Packing of Graphic n-tuples

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Abstract

An n-tuple π (not necessarily monotone) is *graphic* if there is a simple graph G with vertex set $\{v_1, \ldots, v_n\}$ in which the degree of v_i is the ith entry of π. Graphic n-tuples $(d_1^{(1)}, \ldots, d_n^{(1)})$ and $(d_1^{(2)}, \ldots, d_n^{(2)})$ *pack* if there are edge-disjoint n-vertex graphs G_1 and G_2 such that $d_{G_1}(v_i) = d_i^{(1)}$ and $d_{G_2}(v_i) = d_i^{(2)}$ for all i. We prove that graphic n-tuples π_1 and π_2 pack if $\Delta \leq \sqrt{2\delta n} - (\delta - 1)$, where Δ and δ denote the largest and smallest entries in $\pi_1 + \pi_2$ (strict inequality when $\delta = 1$); also, the bound is sharp.

Kundu and Lovász independently proved that a graphic n-tuple π is realized by a graph with a k-factor if the n-tuple obtained by subtracting k from each entry of π is graphic; for even n we conjecture that in fact some realization has k edge-disjoint 1-factors. We prove the conjecture in the case where the largest entry of π is at most $n/2 + 1$ and also when $k \leq 3$.

Keywords: Degree sequence, graphic sequence, graph packing, k-factor, 1-factor

1 Introduction

An integer n-tuple π is *graphic* if there is a simple graph G with vertex set $\{v_1, \ldots, v_n\}$ such that $d_G(v_i) = d_i$, where $\pi = (d_1, \ldots, d_n)$ and $d_G(v)$ denotes the degree of vertex v in graph G. Such a graph G *realizes* π. Two n-vertex graphs G_1 and G_2 *pack* if they can be expressed as

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edge-disjoint subgraphs of the complete graph K_n. We study an analogue of graph packing for graphic n-tuples. Let π_1 and π_2 be graphic n-tuples, with $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ (they need not be monotone). We say that π_1 and π_2 *pack* if there exist edge-disjoint graphs G_1 and G_2 with vertex set $\{v_1, \ldots, v_n\}$ such $d_{G_1}(v_i) = d_i^{(1)}$ and $d_{G_2}(v_i) = d_i^{(2)}$ for all i. In graph packing, vertices may be reordered, but in packing of graphic n-tuples no reordering of the indices is allowed. Graphic n-tuples are often called *graphic sequences*; we use "n-tuple" partly to emphasize that the order of entries matters. When not specifying the length, we use "list".

The condition that $\pi_1 + \pi_2$ is graphic is obviously necessary for π_1 and π_2 to pack, but the following small example shows that it is not sufficient.

Example 1.1. Let $\pi_1 = (3, 1, 2, 2, 0, 0)$ and $\pi_2 = (1, 3, 0, 0, 2, 2)$, with sum $(4, 4, 2, 2, 2, 2)$. Both π_1 and π_2 are graphic, and the complete bipartite graph $K_{2,4}$ realizes their sum. However, in every realization of π_j, the vertex v_j of degree 3 has three nonisolated neighbors. Thus v_1 and v_2 are adjacent in every realization of π_1 or π_2, and the lists do not pack. $\square$

In fact, Dürr, Guiñez, and Matamala [4] showed that determining whether two graphic n-tuples pack is NP-complete. Hence we focus on finding sharp sufficient conditions. In 1978, Sauer and Spencer [14] published the classical result that n-vertex graphs G_1 and G_2 pack if $\Delta(G_1)\Delta(G_2) < n/2$, where $\Delta(G)$ denotes the maximum vertex degree in G. In Section 2, we prove an analogue for n-tuples, showing that graphic n-tuples π_1 and π_2 pack if $\Delta \le \sqrt{2\delta n} - (\delta - 1)$, where Δ and δ denote the largest and smallest values in $\pi_1 + \pi_2$, except that strict inequality is needed when $\delta = 1$. Furthermore, the bound is sharp; we construct lists that do not pack when the maximum entry in the sum is larger by 1. We conjecture the stronger statement that two graphic n-tuples pack if the product of corresponding terms is always less than $n/2$; this would be a more direct analogue of the Sauer–Spencer Theorem.

