Painlevé I asymptotics for orthogonal polynomials with respect to a varying quartic weight

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Abstract

We study polynomials that are orthogonal with respect to a varying quartic weight $\exp(-N(x^2/2+tx^4/4))$ for $t<0$, where the orthogonality takes place on certain contours in the complex plane. Inspired by developments in $2D$ quantum gravity, Fokas, Its, and Kitaev, showed that there exists a critical value for t around which the asymptotics of the recurrence coefficients are described in terms of exactly specified solutions of the Painlevé I equation. In this paper, we present an alternative and more direct proof of this result by means of the Deift/Zhou steepest descent analysis of the Riemann-Hilbert problem associated with the polynomials. Moreover, we extend the analysis to non-symmetric combinations of contours. Special features in the steepest descent analysis are a modified equililbrium problem and the use of Ψ-functions for the Painlevé I equation in the construction of the local parametrix.

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1 Introduction and statement of results

1.1 Introduction

Let $V = V_t$ be the quartic polynomial defined by

$$V(x) = V_t(x) = x^2/2 + tx^4/4, \tag{1.1}$$

where $t \in \mathbb{R}$ is a parameter. In this paper we study orthogonal polynomials with respect to the exponential weight $\mathrm{e}^{-NV_t(x)}$ where $N \in \mathbb{N}$, in cases where t is negative. Since for negative t integrals like

$$\int_{\mathbb{R}} x^k \mathrm{e}^{-NV_t(x)} \mathrm{d}x$$

are divergent, we will have to define what we mean by orthogonal polynomials in this case. Before doing so, and in order to motivate what we are going to do, we discuss the case $t \geq 0$ first.

For $t \geq 0$, the monic orthogonal polynomial π_n of degree $n \in \mathbb{N}$ satisfies

$$\int_{\mathbb{R}} \pi_n(x) x^k \ \mathrm{e}^{-NV_t(x)} \mathrm{d}x = 0, \tag{1.2}$$

for $k = 0, 1, \ldots, n-1$. An important feature of these polynomials is that the recurrence coefficients a_n in the three term recurrence $\pi_{n+1}(x) = x\pi_n(x) - a_n\pi_{n-1}(x)$ satisfied by these polynomials satisfy

$$a_n + ta_n(a_{n-1} + a_n + a_{n+1}) = \frac{n}{N}. \tag{1.3}$$

The nonlinear difference equation (1.3) is referred to as the string equation or the Freud equation. It is also known as an example of a discrete Painlevé equation, see [26] and the references cited therein. Of course the polynomials π_n as well as the recurrence coefficients a_n depend on N and t. If we want to emphasize this dependence we write $\pi_{n,N,t}$ and $a_{n,N}(t)$.

An important tool in the study of the asymptotic behavior of $\pi_{n,n,t}$ and $a_{n,n}(t)$ as $n \to \infty$ is played by the equilibrium measure in the external field V_t [28]. This is the unique Borel probability measure $\mu = \mu_t$ on $\mathbb{R}$ that minimizes

$$I_{V_t}(\mu) = \iint \log \frac{1}{|x-y|} \mathrm{d}\mu(x)\mathrm{d}\mu(y) + \int V_t(x)\mathrm{d}\mu(x) \tag{1.4}$$

among all Borel probability measures on $\mathbb{R}$. The measure μ_t can be calculated explicitly. It is supported on the interval $[-c_t, c_t]$ where

$$c_t^2 = \frac{2}{3t}\left(\sqrt{1+12t}-1\right), \tag{1.5}$$

and it has a density given by

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}x} = \frac{t}{2\pi}(x^2-d_t^2)\sqrt{c_t^2-x^2}, \qquad \text{for } x \in [-c_t, c_t], \tag{1.6}$$

where

$$d_t^2 = -\frac{1}{3t}\left(\sqrt{1+12t}+2\right). \tag{1.7}$$

The formula (1.7) may look a bit awkward since $d_t^2 < 0$ for $t > 0$, but we have written it this way, since we will mainly use (1.7) for $t < 0$ and then $d_t^2 > 0$.

The limiting behavior of the recurrence coefficients $a_{n,n}(t)$ as $n \to \infty$ is directly related to the support $[-c_t, c_t]$ of the equilibrium measure, since we have for $t \geq 0$ that

$$a_{n,n}(t) = \frac{c_t^2}{4} + \mathcal{O}(n^{-1}) = \frac{\sqrt{1+12t}-1}{6t} + \mathcal{O}(n^{-1}) \tag{1.8}$$

as $n \to \infty$. The asymptotics (1.8) follows from far more general results in [11] where it is also shown that $\mathcal{O}(1/n)$ term can be written as a full asymptotic expansion in powers of $1/n$.

1.2 Critical behavior of recurrence coefficients

As said before, we are going to consider the case $t < 0$. Although in this case the orthogonal polynomials are not well-defined by (1.2), the Freud equation (1.3) makes perfect sense. If $t \geq -1/12$ also the measure (1.6) with $c_t > 0$ and $d_t > 0$ given by (1.5) and (1.7) is well-defined and gives a probability measure on $\mathbb{R}$. This measure does not minimize (1.4) among all Borel probability measure on $\mathbb{R}$ (in fact there is no such minimizer), but it does minimize (1.4) among all Borel probability measures on $[-c_t, c_t]$, or among all Borel probability measures on the larger interval $[-d_t, d_t]$. The value $t = -1/12$ is critical, since for $t < -1/12$ the measure (1.6) is not well-defined anymore. Note that for $t = t_{cr} = -1/12$, we have $c_t^2 = d_t^2 = 8$ and we find the critical measure $\mu_{-1/12} = \mu_{cr}$ where

$$\frac{\mathrm{d}\mu_{cr}}{\mathrm{d}x} = \frac{1}{24\pi}(8 - x^2)^{3/2}, \qquad \text{for } x \in [-\sqrt{8}, \sqrt{8}]. \tag{1.9}$$

By formal calculations based on the Freud equation (1.3) it was shown in the physics literature, see e.g. the surveys [15, 16], that the recurrence coefficients $a_{n,n}(t)$ have very interesting limit behavior as $n \to \infty$ for t near the critical value $t_{cr} = -1/12$. Namely, if t depends on n and tends to t_{cr} as $n \to \infty$ in such a way that

$$n^{4/5}(t - t_{cr}) = -c_1 x \tag{1.10}$$

remains fixed, then

$$\lim_{n\to\infty} n^{2/5}(a_{n,n}(t) - 2) = c_2 y(x) \tag{1.11}$$

where $y(x)$ is a solution of the Painlevé I equation

$$y'' = 6y^2 + x. \tag{1.12}$$

In (1.10) and (1.11) we have that c_1 and c_2 are certain explicit positive constants.

In two very important papers [17, 18] Fokas, Its, and Kitaev were able to prove this result in a mathematically rigorous way. First of all they made it clear how the orthogonal polynomials with respect to $\mathrm{e}^{-NV_t(x)}$ should be defined in case $t < 0$. The solution is to consider the orthogonality not on $\mathbb{R}$ but on a contour Γ in the complex plane chosen so that $\mathrm{Re}\, V_t(z) \to +\infty$ as $z \to \infty$ along the contour. For symmetry reasons they chose a contour Γ that consists of the two lines parametrized by $z = r\mathrm{e}^{i\pi/4}$ and $z = r\mathrm{e}^{-i\pi/4}$ with $-\infty < r < \infty$. Then the orthogonality property for the polynomials π_n is

$$\int_\Gamma \pi_n(z) z^k \mathrm{e}^{-NV_t(z)}\mathrm{d}z = 0, \qquad \text{for } k = 0, 1, \ldots, n-1, \tag{1.13}$$

and the integrals are well-defined. Note that the bilinear form

$$\langle p, q\rangle = \int_\Gamma p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z \tag{1.14}$$

is not positive definite, so that (1.17) is an example of non-Hermitian orthogonality. As a result it is not automatic that a unique monic polynomial π_n of degree n exists that satisfies (1.13). However, it follows from the analysis of [17, 18] that in the asymptotic regime given by (1.10) the monic polynomials $\pi_{n+1,n,t}, \pi_{n,n,t}, \pi_{n-1,n,t}$ exist for n large enough, they satisfy a three-term recurrence relation with recurrence coefficient $a_{n,n}(t)$ that satisfies (1.11) for a uniquely specified solution $y(x)$ of the Painlevé I equation (1.12).

The analysis in [17, 18] is based on a characterization of the orthogonal polynomials by a Riemann-Hilbert problem. The Riemann-Hilbert problem admits a Lax pair formulation, and the Freud equation for the recurrence coefficients follows from the compatibility equation for the Lax pair. In [17] the Painlevé I asymptotic behavior is proved via a WKB analysis of the Riemann-Hilbert problem and the associated Lax pair. It is interesting to note that the Riemann-Hilbert problem for orthogonal polynomials was stated for the first time in [18]. It should also be noted that [18] is not restricted to the quartic potential (1.1) but more general polynomial potentials are considered as well.

Subsequent developments in the asymptotic analysis of Riemann-Hilbert problems made us consider this problem again. The main new development is the application of the powerful Deift/Zhou steepest descent analysis for Riemann-Hilbert problems [14] to the Riemann-Hilbert problem for orthogonal polynomials by Deift et al. [11, 12] which had a major impact on the asymptotic theory of orthogonal polynomials.

In this paper we give an alternative proof of the Painlevé I asymptotics using these new techniques. The proof we thus obtain is a direct proof, in the sense that the analysis only deals with the Riemann-Hilbert problem and not the Lax pair. We derive asymptotics for the recurrence coefficients without considering the Freud equation in an explicit way. For clarity of presentation we restrict ourselves to the quartic potential (1.1) but extension to more general potentials is possible. But even in the context of the quartic potential our method allows an extension to a more general model which we describe now.

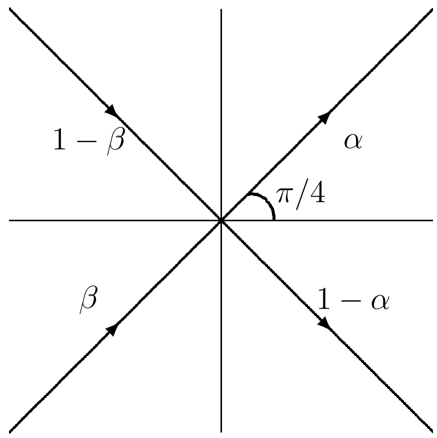

Figure 1: The contour Γ with the weights on the different parts of Γ

1.3 Statement of results

The orthogonality relation with parameters α and β

Instead of the bilinear form (1.14) we put weights α_i on the different parts of the contour Γ and consider a bilinear form

$$\langle p, q\rangle = \sum_{i=1}^{4} \alpha_i \int_{\Gamma_i} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z \tag{1.15}$$

where Γ_i is the part of Γ in the ith quadrant. Not all choices of weights are relevant. We impose $\alpha_1 + \alpha_4 = 1$ and $\alpha_2 + \alpha_3 = 1$. Then putting $\alpha = \alpha_1$ and $\beta = \alpha_3$ we obtain the situation indicated in Figure 1. The bilinear form then depends on the two parameters α and β and we denote it by $\langle p, q\rangle_{\alpha,\beta}$ so that

$$\begin{aligned}\langle p, q\rangle_{\alpha,\beta} = \alpha \int_{\Gamma_1} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z + (1-\beta) \int_{\Gamma_2} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z \\ + \beta \int_{\Gamma_3} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z + (1-\alpha) \int_{\Gamma_4} p(z)q(z)\mathrm{e}^{-NV_t(z)}\mathrm{d}z.\end{aligned} \tag{1.16}$$

The choice $\alpha = \beta$ corresponds to the situation considered by Fokas, Its, and Kitaev [18].

The orthogonal polynomial π_n of degree n is now defined by the relations

$$\langle \pi_n(z), z^k\rangle_{\alpha,\beta} = 0, \qquad \text{for } k = 0, 1, \ldots, n-1. \tag{1.17}$$

The polynomial clearly depends on α and β. It is again an example of non-Hermitian orthogonality, so that uniqueness and existence are not immediate. However if three consecutive monic polynomials π_{n+1}, π_n and π_{n-1} exist then they are connected by a three-term recurrence relation of the form

$$\pi_{n+1}(z) = (z - b_n)\pi_n(z) - a_n\pi_{n-1}(z) \tag{1.18}$$

where we now have two recurrence coefficients a_n and b_n. The b_n coefficient vanishes only if $\alpha = \beta$, since then the bilinear form is even. In that case a_n satisfies the Freud equation (1.3). For $\alpha \neq \beta$ the bilinear form is not even and we have the general three-term recurrence (1.18). Then (1.3) is not satisfied, but instead the a_n and b_n satisfy a more complicated system of nonlinear difference equations.

We use $a_{n,N}(t)$ and $b_{n,N}(t)$ to indicate the dependence on N and t. The recurrence coefficients also depend on α and β but we will not indicate that in the notation.

The regular case $-1/12 < t < 0$

We refer to the case where t is fixed with $-1/12 < t < 0$ as the regular case. In the regular case we will find that the asymptotic formula for $a_{n,n}(t)$ is a straightforward continuation of the one valid for $t \geq 0$, see (1.8). In addition, the recurrence coefficient $b_{n,n}(t)$ is exponentially small. This is our first result.

