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ORIGINAL PAPER

Packing Tree Degree Sequences

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Abstract

A degree sequence is a list of non-negative integers, $D = d_1, d_2, \ldots, d_n$. It is called graphical if there exists a simple graph G such that the degree of the ith vertex is d_i; G is then said to be a realization of D. A tree degree sequence is one that is realized by a tree. In this paper we consider the problem of packing tree degree sequences: given k tree degree sequences, do they have simultaneous (i.e. on the same vertices) edge-disjoint realizations? We conjecture that this is true for any arbitrary number of tree degree sequences whenever they share no common leaves (degree-1 vertices). This conjecture is inspired by work of Kundu (SIAM J Appl Math 28:290–302, 1975) that showed it to be true for 2 and 3 tree degree sequences. In this paper, we give a proof for 4 tree degree sequences and a computer-aided proof for 5 tree degree sequences. We also make progress towards proving our conjecture for arbitrary k. We prove that k tree degree sequences without common leaves and at least $2k - 4$ vertices which are not leaves in any of the trees always have edge-disjoint tree realizations. Additionally, we show that to prove the conjecture, it suffices to prove it for $n \leq 4k - 2$ vertices. The main ingredient in all of the presented proofs is to find rainbow matchings in certain configurations.

Keywords Degree constrained edge partitioning · Tree sequence packing keyword · Rainbow matching

Mathematics Subject Classification 05C05 · 05C07 · 05C70

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1 Introduction

The graph realization problem [6] is a classic problem that asks whether a given degree sequence is graphical. In this paper we will be concerned with the more general problem of edge-disjoint graph realizations, or packing degree sequences, which asks whether a collection of degree sequences have simultaneous edge-disjoint realizations, i.e. a collection of edge-disjoint graphs (on the same vertices) whose degrees realize the given sequences. Our focus will be on giving sufficient conditions for the case of tree degree sequences.

This problem has appeared in various forms over the years, including edge packing [4], edge-disjoint realizations [7], degree constrained edge partitioning [2], and the colored degree matrix problem [8]. The general problem is known to be NP-complete [7], but certain special cases are easy. These special cases include the case when one of the degree sequences is almost regular and there are only two degree sequences [9], or, equivalently, when the element-wise sum of two degree sequences is almost regular [5], or when the degrees are sparse [2, 8].

One reason edge-disjoint realizations are of interest is the following simple observation: if k degree sequences admit edge-disjoint realizations, then their sum must certainly be graphical (simply take the union of the edge-disjoint realizations). Kundu proved in fact that two tree degree sequences have edge-disjoint tree realizations if and only if their sum is graphical [10]. That is, the necessary condition (the sum being graphical) is also sufficient in the case of two tree degree sequences. On the other hand, this characterization is not true of three such sequences: there exist three tree degree sequences such that any two of them have a sum which is graphical, and the sum of all three is also graphical, yet they do not have edge-disjoint tree realizations [11]. However, three tree degree sequences do have edge-disjoint tree realizations when their sum is graphical and the sum of the degrees of any vertex is at least 5 [11]. This minimal degree condition includes the case when the degree sequences have no common leaves, that is, when every vertex has degree 1 in at most one of the degree sequences.

It is easy to see that the sum of two degree sequences of trees is always graphical if they do not have common leaves [3]. This fact and Kundu's theorem for three sequences mean that k tree degree sequences always have edge-disjoint tree realizations if they do not have common leaves for $k = 2, 3$. A natural question now is to ask if this statement is true for arbitrary k. In this paper we conjecture that it is true, and prove it for $k = 4$. For $k = 5$, we prove that the conjecture is true if it is true up to 18 vertices. Computer-aided search then confirms that it is indeed true up to 18 vertices. We also prove the conjecture for arbitrary k in a special case, when there are a prescribed number of vertices which are not leaves in any of the degree sequences. All the presented proofs are based on induction, and the key point in the inductive steps is to find rainbow matchings in certain configurations.

2 Preliminaries

In this section, we give some necessary definitions and notation, as well as state the conjecture that we prove for some special cases.

Definition 1 A degree sequence $D = d_1, d_2, \ldots, d_n$ is a *tree degree sequence* if all degrees are positive and $\sum_{i=1}^{n} d_i = 2n - 2$. A degree sequence is a *path degree sequence* if two of its degrees are 1 and all other degrees are 2. A vertex with degree 1 is called a *leaf*.

It is easy to see that a tree degree sequence is always graphical and there is a realization of it that is a tree.

Definition 2 Let $D_1, D_2, \ldots, D_k$ be degree sequences of equal length. We say that this collection of degree sequences have *edge-disjoint realizations* if there exists a collection of edge-disjoint graphs, $G_1, G_2, \ldots, G_k$ such that for each i, G_i is a realization of the degree sequence D_i. Such a collection of graphs is a *realization* of $D_1, D_2, \ldots, D_k$. The degree of vertex v in degree sequence D_i is denoted by $d_v^{(i)}$.

Alternatively, an edge-colored simple graph is also called a realization of $D_1, D_2, \ldots, D_k$ if it is colored with k colors and for each color c_i, the subgraph containing the edges with color c_i is G_i.

Remark 1 Throughout the paper, degree sequences within any particular collection are assumed to be on the same set of vertices.

Definition 3 In an edge-colored graph, we say that the *$c_1 \ldots c_m$-degree* of a vertex v is the number of edges incident to v whose color is one of $c_1, \ldots, c_m$. Likewise, we say an edge is a *$c_1 \ldots c_m$-edge* if its color is one of $c_1, \ldots, c_m$.

Definition 4 If two edges e_1 and e_2 do not share a vertex, we might say that e_1 and e_2 are *disjoint*, or *vertex-disjoint*. If two edges e_3 and e_4 share a vertex, then we say that e_3 *covers* or *blocks* e_4 (and vice versa).

Definition 5 Given a collection of degree sequences $D_1, \ldots, D_k$, a *common leaf* is a vertex v such that $d_v^{(i)} = d_v^{(j)} = 1$ for some $i \neq j$. If there are no such vertices v, then we say that the degree sequences have *no common leaves*.

In this paper, we make the following conjecture.

Conjecture 1 *Let $D_1, D_2, \ldots, D_k$ be tree degree sequences without common leaves. Then they have edge-disjoint tree realizations.*

We will provide a constructive proof that this conjecture holds for $k = 4$. The constructions use the existence of rainbow matchings, defined below.