Kundu's Theorem [9], published in 1973 and proved independently by Lovász [10] at about the same time, characterizes when a graphic n-tuple has a realization containing a spanning subgraph that is "almost" k-regular. In the language of packing, the result states that if π_1 is graphic and each term in π_2 is k or $k - 1$, then π_1 and π_2 pack if $\pi_1 + \pi_2$ is graphic.

In Section 3, we consider extensions of the k-factor case of Kundu's Theorem, where a k-*factor* of a graph is a spanning k-regular subgraph. Kundu's Theorem implies that a graphic n-tuple π is realizable by a graph having a k-factor if the list obtained by subtracting k from each entry is graphic. We conjecture the stronger statement that in fact when n is even there is a realization containing k edge-disjoint 1-factors (that is, a k-edge-colorable k-factor). We prove the conjecture when the largest entry is at most $n/2 + 1$. We also prove the more difficult result that the conjecture holds when $k \le 3$, by proving in general that there is a realization containing a k-factor that has two edge-disjoint 1-factors.

2 An Analogue of the Sauer-Spencer Theorem

The Sauer–Spencer Theorem immediately implies that n-vertex graphs G_1 and G_2 pack when their maximum degrees sum to less than $\sqrt{2n}$. Chen [2] gave a short proof of Kundu's Theorem; we use a similar technique to prove our result for packing of graphic n-tuples. When the least entry in the sum is 1, the maximum allowed by the hypothesis is the same as in the Sauer–Spencer Theorem. Note that when we prove directly that π_1 and π_2 pack, it follows immediately that $\pi_1 + \pi_2$ is graphic.

Again let Δ and δ denote the largest and smallest entries in $\pi_1 + \pi_2$. Before proving that our condition is sufficient for π_1 and π_2 to pack, we present a simple construction that proves sharpness when $\delta = 1$ (see also Remark 2 of [10]). We later obtain sharpness for $\delta \ge 2$ via a slight modification of this construction.

Example 2.1. For $\delta, m \in \mathbb{N}$ with $m > 1$, let $n = 2\delta m^2$. We construct graphic n-tuples π_1 and π_2 with $\Delta = \sqrt{2\delta n}$ that do not pack. Let

$$\pi_1 = (\delta m, \delta m, (2\delta m)^{\delta(m-1)}, 0^{\delta(m-1)}, (\delta m)^{\delta-1}, 0^{\delta-1}, \delta^{\delta(m^2-m)}, 0^{\delta(m^2-m)})$$

and

$$\pi_2 = (\delta m, \delta m, 0^{\delta(m-1)}, (2\delta m)^{\delta(m-1)}, 0^{\delta-1}, (\delta m)^{\delta-1}, 0^{\delta(m^2-m)}, \delta^{\delta(m^2-m)}),$$

where the exponents denote multiplicity (lengths of constant sublists). The lists have length $2\delta m^2$, as desired. Also, the largest and smallest entries in $\pi_1 + \pi_2$ are $2\delta m$ and δ, respectively, so $\Delta = \sqrt{2\delta n}$. (The Erdős–Gallai conditions [6] readily imply that $\pi_1 + \pi_2$ is graphic, but this is not important). It remains to show that π_1 and π_2 are graphic but do not pack.

To show that π_i is graphic, start with $K_{\delta m+1}$, split its vertices into sets $V_1, \ldots, V_{m-1}$ of size δ plus $\delta + 1$ leftover vertices, for each i make the vertices of V_i adjacent to a set X_i of δm new vertices, and add to these $\delta m^2 + 1$ vertices a set of $\delta m^2 - 1$ isolated vertices.

Given any realization of π_1, let S be the set of $\delta m + 1$ vertices with degree exceeding δ. Their degree-sum is $2\delta^2 m^2 - \delta m(\delta - 1)$, which equals $2\binom{\delta m+1}{2} + \delta^2(m^2 - m)$. To reach this total, S must induce a complete graph, and all other edges must join S to vertices of degree δ. Thus v_1 and v_2 are adjacent in every realization of π_1. The same argument applies to π_2; again v_1 and v_2 are adjacent in every realization. Since v_1 and v_2 are adjacent in all realizations of both lists, π_1 and π_2 do not pack. □

Given a graph G and a set $S \subseteq V(G)$, let $G[S]$ denote the induced subgraph of G with vertex set S, and let $N_G(S)$ be the set of vertices having a neighbor in S. A *clique* is a pairwise adjacent set of vertices.

