2 \notin \mathcal{L}_1^F \cup \mathcal{L}_5^F$, so by Lemma 9 and by symmetry, we may assume that $Q_1 \in \mathcal{O}_{2,1}^F$ and $Q_2 \in \mathcal{O}_{4,2}^F$. We also may assume that Q_1 has its ends on P_{14}^F . By Lemma 13, Q_2 intersects $Q_1 \cap P_{14}^F$. From this and as $\Sigma \subseteq (\delta_G(r_1^F) \cup \delta_G(r_2^F)) - B$, one easily deduces a contradiction against Lemma 10.

So we may assume that P_{12}^F does not consist of a single edge. As G is 3-connected, $\mathcal{L}_5^F \neq \emptyset$. So we may as well assume that $Q_1 \in \mathcal{L}_{5,1}^F$. By Lemmas 12 and 11 there exists a link $Q \in \mathcal{L}_{3,1}^F \cup \mathcal{L}_{4,1}^F$. Hence, $\mathcal{L}_{5,1}^F$ and $\mathcal{L}_{3,1}^F \cup \mathcal{L}_{4,1}^F$ are nonempty, so by Lemma 9, $\Lambda_2^F = \emptyset$. This implies that $Q_2 \in \mathcal{L}_{1,2}^F$. By Lemma 7, Q is vertex disjoint with Q_1 , and as $\mathcal{L}_{5,2}^F \subseteq \Lambda_2^F = \emptyset$, Q is also disjoint with Q_2 . Contract all edges in $P_{12}^F \cup Q_1 \cup Q_2$ that are not incident with $\{r_1^F, r_2^F\}$ and not incident with a vertex on P_{14}^F ; they are all even. The resulting signed graph has the signed graph in Figure 14(a) as a minor. As illustrated in Figure 14, that signed graph has a $K_{3,3}^2$ -minor. This contradiction proves (55).

We may assume that $\Lambda_2^F = \mathcal{O}_{4,2}^F$. Indeed, by Lemma 9 and 12-symmetry we may assume that $\Lambda_2^F = \emptyset$ or $\Lambda_2^F = \mathcal{O}_{4,2}^F$. As by definition, $\mathcal{O}_{4,2}^F$ is contained in Λ_2^F , which means the sets are equal.

$$(56) \quad \text{If } B \text{ has an odd } r_2^F \text{ } p\text{-link with } p \neq r_1^F, \text{ then } P_{12}^F \text{ is a single edge.}$$

Assume that P_{12}^F is not an edge. Then, as G is 3-connected, $\mathcal{L}_5^F \neq \emptyset$. So as $\Lambda_2^F = \mathcal{O}_{4,2}^F$, we have that $\mathcal{L}_{5,2}^F = \emptyset$, so $\mathcal{L}_{5,1}^F \neq \emptyset$. Hence, by Lemma 12, there exists a link

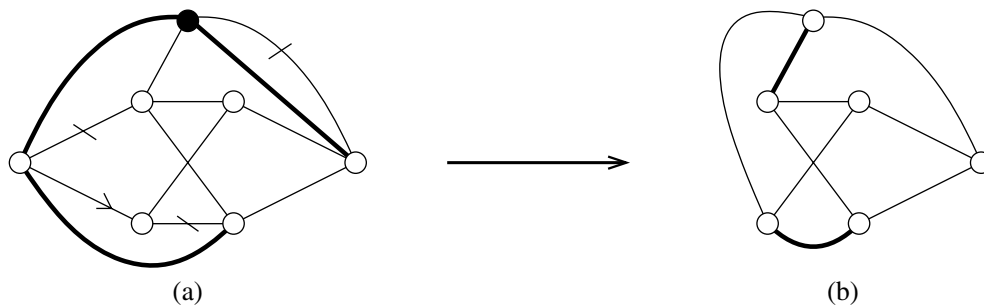


FIG. 14. Bold edges are odd; thin edges are even. To obtain (b) from (a), resign on the black vertex, delete the “crossed” edges, and contract the “directed” edge.