Theorem 1.1. *Let $-1/12 < t < 0$ be fixed. Then there exists an $n_0 = n_0(t)$ such that the recurrence coefficients $a_{n,n}(t)$ and $b_{n,n}(t)$ exist for all $n \geq n_0$, and*

$$a_{n,n}(t) = \frac{\sqrt{1+12t} - 1}{6t} + \mathcal{O}(1/n), \qquad n \to \infty, \tag{1.19}$$

and there exists a constant $C_t > 0$ such that

$$b_{n,n}(t) = \mathcal{O}(\exp(-C_t n)), \qquad n \to \infty. \tag{1.20}$$

The recurrence coefficient $a_{n,n}(t)$ has a full asymptotic expansion in powers of n^{-1} and all terms in the asymptotic expansion are independent of the choice of α and β.

The proof of Theorem 1.1 is given in Section 3.

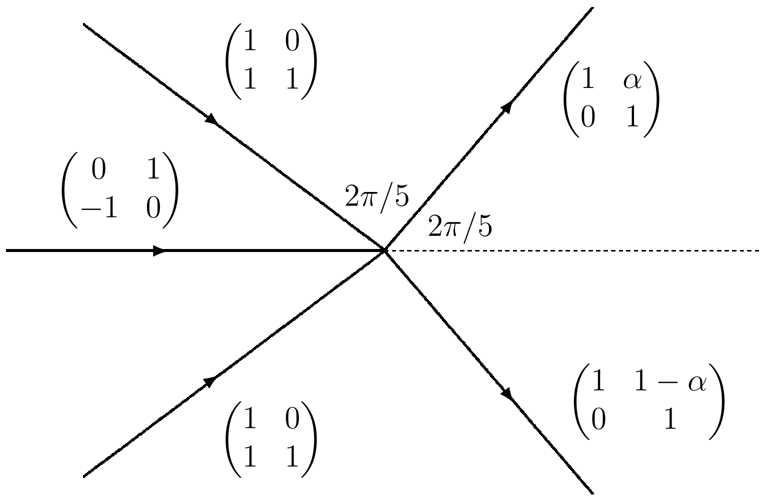

Figure 2: Jump matrices for the Ψ-function associated to the solution y_α of the Painlevé I equation

The critical case

The main purpose of the present paper is to deal with the critical case where t depends on n such that $t \to -1/12$ as $n \to \infty$ at a critical rate. It is only in the critical case that the dependence on the parameters α and β plays a role. In our main result (Theorem 1.2 below) we will give asymptotic formulas for $a_{n,n}(t)$ and $b_{n,n}(t)$ as $n \to \infty$ in terms of special solutions y_α and y_β of the Painlevé I equation (1.12). We describe these special solutions first.

Solutions y_α of the Painlevé I equation

To explain how the solution y_α depends on α we recall the Riemann-Hilbert problem associated with Painlevé I. Let $\Psi(\zeta; x)$ be a 2×2 matrix valued function that is analytic in the complex ζ-plane except for jumps on the contours $\arg \zeta = \pm 2\pi/5$, $\arg \zeta = \pm 4\pi/5$, and $\arg \zeta = \pi$. The contours are oriented as in Figure 2. The orientation induces a $+$ and $-$ side on each part of the contour, where the $+$ side ($-$ side) is on the left (right) if one traverses the contour according to the orientation. For ζ on the contour the limiting values $\Phi_\pm(\zeta;x)$ from both sides exist and are connected by

$$\Psi_+(\zeta;x) = \Psi_-(\zeta;x)A_\alpha$$

where the jump matrices A_α are given in Figure 2. Furthermore, Ψ has asymptotics

$$\Psi(\zeta;x) = \frac{\zeta^{\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & -\mathrm{i}\\ 1 & \mathrm{i}\end{pmatrix}\left(I + \frac{\Psi_1(x)}{\zeta^{1/2}} + \frac{\Psi_2(x)}{\zeta} + \mathcal{O}(\zeta^{-3/2})\right)\mathrm{e}^{\theta(\zeta,x)\sigma_3}, \quad (1.21)$$

for $\zeta \to \infty$, where $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and θ is defined by

$$\theta(\zeta,x) = \frac{4}{5}\zeta^{5/2} + x\zeta^{1/2}. \quad (1.22)$$

Then it is known that

$$y_\alpha(x) = 2\mathrm{i}\left(\Psi_2(x)\right)_{12} \quad (1.23)$$

is a solution of the Painlevé I equation, and it is this solution y_α that will appear in our main result below.

As is the case for any solution of the Painlevé I equation we have that y_α is a meromorphic function with an infinite number of poles in the complex plane. The Riemann-Hilbert problem for Ψ has a solution if and only if x is not a pole of y_α.

The above Riemann-Hilbert problem does not generate all solutions of the Painlevé I equation, but only those solutions that satisfy

$$y(x) = \sqrt{-x/6}(1 + o(1)) \qquad \text{as } x \to -\infty. \quad (1.24)$$

All y_α satisfy (1.24) and they have a common asymptotic series

$$y_\alpha(x) \sim \sqrt{-x/6}\left[1 + \sum_{k=1}^{\infty} a_k(-x)^{-5k/2}\right] \qquad \text{as } x \to -\infty \quad (1.25)$$

for certain coefficients a_k.

For the special value $\alpha = 1$ we have that the asymptotic series (1.25) is valid as $|x| \to \infty$ with $\arg x \in [3\pi/5, \pi]$. The other solutions differ from y_1 by exponentially small terms only. It follows from results of Kapaev [21] that

$$y_\alpha(x) = y_1(x) - \frac{\mathrm{i}(\alpha - 1)}{\sqrt{\pi}2^{11/8}3^{1/8}(-x)^{1/8}}\mathrm{e}^{-\frac{1}{5}2^{11/4}3^{1/4}(-x)^{5/4}}\left(1 + \mathcal{O}(x^{-3/8})\right) \quad (1.26)$$

as $x \to -\infty$. The behavior (1.26) characterizes y_α.

Main result

Our main result is the following theorem.

Theorem 1.2. *Let $\alpha, \beta \in \mathbb{C}$, and let t vary with n such that*

$$n^{4/5}(t + 1/12) = -c_1 x, \qquad c_1 = 2^{-9/5} 3^{-6/5}, \tag{1.27}$$

remains fixed, where x is not a pole of y_α and y_β. Then, for large enough n, the recurrence coefficients $a_{n,n}(t)$ and $b_{n,n}(t)$ associated with the orthogonal polynomials with respect to the bilinear form (1.16) *exist and they satisfy*

$$a_{n,n}(t) = 2 - c_2 \left(y_\alpha(x) + y_\beta(x)\right) n^{-2/5} + \mathcal{O}(n^{-3/5}), \qquad c_2 = 2^{3/5} 3^{2/5}, \tag{1.28}$$

and

$$b_{n,n}(t) = c_3 \left(y_\beta(x) - y_\alpha(x)\right) n^{-2/5} + \mathcal{O}(n^{-3/5}), \qquad c_3 = 2^{1/10} 3^{2/5}, \tag{1.29}$$

as $n \to \infty$. The expansions (1.28) *and* (1.29) *hold uniformly for x in compact subsets of $\mathbb{R}$ not containing any of the poles of y_α and y_β, and the $\mathcal{O}$ terms can be expanded into a full asymptotic expansion in powers of $n^{-1/5}$.*

Remark 1.3. It is quite remarkable that we do not have a term of order $n^{-1/5}$ in (1.28) and (1.29). The terms in these expansions ultimately come from the terms in the asymptotic expansion (1.21) of $\Psi(\zeta; x)$, as we will show. Terms of order $n^{-1/5}$ could have appeared because of the term $\Psi_1(x)/\zeta^{1/2}$ in (1.21). However, it turns out that the entries of $\Psi_1(x)$ cancel out in the calculations. These entries also do not contribute to the $n^{-2/5}$ term.

Remark 1.4. Not much is known about the precise location of the poles of y_α on the real line, however see [8, 20]. Because of (1.24) we know that there can be only a finite number of poles on the negative real axis. Joshi and Kitaev [20, Prop. 3] showed that every real valued solution of (1.12) has at least one pole on the positive real axis, so this applies in particular to our solutions y_α with Re $\alpha = 1/2$.

Remark 1.5. Even though we will concentrate on the quartic potential, it will be obvious that our method generalizes to higher degree potentials. The quartic potential serves as a generic example for all polynomial potentials for which the density of an associated critical equilibrium measure vanishes with an exponent 3/2 at the endpoints of the support as in (1.9).

Note that this type of vanishing is not possible in the case of usual orthogonality with respect to varying exponential weights on the real line, since then it is only possible that the density of the equilibrium measure vanishes at an endpoint with an exponent $(4k+1)/2$ with $k \in \mathbb{N}_0$. See [10, 11] for more details. The generic case $k = 0$ leads to Airy functions. The first critical case $k = 1$ is described in terms of the second member of the Painlevé I hierarchy [7].

Remark 1.6. The general symmetric quartic potential $V(x) = gx^2/2 + tx^4/4$ exhibits another type of critical behavior for $g < 0$ and $t > 0$. Here the zeros of the orthogonal polynomial π_n with respect to $\mathrm{e}^{-nV(x)}$ may accumulate either on one or on two intervals, depending on the values of g and t, and the same holds true for the eigenvalues of the unitary random matrix ensemble $(1/Z_n)\mathrm{e}^{-n \operatorname{Tr} V(M)} dM$ on $n \times n$ Hermitian matrices M. Bleher and Its [2, 3] showed that the transition is described by the Hastings-Mcleod solution of the Painlevé II equation. This result was generalized in [5, 6] to more general potentials.

Overview of the rest of the paper

The proofs of Theorems 1.1 and 1.2 are based on a steepest descent analysis of the RH problem that characterizes the orthogonal polynomial π_n. This RH problem is due to Fokas, Its, and Kitaev [18]. We discuss it in detail in Section 2.

In Section 3 we prove Theorem 1.1. After the appropriate deformation of contours, the steepest descent analysis is similar to the analysis in Deift et al. [11] and therefore we will not give all the details of the proof here.

The major part of the paper is devoted to the proof of Theorem 1.2 which is given in Section 4. Here the critical measure (1.9) corresponding to the critical value $t = -1/12$ comes into play. For $t \neq -1/12$, we will not use the equilibrium measure μ_t from (1.6) in the steepest descent analysis, but instead a modified equilibrium measure ν_t (in fact a signed measure). Similar modified equilibrium measures were used before in [4, 5, 6] for double scaling limits arising in critical random matrix models. The modified equilibrium problem is discussed in Section 4.2. It leads to a modified g-function [13, 11] which is used in the first transformation of the RH problem.

The steepest descent analysis then proceeds as in [11]. We open lenses around the critical interval $[-\sqrt{8}, \sqrt{8}]$ and then construct a parametrix for the

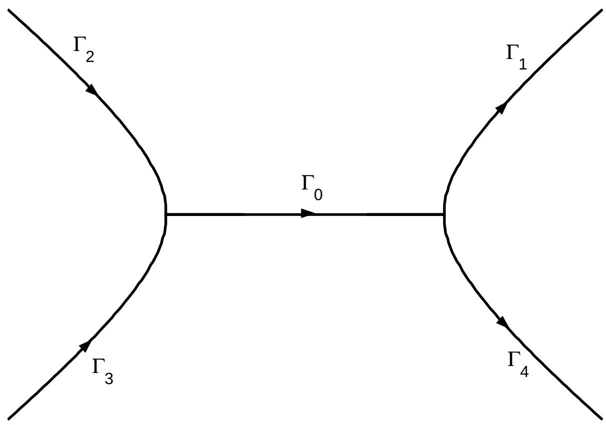

Figure 3: Deformation of the contour

resulting RH problem. The Ψ-functions associated with Painlevé I appear in the construction of the local parametrices around the endpoints $\pm\sqrt{8}$. This construction is analogous to the construction in [5, 6], where a local parametrix was built out of the Ψ-functions associated with Painlevé II. The final transformation of the RH problem is given in Section 4.8. It leads to a RH problem which for large n can be solved explicitly in a series involving powers of $n^{-1/5}$. We calculate the terms up to order $n^{-2/5}$ explicitly. We also show how the recurrence coefficients can be expressed in terms of the final RH problem and then the final computations in Section 4.11 lead to the proof of Theorem 1.2.

2 The Riemann-Hilbert problem

By analyticity, a deformation of the contour Γ does not change the polynomial π_n as defined in (1.17), provided that each part Γ_j extends to infinity in the sector

$$S_j = \{z \in \mathbb{C} \mid (4j-3)\pi/8 < \arg(z) < (4j-1)\pi/8\}. \tag{2.1}$$

Figure 3 shows a particular deformation of Γ, which will turn out to be convenient for our analysis. The contours are deformed so that they contain the part $[-d_t, d_t]$ on the real axis where d_t is defined by (1.7). At the points

$\pm d_t$ the contours separate and extend to infinity in the sectors S_j. The precise setting of this deformation will be given later. Changing our previous notation, we will now use $\Gamma_1, \dots, \Gamma_4$ to denote the contours as in Figure 3. We also put $\Gamma_0 = [-d_t, d_t]$. The contours are oriented as indicated in Figure 3.