Theorem 2.2. *Let π_1 and π_2 be graphic n-tuples. If*

$$\Delta \le \sqrt{2\delta n} - (\delta - 1),$$

where Δ and δ denote the maximum and minimum values in $\pi_1 + \pi_2$, then π_1 and π_2 pack, except that strict inequality is required when $\delta = 1$.

Proof. Let π_1 and π_2 be graphic n-tuples. If $\delta = 0$, then $\Delta \le \sqrt{2\delta n} - (\delta - 1)$ implies that realizations are edgeless or consist of matchings on disjoint vertex sets, so π_1 and π_2 pack. Therefore, we may assume $\delta \ge 1$. We prove that if π_1 and π_2 fail to pack, then $\Delta \ge \sqrt{2\delta n} - (\delta - 1)$, with strict inequality when $\delta > 1$.

Among realizations of π_1 and π_2 on vertices $v_1, \ldots, v_n$ that have the required degrees at each vertex, choose G_1 and G_2 to minimize the number of edges that appear in both graphs. Since π_1 and π_2 do not pack, we may consider an edge xy in $E(G_1) \cap E(G_2)$.

Let $G = G_1 \cup G_2$, and let $I = V(G) - (N_G(x) \cup N_G(y))$. With $\delta \ge 1$, we have $\Delta < \sqrt{n}$, so $I \ne \varnothing$. Let $Q = N_G(I)$. Suppose that G_1 or G_2 has an edge uv such that $u \in I$ and $\{x, y\} \not\subseteq N_G(v)$; by symmetry, we may assume $yv \notin E(G)$. Replacing $\{xy, uv\}$ with $\{xu, yv\}$ in that graph reduces the number of shared edges without changing any vertex degrees, contradicting the choice of G_1 and G_2 (see Figure 1a). Thus $Q \subseteq N_G(x) \cap N_G(y)$.

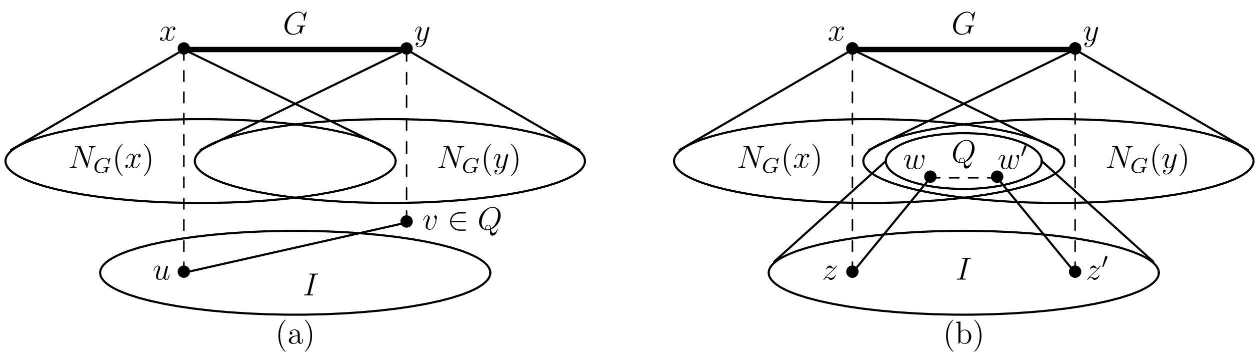

Figure 1: Properties of Q

For $j \in \{1, 2\}$, let $Q_j = N_{G_j}(I)$; we claim that Q_j is a clique in G. Otherwise, choose $w, w' \in Q_j$ with $ww' \notin E(G)$. Let z and z' be (not necessarily distinct) vertices in I such that $zw, z'w' \in E(G_j)$. Since $ww' \notin E(G_j)$, replacing $\{z'w', wz, xy\}$ with $\{w'w, zx, yz'\}$ in $E(G_j)$ reduces the number of shared edges without changing vertex degrees (see Figure 1b).