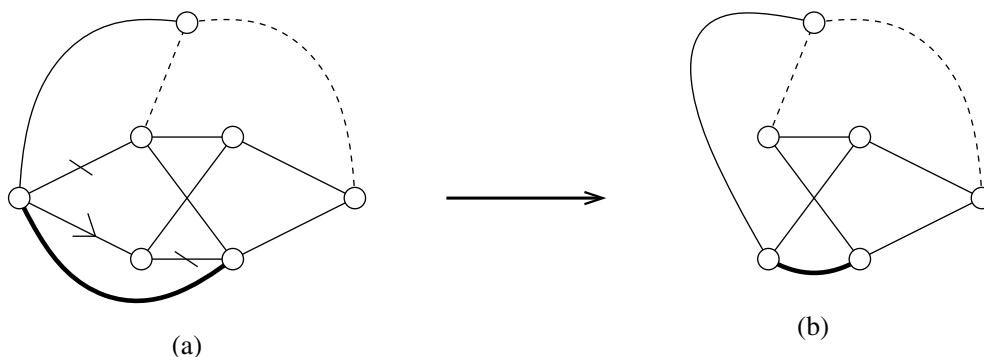


FIG. 15. Bold edges are odd; thin edges are even; and both in (a) and in (b) exactly one of the dashed edges is odd. To obtain (b) from (a), delete the “crossed” edges and contract the “directed” edge.

$R \in \mathcal{L}_{3,1}^F \cup \mathcal{L}_{4,1}^F$. By Lemma 11, $R \in \mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F$. Hence, by Lemma 9, $\mathcal{O}_{4,2}^F = \emptyset$, so $\Lambda_2^F = \emptyset$.

Let P be an odd $r_2^F p$ -link of B with $p \neq r_1^F$. As $\Lambda_2^F = \emptyset$, path P intersects P_{12} . So P contains a link in \mathcal{L}_5^F ; as this collection is equal to $\mathcal{L}_{5,1}^F$ we get that $p \in T_1^F$. Let Q be the shortest path on P_{12} from r_1^F to P . As $\mathcal{L}_{5,2}^F = \emptyset$ and as P intersects P_{12} , the subgraphs R and $P \cup Q$ share no other vertex than r_1^F . Hence, (G, Σ) has a minor as in Figure 15(a), which has a $K_{3,3}^2$ -minor. This contradiction proves (56).

$$(57) \quad \text{There exists a vertex } p \notin \{r_1^F, r_2^F\} \text{ such that each path in } G - r_1^F - r_2^F \text{ from } B \text{ to } C \text{ contains } p.$$

If not, then in $G - r_1^F - r_2^F$ there exist two vertex disjoint paths from C to B . So B has an odd link J contained in $G - r_1^F - r_2^F$. As F has no handle, it follows from Lemma 7 that J is not a link of F , so J intersects P_{12} . But this implies that P_{12} is not an edge and that its union with J contains an odd $r_2^F p$ -link with $p \neq r_1^F$. As this contradicts (56), (57) follows.

Let \mathcal{B} be the union of the bridges of $\{r_1^F, r_2^F, p\}$ that contain edges of B . Assume p is chosen such that \mathcal{B} is as small as possible. Note that \mathcal{B} is 2-connected and that $\mathcal{B} - r_1^F - r_2^F$ is connected. Let P_1, P_2 , and P_3 be three vertex disjoint paths from C to $\{r_1^F, r_2^F, p\}$. Take a path P' from p to $\mathcal{B} - r_1^F - r_2^F$ with no internal vertices in B ;

let u be its end vertex in B .

$$(58) \quad P_{12} \text{ is a single edge.}$$

This follows from (56), as $C \cup P_1 \cup P_2 \cup P_3 \cup P'$ contains an odd $r_2^F p$ -link with $p \neq r_1^F$.

So each link of B , except P_{12} , is a link of F .