The parts $\Gamma_1, \dots, \Gamma_4$ carry the weights as before, and Γ_0 has the weight 1. If then for $z \in \Gamma$ we define

$$\alpha(z) = \begin{cases} 1, & z \in \Gamma_0, \\ \alpha, & z \in \Gamma_1, \\ 1-\beta, & z \in \Gamma_2, \\ \beta, & z \in \Gamma_3, \\ 1-\alpha, & z \in \Gamma_4, \end{cases}$$

then for polynomials p and q, the bilinear form (1.15) is equal to

$$\langle p, q \rangle_{\alpha,\beta} = \int_\Gamma p(z) q(z) \alpha(z) \mathrm{e}^{-NV_t(z)} \mathrm{d}z. \tag{2.2}$$

Consider the following Riemann-Hilbert (RH) problem for a 2×2 matrix-valued function $Y : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$.

$$\begin{cases} Y(z) \text{ is analytic in } \mathbb{C} \setminus \Gamma \\ Y_+(z) = Y_-(z) \begin{pmatrix} 1 & \alpha(z)\mathrm{e}^{-NV_t(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma \\ Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, & z \to \infty, \end{cases} \tag{2.3}$$

By a standard argument one can show that if a solution to the RH problem (2.3) exists, it is unique, see [9, 12]. The existence of Y depends on the existence of polynomials that are orthogonal with respect to the bilinear form (2.2). To be precise, we have

Proposition 2.1. *The RH problem* (2.3) *has a solution if and only if there exist polynomials p and q such that*

(a) *p is monic of degree n and $\langle p(z), z^k \rangle_{\alpha,\beta} = 0$ for $k = 0, 1, \dots, n-1$,*

(b) *q has degree $\leq n-1$, $\langle q(z), z^k \rangle_{\alpha,\beta} = 0$ for $k = 0, 1, \dots, n-2$, and $\langle q(z), z^{n-1} \rangle_{\alpha,\beta} = -2\pi \mathrm{i}$.*

In that case the solution of the RH problem is given by

$$Y(z) = \begin{pmatrix} p(z) & \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_\Gamma \frac{p(s)\alpha(s)\mathrm{e}^{-NV_t(s)}}{s-z} ds \\ q(z) & \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_\Gamma \frac{q(s)\alpha(s)\mathrm{e}^{-NV_t(s)}}{s-z} ds \end{pmatrix}, \qquad z \in \mathbb{C} \setminus \Gamma. \tag{2.4}$$

Proof. The proof is standard, see [18] or [9]. □

Note that the non-Hermitian orthogonality is exactly the right concept for the formulation of a RH problem. This has been exploited before in for example [1, 23, 24].

It follows from Proposition 2.1 that if Y exists, then $p = Y_{11} = \pi_n$ and so the monic orthogonal polynomial π_n exist. The polynomial $q = Y_{21}$ satisfies the orthogonality conditions for π_{n-1}. If q has degree $n-1$, then the monic orthogonal polynomial π_{n-1} exist, and there is a constant $\kappa_{n-1} \neq 0$ such that $q = Y_{21} = \kappa_{n-1}\pi_{n-1}$. It might happen that q has degree $< n-1$, and then π_{n-1} does not exist.

We use $Y^{(n+1)}$, $Y^{(n)}$, $Y^{(n-1)}$ to denote the solutions of the RH problem (2.3) for the values $n+1$, n, and $n-1$, respectively (with fixed N). If all three RH problems have a solution then π_{n+1}, π_n and π_{n-1} exist. We show that they are related by a three term recurrence relation.

Proposition 2.2. *Suppose that the monic orthogonal polynomials π_{n+1}, π_n, π_{n-1} exist. Then they satisfy*

$$\pi_{n+1}(z) = (z - b_n)\pi_n - a_n \pi_{n-1}(z) \tag{2.5}$$

for certain coefficients a_n and b_n.

Proof. From Proposition (2.1) it follows that the RH problems for $Y^{(n)}$ and $Y^{(n+1)}$ are solvable. From the discussion above we find that the first column of $Y^{(n)}$ consists of $\begin{pmatrix} \pi_n \\ \kappa_{n-1}\pi_{n-1} \end{pmatrix}$ for some non-zero constant κ_{n-1}.

Since the jump matrices in the RH problem (2.3) have determinant one, and $\det Y^{(n)}(z) \to 1$ as $z \to \infty$, it is easy to check that $\det Y^{(n)}(z) \equiv 1$ for all $z \in \mathbb{C} \setminus \Gamma$. Thus $Y^{(n)}$ is invertible. Consider $Y^{(n+1)}\left(Y^{(n)}\right)^{-1}$. Since $Y^{(n)}$ and $Y^{(n+1)}$ have the same jumps, this function has no jump on Γ. Therefore its entries are entire functions. Moreover, from the asymptotic condition at infinity it follows by Liouville's theorem that there exist constants a_n, b_n and c_n such that

$$Y^{(n+1)}(z)\big(Y^{(n)}(z)\big)^{-1} = \begin{pmatrix} z-b_n & -a_n/\kappa_{n-1} \\ c_n & 0 \end{pmatrix}. \tag{2.6}$$

So

$$Y^{(n+1)}(z) = \begin{pmatrix} z-b_n & -a_n/\kappa_{n-1} \\ c_n & 0 \end{pmatrix} Y^{(n)}(z) \tag{2.7}$$

and this is in fact the difference equation in the Lax pair in [18]. The 11–entry of equation (2.7) leads to (2.5). □

The recurrence coefficients a_n and b_n can be expressed directly in terms of the solution $Y^{(n)}$ of the RH problem. This follows as in [9, Section 3.2]. For convenience of the reader, and since the precise form (2.10) for the recurrence coefficient b_n is not given in [9], we include the proof of the following proposition as well.

Proposition 2.3. *If we write*

$$Y^{(n)}(z) = \left(I + Y_1^{(n)}/z + Y_2^{(n)}/z^2 + \mathcal{O}(z^{-3})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \tag{2.8}$$

as $z \to \infty$, *then*

$$a_n = \left(Y_1^{(n)}\right)_{12} \left(Y_1^{(n)}\right)_{21}, \tag{2.9}$$

and

$$b_n = \frac{\left(Y_2^{(n)}\right)_{12}}{\left(Y_1^{(n)}\right)_{12}} - \left(Y_1^{(n)}\right)_{22}. \tag{2.10}$$

Proof. As in the proof of Proposition 2.2 we have that $\big(Y^{(n)}\big)_{21}$ is a polynomial of degree $n-1$ with leading coefficient κ_{n-1}. Thus from (2.8) we obtain

$$\kappa_{n-1} = \left(Y_1^{(n)}\right)_{21}. \tag{2.11}$$

Multiplying both sides of the identity (2.7) with $z^{-n\sigma_3}$ and using (2.8), we find

$$(I + \mathcal{O}(1/z)) \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} z-b_n & -a_n/\kappa_{n-1} \\ c_n & 0 \end{pmatrix} \left(I + \frac{Y_1^{(n)}}{z} + \frac{Y_2^{(n)}}{z^2} + \mathcal{O}(z^{-3})\right)$$

as $z \to \infty$. Taking the 12-entries on both sides, we obtain

$$\mathcal{O}(z^{-2}) = \left(Y_1^{(n)}\right)_{12} - \frac{a_n}{\kappa_{n-1}} + z^{-1}\left(\left(Y_2^{(n)}\right)_{12} - b_n\left(Y_1^{(n)}\right)_{12} - \frac{a_n}{\kappa_{n-1}}\left(Y_1^{(n)}\right)_{22}\right) + \mathcal{O}(z^{-2}).$$

Thus

$$\left(Y_1^{(n)}\right)_{12} = \frac{a_n}{\kappa_{n-1}} \tag{2.12}$$

and

$$\left(Y_2^{(n)}\right)_{12} - b_n\left(Y_1^{(n)}\right)_{12} - \frac{a_n}{\kappa_{n-1}}\left(Y_1^{(n)}\right)_{22} = 0. \tag{2.13}$$

Then (2.11) and (2.12) lead to (2.9), while we obtain (2.10) from solving (2.13) for b_n and using (2.12). □

Proposition 2.3 shows that we can compute the recurrence coefficients from the RH problem for $Y^{(n)}$ alone. In our final proposition of this section we show that if we compute a_n and b_n as in (2.9) and (2.10) and if $a_n \neq 0$, then these are indeed the recurrence coefficients.

Proposition 2.4. *Suppose that the RH problem* (2.3) *has a solution* $Y^{(n)}$ *with expansion* (2.8)*. Let* a_n *and* b_n *be given by* (2.9) *and* (2.10)*. If* $a_n \neq 0$*, then the monic orthogonal polynomials* π_{n+1}*,* π_n *and* π_{n-1} *exist and they are connected by the three-term recurrence* (2.5)*.*

Proof. Since $a_n \neq 0$, we get from (2.9) that

$$\kappa_{n-1} = \left(Y_1^{(n)}\right)_{21} \neq 0. \tag{2.14}$$

This means that the 21-entry of Y is a polynomial of exact degree $n-1$ and therefore the orthogonal polynomial π_{n-1} exists.

Define

$$c_n = \kappa_{n-1}/a_n = \left((Y_1^{(n)})_{12}\right)^{-1} \tag{2.15}$$

and use (2.7) to define $Y^{(n+1)}$. Then $Y^{(n+1)}$ is analytic in $\mathbb{C} \setminus \Gamma$ and satisfies the jump relation of the RH problem (2.3). We show that it satisfies the asymptotic condition with n replaced by $n+1$. We have

$$Y^{(n+1)}(z)\begin{pmatrix} z^{-n-1} & 0 \\ 0 & z^{n+1}\end{pmatrix} = \begin{pmatrix} z-b_n & -a_n/\kappa_{n-1} \\ c_n & 0\end{pmatrix}\left(I + \frac{Y_1^{(n)}}{z} + \frac{Y_2^{(n)}}{z^2} + \mathcal{O}(z^{-3})\right)\begin{pmatrix} z^{-1} & 0 \\ 0 & z\end{pmatrix}$$

as $z \to \infty$. Collecting terms according to powers of z, we find

- there is one term with z^2 which comes with coefficient $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = O$.
- the terms with z come with coefficients

  $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y_1^{(n)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -b_n & -a_n/\kappa_{n-1} \\ c_n & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & (Y_1^{(n)})_{12} - a_n/\kappa_{n-1} \\ 0 & 0 \end{pmatrix}$$

  and this is O because of (2.9) and (2.14).
- the constant terms are

  $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -b_n & -a_n/\kappa_{n-1} \\ c_n & 0 \end{pmatrix} Y_1^{(n)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y_2^{(n)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
  $$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -b_n (Y_1^{(n)})_{12} - a_n/\kappa_{n-1} (Y_1^{(n)})_{22} \\ 0 & c_n (Y_1^{(n)})_{12} \end{pmatrix} + \begin{pmatrix} 0 & (Y_2^{(n)})_{12} \\ 0 & 0 \end{pmatrix}$$

  and using (2.9), (2.10), (2.14), and (2.15), we find that this is the identity matrix I.

It follows from all this that

$$Y^{(n+1)}(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^{n+1} & 0 \\ 0 & z^{-n-1} \end{pmatrix}$$

and so $Y^{(n+1)}$ solves the RH problem (2.3) with n replaced by $n+1$. Then the monic orthogonal polynomial π_{n+1} exists and the recurrence (2.5) holds. □

In our main results we are interested in the asymptotic behavior of $a_{n,n}(t)$ and $b_{n,n}(t)$. As a result of Propositions 2.3 and 2.4 it suffices to take $N = n$ in (2.3) and do a steepest descent analysis for this RH problem.

3 Steepest descent analysis in the regular case and the proof of Theorem 1.1

In the regular case $-1/12 < t < 0$ with t fixed, the steepest descent analysis for the RH problem (2.3) with $N = n$ can be done in a way very similar to the classical steepest analysis for $t \geq 0$ as in [12, 11].

In the first transformation we use the continuation of the equilibrium measure that we discussed in Section 1 for $t > 0$, see (3.1). The local parametrix at the end-points is constructed out of Airy functions, as in the case $t \geq 0$. Therefore the analysis will not be carried out in full detail. We only point out how one gets into a standard situation, by deforming the contour and defining the correct equilibrium measure. Then we only sketch the rest of the analysis and state the result for the recurrence coefficients.

The equilibrium measure

The first transformation is based on the equilibrium measure μ_t defined in (1.6). So we have

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}x} = \frac{t}{2\pi}(x^2 - d_t^2)\sqrt{c_t^2 - x^2}, \qquad \text{for } x \in [-c_t, c_t], \tag{3.1}$$

with c_t and d_t given by (1.5) and (1.7). Since $-1/12 < t < 0$, we have that d_t is real and $0 < c_t < d_t$. Also note that the density (3.1) is positive for $x \in (c_t, c_t)$. The measure μ_t minimizes $I_{V_t}(\mu)$, see (1.4), among all probability measures supported on $[-c_t, c_t]$.

Let $g_t : \mathbb{C} \setminus (-\infty, c_t] \to \mathbb{C}$ be the g-function defined by

$$g_t(z) = \int_{-c_t}^{c_t} \log(z - x) \, \mathrm{d}\mu_t(x), \tag{3.2}$$

where for each $x \in (-c_t, c_t)$ we choose the principal branch of the logarithm $\log(z - x)$. Then g_t can be represented in the following way

$$g_t(z) = -\frac{t}{2} \int_{c_t}^{z} (s^2 - d_t^2)(s^2 - c_t^2)^{1/2} \, \mathrm{d}s + \frac{1}{2} V_t(z) + l_t/2, \tag{3.3}$$

for all $z \in \mathbb{C} \setminus (-\infty, c_t]$ and for some constant l_t. Define $\phi_t : \mathbb{C} \setminus (-\infty, c_t] \to \mathbb{C}$ by

$$\phi_t(z) = -\frac{t}{2} \int_{c_t}^{z} (s^2 - d_t^2)(s^2 - c_t^2)^{1/2} \, \mathrm{d}s. \tag{3.4}$$

In the first transformations the jumps for the transformed RH problem are expressed in terms of ϕ_t.