Since $Q = Q_1 \cup Q_2$, and Q_1 and Q_2 are cliques in G, the complement of $G[Q]$ is bipartite. Letting r be the number of edges in $G[Q]$, we obtain

$$r \geq \binom{|Q|}{2} - \frac{|Q|^2}{4} = \frac{|Q|^2}{4} - \frac{|Q|}{2}. \tag{1}$$

Next, note that $|I| = n - |N_G(x) \cup N_G(y)| = n - |N_G(x)| - |N_G(y)| + |N_G(x) \cap N_G(y)|$. Since xy is a shared edge, $|N_G(x)|$ and $|N_G(y)|$ are at most $\Delta - 1$. With $Q \subseteq N_G(x) \cap N_G(y))$,

$$|I| \geq n - 2\Delta + 2 + |Q|. \tag{2}$$

Each vertex $v \in I$ has at least δ incident edges in G_1 and G_2 together, and each neighbor is in Q. Since $Q \subseteq N_G(x) \cap N_G(y)$, at most $(\Delta - 2)|Q| - 2r$ edges of G_1 and G_2 together have endpoints in I and Q. Therefore,

$$|I| \leq \frac{(\Delta - 2)|Q| - 2r}{\delta}. \tag{3}$$

Together, (2) and (3) yield

$$(\Delta - 2)|Q| - 2r \geq \delta(n - 2\Delta + 2 + |Q|). \tag{4}$$

Using (1) to substitute for r, letting $q = |Q|$, and simplifying brings us to

$$q(\Delta - 1 - \delta - q/2) \geq \delta(n - 2\Delta + 2). \tag{5}$$

The left side is maximized when $q = \Delta - 1 - \delta$. Since the inequality must hold there, $(\Delta - 1 - \delta)^2 \geq 2\delta(n - 2\Delta + 2)$. Adding $4\delta(\Delta - 1)$ to both sides yields $(\Delta - 1 + \delta)^2 \geq 2\delta n$, or

$$\Delta \geq \sqrt{2\delta n} - (\delta - 1). \tag{6}$$

To complete the sufficiency proof, we show that equality cannot hold in (6) when $\delta \geq 2$. Equality in (6) requires equality in the inequalities that produced it. Equality holds in (5) only when $q = \Delta - 1 - \delta$. Equality in (4) (equivalent to (5)) requires equality in (3) and (2). Thus $\delta|I|$ equals both sides of (4), and also $Q = N_G(x) \cap N_G(y)$ and $|N_G(x)| = |N_G(y)| = \Delta - 1$. By this last equality, G_1 and G_2 share no edges incident to x or y except xy.

Equality in (3) requires $N_G(w) = Q$ whenever $w \in I$. Since exactly $(\Delta - 2)|Q| - 2r$ edges have endpoints in Q and I, and by definition $G[Q]$ has r edges, the edges joining Q to $I \cup \{x, y\}$ and within Q exhaust the total degree sum available to vertices of Q. We conclude that in G each vertex of Q has degree Δ and has no neighbor in $N_G(x) \cup N_G(y)$ outside Q.

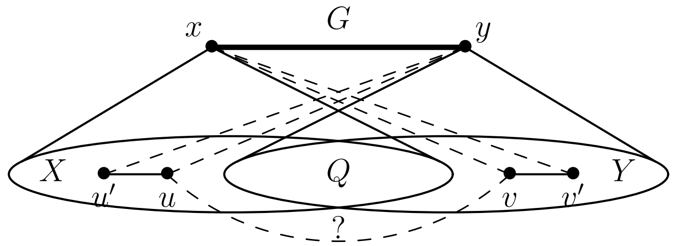

Figure 2: $Q = N_G(I) \subseteq N_G(x) \cap N_G(y)$

Let $X = N_G(x) - N_G(y) - \{y\}$, and let $Y = N_G(y) - N_G(x) - \{x\}$ (see Figure 2). Since $|N_G(x)| = |N_G(y)| = \Delta - 1$ and $|N_G(x) \cap N_G(y)| = q = \Delta - 1 - \delta$, we have $|X| = |Y| = \delta - 1$. If G_j has edges within both X and Y, say $uu' \in G_j[X]$ and $vv' \in G_j[Y]$, then consider whether $uv \in E(G_j)$. If so, then replacing $\{xy, uv\}$ with $\{yu, vx\}$ in G_j reduces the number of shared edges; if not, then replacing $\{vv', xy, u'u\}$ with $\{v'x, yu', uv\}$ does so. Hence by symmetry we may assume that edges of $G[X]$ lie only in G_1 and edges of $G[Y]$ lie only in G_2. Now vertices of X are isolated in G_2 and have at most $\delta - 1$ neighbors in G_1 (including x). If X is nonempty, then this contradicts the definition of δ. Hence equality in (6) requires $X = \varnothing$ and $\delta = 1$. □