$C \cup P_1 \cup P_2 \cup P_3 \cup P'$ contains an odd $r_1^F u$ -link of F and an odd $r_2^F u$ -link of F . So as $\Lambda_2^F = \mathcal{O}_{4,2}^F$, we have that $u \in T_1^F$ and thus that $\mathcal{O}_{2,1}^F$ and $\mathcal{O}_{4,2}^F$ are not empty. Hence, we have by Lemma 9 and (58) that $\Lambda_1^F = \mathcal{O}_{2,1}^F$ and $\Lambda_2^F = \mathcal{O}_{4,2}^F$.

$$(59) \quad \mathcal{B} \text{ contains a link } P \text{ in } \mathcal{O}_{2,1}^F \cup \mathcal{O}_{4,2}^F.$$

As $\{r_1^F, r_2^F, p\}$ is not an improper 3-vertex cutset, \mathcal{B} contains an odd circuit. From this and as \mathcal{B} is 2-connected, it follows that \mathcal{B} contains an odd $r_1^F r_2^F$ -path, say, Q . As $\mathcal{B} - r_1^F - r_2^F$ is connected, it contains a path R that connects $Q - r_1^F - r_2^F$ with $B - r_1^F - r_2^F$. The union of R and Q contains an odd link P of F that has at most one end in $\{r_1^F, r_2^F\}$. By (58), $P \in \Lambda_1^F \cup \Lambda_2^F = \mathcal{O}_{2,1}^F \cup \mathcal{O}_{4,2}^F$. So (59) follows.

Let q be the end of P not in $\{r_1^F, r_2^F\}$. By 46-symmetry, we may assume that $q \in P_{14}^F - r_1^F$. Take the subpath Q of P' from p to $q \in P \cup T_1^F$. Then as Q can be extended to an $r_1^F p$ -link as well as an $r_2^F p$ -link of $F \cup P$ of either parity, it is straightforward to argue from Lemma 13 that $q \in (Q \cup P_{14}^F) - r_1^F$ and from Lemma 10 that $q \in P_{16}^F - r_1^F$. This is absurd. \square

11. Proof of Theorem 3(ii). We finally prove Theorem 3(ii). Assume that (G, Σ) is a 3-connected clean signed graph with no blockvertex and no improper 3-vertex cutset. Let F be an extreme $K_{3,3}^1$ -extension in (G, Σ) . By Lemma 14 and by 12-symmetry, we may assume that F has a handle in $\mathcal{O}_{2,1}^F$.

Let \mathcal{F} be the set of all $K_{3,3}^1$ -extensions F' with $P_{12}^{F'} = P_{12}^F, T_2^{F'} = T_2^F, \text{CAGE}(F') = \text{CAGE}(F)$, and $\{r_4^{F'}, r_6^{F'}\} = \{r_4^F, r_6^F\}$; obviously each $F' \in \mathcal{F}$ is extreme.

$$(60) \quad \text{Each } F' \in \mathcal{F} \text{ has a handle in } \mathcal{O}_{2,1}^{F'}.$$

If not, then by Lemma 14 some $F' \in \mathcal{F}$ has a handle P in $\mathcal{O}_{2,2}^{F'}$. As $\mathcal{O}_{2,1}^{F'} \neq \emptyset$, it follows from Lemma 9 that $\mathcal{O}_{2,2}^{F'} = \emptyset$. Hence, $P \notin \mathcal{O}_{2,2}^{F'}$. Therefore this handle intersects $T_1^{F'} - r_1^{F'}$, and thus it contains a link of F that contradicts Lemma 7. So (60) follows.

Hence, Lemma 9 implies

$$(61) \quad \Lambda_2^{F'} = \mathcal{O}_{4,2}^{F'} \text{ for each } F' \in \mathcal{F}.$$

The *tip* of a link in $\mathcal{O}_{2,1}^F$, so in particular of a handle, is the end that lies farthest from r_1^F on T_1^F .

$$(62) \quad \text{Let } P \text{ be a handle of } F \text{ with tip } s \text{ on } P_{14}^F, \text{ and let } L \in \mathcal{L}_{2,1}^F \text{ with ends } x \text{ in } (P_{14}^F)_{r_1^F s} - r_1^F - s \text{ and } y \text{ in } (P_{14}^F)_{sr_4^F} - s. \text{ Then there exists a } K_{3,3}^1\text{-extension } F' \text{ in } \mathcal{F} \text{ with } P_{16}^{F'} = P_{16}^F \text{ and } (P_{14}^{F'})_{yr_4^F} = (P_{14}^F)_{yr_4^F} \text{ that has a handle with tip } y.$$

In proving this we clearly may assume that L consists of a path that is internally disjoint with P and possibly a part of P . If L is odd, then it is a handle of F with tip y . Hence, we may assume that L is even. We may also assume that the only odd