Definition 6 A *matching* is a set of disjoint edges. In an edge-colored graph, a *rainbow matching* is a matching in which no two edges have the same color. A *$c_1 \ldots c_m$-rainbow matching* is a rainbow matching consisting of an edge of each of the colors $c_1, \ldots, c_m$.

Definition 7 A matching *avoids* a vertex v if no edge in the matching is incident to v. (As a special case, an edge avoids a vertex when it is not incident to the vertex.)

As we observed earlier, a trivially necessary condition for a collection of degree sequences to be graphical is that the sum of the degree sequences is graphical. Therefore, Conjecture 1 would imply the that sum of tree degree sequences without common leaves is always graphical. This corollary can in fact be proven in the general case independent of the conjecture. For the proof we will first need to recall the Erdős–Gallai theorem, which provides a characterization of which degree sequences are graphical.

Theorem 1 [6] *A degree sequence $f_1 \geq f_2 \geq \cdots \geq f_n$ is graphical if and only if the sum of the degrees is even and for each $1 \leq s \leq n$ the inequality*

$$\sum_{i=1}^{s} f_i \leq s(s-1) + \sum_{j=s+1}^{n} \min\{s, f_j\} \tag{1}$$

holds.

We will also need the following lemma, which says that only the first few inequalities (1) in the Erdős–Gallai theorem have to be checked if the given degree sequence is the sum of tree degree sequences. This lemma is of some interest on its own, since it relates to ongoing research on how to characterize graphical degree sequences (see for example [12, 14]), and on how many Erdős–Gallai inequalities have to be checked for a degree sequence to be graphical [13].

Lemma 1 *Let $F = f_1 \geq f_2 \geq \cdots \geq f_n$ be the sum of k arbitrary tree degree sequences. Then the Erdős–Gallai inequalities (1) hold for every $s \geq 2k$.*

Proof To obtain an upper bound on the left-hand side of (1), we observe that each f_j, being the sum of k degrees, is at least k, and the sum of each sequence is $2n-2$, so that $\sum_j f_j = k(2n-2)$. As a consequence, it holds that

$$\sum_{i=1}^{s} f_i \leq k(2n-2) - (n-s)k.$$

Furthermore, we can give a lower bound on the right-hand side of (1). Indeed, if $s \geq 2k$, then

$$s(s-1) + (n-s)k \leq s(s-1) + \sum_{j=s+1}^{n} \min\{s, f_j\}.$$

Therefore, it is sufficient to prove that

$$k(2n-2) - (n-s)k \leq s(s-1) + (n-s)k.$$

Rearranging this, we get that

$$2k(s-1) \leq s(s-1),$$

which is true when $s \geq 1$ and $2k \leq s$. □

We use this lemma to prove the following theorem—now on tree degree sequences without common leaves.

Theorem 2 *Let $D_1, D_2, \ldots, D_k$ be tree degree sequences without common leaves. Then their sum is graphical.*

Proof Let $F = f_1, f_2, \ldots, f_n$ denote the sum of the degrees in decreasing order. We use the Erdős–Gallai theorem presented in Theorem 1. By Lemma 1, it is sufficient to prove the inequalities for $s \leq 2k-1$.

Since there are no common leaves, each f_j is at least $2k-1$, therefore $\min\{s, f_j\}$ is s for any $s \leq 2k-1$. This means inequality (1) simplifies to

$$\sum_{i=1}^{s} f_i \leq s(s-1) + (n-s)s = s(n-1).$$

And this is in fact the case since we claim that the sum of the degrees cannot be more than $n-1$ on any vertex. Indeed, $d_v^{(i)} = l$ means there are at least l leaf vertices which are not v in tree T_i. This is because any tree with a vertex of degree d contains at least d leaves and furthermore, any tree contains at least two leaves. Since there are no common leaves, and there are $n-1$ vertices when v is excluded, $f_v = \sum_{i=1}^{k} d_v^{(i)} \leq n-1$. So this inequality holds for $s \leq 2k-1$. □

We now present partial results on Conjecture 1. The results are obtained by inductive proofs in which larger realizations are constructed from smaller realizations.

3 The Theorem for 4 Tree Degree Sequences

In this section, we are going to prove that 4 tree degree sequences always have edge-disjoint realizations if they do not have common leaves. The proof is based on induction. We will need several lemmas: Lemmas 2 and 3 do the bulk of the work needed for the inductive step, while Lemmas 4 and 5 provide the base cases.

Lemma 2 *Let $D_1, D_2, \ldots, D_k$ be tree degree sequences without common leaves such that not all of them are degree sequences of paths. Then there exist vertices v, w and an index i such that $d_v^{(i)} = 1$, $d_v^{(j)} = 2$ for all $j \neq i$, and $d_w^{(i)} > 2$.*

Proof Arrange the sequences into a $k \times n$ matrix where the rows are the sequences and each column corresponds to a vertex. First, observe that the smallest column sum is $2k-1$, since there are no common leaves. Since the sequences are tree degree sequences, each row has sum exactly $2n-2$, therefore the total sum of all degrees is $k(2n-2)$. Observe that there must be at least $2k$ columns with column sum $2k-1$, otherwise the total sum would be at least

$$(n - (2k - 1))2k + (2k - 1)(2k - 1) = k(2n - 2) + 1,$$

a contradiction. However, there are by assumption at most $k - 1$ path sequences, therefore at most $2k - 2$ of these columns have their degree 1 entry in a path sequence. It follows that there is a column with sum $2k - 1$ whose degree 1 entry is in a row of a non-path sequence. □

Lemma 3 *Let $G = (V, E)$ be an edge-colored graph satisfying the following conditions*:

- $|V| \geq 10$
- *Each edge is colored exactly one of four colors*: *say*, *red*, *blue*, *green*, *or yellow.*
- *The subgraph corresponding to any one of the four colors is a tree on n vertices*, *and these trees have no common leaves.*

Let $v_0 \in V$ be an arbitrary vertex. Then for any 3 of the 4 colors, *G contains a rainbow matching* (*of those three colors*) *that avoids v_0.*

Proof Fix a vertex $v_0 \in V$; we will show that there exists an RBG-rainbow matching in G which avoids v_0.

Let v be a vertex with at least one edge of each color (red, blue, and green) going to a vertex that is not v_0; such a vertex can easily be seen to exist. Indeed, just pick any vertex adjacent to v_0 by a yellow edge. Hence, let vertices v, u_1, u_2, u_3 be such that (v, u_1) is blue, (v, u_2) is green and (v, u_3) is red. We will refer to the set of these four vertices as "the complex".

Let $W = V \setminus \{v_0, v, u_1, u_2, u_3\}$; notice that $|W| \geq 5$ since $|V| \geq 10$. We make some important observations.