Theorem 2.3. *The result of Theorem 2.2 is sharp: for $\delta, m \in \mathbb{N}$ with $m \geq \delta \geq 2$, there exist π_1 and π_2 with $n = 2\delta m^2$ such that $\Delta = \sqrt{2\delta n} - (\delta - 2)$ but π_1 and π_2 do not pack.*

Proof. We consider only $\delta \geq 2$ since the construction in Example 2.1 proves sharpness for $\delta = 1$. Choose $m \in \mathbb{N}$ with $m \geq \delta$, and let $n = 2\delta m^2$. Let G be the construction using these parameters in Example 2.1. We modify G to reduce the maximum degree by $\delta - 1$. This will also reduce Δ by $\delta - 1$ in the sum of two specified orderings of the vertex degrees.

Recall that the construction of G begins with a complete graph $K_{\delta m+1}$ whose vertex set is composed of sets $V_1, \ldots, V_{m-1}$ of size δ plus $\delta + 1$ additional vertices. For each i the set V_i is adjacent to a set X_i of δm new vertices, and there are $\delta m^2 - 1$ additional isolated vertices. Each vertex in $\bigcup_i V_i$ has degree $2\delta m$, each extra vertex in the clique has degree δm, and $\delta m(m-1)$ vertices outside the clique have degree δ.

Modify G by removing $\delta - 1$ of the extra vertices from the clique, reducing the degrees of the other vertices by $\delta - 1$. For $1 \leq i \leq \delta - 1$, put one of the removed vertices into X_i. Hence the number of vertices remains $2\delta m^2$, the vertices of $V_1, \ldots, V_{\delta-1}$ have degree $2\delta m - \delta + 2$, those of $V_\delta, \ldots, V_{m-1}$ have degree $2\delta m - \delta + 1$, the two unmoved extra vertices have degree $\delta m - \delta + 1$, and the remaining vertices have degree δ. The new graph G' realizes the n-tuples

π_1' and π_2' given by

$$((\delta m-\delta+1)^2, (2\delta m-\delta+2)^{\delta(\delta-1)}, 0^{\delta(\delta-1)}, (2\delta m-\delta+1)^{\delta(m-\delta)}, 0^{\delta(m-\delta)}, \delta^{\delta(m^2-m)+\delta-1}, 0^{\delta(m^2-m)+\delta-1})$$

and

$$((\delta m-\delta+1)^2, 0^{\delta(\delta-1)}, (2\delta m-\delta+2)^{\delta(\delta-1)}, 0^{\delta(m-\delta)}, (2\delta m-\delta+1)^{\delta(m-\delta)}, 0^{\delta(m^2-m)+\delta-1}, \delta^{\delta(m^2-m)+\delta-1}).$$

By construction, π_1' and π_2' are graphic. To show that they do not pack, we argue as in Example 2.1. In any realization of π_1', let S be the set of $\delta m - \delta + 2$ vertices with degrees exceeding δ. Their degrees sum to

$$2\delta m\delta(m-1) - (\delta-1)\delta(m-1) + \delta(\delta-1) + 2\delta m - 2(\delta-1),$$

which equals $2\binom{\delta m-\delta+2}{2} + \delta^2(m^2-m) + \delta(\delta-1)$. To achieve this total, again S must be a clique. As in Example 2.1, v_1 and v_2 must be adjacent in all realizations of both graphs; hence π_1 and π_2 do not pack. □

If $a+b < \sqrt{2n}$, then also $ab < n/2$. Hence the conjecture below would strengthen Theorem 2.2 when $\delta = 1$ and provide a more direct analogue to the Sauer-Spencer Theorem.

Conjecture 2.4. *Let π_1 and π_2 be graphic n-tuples, with δ the least entry in $\pi_1 + \pi_2$. If $\delta \geq 1$ and the product of corresponding entries in π_1 and π_2 is always less than $n/2$, then π_1 and π_2 pack.*

For fixed δ, a suitable bound on the product of corresponding entries to guarantee packing may be something like $\delta n/2 - O(\delta\sqrt{\delta n})$.