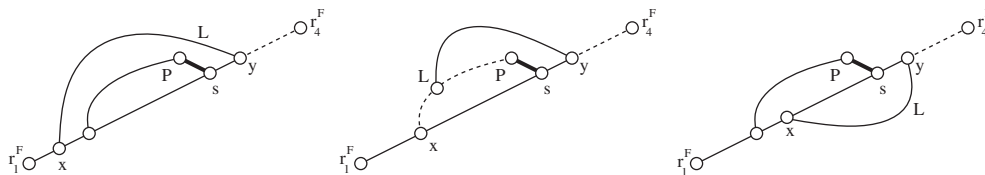


FIG. 16. Bold edges are odd; thin edges are even; and dashed edges may have length zero.

edge on $P \cup L$ is the edge of P incident with s . Figure 16 depicts the three possible arrangements of P and L along P_{14}^F . Let F' be the $K_{3,3}^1$ -extension obtained from F by replacing $(P_{14}^F)_{xy}$ with L . One easily checks in Figure 16 that F' satisfies all claims in (62).

A *single border* of F is any pair (r_1^F, s) where s is the tip of a handle. A pair (r, s) is a *linked border* of F if s is the tip of a handle and there exists an rr' -link in $\mathcal{L}_{6,1}^F$ with $r' \in (T_1^F)_{r_1^F s} - s$; any such rr' -link is a *join for the linked border* (r, s) . A pair (r, s) is a *double border* of F if r and s are both tips of a handle, one lying in P_{14}^F and the other in P_{16}^F , and there exists a link in $\mathcal{L}_{6,1}^F$ with both ends in $(T_1^F)_{rs} - r - s$; any such link is a *join for the double border* (r, s) . A *border* of F is a single, linked, or double border of F . Note that if (r, s) is a border, then one among r and s lies on P_{14}^F and the other on P_{16}^F . Moreover, $s \neq r_1^F$ and $r = r_1^F$ exactly when (r, s) is a single border. Note that by Lemma 7, joins for borders are even.

If (r, s) is a border, let $B[r, s] = F - (T_1^F)_{rs}$, and let $\mathcal{L}[r, s]$ be the collection of links of F with one end in $B[r, s] - r_1^F - r - s$ and the other end in $(T_1^F)_{rs} - r_1^F - r - s$.

$$(63) \quad \text{If } (r_1^F, s) \text{ is a single border of } F \text{ with } \mathcal{L}[r_1^F, s] \not\subseteq \mathcal{L}_{2,1}^F \cup \mathcal{L}_{6,1}^F, \\ \text{then } \mathcal{L}[r_1^F, s] \cap \mathcal{O}_{4,2}^F \neq \emptyset \text{ and } \Lambda_1^F = \mathcal{O}_{2,1}^F.$$

To prove this, let Q be a handle with end s , and let $P \in \mathcal{L}[r_1^F, s] \setminus (\mathcal{L}_{2,1}^F \cup \mathcal{L}_{6,1}^F)$. Then, by Lemma 7, P has an end on $P_{12}^F - r_1^F$. Let P' be the shortest subpath of P from P_{12}^F to $Q \cup T_1^F$. Clearly, by changing P if necessary, we may assume that P consists of P' and possibly a subpath of Q . If P was even, (G, Σ) would have the signed graph in Figure 17(a) as a minor. As illustrated in Figure 17 that signed graph has a $K_{3,3}^{1,1}$ -minor. So P is odd. As by Lemma 7, $\mathcal{O}_{5,1}^F = \emptyset$, this means that $P \in \mathcal{O}_{4,2}^F$. So $\mathcal{L}[r_1^F, s] \cap \mathcal{O}_{4,2}^F \neq \emptyset$ indeed. Moreover, as $\mathcal{O}_{4,2}^F \neq \emptyset$, it follows from Lemma 9 that $\Lambda_1^F = \mathcal{O}_{2,1}^F \cup \mathcal{L}_{5,1}^F$. In other words, $\mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F = \emptyset$. So, as $\mathcal{O}_{2,1}^F \neq \emptyset$ it follows from Lemma 12 that $\mathcal{E}_{3,1}^F \cup \mathcal{E}_{4,1}^F \neq \emptyset$. Hence, by Lemma 11, $\mathcal{L}_{5,1}^F$ is empty. Thus $\Lambda_1^F = \mathcal{O}_{2,1}^F$ indeed, and (63) follows.