1. Each vertex $w \in W$ has c-degree at least 1 for every color c, and because of the no-common-leaves condition, it can in fact have degree 1 in at most one color. Therefore, each $w \in W$ has RBG-degree at least 5. A given vertex may be adjacent to v_0 in some color, so each $w \in W$ has at least four incident RBG-edges which avoid v_0.
2. By similar reasoning, we see that for any color, each $w \in W$ is incident to at least two edges not of this color that avoid v_0.

Our proof will have two cases distinguished by the size of the largest RBG-matching (note: not necessarily rainbow matching) within W: exactly one, or at least two. But first we will show that we do not need to consider the case in which the largest matching in W is of size 0 (i.e. there are no edges at all in W), because this is not possible.

Indeed, suppose for contradiction that there are no edges within W. We will count RBG-edges to obtain a contradiction. There are three RBG-edges within the complex. Additionally, each vertex in W has RBG-degree at least 5 (by Observation 1 at the start of this proof), including edges incident to v_0. Therefore G has at least $5|W| + 3$ RBG-edges. Then, w.l.o.g., we can assume that G has at least $\frac{5}{3}|W| + 1$ red edges. Recall that, by definition of W, G has $|W| + 5$ vertices and $|W| \geq 5$. But notice that $\frac{5}{3}|W| + 1 > |W| + 4$ for all $|W| \geq 5$, showing that there are too many red edges for the red subgraph to be a tree. This contradiction shows that there must be at least one edge within W. We are now ready to examine our two cases.

Case 1: There exists an RBG-matching of size 2 within W. Let two vertex-disjoint RBG-edges be (w_1, w_2) and (w_3, w_4). We may assume they are the same color because otherwise we have an RBG-rainbow matching easily by taking an edge from the complex; say they are red. Now, by Observation 2 from the start of the proof, w_5 is incident to at least 2 BG-edges. And we claim that we may assume that each of w_5's BG-edges go to the complex because if w_5 were incident to a BG-edge e which did not go to the complex, then since it can only cover at most one of (w_1, w_2) and (w_3, w_4), we'd have an RBG-rainbow matching consisting of e, either (w_1, w_2) or (w_3, w_4), and an edge from the complex of the appropriate color.

Thus, we may assume that w_5 has 2 BG-edges going to the complex. Clearly at most one of these goes to v; w.l.o.g., say one which does not go to v is blue—then we may assume that it is (w_5, u_2), as otherwise, we have an RBG-rainbow matching.

We have now identified four pairwise vertex-disjoint edges: blue (v, u_1), blue (w_5, u_2), red (w_1, w_2), and red (w_3, w_4). We also have the additional red edge (v, u_3). We claim that any green edge not blocking both of the aforementioned blue edges can be extended to an RBG-rainbow matching. Indeed, if a green edge blocks both red edges in W, then that green edge, along with (v, u_3) and (w_5, u_2), is an RBG-rainbow matching. If a green edge does not block both red edges in W, and does not block both blue edges (as we assume), then that green edge, a red edge in W, and one of the blue edges will make an RBG-rainbow matching.

And only 3 green edges can block both blue edges, as otherwise there would be a green cycle. If all green edges in G that avoid v_0 block both blue edges, then v_0 has a green edge going to all but three of the other vertices in G. This means that the RBY-degree of v_0 is at most 3, in violation of Observation 1. Therefore there is at least one green edge that avoids v_0 and does not block both blue edges, and our argument for Case 1 is complete.

Case 2: The largest matching within W is of size 1. The edges within W must form either a star or a triangle, as these are the only configurations which do not yield a matching of size 2.

Case 2.1: the edges within W form a star. W.l.o.g., say w_1 is the center of the star. We claim that, after possibly re-labeling the colors, we can find $w_2, w_3 \in W \backslash \{w_1\}$ so that (w_1, w_2) is a red edge and w_3 sends two BG-edges to the complex.

First, suppose each $w \in W \backslash \{w_1\}$ has an edge going to w_1. Then, since $|W| \geq 5$, by the pigeonhole principle, we can find $w_2, w_3 \in W \backslash \{w_1\}$ such that (w_1, w_2) and (w_1, w_3) are edges of the same color (say red). Notice then that w_3 is incident to at least two BG-edges, by Observation 2, and each of these must go to the complex since w_1 is the center of the star.

Suppose on the other hand that there exists $w_3 \in W \backslash \{w_1\}$ such that (w_1, w_3) is not an edge. Then fix such a w_3, and choose w_2 so that (w_1, w_2) is an edge (which is possible since there's at least one edge within W, and w_1 is the center of our star). W.l.o.g., let the color of (w_1, w_2) be red, and notice that w_3 must send at least two BG-edges to the complex, again by Observation 2. Thus the claim is true.

So now we have that (w_1, w_2) is a red edge and w_3 sends two BG-edges to the complex. In particular w_3 sends a BG-edge to the complex that does not go to v; w.l.o.g., say it is blue. Then it must be (w_3, u_2), or else we have an RBG-rainbow matching. Now consider w_4 and $w_5 \in W \backslash \{w_1, w_2, w_3\}$: notice that if either one is incident to a green edge which avoids v_0, we are done. Therefore we may assume they both have no green edges that avoid v_0, meaning that their only green edge (there must be at least one) goes to v_0. Recalling Observation 1, this means that each of w_4 and w_5 has one green edge going to v_0, and at least two blue and at least two red edges (which avoid v_0).

We will now build an RBG-rainbow matching containing the green edge (v, u_2) by considering where the BG edges leaving w_4 and w_5 go. At most two edges from each of w_4 and w_5 cover the green edge (v, u_2), and since we have no monochromatic cycles, at most three of either color do. Therefore we can, w.l.o.g., choose a red edge from w_4 and a blue edge from w_5 which are both vertex-disjoint from the green (v, u_2). These edges must have the same endpoint, or else we have an RBG-rainbow matching. Moreover, this endpoint must be w_1 because otherwise we could choose the red (w_1, w_2) along with the blue edge from w_5 which is vertex-disjoint from the green (v, u_2), which would give an RBG-rainbow matching. Thus we can assume we have a red (w_4, w_1) and a blue (w_5, w_1).

Now, w_4 is incident to at least 3 additional RB-edges. At least one of these is vertex-disjoint from both v and u_2, and therefore this edge gives an RBG-rainbow matching along with the green (v, u_2) and either (w_1, w_2) or (w_1, w_5). This completes the star case.