3 Extensions of Kundu's Theorem

Let $D_k(\pi)$ denote the n-tuple obtained from an n-tuple π by subtracting k from each entry. The "regular" case of Kundu's Theorem states that if π and $D_k(\pi)$ are graphic, then some realization of π has a k-factor. To extend the theorem, one could try to guarantee that some realization of π has edge-disjoint regular factors of degrees $k_1, \ldots, k_t$, where $\sum_{i=1}^t k_i = k$.

When n is odd, no regular n-vertex graph has odd degree, so existence requires all $k_1, \ldots, k_t$ even. In that case, existence then follows immediately from Kundu's Theorem and Petersen's 2-Factor Theorem [12]; the latter states that every $2r$-regular graph decomposes into 2-factors. It remains to consider even n.

Conjecture 3.1. *Let n be an even integer. If π is a graphic n-tuple such that $D_k(\pi)$ is also graphic, and $k_1, \ldots, k_t$ are positive integers with sum k, then some realization of π has edge-disjoint regular factors with degrees $k_1, \ldots, k_t$.*

Conjecture 3.1 is immediately equivalent to the following conjecture.

Conjecture 3.2. *Let n be an even integer. If π is a graphic n-tuple such that $D_k(\pi)$ is also graphic, then some realization of π has k edge-disjoint 1-factors.*

Our main result (Theorem 3.9) toward Conjecture 3.2 combines with Petersen's Theorem to yield Conjecture 3.1 when k is even and at most two of $k_1, \ldots, k_t$ are odd, and when k is odd and at most one of $k_1, \ldots, k_t$ is odd.

We have proved several special cases of Conjecture 3.2. The first uses a lemma proved by A.R. Rao and S.B. Rao [13] in their study of what was called the "k-Factor Conjecture" before it became Kundu's Theorem.

Lemma 3.3. *Fix $k \in \mathbb{N}$, and let π be a graphic n-tuple such that $D_k(\pi)$ is also graphic. If r is a positive integer such that $r \leq k$ and rn is even, then $D_r(\pi)$ is also graphic.*

Let $\Delta(G)$ and $\delta(G)$ denote the largest and smallest vertex degrees in a graph G.

Theorem 3.4. *Fix $k, n \in \mathbb{N}$ with n even, and let π be a graphic n-tuple such that $D_k(\pi)$ is also graphic. If every entry in π is at most $n/2 + 1$, then some realization of π has k edge-disjoint 1-factors.*

Proof. The proof is by induction on k. For $k = 0$, the statement is vacuous, and the case $k = 1$ is a special case of Kundu's Theorem. Suppose then that $k \geq 2$ and that $D_k(\pi)$ is graphic. By Lemma 3.3, $D_2(\pi)$ is graphic, and since $D_k(\pi)$ is graphic the induction hypothesis implies that there is a realization G of $D_2(\pi)$ having $k - 2$ disjoint 1-factors.

The hypothesis on π yields $\Delta(G) \leq n/2 - 1$, so $\delta(\overline{G}) \geq n/2$. Dirac's Theorem [3] now implies that $\overline{G}$ has a spanning cycle C. Since n is even, C decomposes into two edge-disjoint 1-factors. Therefore, $G \cup C$ is a realization of π having k edge-disjoint 1-factors. □

We also obtain Conjecture 3.2 in those cases where every entry in π is large, by applying Theorem 3.4 to the n-tuple obtained by subtracting every entry of $D_k(\pi)$ from $n - 1$.

Corollary 3.5. *Fix $k, n \in \mathbb{N}$ with n even, and let π be a graphic n-tuple such that $D_k(\pi)$ is also graphic. If every entry in π is at least $n/2 + k - 2$, then some realization of π has k edge-disjoint 1-factors.*

Our main result in this section is that, under the conditions of Conjecture 3.2, there is a realization of π having edge-disjoint factors M_1, M_2, F that are regular of degrees 1, 1, and $k-2$. This implies Conjecture 3.2 for $k \le 3$; for Conjecture 3.1, it allows one or two of $k_1, \dots, k_t$ to be odd when k is odd or even, respectively.