Note that the integrand in (3.4) is negative for $s \in (c_t, d_t)$. Therefore

$$\phi_t(z) < 0, \qquad \text{for } z \in (c_t, d_t). \tag{3.5}$$

By symmetry we have

$$\phi_{t+}(z) + \phi_{t-}(z) < 0, \qquad \text{for } z \in (-d_t, -c_t). \tag{3.6}$$

Then it follows from (3.3), (3.5) and (3.6) that

$$\begin{cases} g_{t+}(z) + g_{t-}(z) - V_t(z) = l_t, & z \in (-c_t, c_t), \\ g_{t+}(z) + g_{t-}(z) - V_t(z) < l_t, & z \in (-d_t, -c_t) \cup (c_t, d_t). \end{cases} \tag{3.7}$$

The transformation $Y \mapsto T$

The transformation $Y \mapsto T$ normalizes the condition at infinity and serves as a first step to get the jumps close to the identity. Define $T : \mathbb{C} \setminus \Gamma$ by

$$T(z) = \mathrm{e}^{-nl_t\sigma_3/2} Y(z) \mathrm{e}^{-ng_t(z)\sigma_3/2} \mathrm{e}^{nl_t\sigma_3/2}, \tag{3.8}$$

for all $z \in \mathbb{C} \setminus \Gamma$. Then T satisfies the following RH problem

$$\begin{cases} T(z) \text{ is analytic in } \mathbb{C} \setminus \Gamma \\ T_+(z) = T_-(z) \begin{pmatrix} \mathrm{e}^{-n(g_{t+}(z)-g_{t-}(z))} & \alpha(z)\mathrm{e}^{-n(V_t(z)+l_t-g_{t+}(z)-g_{t-}(z))} \\ 0 & \mathrm{e}^{n(g_{t+}(z)-g_{t-}(z))} \end{pmatrix}, & z \in \Gamma \\ T(z) = I + \mathcal{O}(1/z), & z \to \infty. \end{cases} \tag{3.9}$$

Since g_t is analytic in $\mathbb{C} \setminus (-\infty, c_t]$, one might expect a jump on $(-\infty, -d_t)$. However, since $g_{t+}(z) - g_{t-}(z) = 2\pi\mathrm{i}$ for $z \in (-\infty, -d_t)$, the function $\mathrm{e}^{n(g_{t+}(z)-g_{t-}(z))}$ is analytic for $z \in (-\infty, -d_t)$ and therefore there is no jump for S on $(-\infty, -d_t)$.

Now from (3.3) and (3.4) it follows that the jump matrix can be rewritten as

$$T_+(z) = T_-(z) \begin{cases} \begin{pmatrix} 1 & \alpha(z)\mathrm{e}^{2n\phi_t(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma \setminus [-c_t, c_t], \\ \begin{pmatrix} \mathrm{e}^{-2n\phi_{t+}(z)} & 1 \\ 0 & \mathrm{e}^{-2n\phi_{t-}(z)} \end{pmatrix}, & z \in (-c_t, c_t). \end{cases} \tag{3.10}$$

For $z \in (-c_t, c_t)$ we get from (3.4) that $\phi_{t\pm}(z) \in \mathrm{i}\mathbb{R}$. The diagonal entries of the jump matrix on $(-c_t, c_t)$ are therefore oscillating functions. By (3.5) and (3.6) the jump matrix converges pointwise to the identity for $z \in (-d_t, -c_t) \cup (c_t, d_t)$. We first have to specify the precise deformation of the contour to control the behavior of the jump matrices on $\Gamma_1, \ldots, \Gamma_4$. We will deform the contour such that the jump matrix converges pointwise to the identity for $z \in \Gamma \setminus [-c_t, c_t]$.

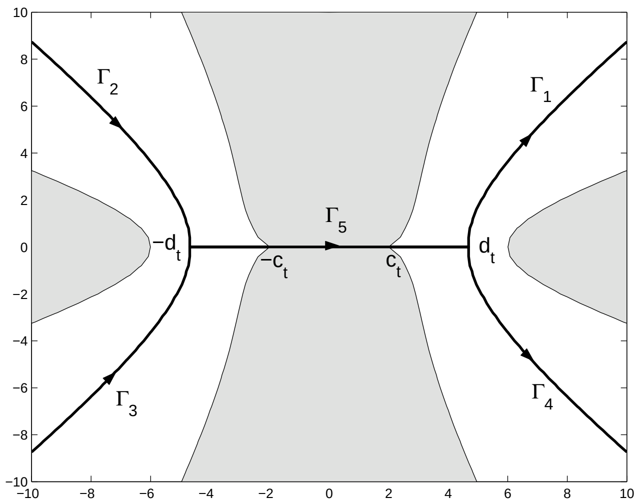

Figure 4: The shaded region is the region where Re $\phi_t > 0$ in the regular case. The curves are steepest descent curves for ϕ_t. The figure corresponds to the value $t = -1/24$.

Deformation of the contour

In view of the jump matrix (3.10) on $[-c_t, c_t]$ we want that the real part of ϕ_t is negative on $\Gamma \setminus [-c_t, c_t]$. The shaded region in Figure 4 is the region where Re $\phi_t(z) > 0$. In the white unshaded region we have Re $\phi_t(z) < 0$. So we want that each Γ_j stays in the white region. One possible way to define Γ_j is to take the steepest descent curve for ϕ_t through $\pm d_t$ in the k-th quadrant, where we have that Im ϕ_t is constant, and Re $\phi_t(z) \to -\infty$ as $z \to \infty$ along the steepest descent curve, see Figure 4.

Summary of the rest of the steepest descent analysis

In the remaining part of the steepest descent analysis we open a lens around $[-c_t, c_t]$ to turn the oscillating jumps on $[-c_t, c_t]$ into a constant jump at $[-c_t, c_t]$ and a jump that converges pointwise to the identity on the upper and lower lips of the lens. In Figure 5 the contour is shown after opening of the lens. This lens can be defined in a completely similar way as in the case $t \geq 0$. After this, we construct a local solution, called parametrix, around the endpoints $\pm c_t$. But locally around $\pm c_t$, our situation does not differ from the case $t \geq 0$: the equilibrium measure vanishes with an exponent 1/2 at

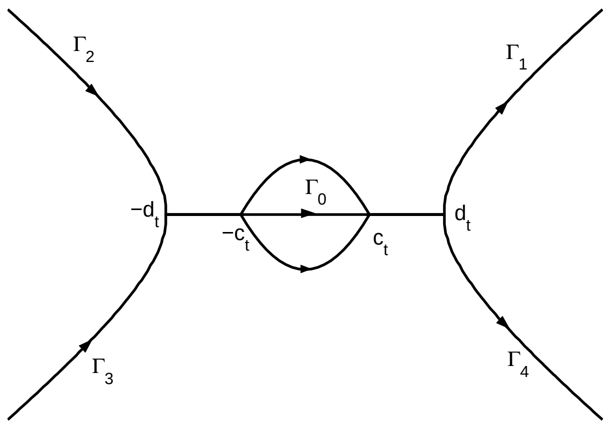

Figure 5: Opening of the lenses in the regular case. The lens around $[-c_t, c_t]$ is contained in the shaded region of Figure 4

$\pm c_t$ and the contours together with the jump matrices locally have exactly the same shape as in the case $t \geq 0$. So the parametrix is constructed by means of Airy functions as in [11].

Proof of Theorem 1.1

The result of the steepest descent analysis, which we will not perform in more detail here, is that the RH problem for $Y^{(n)}$ is solvable if n is large enough, so that the coefficients $a_{n,n}(t)$ and $b_{n,n}(t)$ can be computed in terms of $Y^{(n)}$ as in Proposition 2.3. It also follows that the asympotic formula (1.8) for the recurrence coefficient $a_{n,n}(t)$ which holds for $t \geq 0$ continues to hold for $-1/12 < t < 0$. Then $a_{n,n}(t) \neq 0$ for n large enough, so that $a_{n,n}(t)$ and $b_{n,n}(t)$ are indeed the recurrence coefficients by Proposition 2.4. As in [11] we also find a full asymptotic expansion for $a_{n,n}(t)$ in powers of n^{-1}. In addition, the recurrence coefficient $b_{n,n}(t)$ is exponentially small.

This completes the proof of Theorem 1.1.

4 Steepest descent analysis in the critical case and the proof of Theorem 1.2

The rest of the paper is devoted to the critical case. We take $N = n$ in (2.3) and analyze the RH problem in the double scaling limit. First we let t be a fixed value close to $-1/12$, which could be less than $-1/12$. Later we let $t \to -1/12$ and $n \to \infty$ simultaneously and show that the Painlevé I equation appears.

In the critical case the measure μ_{cr} is given by

$$\frac{\mathrm{d}\mu_{cr}}{\mathrm{d}x}(x) = \frac{(8-x^2)^{3/2}}{24\pi}, \tag{4.1}$$

for $x \in [-\sqrt{8}, \sqrt{8}]$. The measure vanishes with an exponent $3/2$ at $x = \pm\sqrt{8}$. A consequence of this phenomenon is that the parametrix with Airy functions which we could use in the regular case fails to work in this case.

4.1 Deformation of the contour

As in the regular case we begin by deforming the contour Γ. The deformed contour is constructed in terms of ϕ_{cr} and will therefore not depend on t. Here ϕ_{cr} is

$$\phi_{cr}(z) = \phi_{-1/12}(z) = \frac{1}{24}\int_{c_{cr}}^{z} (s^2-8)^{3/2}\mathrm{d}s \tag{4.2}$$

and $c_{cr} = \sqrt{8}$.

The shaded region in Figure 6 represents all $z \in \mathbb{C}$ for which $\mathrm{Re}\ \phi_{cr}(z) \geq 0$. The white region represents all $z \in \mathbb{C}$ for which $\mathrm{Re}\ \phi_{cr}(z) < 0$. As in the regular case, we want the Γ_j to be contained in the white region. Comparing with Figure 4 we see that the shaded parts meet at $\pm\sqrt{8}$, which is a direct consequence of the exponent $3/2$ vanishing of the measure at $\pm c_{cr} = \pm\sqrt{8}$. Hence a major difference with the regular case is the fact that the contours have to leave the real axis at the point $\pm c_{cr} = \pm\sqrt{8}$ to continue in the complex plane, whereas in the regular case we can deform the contour such that it contains the interval $[-d_t, d_t]$ with $d_t > c_t$.

Now define Γ_j, $j = 1, \ldots, 4$ as the steepest descent curve for ϕ_{cr} through $\pm\sqrt{8}$ in the j-th quadrant, where we have that $\mathrm{Im}\ \phi_{cr}$ is constant, and $\mathrm{Re}\ \phi_{cr}(z) \to -\infty$ as $z \to \infty$ along the steepest descent curves. We also put $\Gamma_0 = [-\sqrt{8}, \sqrt{8}]$, see Figure 6.

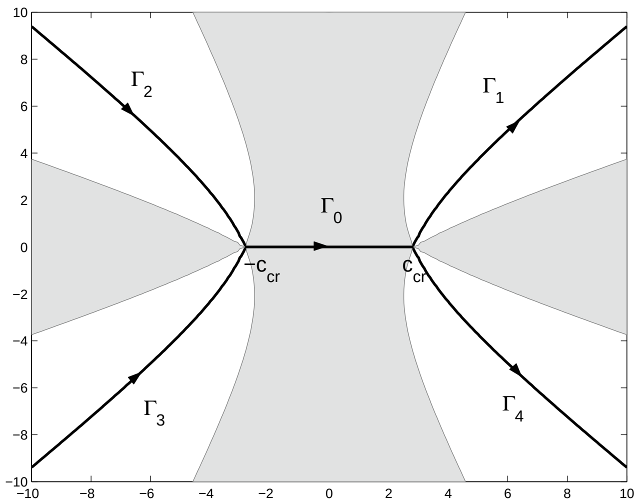

Figure 6: The shaded region is the region where Re $\phi_{cr} > 0$. The curves $\Gamma_1, \ldots, \Gamma_4$ are the steepest descent curves for ϕ_{cr} that contain $\pm c_{cr} = \pm\sqrt{8}$.

4.2 Modified equilibrium problem

An important feature of our analysis is the use of a modified equilibrium problem. This is inspired on the papers [5, 6], see also [4], where the authors used a modification of a similar equilibrium problem to handle a double scaling limit.

Recall the energy functional I_{V_t} as defined in (1.4). Instead of minimizing I_{V_t} among Borel probability measures on $\mathbb{R}$, we consider the following extremal problem. Minimize

$$I_{V_t}(\nu) = \iint \log \frac{1}{|x-y|} \, \mathrm{d}\nu(x) \, \mathrm{d}\nu(y) + \int V_t(x) \, \mathrm{d}\nu(x) \tag{4.3}$$

among all *signed* measures ν such that

$$\int \mathrm{d}\nu = 1, \qquad \text{and} \qquad \operatorname{supp}(\nu) \subset [-\sqrt{8}, \sqrt{8}]. \tag{4.4}$$

From general arguments of potential theory [28], there exists a unique minimizer ν_t of I_{V_t}. We will use this measure in the steepest descent analysis. Note that its properties differ on two crucial points from the equilibrium measure μ_t as defined in (3.1). It is forced to have support within $[-\sqrt{8}, \sqrt{8}]$, and it is not necessarily positive everywhere. Only for $t = t_{cr} = -1/12$, we have $\nu_t = \mu_t$.

The use of this modified equilibrium measure is not without a price. If $t > -1/12$ the density of the equilibrium measure is negative near the endpoints $\pm\sqrt{8}$, which causes a problem when we open the lens. Instead of exponential decay, we get exponential growth in a neighborhood of $\pm\sqrt{8}$. This neighborhood becomes smaller when t tends to $-1/12$ and will eventually fall inside the region where we are going to make a special parametrix anyway. On the other hand, if $t < -1/12$ the measure ν_t is positive and the opening of the lens causes no problem, but we still get exponential growth in a neighborhood of the endpoints $\pm\sqrt{8}$ which now follows from the behavior of its associated g-function on the contours Γ_j, $j = 1, \ldots, 4$. However, when t tends to $-1/12$, this neighborhood will eventually fall inside the region again where we make the special parametrix.