Case 2.2: the edges within W form a triangle. Say the triangle is between w_1, w_2, and w_3. Then all edges with an endpoint in $W \backslash \{w_1, w_2, w_3\}$ have their other endpoint in the complex. Clearly the edges in the triangle cannot all be the same color because that would make a cycle. If we have one edge of each color, then we are done easily: w.l.o.g., say we have red (w_1, w_2), blue (w_1, w_3), and green (w_2, w_3). Then choose any edge from some $w \in W \backslash \{w_1, w_2, w_3\}$ which does not go to v; w.l.o.g., say this edge is red. It blocks at most one of the blue (v, u_1) and the green (v, u_2), so we can pick one of these along with our red edge, and then complete our RBG-rainbow matching with an edge from the triangle.

So we may now assume that the triangle has two edges of one color, and the third edge is a different color. We assume w.l.o.g. that (w_1, w_2) and (w_2, w_3) are red and (w_1, w_3) is blue. Fix some $w_4, w_5 \in W \backslash \{w_1, w_2, w_3\}$ and notice that w_4 and w_5 each send at least 2 BG-edges to the complex, by Observation 2. If all of these BG-edges are blue, then at least one is vertex-disjoint from the green (v, u_2) (otherwise we'd have a blue cycle), and we're done. So, we may assume that at least one of these BG-edges is green. If this green edge does not go to v, it is disjoint from either the red (v, u_3) or the blue (v, u_1), and we finish the RBG-rainbow matching with an appropriate edge from the triangle. So we may assume the green edge goes to v: w.l.o.g., say it is (w_4, v). Now look at any BG-edge from w_5 which does not go to v. If it's green, we're done, as argued above. And if it's blue, then we take it along with the green (w_4, v) and a red edge from the triangle. So in either case, we have an RBG-rainbow matching, and the triangle case is done. This completes the proof of the lemma. □

The following two lemmas establish the base cases of the induction. The first lemma is stated and proved for an arbitrary number of path degree sequences; later we use the general version in the proof of our conditional Theorem 6.

Lemma 4 *Let $D_1, D_2, \ldots D_k$ be path degree sequences without common leaves. They have edge-disjoint realizations.*

Proof The proof is by construction. This construction is known as the "Walecki construction"; see for example [1]. For clarity, we briefly describe the construction.

It should be clear that $n \geq 2k$, since any tree contains at least two leaves. We can say w.l.o.g. that the leaves in the i path have indexes i and $\lceil \frac{n}{2} \rceil + i$. Then the ith path contains the edges $(i, n-1+i)$, $(n-1+i, 1+i)$, $(1+i, n-2+i)$, $(n-2+i, 2+i)$, …, where the indexes are modulo n shifted by 1, that is, between 1 and n. The last edge is $\left(\lceil \frac{n}{2} \rceil + i - 1, \lceil \frac{n}{2} \rceil + i\right)$ if n is even, and $\left(\lceil \frac{n}{2} \rceil + i + 1, \lceil \frac{n}{2} \rceil + i\right)$ if n is odd. Figure 1 shows an example for 8 vertices. It is easy to see that there are no parallel edges if such a path is rotated with at most $\lfloor \frac{n}{2} \rfloor$ vertices. □

Lemma 5 *Let D_1, D_2, D_3, D_4 be tree degree sequences on at most 10 vertices, without common leaves. They have edge-disjoint realizations.*

Proof Up to isomorphism, there are only 14 possible such degree sequence quartets. The appendix contains a realization for each of them. □

Now we are ready to prove the main theorem.

Theorem 3 *Let D_1, D_2, D_3, D_4 be tree degree sequences without common leaves. They have edge-disjoint tree realizations.*

Proof The proof is by induction; the base cases are the degree sequences on at most 10 vertices and the path degree sequence quartets. They all have edge-disjoint realizations, based on Lemmas 4 and 5.

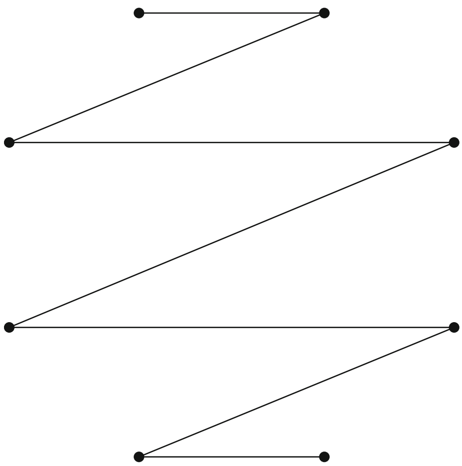

Fig. 1 An example Hamiltonian path on 8 vertices. See the text for details

So assume that D_1, D_2, D_3, D_4 are tree degree sequences on more than 10 vertices and at least one of them is not a path degree sequence. Then there exist vertices v and w and an index i such that $d_v^{(i)} = 1$ for all $j \neq i$, $d_v^j = 2$ and $d_w^{(i)} > 2$, according to Lemma 2. Consider the degree sequences D'_1, D'_2, D'_3, D'_4 which is obtained by deleting vertex v and subtracting 1 from $d_w^{(i)}$. These are tree degree sequences without common leaves, and based on the inductive assumption, they have edge-disjoint realizations. Let G be the colored graph representing these edge-disjoint realizations and permute the degree sequences (and the colors accordingly) so that D_i is moved to the fourth position.

Let the subgraphs of G corresponding to D'_1, D'_2, D'_3, and D'_4 be colored in red, blue, green, and yellow respectively. Since G contains at least 10 vertices, it has an RBG-rainbow matching which avoids w, according to Lemma 3. The realizations of D_1, D_2, D_3 and D_4 are obtained by the following way. Take the realizations represented by G. Add vertex v. Connect v with w in the first tree, delete the edges of the RBG-rainbow matching, and connect v to all the vertices incident to the edges of the RBG-rainbow matching, 2 edges for each tree, according to the color of the deleted edge. □

4 Some Results in the General Case

We now present some results in the general case, i.e. for an arbitrary number k of tree degree sequences. First we show that $n \geq 4k - 2$ suffices to guarantee a rainbow matching. However for our original purpose of finding edge-disjoint realizations via the inductive proof, this is not sufficient to show that the induction step goes through every time, since our base case is $n = 2k$. We need something else to bridge the gap between $n = 2k$ and $n = 4k - 2$. This is accomplished by adding an extra condition: we show that if we have at least $2k - 4$ vertices that are not leaves in any tree, then we are indeed guaranteed edge-disjoint realizations. We then use these results to reduce the $k = 5$ case to one that is verifiable by a computer search.