We use a well-known description of the maximum matchings in a graph. Say that a matching M *avoids* a vertex x if M has no edge incident to x. The *Gallai–Edmonds decomposition* of a graph G is a partition of $V(G)$ into three sets defined as follows (the presentation by Lovász and Plummer [11] uses (D, A, C) instead of our (A, B, C)):

$$A = \{x \in V(G)\colon \text{ some maximum matching avoids } x\},$$
$$B = \{x \in V(G) - A\colon\ x \text{ has a neighbor in } A\},$$
$$C = V(G) - (A \cup B).$$

A *near-perfect* matching in G is a matching that avoids exactly one vertex. A graph is *factor-critical* if each vertex is avoided by some near-perfect matching. The *deficiency* $\mathrm{def}(G)$ of a graph G is defined to be $\max_{X \subseteq V(G)} (o(G - X) - |X|)$, where $o(H)$ is the number of odd components (odd number of vertices) in H. It is immediate that every matching in G avoids at least $\mathrm{def}(G)$ vertices, and the Berge–Tutte Formula [1] states that equality holds for a maximum matching.

The Gallai–Edmonds Structure Theorem [5, 7, 8] describes the maximum matchings in a graph in terms of its Gallai–Edmonds Decomposition. We state only the parts we need.

Theorem 3.6. *If (A, B, C) is the Gallai–Edmonds Decomposition of a graph G, then (a) the components of $G[A]$ are factor-critical, and (b) every maximum matching in G consists of a near-perfect matching in each component of $G[A]$, a perfect matching in $G[C]$, and a matching of B into vertices in distinct components of $G[A]$.*

Consider the decomposition (A, B, C) of a graph G having an even number of vertices but no 1-factor. Say that a component of $G[A]$ is *missed* by a matching M if it has no vertex matched with a vertex of B in M. By Theorem 3.6, a maximum matching in G misses at least two components of $G[A]$. Our structural lemma, which may be of independent interest, is that when G is regular we can ensure that two such components will be nontrivial, where a graph is *nontrivial* if it has at least one edge.

Lemma 3.7. *Let (A, B, C) be the Gallai–Edmonds decomposition of a regular graph F with an even number of vertices. If F does not have a 1-factor, then some maximum matching in F misses two nontrivial components of $F[A]$.*

Proof. When F has no 1-factor, the set A is nonempty. Let S be the set of isolated vertices in $F[A]$. By Theorem 3.6, every maximum matching in F pairs B with vertices of distinct components of $F[A]$. Since the number of vertices is even, every maximum matching misses at least two components of $F[A]$.

Among the maximum matchings in F, choose M to miss the most nontrivial components of $F[A]$. If M does not miss two such components, then $|S| \geq 1$ and M misses at least one vertex a of S. Let B_S be the subset of B matched into S by M, and let $B'_S = B - B_S$.

Let R be the set of vertices reachable from a by M-alternating paths in F. Since M matches B into A, such paths move from A to B by edges not in M and return to A via M. If $a, \ldots, b, a'$ are the vertices of such a path with $b \in B'_S$, then exchanging membership between M and $E(F) - M$ along the path produces a new matching M' that misses one more nontrivial component of $F[A]$ than M. The choice of M thus implies $R \subseteq S \cup B_S$.

As we explore M-augmenting paths from a, reaching a vertex in B_S also immediately adds a new vertex of S. Thus $|R \cap S| = |R \cap B_S| + 1$. This contradicts k-regularity, since $N(R \cap S) \subseteq R \cap B_S$. We conclude that a maximum matching missing the most nontrivial components must miss at least two. □

Our second lemma concerns an auxiliary graph used in the proof of the theorem.

Lemma 3.8. *Let l and m be positive odd integers. Let H be the graph with vertices $v_{i,j}$ for $i \in \mathbb{Z}_l$ and $j \in \mathbb{Z}_m$ such that each $v_{i,j}$ is adjacent to the four vertices of the form $v_{i\pm1,j\pm1}$. Let S be an independent set in H. If the first coordinates of the vertices in S are distinct, and the second coordinates of the vertices in S are distinct, then $H - S$ contains an odd cycle.*

Proof. When we arrange the vertices in the natural l-by-m grid, the condition on S implies each row and column has at most one vertex of S. It suffices to find an odd closed walk avoiding S. The vertices $v_{1,1}, \ldots, v_{lm,lm}$ form an odd closed walk; it suffices unless $v_{r,r} \in S$ for some r. Since S is independent, $v_{r-1,r+1} \notin S$. Also, $v_{r-2,r}, v_{r,r+2} \notin S$. Replacing $v_{r,r}$ with $v_{r-2,r}, v_{r-1,r+1}, v_{r,r+2}$ increases the length of the walk by 2 but decreases the number of vertices of S on it by 1. Doing this independently for each vertex of S on it yields an odd closed walk avoiding S. □

We can now prove the main result of this section.