We will derive an explicit expression for the minimizer. The Euler-Lagrange equations for the minimization problem yield that for some constant l_t we have

$$2\int \log\frac{1}{|x-y|}\mathrm{d}\nu_t(y) + V_t(x) + l_t = 0$$

for $x \in [-\sqrt{8}, \sqrt{8}]$. Taking a derivative with respect to x, we get a singular integral equation for the density v_t of ν_t,

$$2PV\int_{-\sqrt{8}}^{\sqrt{8}} \frac{v_t(y)}{x-y}dy = V_t'(x) = V_{cr}'(x) + (t+1/12)x^3, \qquad \text{for } x \in (-\sqrt{8}, \sqrt{8}).$$

Thus

$$\frac{\mathrm{d}\nu_t}{\mathrm{d}x}(x) = v_t(x) = v_{cr}(x) + (t+1/12)v^\circ(x)$$

where

$$v_{cr}(x) = \frac{\mathrm{d}\mu_{cr}}{\mathrm{d}x}(x) = \frac{(8-x^2)^{3/2}}{24\pi}, \qquad x \in [-\sqrt{8}, \sqrt{8}),$$

and $v^\circ(x)$ is such that

$$\int_{-\sqrt{8}}^{\sqrt{8}} v^\circ(x)\mathrm{d}x = 0 \tag{4.5}$$

and

$$PV\int_{-\sqrt{8}}^{\sqrt{8}} \frac{v^\circ(y)}{x-y}dy = x^3/2, \qquad x \in (-\sqrt{8}, \sqrt{8}). \tag{4.6}$$

We determine v° from (4.5) and (4.6).

Lemma 4.1. *We have that v° is given by*

$$v^\circ(x) = \frac{8 + 4x^2 - x^4}{2\pi\sqrt{8 - x^2}}, \qquad x \in (-\sqrt{8}, \sqrt{8}). \tag{4.7}$$

Proof. Define

$$h(z) = \frac{8 + 4z^2 - z^4}{2(z^2 - 8)^{1/2}}, \qquad z \in \mathbb{C} \setminus [\sqrt{8}, \sqrt{8}],$$

where the branch of the square root is chosen which is positive for real $z > \sqrt{8}$. Then,

$$\frac{8 + 4y^2 - y^4}{\sqrt{8 - y^2}} = \frac{1}{\mathrm{i}}(h_-(y) - h_+(y)), \qquad y \in (-\sqrt{8}, \sqrt{8}),$$

which implies that

$$\int_{-\sqrt{8}}^{\sqrt{8}} \frac{8 + 4y^2 - y^4}{2\pi\sqrt{8 - y^2}}\, \mathrm{d}y = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} h(z)\, \mathrm{d}z$$

and

$$PV \int_{-\sqrt{8}}^{\sqrt{8}} \frac{8 + 4y^2 - y^4}{2\pi\sqrt{8 - y^2}(x - y)} \mathrm{d}y = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{h(z)}{x - z}\, \mathrm{d}z, \qquad x \in (-\sqrt{8}, \sqrt{8}),$$

where $\mathcal{C}$ is a closed contour around $[-\sqrt{8}, \sqrt{8}]$ with counterclockwise orientation. Since

$$h(z) = \frac{8 + 4z^2 - z^4}{2(8 - z^2)^{1/2}} = -z^3/2 + \mathcal{O}(1/z^3), \qquad z \to \infty,$$

which we can check by straightforward calculation, we find by the residue theorem

$$\int_{-\sqrt{8}}^{\sqrt{8}} \frac{8 + 4y^2 - y^4}{2\pi\sqrt{8 - y^2}}\, \mathrm{d}y = -\operatorname*{Res}_{z=\infty} h(z) = 0,$$

$$PV \int_{-\sqrt{8}}^{\sqrt{8}} \frac{8 + 4y^2 - y^4}{2\pi\sqrt{8 - y^2}(x - y)}\, \mathrm{d}y = -\operatorname*{Res}_{z=\infty} \frac{h(z)}{x - z} = x^3/2.$$

Thus v° defined by (4.7) satisfies (4.5) and (4.6). This proves the lemma. □

4.3 Modified g_t and ϕ_t functions

We use g_t to denote the g-function associated with ν_t. So $g_t : \mathbb{C} \setminus (-\infty, \sqrt{8}]$ is defined by

$$g_t(z) = \int \log(z-x) \mathrm{d}\nu_t(x) \tag{4.8}$$

and it satisfies

$$g_{t+}(z) + g_{t-}(z) - V_t(z) = l_t, \qquad z \in (-\sqrt{8}, \sqrt{8}) \tag{4.9}$$

and

$$g_t(z) = \log z + \mathcal{O}(1/z), \qquad z \to \infty. \tag{4.10}$$

We also define $\phi_t : \mathbb{C} \setminus (-\infty, \sqrt{8}]$ by

$$\phi_t(z) = \phi_{cr}(z) + (t + 1/12)\phi^\circ(z) \tag{4.11}$$

where

$$\phi^\circ(z) = \frac{1}{2} \int_{\sqrt{8}}^{z} \frac{8 + 4s^2 - s^4}{(s^2-8)^{1/2}} \, \mathrm{d}s. \tag{4.12}$$

Then

$$\phi_t(z) = -\frac{1}{2}V_t(z) + g_t(z) - \frac{1}{2}l_t, \qquad z \in \mathbb{C} \setminus (-\infty, \sqrt{8}], \tag{4.13}$$

which will be used in the first transformation.

Observe that g_t and ϕ_t are different from the functions with the same name that were used in Section 3 in the steepest descent analysis in the regular case. Since we will not use the results of Section 3 in what follows, we trust that this will not cause any confusion.

4.4 The transformations $Y \mapsto T \mapsto S$

Define $T : \mathbb{C} \setminus \Gamma$ by

$$T(z) = \mathrm{e}^{-nl_t\sigma_3/2} Y(z) \mathrm{e}^{-ng_t(z)\sigma_3/2} \mathrm{e}^{nl_t\sigma_3/2}, \tag{4.14}$$

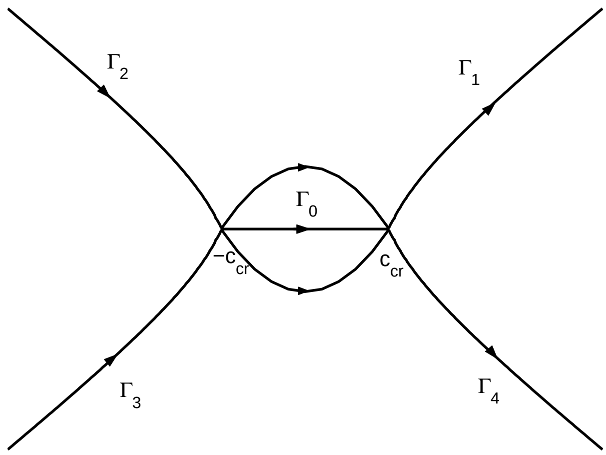

Figure 7: The contour Σ_S that arises after opening of the lens around $[-c_{cr}, c_{cr}]$ where $c_{cr} = \sqrt{8}$.

for all $z \in \mathbb{C} \setminus \Gamma$. Then T satisfies the following RH problem

$$\begin{cases} T(z) \text{ is analytic in } \mathbb{C} \setminus \Gamma \\ T_+(z) = T_-(z) \begin{pmatrix} 1 & \alpha(z) \mathrm{e}^{2n\phi_t(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma_j, \quad j = 1, 2, 3, 4, \\ T_+(z) = T_-(z) \begin{pmatrix} \mathrm{e}^{-2n\phi_{t+}(z)} & 1 \\ 0 & \mathrm{e}^{-2n\phi_{t-}(z)} \end{pmatrix}, & z \in (-\sqrt{8}, \sqrt{8}), \\ T(z) = I + \mathcal{O}(1/z), & z \to \infty. \end{cases} \tag{4.15}$$

Since $\phi_{t\pm}(z) \in \mathrm{i}\mathbb{R}$ for $z \in (-\sqrt{8}, \sqrt{8})$, the jump matrix is oscillatory on this interval. In the second transformation we open a lens. This lens will not depend on t. Add two curves to the contour such that both contour go from $-\sqrt{8}$ to $\sqrt{8}$, one below and the other one above this interval, such that

$$\operatorname{Re} \phi_{cr}(z) > 0, \tag{4.16}$$

holds on the lips of the lens. Figure 6 indicates that this can be done. It leads to a contour Σ_S as in Figure 7 where the upper and lower lips of the lens are in the shaded region of Figure 6. Define

$$\begin{cases} S(z) = T(z) \begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{-2n\phi_t(z)} & 1 \end{pmatrix}, & \text{in the upper part of the lense,} \\ S(z) = T(z) \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-2n\phi_t(z)} & 1 \end{pmatrix}, & \text{in the lower part of the lense,} \\ S(z) = T(z), & \text{elsewhere.} \end{cases} \tag{4.17}$$

Then S satisfies the following RH problem on the contour Σ_S consisting of Γ and the upper and lower lips of the lense.

$$\begin{cases} S(z) \text{ is analytic in } \mathbb{C} \setminus \Sigma_S \\ S_+(z) = S_-(z) \begin{pmatrix} 1 & \alpha(z)\mathrm{e}^{2n\phi_t(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma_j, \quad j = 1,2,3,4, \\ S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-2n\phi_t(z)} & 1 \end{pmatrix}, & \text{on the lips of the lense,} \\ S_+(z) = S_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\sqrt{8}, \sqrt{8}), \\ S(z) = I + \mathcal{O}(1/z), & z \to \infty. \end{cases} \tag{4.18}$$

The contour Σ_S is constructed such that $\mathrm{Re}\ \phi_{cr}(z) < 0$ on Γ_j, $j = 1,2,3,4$ and $\mathrm{Re}\ \phi_{cr}(z) > 0$ on the upper and lower lips of the lens. So the jump matrices in the RH problem for S would have the correct decay properties in case $t = -1/12$, since then $\phi_t = \phi_{cr}$. However, if $t \neq -1/12$, then ϕ_t is different from ϕ_{cr}, and $\mathrm{Re}\ \phi_t(z)$ has an incorrect sign on some of the contours. In Figure 8 the contour Σ_S is shown near $\sqrt{8}$. The shaded regions are the regions where $\mathrm{Re}\ \phi_t > 0$. The left picture shows the behavior of $\mathrm{Re}\ \phi_t$ for $t < -1/12$ and the right picture the behavior for $t > -1/12$. In the left picture we see that $\sqrt{8}$ is inside the shaded region, whereas it is inside the white region in the right picture. Therefore, $\mathrm{Re}\ \phi_t$ has the wrong sign on (small) parts of Γ_j, $j = 1,2,3,4$ in the left picture. In the right picture it has the wrong sign on (small) parts of the lips of the lens. However, the following proposition states that this behavior only occurs in small neighborhoods of $\pm\sqrt{8}$.

Proposition 4.2. *Let U and $\widehat{U}$ be neighborhoods of $\sqrt{8}$ and $-\sqrt{8}$, respectively. Then there exist $\delta > 0$ and $\varepsilon > 0$ such that for all $t \in \mathbb{R}$ with $|t + 1/12| < \delta$, we have* $\mathrm{Re}\ \phi_t > \varepsilon$ *on the upper and lower lips of the lens outside $U \cup \widehat{U}$ and* $\mathrm{Re}\ \phi_t(z) < -\varepsilon|z|^4$ *on $\Gamma_j \setminus (U \cup \widehat{U})$ for $j = 1,2,3,4$.*

Proof. There exists an $\varepsilon > 0$ such that $\mathrm{Re}\ \phi_{cr} > \varepsilon$ on the lips of the lens outside U and $\mathrm{Re}\ \phi_{cr}(z) < -\varepsilon|z|^4$ on the contours Γ_j outside U. The latter

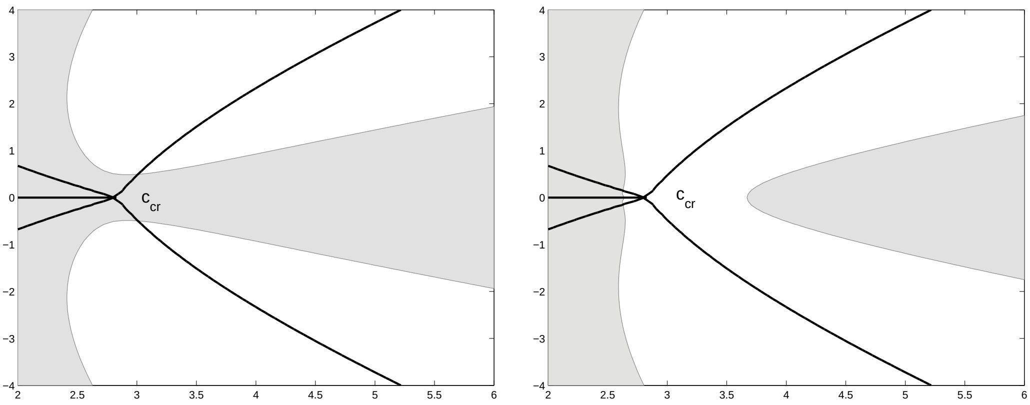

Figure 8: Behavior of Re ϕ_t in a neighborhood of $c_{cr} = \sqrt{8}$ for $t < -1/12$ (left) and $t > -1/12$ (right). The shaded region is where Re $\phi_t > 0$. The real part of ϕ_t has the wrong sign on some parts of Σ_S.