The *value for F* of a border (r, s) is defined as the number of edges in $B[r, s]$. Choose $F \in \mathcal{F}$ and a border (r, s) for F such that

$$(64) \quad \text{the value for } F \text{ of } (r, s) \text{ is as small as possible.}$$

By 46-symmetry assume that s lies on P_{14}^F and that r lies on P_{16}^F . Then we have the following:

$$(65) \quad \mathcal{L}[r, s] \cap \mathcal{L}_{2,1}^F = \emptyset.$$

Suppose this is not true; let $L \in \mathcal{L}[r, s] \cap \mathcal{L}_{2,1}^F$. Let x be the end of L in $(T_1^F)_{rs}$, and let y be the other end of L . If x and y lie on P_{14}^F , then by (62) there exists a

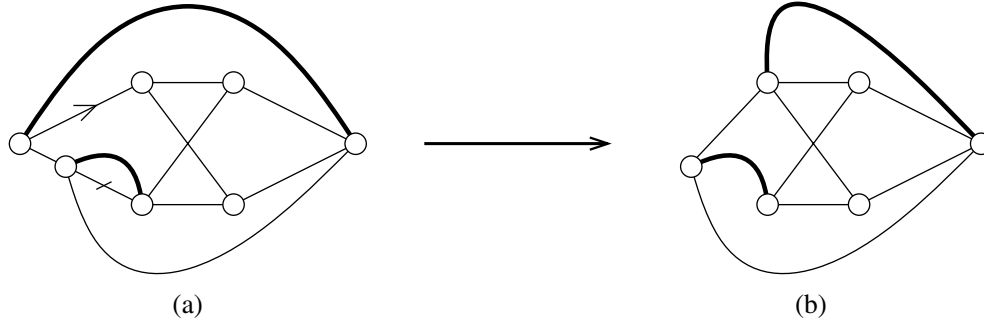


FIG. 17. Bold edges are odd; thin edges are even. To obtain (b) from (a), delete the “crossed” edge and contract the “directed” edge.

$K_{3,3}^1$ -extension F' such that (r, y) is a border of F' . The value for F' of (r, y) is clearly smaller than the value for F of (r, s) . By (64) this is impossible, so x and y lie on P_{16}^F . In fact, by 46-symmetry and symmetry between r and s , this also means that (r, s) is not a double border. Hence, as $s \in P_{14}^F$, (r, s) is a linked border. Let P be a join for (r, s) .

If L intersected P , it would do so internally and (y, s) would be a linked border for F (with a join in $L \cup P$). As the value of (y, s) is smaller than that of (r, s) , it follows from (64) that this is impossible, so L and P are disjoint.

If L was odd, it would be a handle and (y, s) would be a double border, again contradicting (64). So L is even. Let F' be the $K_{3,3}^1$ -extension obtained from F by replacing $(P_{16}^F)_{xy}$ with L . Clearly, $F' \in \mathcal{F}$. Now (y, s) is a linked border of F' . The value for F' of (y, s) is clearly smaller than the value for F of (r, s) . By (64) this is impossible, so (65) follows.

$$(66) \quad \mathcal{L}[r, s] \cap \mathcal{L}_{6,1}^F = \emptyset.$$

Suppose this is not true; let $L \in \mathcal{L}[r, s] \cap \mathcal{L}_{6,1}^F$. Let y be the end of L in $B[r, s]$. If y lies on P_{16}^F , then (y, s) would be a linked border of F that has a smaller value than (r, s) , contradicting (64) (L would be a join for that border). So, $y \in P_{14}^F$. By 46-symmetry and symmetry between r and s , this also implies that (r, s) is not a double border. Now, as $L \in \mathcal{L}_{6,1}^F$, (r, s) is a linked border; let R be a join for (r, s) , and let Q be a handle with tip s . By (65), L and R are internally vertex disjoint, and by construction they do not share any end. By Lemma 7, L and R are both even. Moreover, both these paths are internally disjoint with Q ; otherwise, we would have a link in $\mathcal{O}_{6,1}^F$. Now, let $K \in \mathcal{K}(F)$, and let K' be the $K_{3,3}^1$ -subdivision obtained from K by replacing P_{45}^K and P_{63}^K with L and R . Then K' is extreme in $K' \cup Q$. As Q is a link of K' that violates Lemma 7 with respect to K' , (66) follows.