Theorem 3.9. *Fix $n, k \in \mathbb{N}$ with n even and $k \geq 2$. If π is a graphic n-tuple such that $D_k(\pi)$ is also graphic, then some realization of π has a k-factor with two edge-disjoint 1-factors.*

Proof. Since $D_k(\pi)$ is graphic, Kundu's Theorem provides a realization of π with a k-factor. Among all such realizations, choose a realization G and k-factor F in it to lexicographically maximize (r, s), where r is the maximum number of edge-disjoint 1-factors in F and s is the maximum size of a maximum matching in the graph $\hat{F}$ left by deleting r 1-factors from F.

If $r \geq 2$, then the claim holds. Otherwise, $r \leq 1$ and $0 < s < n/2$. If $r = 1$, let $\hat{M}$ be the specified matching; if $r = 0$, then $\hat{M} = \varnothing$. View $\overline{G}$, $\hat{M}$, $\hat{F}$, and $G - E(F)$ as a decomposition of K_n into edge-disjoint subgraphs.

Let (A, B, C) be the Gallai–Edmonds decomposition of $\hat{F}$. By Lemma 3.7, $\hat{F}$ has a maximum matching M that misses two nontrivial components of $\hat{F}[A]$; call them Q and Q'. Since Q and Q' are components of $\hat{F}[A]$, each edge of K_n joining them is not in $\hat{F}$.

If edges xy in Q and $x'y'$ in Q' exist such that xx' and yy' lie in the same graph among $\{\overline{G}, \hat{M}, G - E(F)\}$, then switching $\{xy, x'y'\}$ into it and $\{xx', yy'\}$ into $\hat{F}$ yields a realization G' of π with a k-factor F' (having a 1-factor if $r = 1$). Since Q and Q' are factor-critical (by Theorem 3.6), $Q - x$ and $Q' - x'$ have 1-factors. Since M misses Q and Q', replacing the edges of M in Q and Q' with xx' and 1-factors of $Q - x$ and $Q' - x'$ yields a matching M' in F' that is larger than M. By the choice of G and F, no such xx' and yy' exist.

Being factor-critical and nontrivial, Q and Q' are nonbipartite; hence each contains an odd cycle. Let $\{u_1, \ldots, u_l\}$ and $\{w_1, \ldots, w_m\}$ be the vertices along odd cycles chosen in Q and Q', respectively. Form the auxiliary graph H of Lemma 3.8, with vertices $v_{i,j}$ for $i \in \mathbb{Z}_l$ and $j \in \mathbb{Z}_m$. Let S be the subset of $V(H)$ corresponding to edges of the form $u_i w_j$ that belong to $\hat{M}$. If $r = 0$, then S is empty; if $r = 1$, then S has at most one vertex in each row and column, because $\hat{M}$ is a matching.

The vertices of $H - S$ correspond to other edges $u_i w_j$ in K_n, each belonging to $\overline{G}$ or to $G - E(F)$. By Lemma 3.8, $H - S$ contains an odd cycle, and hence two adjacent vertices in $H - S$ correspond to edges from the same subgraph. These edges have the form xx' and yy' previously forbidden. We conclude that $r \geq 2$, as desired. □

Corollary 3.10. *Conjecture 3.2 is true for $k \leq 3$.*

We believe that the conclusion of Lemma 3.8 remains true when two such independent sets S and S' are deleted. This would improve Theorem 3.9 to produce a realization having a k-factor with three edge-disjoint 1-factors, yielding Conjecture 3.2 for $k \leq 4$ and Conjecture 3.1 with one more odd value in $k_1, \ldots, k_t$ than allowed by Theorem 3.9. The method cannot extend beyond that, because when $l = m = 3$ there may be three independent sets of size 1 in H that together occupy one column, and then what remains is bipartite.

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