inequality holds since $\phi_{cr}(z) = (1/96)z^4 + \mathcal{O}(z^3)$ as $z \to \infty$ by (4.2) and Γ_j is a steepest descent curve for ϕ_{cr}. Since

$$\phi_t = \phi_{cr} + (t + 1/12)\phi^\circ.$$

and $\phi^\circ(z) = -(1/8)z^4 + \mathcal{O}(z^3)$ as $z \to \infty$, see (4.12), it follows by continuity that the same inequalities hold for ϕ_t if t is sufficiently close to $-1/12$. □

4.5 The parametrix M away from the endpoints

The 2×2 matrix valued function $M : \mathbb{C} \setminus [-\sqrt{8}, \sqrt{8}] \to \mathbb{C}^{2\times 2}$ defined by

$$M(z) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} \left(\frac{z - \sqrt{8}}{z + \sqrt{8}} \right)^{\sigma_3/4} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}, \tag{4.19}$$

is a solution of the RH problem

$$\begin{cases} M \text{ is analytic in } \mathbb{C} \setminus [-\sqrt{8}, \sqrt{8}], \\ M_+(z) = M_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\sqrt{8}, \sqrt{8}), \\ M(z) = I + \mathcal{O}(1/z), & z \to \infty. \end{cases} \tag{4.20}$$

Thus M has the same jumps as S on the interval $(-\sqrt{8}, \sqrt{8})$. We will use M as a parametrix for S away from the endpoints. That is, we use M as an

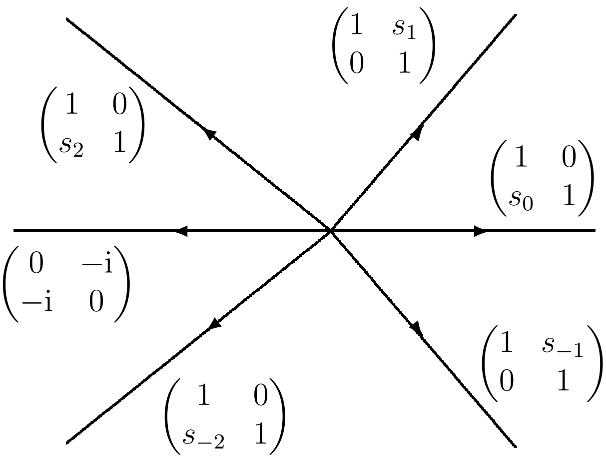

Figure 9: The contour Σ_Ψ and the jump matrices A_0 for the RH problem for $\Psi^{(0)}$.

approximation for S away from $\pm\sqrt{8}$. The approximation will be good if n is large, and t is sufficiently close to $-1/12$. The approximation will not be good near the endpoints. There we construct a local parametrix with the aid of the RH problem for Painlevé I which was already given in Section 1, but which will be discussed in more detail next.

4.6 The RH problem for Painlevé I

In the literature one can find several equivalent versions of the RH problem for Painlevé I [19, 22]. Here we start from the RH problem as formulated by Kapaev [21].

Let Σ_Ψ be the collection of rays $\arg(\zeta) = 2k\pi/5$ for $k = -2, -1, 0, 1, 2$ together with the negative real axis $\arg(\zeta) = \pi$ with orientation as in Figure 9. Let $A_0(\zeta)$ for $\zeta \in \Sigma_\Psi$ be as indicated in Figure 9, that is,

$$A_0(\zeta) = \begin{cases} \begin{pmatrix}1 & s_k\\0 & 1\end{pmatrix} & \arg(\zeta) = 2k\pi/5, \quad k = -1, 1\\ \begin{pmatrix}1 & 0\\s_k & 1\end{pmatrix} & \arg(\zeta) = 4k\pi/5, \quad k = -2, 0, 2\\ \begin{pmatrix}0 & -\mathrm{i}\\-\mathrm{i} & 0\end{pmatrix} & \arg(\zeta) = \pi, \end{cases} \tag{4.21}$$

where the Stokes multipliers s_k are complex numbers such that

$$1 + s_k s_{k+1} = -\mathrm{i} s_{k+3}, \qquad s_{k+5} = s_k, \qquad k \in \mathbb{Z}. \tag{4.22}$$

Consider the RH problem for a 2×2 matrix valued function $\Psi^{(0)}$ depending on a parameter x,

$$\begin{cases} \Psi^{(0)}(\cdot; x) \text{ is analytic in } \mathbb{C} \setminus \Sigma_\Psi, \\ \Psi^{(0)}_+(\zeta; x) = \Psi^{(0)}_-(\zeta, x) A_0(\zeta), & \zeta \in \Sigma_\Psi, \\ \Psi^{(0)}(\zeta; x) = \frac{\zeta^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(I + \mathcal{O}(\zeta^{-1/2}) \right) \mathrm{e}^{\theta(\zeta, x)\sigma_3}, & \zeta \to \infty \end{cases} \tag{4.23}$$

where θ is given as before by (1.22). Then $\Psi^{(0)}$ has an expansion of the form

$$\Psi^{(0)}(\zeta; x) = \frac{\zeta^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(I + \begin{pmatrix} -\mathcal{H} & 0 \\ 0 & \mathcal{H} \end{pmatrix} \zeta^{-1/2} + \frac{1}{2} \begin{pmatrix} \mathcal{H}^2 & y \\ y & \mathcal{H}^2 \end{pmatrix} \zeta^{-1} + \mathcal{O}(\zeta^{-3/2}) \right) \mathrm{e}^{\theta(\zeta, x)\sigma_3}, \tag{4.24}$$

where $\mathcal{H}$ and y depend on x. The function $y = y(x)$ is a solution of the Painlevé I equation (1.12) and $\mathcal{H}(x) = \frac{1}{2}(y'(x))^2 - 2y^3(x) - yx$ is the Hamiltonian. Note that $\mathcal{H}'(x) = -y(x)$. To every set of Stokes multipliers s_k satisfying (4.22), there corresponds a unique solution of (1.12) and vice versa. The RH problem for $\Psi^{(0)}$ has a solution if and only if x is not a pole of y.

For our purposes we need a special choice of Stokes multipliers. We take $s_0 = 0$ and $s_1 = \mathrm{i}\alpha$. This determines the other Stokes multipliers by (4.22) and it follows that

$$s_0 = 0, \quad s_1 = \mathrm{i}\alpha, \quad s_{-1} = \mathrm{i}(1 - \alpha), \quad s_2 = s_{-2} = \mathrm{i}. \tag{4.25}$$

We will denote this special solution by $\Psi^{(0)}(\zeta; x, \alpha)$. The corresponding solution of the Painlevé I equation and its Hamiltonian will be denoted by y_α and $\mathcal{H}_\alpha$.

Before we proceed with the parametrix we first modify the RH problem for $\Psi^{(0)}$ by defining

$$\Psi(\zeta; x, \alpha) = \Psi^{(0)}(\zeta; x, \alpha) \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \tag{4.26}$$

and we reverse the orientation on the contours $\arg \zeta = \pi$ and $\arg \zeta = \pm 4\pi/5$.

Then the RH problem for Ψ reads

$$\begin{cases} \Psi(\cdot;x,\alpha) \text{ is analytic in } \mathbb{C}\setminus\Sigma_\Psi \\ \Psi_+(\zeta;x,\alpha) = \Psi_-(\zeta;x,\alpha)A(\zeta), & \zeta\in\Sigma_\Psi, \\ \Psi(\zeta;x,\alpha) = \frac{\zeta^{\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i}\end{pmatrix}\left(I + \frac{\Psi_1(x)}{\zeta^{1/2}} + \frac{\Psi_2(x)}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^{3/2}}\right)\right)\mathrm{e}^{\theta(\zeta,x)\sigma_3}, & \zeta\to\infty. \end{cases} \tag{4.27}$$

where the jumps $A(\zeta)$ are as indicated in Figure 2 in Section 1 and

$$\Psi_1 = \begin{pmatrix} -\mathcal{H}_\alpha & 0 \\ 0 & \mathcal{H}_\alpha \end{pmatrix} = -\mathcal{H}_\alpha\sigma_3, \tag{4.28}$$

$$\Psi_2 = \frac{1}{2}\begin{pmatrix} \mathcal{H}_\alpha^2 & -\mathrm{i}y_\alpha \\ \mathrm{i}y_\alpha & \mathcal{H}_\alpha^2 \end{pmatrix} = \frac{1}{2}\mathcal{H}_\alpha^2 I + \frac{1}{2}y_\alpha\sigma_2, \tag{4.29}$$

where we use

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to denote the Pauli spin matrices.

We note that $\zeta^{-\sigma_3/4}\Psi^{(0)}\mathrm{e}^{-\theta\sigma_3}$ has a full asymptotic series in powers of $\zeta^{-1/2}$, see [19, 27]. Therefore Ψ has also an asymptotic series

$$\Psi(\zeta;x,\alpha) \sim \frac{\zeta^{\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i}\end{pmatrix}\left(I + \sum_{k=1}^{\infty}\Psi_k(x,\alpha)\zeta^{-k/2}\right)\mathrm{e}^{\theta(\zeta,x)\sigma_3} \tag{4.30}$$

for $\zeta\to\infty$. For our purposes it is sufficient to work with the terms up to order $\mathcal{O}(\zeta^{-1})$. It is possible to derive a full asymptotic expansion for the recurrence coefficients if one works with the full expansion (4.30).

4.7 Construction of the local parametrix

We will define local parametrices P and $\widehat{P}$ for the RH problem for S around the endpoints $\pm\sqrt{8}$. The jumps for S are given in (4.18) and we want that P has the same jumps in a neighborhood of $\sqrt{8}$. In addition we want that P matches with M that we constructed in (4.19). So in a disk U around $\sqrt{8}$

we want that P satisfies the following RH problem.

$$\begin{cases} P(z) \text{ is analytic in } U \setminus \Sigma_S \\ P_+(z) = P_-(z) \begin{pmatrix} 1 & \alpha e^{2n\phi_t(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma_1 \cap U, \\ P_+(z) = P_-(z) \begin{pmatrix} 1 & (1-\alpha) e^{2n\phi_t(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma_4 \cap U, \\ P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 \\ e^{-2n\phi_t(z)} & 1 \end{pmatrix}, & \text{on the lips of the lens inside } U, \\ P_+(z) = P_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\sqrt{8}, \sqrt{8}) \cap U, \\ P(z) = M(z)(I + \mathcal{O}(n^{-1/5})), & \text{as } n \to \infty, \text{ uniformly for } z \in \partial U. \end{cases} \tag{4.31}$$

The construction of P is based on the following observation.

Lemma 4.3. *Let $P : U \setminus \Sigma_S \to \mathbb{C}^{2\times 2}$ be defined by*

$$P(z) = E(z)\Psi(n^{2/5} f(z); n^{4/5} u_t(z), \alpha) e^{-n\phi_t(z)\sigma_3} \tag{4.32}$$

where

(1) *$\Psi(\cdot; x, \alpha)$ is the solution of the RH problem* (4.27) *that is associated with y_α;*

(2) *f is a conformal map from U to a neighborhood of* 0 *such that $\Sigma_S \cap U$ is mapped to (part of) the contour Σ_Ψ;*

(3) *$u_t : U \to \mathbb{C}$ is analytic and $n^{4/5} u_t(U)$ does not contain any poles of y_α;*

(4) *$E : U \to \mathbb{C}^{2\times 2}$ is analytic.*

Then P is analytic in $U \setminus \Sigma_S$ and satisfies the jump conditions in (4.31)*.*

Proof. If we put $\widetilde{P} = P e^{n\phi_t\sigma_3}$ then the jump properties for P in (4.31) translate into

$$\begin{cases} \widetilde{P}_+(z) = \widetilde{P}_-(z) \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, & z \in \Gamma_1 \cap U, \\ \widetilde{P}_+(z) = \widetilde{P}_-(z) \begin{pmatrix} 1 & 1-\alpha \\ 0 & 1 \end{pmatrix}, & z \in \Gamma_4 \cap U, \\ \widetilde{P}_+(z) = \widetilde{P}_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{on the lips of the lens inside } U, \\ \widetilde{P}_+(z) = \widetilde{P}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\sqrt{8}, \sqrt{8}) \cap U. \end{cases} \tag{4.33}$$

Here we used the fact that ϕ_t is analytic in $\mathbb{C} \setminus (-\infty, \sqrt{8}]$ and $\phi_{t+} + \phi_{t-} = 0$ on $(-\sqrt{8}, \sqrt{8})$. The jump matrices in (4.33) are exactly the same as the ones that appear in the RH problem for Ψ, see Figure 2, except that these jumps are on different contours. Since f provides a conformal map between the respective contours, it is then immediate that $\Psi(n^{2/5} f(z); x, \alpha) \mathrm{e}^{-n\phi_t(z)\sigma_3}$ has the required jump properties on $\Sigma_S \cap U$, for every x that is not a pole of y_α. Since the jumps do not depend on x, the jump properties will not be affected if we let $x = n^{4/5} u_t(z)$ where u_t is analytic and $n^{4/5} u_t(z)$ is not a pole of y_α for every $z \in U$. Finally, we note that the multiplication on the left by an analytic factor $E(z)$ does not change the jumps either, so that (4.32) has indeed the required jump properties on $\Sigma_S \cap U$. □

What remains to be shown is that we can find f, u_t and E with the properties stated in Lemma 4.3 such that the matching condition

$$P(z) = M(z)(I + \mathcal{O}(n^{-1/5})), \qquad \text{as } n \to \infty, \text{ uniformly for } z \in \partial U, \quad (4.34)$$

is satisfied as well. In the next lemma we will show that this is indeed satisfied if we define

$$f(z) = \left[\frac{5}{4}\phi_{cr}(z)\right]^{2/5}, \tag{4.35}$$

$$u_t(z) = (4/5)^{1/5} \frac{\phi_t(z) - \phi_{cr}(z)}{(\phi_{cr}(z))^{1/5}}, \tag{4.36}$$

and

$$E(z) = M(z) \left[\frac{(n^{2/5} f(z))^{\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}\right]^{-1} = \frac{1}{\sqrt{2}} M(z) \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} (n^{2/5} f(z))^{-\sigma_3/4}. \tag{4.37}$$

Recall that ϕ_{cr} is given by (4.2), ϕ_t is given by (4.11), and M is given by (4.19). The branches of the fractional exponents in (4.35)-(4.37) are all taken to be positive for $z - \sqrt{8}$ real and positive and sufficiently small.