4.1 Rainbow Matchings from Matchings: $n \geq 4k - 2$ Guarantees a Rainbow Matching of Size $k - 1$ Avoiding a Given Color and a Given Vertex

We now show that $n \geq 4k - 2$ suffices to guarantee a rainbow matching. The broad line of attack will be to stitch a rainbow matching together from regular (singly-colored, and large, but not necessarily perfect) matchings. A crucial ingredient in guaranteeing large matchings will be the fact that a tree with m non-leaves must contain a matching of size roughly $m/2$. The idea will be that when n is large enough, the no-common-leaves condition guarantees a large number of non-leaves in each color, which then guarantees large matchings in each color, which can then be stitched together into a rainbow matching. We formalize the main ingredients as the following lemmas.

Lemma 6 *Let G be an edge colored graph such that for each color c_i, $i = 1, 2, \ldots k$ there is a matching of size $2i$ in the subgraph of color c_i. Let v be an arbitrary vertex. Then G has a rainbow matching of size k that avoids v.*

Proof The proof is by induction using the Pigeonhole Principle. Since there are 2 disjoint edges of the first color in G, at least one of them is not incident to v. Take that edge to be in the rainbow matching.

Assume that we have already found a rainbow matching of size i. There is a matching of size $2i + 2$ in the subgraph of color c_{i+1}. At most $2i$ of these edges are blocked by the rainbow matching of size i, and at most one of them is incident to v. Thus, there is an edge of color $i + 1$ which is disjoint from the rainbow matching of size i and not incident to v. Extend the rainbow matching with this edge. □

Lemma 7 *A tree with at least one edge and m internal nodes contains a matching of size at least* $\lceil \frac{m+1}{2} \rceil$.

Proof The proof is by induction. The base cases are the trees with 2 and 3 vertices. They have 0 and 1 internal nodes (i.e. non-leaves) respectively, and they each have an edge, which is a matching of size 1.

Now assume that the number of vertices in tree T is more than 3, and the number of internal nodes in it is m. Take any leaf and its incident edge e. There are two cases.

1. The non-leaf vertex of e has degree more than 2. Then $T' = T \setminus \{e\}$ has the same number of internal nodes as T. By the inductive hypothesis, T' has a matching of size $\lceil \frac{m+1}{2} \rceil$, so T does also.
2. The non-leaf vertex of e has degree 2. Let its other edge be denoted by f. Then the internal nodes in $T' = T \setminus \{e, f\}$ is the internal nodes in T minus at most 2. Thus T' has a matching M of size $\lceil \frac{m-1}{2} \rceil$. $M \cup \{e\}$ is a matching in T with size $\lceil \frac{m+1}{2} \rceil$.

□

We now show that $n \geq 4k - 2$ suffices to guarantee a rainbow matching.

Theorem 4 *Let k trees be given on n vertices,* $k \geq 5$, *having no common leaves. Let w be an arbitrary vertex and let c be an arbitrary color. Then if the number of vertices are greater or equal than* $4k - 2$, *we can find a rainbow matching of all colors except c that avoids w.*

Proof Arrange the $k - 1$ trees corresponding to colors other than c in increasing order of number of internal nodes. We would like to show that the ith tree has a matching of size $2i$. This is sufficient to find a rainbow matching, according to Lemma 6.

Since internal nodes are exactly the vertices of a tree which are not leaves, we have also arranged the trees in decreasing order of number of leaves. Each tree has at least 2 leaves, therefore in the $k - 1 - i$ trees above the ith tree and in the kth tree there are altogether at least $2(k - i)$ leaves. Since no vertex is a leaf in more than one tree, there remain only at most $n - 2(k - i)$ vertices that might still be leaves in the the ith tree and the $i - 1$ trees below. And since the number of leaves in the trees below is no less than in the ith tree, the ith tree contains at most

$$\left\lfloor \frac{n-2(k-i)}{i} \right\rfloor$$

leaves, and thus at least

$$n - \left\lfloor \frac{n-2(k-i)}{i} \right\rfloor = \left\lceil \frac{(i-1)n+2(k-i)}{i} \right\rceil$$

internal nodes. If $n \geq 4k-2$, this means at least

$$\left\lceil \frac{(i-1)(4k-2)+2(k-i)}{i} \right\rceil = \left\lceil \frac{4ki-2k-4i+2}{i} \right\rceil = 4k-4-\left\lfloor \frac{2k-2}{i} \right\rfloor$$

internal nodes. Lemma 7 gives a lower-bound for the size of the largest matching in the ith tree, which, recall, we want to show is at least $2i$. That is, we want to show that:

$$\left\lceil \frac{4k-4-\left\lfloor \frac{2k-2}{i} \right\rfloor+1}{2} \right\rceil \geq 2i. \tag{2}$$

When $i = k-1$, the left hand side is

$$\left\lceil \frac{4k-4-2+1}{2} \right\rceil = 2(k-1) = 2i.$$

For $i < k-1$, it is sufficient to show that

$$\frac{4k-3-\frac{2k-2}{i}}{2} \geq 2i.$$

After rearranging, we get that

$$0 \geq 4i^2-(4k-3)i+2k-2$$

Solving the second order equation, we get that

$$\frac{4k-3-\sqrt{(4k-7)^2-8}}{8} \leq i \leq \frac{4k-3+\sqrt{(4k-7)^2-8}}{8}.$$

Rounding the discriminant knowing that $k \geq 5$, we get that

$$\frac{4k-3-(4k-8)}{8} \leq i \leq \frac{4k-3+4k-8}{8}.$$

namely,

$$\frac{5}{8} \leq i \leq k - \frac{11}{8}$$

which holds since $1 \leq i \leq k-2$. Therefore, in the ith tree there is a matching of size at least $2i$, which is sufficient to have the prescribed rainbow matching. □

4.2 Edge-Disjoint Realizations Under a Condition on the Degree Distribution

Theorem 4 is not strong enough to prove the full conjecture of edge-disjoint realizations, Conjecture 1, since in our inductive proof we need to find rainbow matchings at each inductive step, starting from $n = 2k$. But by adding an extra condition to the degree distribution, and showing that this condition is maintained throughout the induction process, we are successfully able to guarantee edge-disjoint realizations.

Definition 8 Define a *never-leaf* to be a vertex that is not a leaf in any tree.

Theorem 5 *k tree degree sequences without common leaves and with at least $2k-4$ never-leaves always have edge-disjoint realizations.*

Proof We will use the same inductive process presented in Theorem 3. The crucial observation about that proof is that nowhere during the inductive step do we create any new leaves in any tree. This means the number of never-leaves does not decrease during the inductive step, and so at each step, we have at least $2k-4$ never-leaves.