$$(67) \quad (r, s) \text{ is a linked or double border of } F.$$

Suppose this is not true; then (r, s) is a single border and $r = r_1^F$. As G is 3-connected, $\{r_1^F, s\}$ is not a 2-vertex cutset, so $\mathcal{L}[r_1^F, s] \neq \emptyset$. By (65), (66), and (63), there exists an $L \in \mathcal{L}[r_1^F, s] \cap \mathcal{O}_{4,2}^F$, and $\Lambda_1^F = \mathcal{O}_{2,1}^F$. In particular, $\mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F = \emptyset$, so by Lemma 12, $\mathcal{E}_{3,1}^F \cup \mathcal{E}_{4,1}^F \neq \emptyset$. As also $\mathcal{L}_{5,1}^F = \emptyset$, it follows from (61) that $\mathcal{L}_5^F = \emptyset$. So P_{12}^F is a single edge. From this, (65), and (66), it follows that the bridge, say, \mathcal{B} , of $\{r_1^F, s, r_2^F\}$ containing $\text{CAGE}(F)$ is distinct from the bridge, say, \mathcal{A} , of $\{r_1^F, s, r_2^F\}$ containing

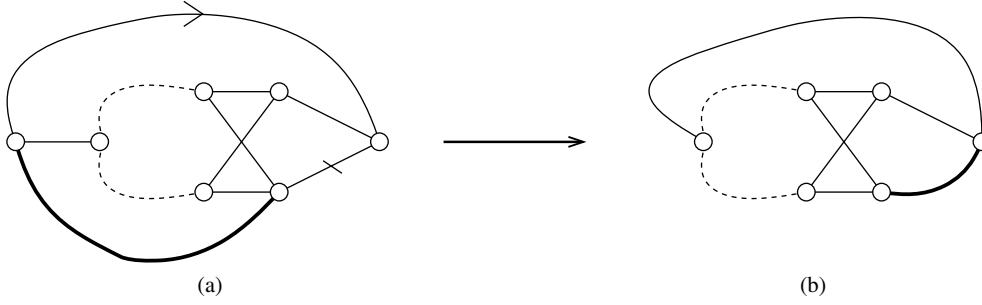


FIG. 18. Bold edges are odd; thin edges are even; and both in (a) and in (b) exactly one of the dashed edges is odd. To obtain (b) from (a), delete the “crossed” edge and contract the “directed” edge.

$(T_{12}^F)_{r_1^F s}$. Hence, as (G, Σ) has no improper 3-vertex cutset, \mathcal{B} is not bipartite. By Lemma 8, \mathcal{B} contains an odd path P from s to one of r_1^F and r_2^F that misses the other vertex in $\{r_1^F, r_2^F\}$. As P_{12}^F is a single edge, P_{12}^F is not contained in \mathcal{B} . Therefore P contains a link $Q \in \mathcal{O}_{2,1}^F \cup \mathcal{O}_{4,2}^F$.

Let R be a handle with tip s . Then R lies in \mathcal{A} . As $Q \in \mathcal{B}$, links R and Q are internally disjoint. This means that if $Q \in \mathcal{O}_{4,2}^F$, then, by Lemma 13, Q has an end in P_{14}^F . However, then links Q and R contradict Lemma 10. So $Q \in \mathcal{O}_{2,1}^F$.

As L lies in \mathcal{A} and Q lies in \mathcal{B} , these links are internally vertex disjoint. Since L has an end in P_{14}^F , it follows from Lemma 13 that Q has its ends in P_{14}^F . As Q lies in \mathcal{B} its tip, say, y , lies in $(P_{14}^F)_{sr_4^F} - s$. Hence, by (64), Q is not a handle. So the other end of Q is r_1^F . But then Q and L violate Lemma 10. This proves (67).

- (68) $B[r, s]$ has an odd rs -link T with the following three properties:
 T intersects $(P_{14}^F)_{r_1^F s}$ internally; r_1^F does not lie on T ; and if (r, s) is a linked border, then T intersects P_{16}^F only in r .

Indeed such a path is contained in the union of a handle with tip s , a join for (r, s) and T_1^F .