Lemma 4.4. *Assume that $x \in \mathbb{R}$ is not a pole of y_α and let*

$$t = t_{cr} - c_1 x n^{-4/5}, \qquad c_1 = 2^{-9/5} 3^{-6/5}. \tag{4.38}$$

Let f, u_t, and E be as in (4.35)-(4.37). Then there is a disk U around $\sqrt{8}$, depending only on x and α but not on n, such that the properties (2)–(4) *stated in Lemma* 4.3 *are satisfied and such the matching condition* (4.34) *holds.*

Proof. We begin with the properties of f. It follows from the definition (4.2) of ϕ_{cr} that

$$\phi_{cr}(z) = \frac{2^{7/4}}{15}(z - \sqrt{8})^{5/2}(1 + h_{cr}(z)) \tag{4.39}$$

with h_{cr} an analytic function in a neighborhood of $\sqrt{8}$ with $h_{cr}(\sqrt{8}) = 0$. Thus (4.35) defines a conformal map $\zeta = f(z)$ from a small enough disk U_1 around $\sqrt{8}$ to a neighborhood of $\zeta = 0$. Then $f(z)$ is real for real $z \in U_1$ and since by (4.39) and (4.35) we have

$$f(z) = 2^{-1/10}3^{-2/5}(z - \sqrt{8}) + \mathcal{O}\big((z - \sqrt{8})^2\big), \tag{4.40}$$

as $z \to \sqrt{8}$, we see that f maps $(-\sqrt{8}, \sqrt{8}) \cap U_1$ to a part of the negative real axis.

Since Γ_1 and Γ_4 are defined as the steepest descent curves of ϕ_{cr} that start from $\sqrt{8}$, we have that ϕ_{cr} is real and negative on Γ_1 and Γ_4. Then it follows from (4.35) that f maps these contours onto the rays $\arg \zeta = 2\pi/5$ and $\arg \zeta = -2\pi/5$, respectively.

Finally, to have all the mapping properties stated in (2) in Lemma 4.3, we want that f maps the upper and lower lips of the lens that are inside U_1 to parts of the rays $\arg \zeta = \pm 4\pi/5$. Here we use the additional freedom we have in choosing the exact location of the lips of the lense. So far we have only specified that they should be in the region where $\operatorname{Re} \phi_{cr} > 0$. Now we require in addition that in a neighborhood of $\sqrt{8}$ we want the lips to be such that $\arg \phi_{cr}(z) = \pm 2\pi$ for z on the upper and lower lips near $\sqrt{8}$. We can clearly impose this extra requirement. Note that we take $\arg \phi_{cr}(z)$ so that it is continuous for $z \in \mathbb{C} \setminus (-\infty, \sqrt{8}]$ and has the value 0 for $z - \sqrt{8}$ real and positive. Then by (4.35) we have that $\arg f(z) = \pm 4\pi/5$ for $z \in U_1$ on the lips of the lense, and thus all the properties stated in (2) of Lemma 4.3 are satisfied.

Now we turn to the properties of u_t. Because of (4.11) and (4.36) we can write

$$u_t(z) = (t + 1/12)u^\circ(z) \tag{4.41}$$

where

$$u^\circ(z) = (4/5)^{1/5} \frac{\phi^\circ(z)}{(\phi_{cr}(z))^{1/5}}. \tag{4.42}$$

From (4.12) it follows that

$$\phi^\circ(z) = -2^{7/4}3\,(z - \sqrt{8})^{1/2}(1 + h^\circ(z)), \tag{4.43}$$

where h° is analytic in a neighborhood of $\sqrt{8}$ with $h^\circ(\sqrt{8}) = 0$. Then by (4.39), (4.42), and (4.43) we find that u° is analytic in a neighborhood of $\sqrt{8}$ and

$$u^\circ(z) = -2^{9/5}3^{6/5}\left(1 + \mathcal{O}(z - \sqrt{8})\right) \tag{4.44}$$

for $z \to \sqrt{8}$, so that $u^\circ(\sqrt{8}) = -c_1^{-1}$. Combining (4.38), (4.41), and (4.44), we see that $n^{4/5}u_t(\sqrt{8}) = x$. Since x is not a pole of y_α we can then find a disk U_2 around $\sqrt{8}$ with $U_2 \subset U_1$, such that $n^{4/5}u_t(z)$ is not a pole of y_α for every $z \in U_2$. Since $n^{4/5}u_t = -c_1xu^\circ$ depends only on x, the disk U_2 depends only on x. So the properties stated in (3) of Lemma 4.3 are satisfied.

Since M is given by (4.19) we find from (4.37) that

$$E(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}\left(\frac{n^{2/5}f(z)(z+\sqrt{8})}{z-\sqrt{8}}\right)^{-\sigma_3/4}$$

which implies that E is analytic in a disk $U \subset U_2$ around $\sqrt{8}$, as $f(z)$ has a simple zero at $z = \sqrt{8}$, see (4.40). This shows that property (4) of Lemma 4.3 is satisfied.

We finally show that the matching condition (4.34) is satisfied. Since Ψ has asymptotics (1.21), we find from (4.32) that uniformly for $z \in \partial U$,

$$\begin{aligned} P(z) = & E(z)\frac{(n^{2/5}f(z))^{\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}\left(I + \mathcal{O}(n^{-1/5})\right) \\ & \times \exp\left((\theta(n^{2/5}f(z), n^{4/5}u_t(z)) - n\phi_t(z))\sigma_3\right) \end{aligned} \tag{4.45}$$

as $n \to \infty$. Since $\theta(\zeta, x) = \frac{4}{5}\zeta^{5/2} + x\zeta^{1/2}$, we see that

$$\theta(n^{2/5}f(z), n^{4/5}u_t(z)) = n\left[\frac{4}{5}(f(z))^{5/2} + u_t(z)(f(z))^{1/2}\right] = n\phi_t(z) \tag{4.46}$$

where in the last equality we used the formulas (4.35) and (4.36) for f and u_t. Hence we can forget about the exponential factor in (4.45). Because of the definition (4.37) of E we see that (4.45) leads to the matching condition (4.34).

This completes the proof of Lemma 4.4. □

Remark 4.5. The neighborhood U in Lemma 4.4 depends on x, but an inspection of the proof shows that that we can take U independent of x if x is allowed to vary in a compact set that does not contain any poles of y_α. Then also the matching condition (4.34) is uniformly valid for such x.

Parametrix around $-\sqrt{8}$

The local parametrix $\widehat{P}$ in a disk $\widehat{U}$ around the other endpoint $-\sqrt{8}$ should satisfy the following RH problem.

$$\begin{cases} \widehat{P}(z) \text{ is analytic in } \widehat{U} \setminus \Sigma_S \\ \widehat{P}_+(z) = \widehat{P}_-(z) \begin{pmatrix} 1 & (1-\beta)\mathrm{e}^{2n\phi_t(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma_2 \cap \widehat{U}, \\ \widehat{P}_+(z) = \widehat{P}_-(z) \begin{pmatrix} 1 & \beta\mathrm{e}^{2n\phi_t(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma_3 \cap \widehat{U}, \\ \widehat{P}_+(z) = \widehat{P}_-(z) \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-2n\phi_t(z)} & 1 \end{pmatrix}, & \text{on the lips of the lens inside } \widehat{U}, \\ \widehat{P}_+(z) = \widehat{P}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\sqrt{8}, \sqrt{8}) \cap \widehat{U}, \\ \widehat{P}(z) = M(z)(I + \mathcal{O}(n^{-1/5})), & \text{as } n \to \infty, \text{ uniformly for } z \in \partial\widehat{U}. \end{cases} \tag{4.47}$$

It turns out that $\widehat{P}$ can be expressed directly in terms of P. Let us write $P = P_\alpha$ to emphasize that the solution of (4.31) depends on α. Then it follows that

$$\widehat{P}(z) = \sigma_3 P_\beta(-z)\sigma_3 \tag{4.48}$$

solves (4.47) on the neighborhood $\widehat{U} = -U_\beta$ of $-\sqrt{8}$. Indeed, the identity (4.48) is an immediate consequence of the RH problems (4.31) and (4.47), and the facts that $M(-z) = \sigma_3 M(z)\sigma_3$ and $\phi_t(-z) = \phi_t(z) \pm \pi\mathrm{i}$. It is well-defined if x is not a pole of y_β.

Since f, u_t, and E do not depend on α, it follows from (4.32) and (4.48) that

$$\widehat{P}(z) = \sigma_3 E(-z)\Psi(n^{2/5}f(-z); n^{4/5}u_t(-z), \beta)\mathrm{e}^{-n\phi_t(-z)\sigma_3}\sigma_3 \tag{4.49}$$

for $z \in \widehat{U}$.

Matching condition

We obtain a more precise matching condition than (4.34) if we use the asymptotic series (4.30) in (4.32). Then it follows as in the proof of Lemma 4.4 that

$$P(z) \sim M(z)\left(I + \sum_{k=1}^{\infty} \Psi_k(n^{4/5}u_t(z), \alpha) f(z)^{-k/2} n^{-k/5}\right) \tag{4.50}$$

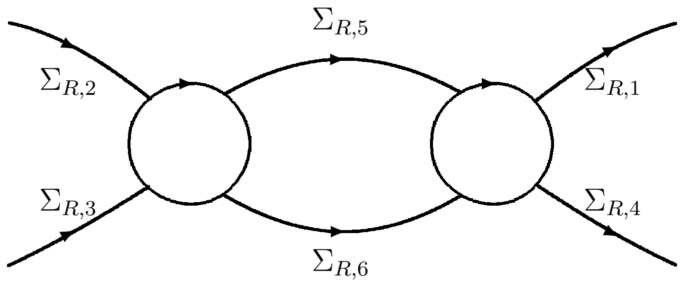

Figure 10: Contour Σ_R for the RH problem for R

as $n \to \infty$, uniformly for $z \in \partial U$, and

$$\widehat{P}(z) \sim M(z)\sigma_3 \left(I + \sum_{k=1}^{\infty} \Psi_k(n^{4/5}u_t(-z), \beta) f(-z)^{-k/2} n^{-k/5} \right) \sigma_3 \tag{4.51}$$

as $n \to \infty$, uniformly for $z \in \partial\widehat{U}$, where in the last identity we also used that $\sigma_3 M(-z)\sigma_3 = M(z)$.

For future reference we recall that

$$n^{4/5}u_t(z) = -c_1 x u^o(z) \tag{4.52}$$

where u^o is an analytic function with

$$u^o(\sqrt{8}) = -c_1^{-1}. \tag{4.53}$$

4.8 Third transformation $S \mapsto R$

Now let x be fixed and assume that x is not a pole of y_α and y_β. As before we let t vary with n such that

$$t = -1/12 - c_1 x n^{-4/5}. \tag{4.54}$$

Then the local parametrices P and $\widehat{P}$ are defined in U and $\widehat{U}$ and they satisfy (4.31) and (4.47). We also have M as a parametrix for S away from the endpoints.

Define R as

$$R(z) = \begin{cases} S(z)M^{-1}(z), & z \in \mathbb{C} \setminus \left(\overline{U} \cup \overline{\widehat{U}} \cup \Sigma_S\right), \\ S(z)P^{-1}(z), & z \in U \setminus \Sigma_S, \\ S(z)\widehat{P}^{-1}(z), & z \in \widehat{U} \setminus \Sigma_S. \end{cases} \tag{4.55}$$

Then R satisfies a RH problem on the contour Σ_R that is shown in Figure 10. Σ_R consists of the curves $\Sigma_{R,j}$ for $j = 1, 2, 3, 4$ which are the parts of the curves Γ_j outside the disks U and $\widehat{U}$, of the curves $\Sigma_{R,5}$ and $\Sigma_{R,6}$ which are the parts of the upper and lower lips of the lens that are outside the two disks, and of the circles ∂U and $\partial\widehat{U}$. The orientation of Σ_R is as indicated in Figure 10. In particular we choose clockwise orientation on the two circles.

Lemma 4.6. *With Σ_R as described above we have that R satisfies the RH problem*

$$\begin{cases} R \text{ is analytic in } \mathbb{C} \setminus \Sigma_R \\ R_+(z) = R_-(z)A_R(z), \quad z \in \Sigma_R, \quad j = 1, 2, 3, 4, \\ R(z) = I + \mathcal{O}(1/z), \qquad z \to \infty, \end{cases} \tag{4.56}$$

where

$$A_R(z) = M(z) \begin{pmatrix} 1 & \alpha(z)e^{2n\phi_t(z)} \\ 0 & 1 \end{pmatrix} M^{-1}(z), \qquad z \in \Sigma_{R,j}, \quad j = 1, 2, 3, 4, \tag{4.57}$$

$$A_R(z) = M(z) \begin{pmatrix} 1 & 0 \\ e^{-2n\phi_t(z)} & 1 \end{pmatrix} M^{-1}(z), \qquad z \in \Sigma_{R,5} \cup \Sigma_{R,6}, \tag{4.58}$$

$$A_R(z) = P(z)M(z)^{-1}, \qquad z \in \partial U, \tag{4.59}$$

$$A_R(z) = \widehat{P}(z)M(z)^{-1}, \qquad z \in \partial\widehat{U}. \tag{4.60}$$

Proof. Note that S and M have the same jumps on $(-\sqrt{8}, \sqrt{8})$, S and P have the same jumps on $\Sigma_S \cap U$, and S and $\widehat{P}$ have the same jumps on $\Sigma_S \cap \widehat{U}$). From this it follows that $R_+ = R_-$ on these contours, so that R is analytic there, and these contours do not appear in Σ_R.