It only remains to be shown, then, that whenever we have $2k-4$ never-leaves we can find a rainbow matching. We claim that in each tree there are at least $4k-6$ internal nodes. Indeed, the $2k-4$ never-leaves are certainly internal nodes in this tree. And in each of the other $k-1$ trees there are at least two leaves, and these leaves are internal nodes in all other trees because no common leaves, giving an additional $2k-2$ internal nodes, altogether $4k-6$ internal nodes. By Lemma 7 this means we have matchings of size at least

$$\left\lceil \frac{4k-5}{2} \right\rceil = 2k-2$$

in each tree, and by Lemma 6 these guarantee a rainbow matching, and we are done. □

4.3 A Conditional Theorem and the $k = 5$ Case

The consequence of Theorem 4 is the following, which says that for any k we only need to prove Conjecture 1 up to $4k-2$ vertices.

Theorem 6 *Fix a k. If all tree degree sequence k-tuples without common leaves on at most $4k-2$ vertices have edge-disjoint realizations, then any tree degree sequence k -tuples without common leaves have edge-disjoint realizations.*

Proof The proof is by induction. The base cases are the path degree sequences, which have edge-disjoint realizations, according to Lemma 4, and the degree sequences on at most $4k-2$ vertices, which have edge-disjoint realizations by hypothesis.

Let $D_1, D_2, \ldots, D_k$ be tree degree sequences without common leaves on more than $4k-2$ vertices. By Lemma 2, there are vertices v, w and an index i such that $d_v^{(i)} = 1$, for all $j \neq i$, $d_v^{(j)} = 2$ and $d_w^{(i)} > 2$. Construct the degree sequences

$D'_1, D'_2, \ldots, D'_k$ by removing v and subtracting 1 from $d_w^{(i)}$. These are tree degree sequences on at least $4k-2$ vertices, and they have edge-disjoint realizations $T'_1, T'_2, \ldots, T'_k$ by the inductive hypothesis. Let G be the edge-colored graph that is the union of $T'_1, T'_2, \ldots, T'_k$, in which the edges of T'_ℓ are colored c_ℓ, for each index ℓ. Then there is a rainbow matching consisting of all colors except c_i which avoids vertex w, according to Theorem 4. Construct a realization of $D_1, D_2, \ldots, D_k$ in the following way. Start with the edge-colored graph G. Add vertex v and connect it to w in T'_i. Delete the edges in the rainbow matching and, for each vertex that had been incident to an edge, say of color c_ℓ, in the rainbow matching, connect it to v by an edge of color c_ℓ. □

When $k = 5$, Theorem 6 says the following: if all tree degree sequence quintets without common leaves and on at most 18 vertices have edge-disjoint realizations, then all tree degree sequence quintets have edge-disjoint realizations. A computer-aided search showed that up to permutation of sequences and vertices, there are at most 592,000 tree degree quintets without common leaves and on at most 18 vertices, and they all have edge-disjoint tree realizations.

5 Discussion

In this paper, we considered the conjecture that any arbitrary number of tree degree sequences without common leaves have edge-disjoint tree realizations. The conjecture has been inspired by Kundu's theorem that 3 tree degree sequences have edge-disjoint tree realizations if the minimum sum of the degrees is 5 [11]. We do not know if this theorem can be generalized to arbitrary number of tree degree sequences, that is, we do not know if k tree degree sequences can always be realized with k edge disjoint trees if the minimum sum of the degrees is at least $2k-1$.

On the other hand, our conjecture seems to be true for an arbitrary number of tree degree sequences. It is always true when $n = 2k$; this is the Walecki construction of decomposing K_{2n} into paths. Here we would like to mention Kundu's conjecture that a collection of tree degree sequences always have edge-disjoint tree realizations if their sum is the degree sequence of K_{2n} [11].

We also showed the following. If our conjecture is not true, then there must be a relatively small counterexample, according to Theorem 6. However, it seems hard to close the gap between $n = 2k$ and $n \geq 4k-2$. On the other hand, in a forthcoming paper of the third author, we will make the conjecture that edge-disjoint caterpillar realizations exist for tree degree sequences without common leaves (recall that a tree is a caterpillar if its non-leaves form a path). Furthermore, such edge-disjoint realizations always exist unconditionally for large, but still $O(k)$, number of vertices. That is, if our conjecture is not true then all counterexamples must be small.

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Appendix

Up to permutations of degree sequences and vertices, there are 14 tree degree sequence quartets on at most 10 vertices without common leaves, which we enumerate below. This appendix gives an example realization for all of them.

If the number of vertices is 8, there is only one possible degree sequence quartet, each degree sequence is a path degree sequence (case 1).

If the number of vertices is 9, there are 2 possible cases: either all degree squences are path degree sequences (case 2) or there is a degree 3 (case 3).

If the number of vertices is 10, there are 11 possible cases: all degree sequences are path degree sequences (case 4), there is a degree 3 which might be on a vertex with a leaf (case 5) or without a leaf (case 6), there is a degree 4 (case 7) or there are 2 degree 3s in the degree sequences (cases 8–14).

The two 3s might be in the same degree sequence, and the leaves on these two vertices might be in the same degree sequence (case 8) or in different degree sequences (case 9).

If the two degree 3s are in different degree sequences, they might be on the same vertex (case 10) or on different vertices.

If the two degree 3s are in different sequences, D_i and D_j, and on different vertices u and v, consider the degrees of u and v in D_i and D_j which are not 3. They might be both 1 (case 11), or else maybe one of them is 1 and the other is 2 (case 12), or else both of them are 2. In this latter case, the degree 1s on u and v might be in the same degree sequence (case 13) or in different degree sequences (case 14).

The realizations are represented with an adjacency matrix, in which 0 denotes the absence of edges, and for each degree sequence D_i, i denotes the edges in the realization of D_i.