- (69) No odd $r_2^F w$ -link of $B[r, s] \cup T$ with $w \in (T_2^F \cup \text{CORE}(F)) - r_2^F$ contains r_1^F .

Assume this is false; let P be an odd $r_2^F w$ -link of $B[r, s] \cup T$ with $w \in (T_2^F \cup \text{CORE}(F)) - r_2^F$ that contains r_1^F . Let Y be the subpath of P_{14}^F from P to T . Note that by (68) Y has neither r_4^F nor r_6^F as one of its ends. By resigning on the vertices of Y , if necessary, we see that (G, Σ) has the signed graph in Figure 18(a) as a minor. As illustrated in Figure 18, that signed graph has $K_{3,3}^2$ as a minor. This contradiction proves (69).

$$(70) \quad \mathcal{E}_{3,1}^F \cup \mathcal{E}_{4,1}^F = \emptyset.$$

Suppose this is false; let $P \in \mathcal{E}_{3,1}^F \cup \mathcal{E}_{4,1}^F$. Paths P and T are disjoint as otherwise F has a link that violates Lemma 7. This means that $P_{12}^F \cup P$ contradicts (69), so (70) follows.

Hence, as $\mathcal{O}_{2,1}^F \neq \emptyset$, it follows from Lemma 12 that $\mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F \neq \emptyset$. Hence, by Lemma 9, $\Lambda_2^F = \emptyset$.

$$(71) \quad \mathcal{L}[r, s] = \emptyset.$$

Suppose this is false; let $L \in \mathcal{L}[r, s]$. By (65), (66), and Lemma 7, L has an end, say, y , on $P_{12}^F - r_1^F$. Let x be the other end of L . By the properties of T listed in (68) we may assume that if L meets T , then $x \in P_{14}^F$ (if not, we can replace L with another path in $T \cup L$ that does end in P_{14}^F). In any case, $L \in \mathcal{L}[r_1^F, t]$ for $t = s$ or $t = r$. As $\mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F \neq \emptyset$, it follows from (63) that t is not the tip of a handle. So $L \in \mathcal{L}[r_1^F, r]$ and (r, s) is a linked border and, as $x \notin P_{14}^F$, the paths T and L are vertex disjoint. Moreover, as $\mathcal{O}_{4,2}^F$ and $\mathcal{O}_{5,1}^F$ are both empty, L is even. Hence, the concatenation of $(P_{12}^F)_{r_2^F y}$, L , $(P_{16}^F)_{x r_1^F}$, and any link in $\mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F$ violates (69). So (71) follows.

As $\{r, r_1^F, s\}$ is not an improper 3-vertex cutset, there exists a link Q of F that closes with $B[r, s]$ an odd circuit. As $\mathcal{E}_{3,1}^F \cup \mathcal{E}_{4,1}^F = \Lambda_2^F = \mathcal{O}_{5,1}^F = \emptyset$, link $Q \in \mathcal{L}_{2,1}^F \cup \mathcal{L}_1^F$. By (64), Q cannot be in $\mathcal{O}_{2,1}^F$. If $Q \in \mathcal{L}_1^F$, then as Q closes with B an odd circuit, $L \cup P_{12}^F$ contains an even $r_2^F r_1^F$ -path, which together with any link in $\mathcal{O}_{3,1}^F \cup \mathcal{O}_{4,1}^F$ forms a link violating (69). So $Q \in \mathcal{E}_{2,1}^F$. As Q closes with $B[r, s]$ an odd circuit, r_1^F is an end of Q . Let q be the other end of Q . Let u be the vertex among r and s that is farthest from q along T_1^F . Let F^* be the $K_{3,3}^1$ -extension in \mathcal{F} obtained from F by replacing $(T_1^F)_{r_1^F q}$ with Q . Vertex u is not the tip of a handle of F , as otherwise (q, u) is a linked border of F^* that has a smaller value than (r, s) has. So u is r , border (r, s) is linked, and q lies on P_{14}^F . By (71), Q and T are disjoint. Hence, by the last property of T listed in (68), $T \cup (P_{14}^F)_{sq} \in \mathcal{O}_{6,1}^{F^*}$. This contradicts Lemma 7, which completes the proof of Theorem 3(ii). \square

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