1.

$$D_1 = 1,2,2,2,1,2,2,2$$
$$D_2 = 2,1,2,2,2,1,2,2$$
$$D_3 = 2,2,1,2,2,2,1,2$$
$$D_4 = 2,2,2,1,2,2,2,1$$
$$\times \begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 \\ 2 & 2 & 0 & 3 & 4 & 4 & 1 & 1 \\ 2 & 3 & 3 & 0 & 4 & 1 & 1 & 2 \\ 3 & 3 & 4 & 4 & 0 & 1 & 2 & 2 \\ 3 & 4 & 4 & 1 & 1 & 0 & 2 & 3 \\ 4 & 4 & 1 & 1 & 2 & 2 & 0 & 3 \\ 4 & 1 & 1 & 2 & 2 & 3 & 3 & 0 \end{pmatrix}$$

2.

$$D_1 = 1,2,2,2,2,1,2,2,2$$
$$D_2 = 2,1,2,2,2,2,1,2,2$$
$$D_3 = 2,2,1,2,2,2,2,1,2$$
$$D_4 = 2,2,2,1,2,2,2,2,1$$
$$\times \begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 0 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 1 \\ 2 & 2 & 0 & 3 & 4 & 4 & 0 & 1 & 1 \\ 2 & 3 & 3 & 0 & 4 & 0 & 1 & 1 & 2 \\ 3 & 3 & 4 & 4 & 0 & 1 & 1 & 2 & 2 \\ 3 & 4 & 4 & 0 & 1 & 0 & 2 & 2 & 3 \\ 4 & 4 & 0 & 1 & 1 & 2 & 0 & 3 & 3 \\ 4 & 0 & 1 & 1 & 2 & 2 & 3 & 0 & 4 \\ 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 0 \end{pmatrix}$$

3.

$$
\begin{aligned}
D_1 &= 1,3,2,2,1,2,2,2,1\\
D_2 &= 2,1,2,2,2,1,2,2,2\\
D_3 &= 2,2,1,2,2,2,1,2,2\\
D_4 &= 2,2,2,1,2,2,2,1,2\\
&\times \begin{pmatrix}
0 & 1 & 0 & 2 & 3 & 3 & 4 & 4 & 2\\
1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1\\
0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2\\
2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 4\\
3 & 3 & 4 & 0 & 0 & 1 & 2 & 2 & 4\\
3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3\\
4 & 4 & 1 & 1 & 2 & 2 & 0 & 3 & 0\\
4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3\\
2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0
\end{pmatrix}
\end{aligned}
$$

4.

$$
\begin{aligned}
D_1 &= 1,2,2,2,2,1,2,2,2,2\\
D_2 &= 2,1,2,2,2,2,1,2,2,2\\
D_3 &= 2,2,1,2,2,2,2,1,2,2\\
D_4 &= 2,2,2,1,2,2,2,2,1,2\\
&\times \begin{pmatrix}
0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 0 & 0\\
1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 0 & 1\\
2 & 2 & 0 & 3 & 4 & 4 & 0 & 0 & 1 & 1\\
2 & 3 & 3 & 0 & 4 & 0 & 0 & 1 & 1 & 2\\
3 & 3 & 4 & 4 & 0 & 0 & 1 & 1 & 2 & 2\\
3 & 4 & 4 & 0 & 0 & 0 & 1 & 2 & 2 & 3\\
4 & 4 & 0 & 0 & 1 & 1 & 0 & 2 & 3 & 3\\
4 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 3 & 4\\
0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 4\\
0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 0
\end{pmatrix}
\end{aligned}
$$

5.

$$D_1 = 1, 3, 2, 2, 2, 1, 2, 2, 2, 1$$
$$D_2 = 2, 1, 2, 2, 2, 2, 1, 2, 2, 2$$
$$D_3 = 2, 2, 1, 2, 2, 2, 2, 1, 2, 2$$
$$D_4 = 2, 2, 2, 1, 2, 2, 2, 2, 1, 2$$
$$\times \begin{pmatrix} 0 & 1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 2 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 1 & 1 \\ 0 & 2 & 0 & 3 & 4 & 4 & 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 0 & 0 & 0 & 1 & 1 & 2 & 4 \\ 3 & 3 & 4 & 0 & 0 & 1 & 1 & 2 & 2 & 4 \\ 3 & 4 & 4 & 0 & 1 & 0 & 2 & 2 & 3 & 0 \\ 4 & 4 & 0 & 1 & 1 & 2 & 0 & 0 & 3 & 3 \\ 4 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 4 & 3 \\ 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 0 & 0 \\ 2 & 1 & 2 & 4 & 4 & 0 & 3 & 3 & 0 & 0 \end{pmatrix}$$

6.

$$D_1 = 1, 2, 2, 2, 3, 1, 2, 2, 2, 1$$
$$D_2 = 2, 1, 2, 2, 2, 2, 1, 2, 2, 2$$
$$D_3 = 2, 2, 1, 2, 2, 2, 2, 1, 2, 2$$
$$D_4 = 2, 2, 2, 1, 2, 2, 2, 2, 1, 2$$
$$\times \begin{pmatrix} 0 & 1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 2 \\ 1 & 0 & 2 & 0 & 3 & 4 & 4 & 0 & 1 & 3 \\ 0 & 2 & 0 & 3 & 4 & 4 & 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 0 & 4 & 0 & 1 & 1 & 2 & 3 \\ 3 & 3 & 4 & 4 & 0 & 1 & 1 & 2 & 2 & 1 \\ 3 & 4 & 4 & 0 & 1 & 0 & 2 & 2 & 3 & 0 \\ 4 & 4 & 0 & 1 & 1 & 2 & 0 & 3 & 3 & 0 \\ 4 & 0 & 1 & 1 & 2 & 2 & 3 & 0 & 0 & 4 \\ 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 4 \\ 2 & 3 & 2 & 3 & 1 & 0 & 0 & 4 & 4 & 0 \end{pmatrix}$$

7.

$$
\begin{aligned}
D_1 &= 1, 4, 2, 2, 1, 2, 2, 2, 1, 1\\
D_2 &= 2, 1, 2, 2, 2, 1, 2, 2, 2, 2\\
D_3 &= 2, 2, 1, 2, 2, 2, 1, 2, 2, 2\\
D_4 &= 2, 2, 2, 1, 2, 2, 2, 1, 2, 2\\
&\times \begin{pmatrix}
0 & 1 & 0 & 0 & 3 & 3 & 4 & 4 & 2 & 2\\
1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 1\\
0 & 2 & 0 & 3 & 0 & 4 & 1 & 1 & 2 & 4\\
0 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 4 & 2\\
3 & 3 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 4\\
3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0\\
4 & 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3\\
4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 & 3\\
2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0\\
2 & 1 & 4 & 2 & 4 & 0 & 3 & 3 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

8.

$$
\begin{aligned}
D_1 &= 1, 3, 2, 2, 1, 3, 2, 2, 1, 1\\
D_2 &= 2, 1, 2, 2, 2, 1, 2, 2, 2, 2\\
D_3 &= 2, 2, 1, 2, 2, 2, 1, 2, 2, 2\\
D_4 &= 2, 2, 2, 1, 2, 2, 2, 1, 2, 2\\
&\times \begin{pmatrix}
0 & 1 & 0 & 2 & 0 & 3 & 4 & 4 & 2 & 3\\
1 & 0 & 0 & 3 & 3 & 4 & 4 & 1 & 1 & 2\\
0 & 0 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 2\\
2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 0 & 4\\
0 & 3 & 4 & 0 & 0 & 1 & 2 & 2 & 4 & 3\\
3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 1\\
4 & 4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 0\\
4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3 & 0\\
2 & 1 & 2 & 0 & 4 & 3 & 0 & 3 & 0 & 4\\
3 & 2 & 2 & 4 & 3 & 1 & 0 & 0 & 4 & 0
\end{pmatrix}
\end{aligned}
$$

9.

$$
\begin{aligned}
D_1 &= 1,3,3,2,1,2,2,2,1,1\\
D_2 &= 2,1,2,2,2,1,2,2,2,2\\
D_3 &= 2,2,1,2,2,2,1,2,2,2\\
D_4 &= 2,2,2,1,2,2,2,1,2,2\\
&\times \begin{pmatrix}
0 & 1 & 0 & 0 & 3 & 3 & 4 & 4 & 2 & 2\\
1 & 0 & 2 & 3 & 3 & 0 & 4 & 1 & 1 & 4\\
0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 1\\
0 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 4 & 2\\
3 & 3 & 4 & 0 & 0 & 1 & 2 & 2 & 4 & 0\\
3 & 0 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 4\\
4 & 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3\\
4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 & 3\\
2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0\\
2 & 4 & 1 & 2 & 0 & 4 & 3 & 3 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

10.

$$
\begin{aligned}
D_1 &= 1,3,2,2,1,2,2,2,1,2\\
D_2 &= 2,1,2,2,2,1,2,2,2,2\\
D_3 &= 2,3,1,2,2,2,1,2,2,1\\
D_4 &= 2,2,2,1,2,2,2,1,2,2\\
&\times \begin{pmatrix}
0 & 1 & 0 & 2 & 3 & 3 & 4 & 0 & 2 & 4\\
1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 3\\
0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 0\\
2 & 3 & 3 & 0 & 0 & 0 & 1 & 2 & 4 & 1\\
3 & 3 & 4 & 0 & 0 & 1 & 0 & 2 & 4 & 2\\
3 & 4 & 4 & 0 & 1 & 0 & 2 & 0 & 3 & 1\\
4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0 & 2\\
0 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3 & 4\\
2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0\\
4 & 3 & 0 & 1 & 2 & 1 & 2 & 4 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

11.

$$
\begin{aligned}
D_1 &= 1,3,2,2,1,2,2,2,1,2\\
D_2 &= 3,1,2,2,2,1,2,2,2,1\\
D_3 &= 2,2,1,2,2,2,1,2,2,2\\
D_4 &= 2,2,2,1,2,2,2,1,2,2\\
&\times\begin{pmatrix}
0 & 1 & 0 & 2 & 3 & 3 & 4 & 4 & 2 & 2\\
1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 0\\
0 & 2 & 0 & 3 & 0 & 4 & 1 & 1 & 2 & 4\\
2 & 3 & 3 & 0 & 0 & 0 & 1 & 2 & 4 & 1\\
3 & 3 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 4\\
3 & 4 & 4 & 0 & 1 & 0 & 2 & 0 & 3 & 1\\
4 & 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3\\
4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 & 3\\
2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0\\
2 & 0 & 4 & 1 & 4 & 1 & 3 & 3 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

12.

$$
\begin{aligned}
D_1 &= 1,3,2,2,1,2,2,2,1,2\\
D_2 &= 2,1,3,2,2,1,2,2,2,1\\
D_3 &= 2,2,1,2,2,2,1,2,2,2\\
D_4 &= 2,2,2,1,2,2,2,1,2,2\\
&\times\begin{pmatrix}
0 & 0 & 0 & 2 & 3 & 3 & 4 & 4 & 2 & 1\\
0 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 1\\
0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 2\\
2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 0 & 4\\
3 & 3 & 4 & 0 & 0 & 1 & 2 & 2 & 4 & 0\\
3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0\\
4 & 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3\\
4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 & 3\\
2 & 1 & 2 & 0 & 4 & 3 & 0 & 3 & 0 & 4\\
1 & 1 & 2 & 4 & 0 & 0 & 3 & 3 & 4 & 0
\end{pmatrix}
\end{aligned}
$$

13.

$$
\begin{aligned}
D_1 &= 1, 3, 2, 2, 1, 2, 2, 2, 1, 2 \\
D_2 &= 2, 1, 2, 2, 2, 1, 2, 2, 2, 2 \\
D_3 &= 2, 2, 1, 2, 2, 3, 1, 2, 2, 1 \\
D_4 &= 2, 2, 2, 1, 2, 2, 2, 1, 2, 2 \\
&\times \begin{pmatrix}
0 & 0 & 0 & 2 & 3 & 3 & 4 & 4 & 2 & 1 \\
0 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 1 \\
0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 0 \\
2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 0 & 4 \\
3 & 3 & 4 & 0 & 0 & 1 & 0 & 2 & 4 & 2 \\
3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 3 \\
4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0 & 2 \\
4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3 & 0 \\
2 & 1 & 2 & 0 & 4 & 3 & 0 & 3 & 0 & 4 \\
1 & 1 & 0 & 4 & 2 & 3 & 2 & 0 & 4 & 0
\end{pmatrix}
\end{aligned}
$$

14.

$$
\begin{aligned}
D_1 &= 1, 3, 2, 2, 1, 2, 2, 2, 1, 2 \\
D_2 &= 2, 1, 2, 2, 2, 1, 2, 2, 2, 2 \\
D_3 &= 2, 2, 1, 3, 2, 2, 1, 2, 2, 1 \\
D_4 &= 2, 2, 2, 1, 2, 2, 2, 1, 2, 2 \\
&\times \begin{pmatrix}
0 & 0 & 0 & 2 & 3 & 3 & 4 & 4 & 2 & 1 \\
0 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 1 \\
0 & 2 & 0 & 3 & 4 & 0 & 1 & 1 & 2 & 4 \\
2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 4 & 3 \\
3 & 3 & 4 & 0 & 0 & 1 & 0 & 2 & 4 & 2 \\
3 & 4 & 0 & 1 & 1 & 0 & 2 & 0 & 3 & 4 \\
4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0 & 2 \\
4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3 & 0 \\
2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0 \\
1 & 1 & 4 & 3 & 2 & 4 & 2 & 0 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

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