The jumps (4.57)-(4.60) are an easy consequence of the definition (4.55) and the jumps satisfied by S, see (4.18). □

The jump matrices (4.57) and (4.58) are exponentially close to the identity matrix as $n \to \infty$. This follows from Proposition 4.2 since $t \to -1/12$ as $n \to \infty$.

Because of (4.50)-(4.51) and (4.59)-(4.60) we have that $A_R(z)$, $z \in \partial U \cup \partial \widehat{U}$, has an asymptotic series in powers of $n^{-1/5}$

$$A_R(z) \sim \begin{cases} I + \displaystyle\sum_{k=1}^{\infty} W^{(k)}(z) n^{-k/5}, & z \in \partial U, \\ I + \displaystyle\sum_{k=1}^{\infty} \widehat{W}^{(k)}(z) n^{-k/5}, & z \in \partial \widehat{U}, \end{cases} \tag{4.61}$$

where

$$W^{(k)}(z) = W_\alpha^{(k)}(z) = M(z) \Psi_k(n^{4/5} u_t(z), \alpha) M^{-1}(z) f(z)^{-k/2}, \tag{4.62}$$

and, see also (4.48),

$$\widehat{W}^{(k)}(z) = \sigma_3 W_\beta^{(k)}(-z) \sigma_3. \tag{4.63}$$

It follows that R admits an asymptotic series in powers of $n^{-1/5}$.

Lemma 4.7. *For large enough n, the RH problem for R has a solution, and R admits an asymptotic expansion*

$$R(z) \sim I + \sum_{k=1}^{\infty} R^{(k)}(z) n^{-k/5}, \tag{4.64}$$

as $n \to \infty$, uniformly for $z \in \mathbb{C} \setminus \Sigma_R$.

The expansion (4.64) *is valid uniformly near infinity in the sense that for every $K \geq 1$ there is a constant $C_K > 0$ such that*

$$\left\| R(z) - I - \sum_{k=1}^{K-1} R^{(k)}(z) n^{-k/5} \right\| \leq C_K |z|^{-1} n^{-K/5} \tag{4.65}$$

holds for every z with $|z| \geq 3$.

Proof. We already observed that the jump matrix A_R for R is close to the identity if n is large. On the parts $\Sigma_{R,j}$ we have by Proposition 4.2 that

$$\|A_R(z) - I\| \leq C e^{-\varepsilon |z|^4}, \qquad z \in \Sigma_{R,j}, \quad j = 1, \ldots, 6, \tag{4.66}$$

for some positive constants C and ε.

Then the lemma follows from (4.66) and the expansions (4.61), cf. also the arguments used in the proof of [12, Theorem 7.81].

The estimate (4.65) follows as in [25, Lemma 8.3]. □

Corollary 4.8. *The RH problem for Y is solvable for sufficiently large n.*

Proof. Since the transformations $Y \mapsto T \mapsto S \mapsto R$ are invertible we can recover Y from R. Since the RH problem for R has a solution for large enough n, it now also follows that the original RH problem for Y has a solution for large enough n. □

From Proposition 2.1 it now follows that the monic orthogonal polynomial $\pi_{n,n}(t)$ exists for n large enough. In the rest of the proof we will calculate the coefficients $a_{n,n}(t)$ and $b_{n,n}(t)$ by means of Proposition 2.3. Then we find that $a_{n,n}(t) \neq 0$ for n large enough, so that by Proposition 2.4 the monic orthogonal polynomials of degrees $n+1$ and $n-1$ exist as well, and that $a_{n,n}(t)$ and $b_{n,n}(t)$ are the recurrence coefficients in the three-term recurrence relation.

4.9 Explicit expressions for $R^{(1)}$ and $R^{(2)}$

The matrix coefficients $R^{(k)}(z)$ that appear in the expansion (4.64) can be found by solving iteratively the following RH problems

$$\begin{cases} R^{(k)} \text{ is analytic in } \mathbb{C} \setminus \left(\partial U \cup \partial \widehat{U}\right) \\ R_+^{(k)}(z) = R_-^{(k)}(z) + \sum_{l=1}^{k} R_-^{(k-l)}(z) W^{(l)}(z), & z \in \partial U, \\ R_+^{(k)}(z) = R_-^{(k)}(z) + \sum_{l=1}^{k} R_-^{(k-l)}(z) \widehat{W}^{(l)}(z), & z \in \partial \widehat{U}, \\ R^{(k)}(z) = \mathcal{O}(1/z), & z \to \infty, \end{cases} \tag{4.67}$$

where $R^{(0)}(z) = I$. These are additive RH problems and therefore can be solved by the Sokhotskii-Plemelj formula. It turns out that the jump matrices have analytic continuations inside U and $\widehat{U}$ with a pole at $\pm\sqrt{8}$. This makes them easy to solve in an explicit way as in [25].

In order to establish (1.28) and (1.29) we have to determine $R^{(1)}$ and $R^{(2)}$ explicitly. First we show that $W^{(1)}$ and $W^{(2)}$ are analytic in $U \setminus \{\sqrt{8}\}$. We recall that $c_3 = 2^{1/10} 3^{2/5}$, which was already defined in Theorem 1.2.

Lemma 4.9. *The functions $W^{(1)}$ and $W^{(2)}$ are analytic in $U \setminus \{\sqrt{8}\}$ with simple poles at $\sqrt{8}$, and residues*

$$\operatorname*{Res}_{z=\sqrt{8}} W^{(1)}(z) = -2^{1/4} c_3^{1/2} \mathcal{H}_\alpha(x) \left(\sigma_3 - \mathrm{i}\sigma_1\right) \tag{4.68}$$

and

$$\operatorname*{Res}_{z=\sqrt{8}} W^{(2)}(z) = \frac{1}{2} c_3 \mathcal{H}_\alpha^2(x) I + \frac{1}{2} c_3 y_\alpha(x) \sigma_2. \tag{4.69}$$

Similarly, we have that $\widehat{W}^{(1)}$ and $\widehat{W}^{(2)}$ are analytic in $\widehat{U} \setminus \{-\sqrt{8}\}$ with simple poles at $-\sqrt{8}$, and

$$\operatorname*{Res}_{z=-\sqrt{8}} \widehat{W}^{(1)}(z) = 2^{1/4} c_3^{1/2} \mathcal{H}_\beta(x)\, (\sigma_3 + \mathrm{i}\sigma_1) \tag{4.70}$$

and

$$\operatorname*{Res}_{z=-\sqrt{8}} \widehat{W}^{(2)}(z) = -\frac{1}{2} c_3 \mathcal{H}_\beta^2(x) I + \frac{1}{2} c_3 y_\beta(x) \sigma_2. \tag{4.71}$$

Proof. Because of (4.28) and (4.52) we obtain from (4.62) that

$$W^{(1)}(z) = -\frac{\mathcal{H}_\alpha(-c_1 x u^\circ(z))}{f(z)^{1/2}} M(z) \sigma_3 M^{-1}(z), \tag{4.72}$$

and

$$W^{(2)}(z) = \frac{1}{2} \frac{\mathcal{H}_\alpha^2(-c_1 x u^\circ(z))}{f(z)} + \frac{1}{2} \frac{y_\alpha(-c_1 x u^\circ(z))}{f(z)} M(z) \sigma_2 M^{-1}(z). \tag{4.73}$$

It turns out to be convenient to rewrite M from (4.19) as

$$M(z) = \frac{1}{2} \left(\frac{z-\sqrt{8}}{z+\sqrt{8}} \right)^{1/4} (I + \sigma_2) + \frac{1}{2} \left(\frac{z+\sqrt{8}}{z-\sqrt{8}} \right)^{1/4} (I - \sigma_2).$$

and $M^{-1} = M^t$ as

$$M^{-1}(z) = \frac{1}{2} \left(\frac{z+\sqrt{8}}{z-\sqrt{8}} \right)^{1/4} (I + \sigma_2) + \frac{1}{2} \left(\frac{z-\sqrt{8}}{z+\sqrt{8}} \right)^{1/4} (I - \sigma_2).$$

Then by straightforward calculations it follows that

$$M(z)\sigma_3 M^{-1}(z) = \frac{1}{2} \left(\left(\frac{z-\sqrt{8}}{z+\sqrt{8}} \right)^{1/2} (\sigma_3 + \mathrm{i}\sigma_1) + \left(\frac{z+\sqrt{8}}{z-\sqrt{8}} \right)^{1/2} (\sigma_3 - \mathrm{i}\sigma_1) \right) \tag{4.74}$$

and

$$M(z)\sigma_2 M^{-1}(z) = \sigma_2. \tag{4.75}$$

By (4.40) we have that f has a simple zero at $\sqrt{8}$, and so we obtain from (4.72)–(4.73) and (4.74)–(4.75) that $W^{(1)}$ and $W^{(2)}$ are analytic in U with simple poles at $\sqrt{8}$. Because of (4.53) and

$$\lim_{z\to\infty} \frac{z-\sqrt{8}}{f(z)} = 2^{1/10} 3^{2/5} = c_3 \tag{4.76}$$

we find the residues (4.68) and (4.69).

The statements (4.70) and (4.71) about $\widehat{W}^{(1)}$ and $\widehat{W}^{(2)}$ then follow from (4.68), (4.69), and (4.63). $\square$

Explicit expression for $R^{(1)}$

The RH problem (4.67) with $k = 1$ reads

$$\begin{cases} R^{(1)} \text{ is analytic in } \mathbb{C} \setminus \left(\partial U \cup \partial \widehat{U}\right), \\ R_+^{(1)}(z) = R_-^{(1)}(z) + W^{(1)}(z), & z \in \partial U, \\ R_+^{(1)}(z) = R_-^{(1)}(z) + \widehat{W}^{(1)}(z), & z \in \partial \widehat{U}, \\ R^{(1)}(z) = \mathcal{O}(1/z), & z \to \infty. \end{cases} \tag{4.77}$$

Lemma 4.10. *The solution to* (4.77) *is given by*

$$R^{(1)}(z) = \begin{cases} \dfrac{1}{z-\sqrt{8}} \operatorname*{Res}_{z=\sqrt{8}} W^{(1)}(z) + \dfrac{1}{z+\sqrt{8}} \operatorname*{Res}_{z=-\sqrt{8}} \widehat{W}^{(1)}(z), & z \in \mathbb{C} \setminus (\overline{U} \cup \overline{\widehat{U}}), \\ \dfrac{1}{z-\sqrt{8}} \operatorname*{Res}_{z=\sqrt{8}} W^{(1)}(z) + \dfrac{1}{z+\sqrt{8}} \operatorname*{Res}_{z=-\sqrt{8}} \widehat{W}^{(1)}(z) - W^{(1)}(z), & z \in U, \\ \dfrac{1}{z-\sqrt{8}} \operatorname*{Res}_{z=\sqrt{8}} W^{(1)}(z) + \dfrac{1}{z+\sqrt{8}} \operatorname*{Res}_{z=-\sqrt{8}} \widehat{W}^{(1)}(z) - \widehat{W}^{(1)}(z), & z \in \widehat{U}. \end{cases} \tag{4.78}$$

Proof. Let $R^{(1)}$ be defined by (4.78). Since $W^{(1)}$ is analytic in U with a simple pole at $\sqrt{8}$, and $\widehat{W}^{(1)}$ is analytic in $\widehat{U}$ with a simple pole at $-\sqrt{8}$, it is then clear that $R^{(1)}$ satisfies all of the properties in (4.77), including the analyticity at $\pm\sqrt{8}$. $\square$

If we use the explicit expressions (4.68) and (4.70) for the residues of $W^{(1)}$ and $\widehat{W}^{(1)}$, we find the following expansion of $R^{(1)}(z)$ as $z \to \infty$.

Corollary 4.11. *We have that*

$$R^{(1)}(z) = \frac{R_1^{(1)}}{z} + \frac{R_2^{(1)}}{z^2} + \mathcal{O}(z^{-3}), \qquad z \to \infty \tag{4.79}$$

where

$$\begin{aligned} R_1^{(1)} &= \operatorname*{Res}_{z=\sqrt{8}} W^{(1)}(z) + \operatorname*{Res}_{z=-\sqrt{8}} \widehat{W}^{(1)}(z) \\ &= 2^{1/4} c_3^{1/2} \left[\left(\mathcal{H}_\beta(x) + \mathcal{H}_\alpha(x)\right) \mathrm{i}\sigma_1 + \left(\mathcal{H}_\beta(x) - \mathcal{H}_\alpha(x)\right) \sigma_3 \right], \end{aligned} \tag{4.80}$$

and

$$\begin{aligned} R_2^{(1)} &= \sqrt{8} \operatorname*{Res}_{z=\sqrt{8}} W^{(1)}(z) - \sqrt{8} \operatorname*{Res}_{z=-\sqrt{8}} \widehat{W}^{(1)}(z) \\ &= 2^{7/4} c_3^{1/2} \left[(\mathcal{H}_\alpha(x) - \mathcal{H}_\beta(x))\, \mathrm{i}\sigma_1 + (\mathcal{H}_\beta(x) + \mathcal{H}_\alpha(x))\, \sigma_3 \right]. \end{aligned} \tag{4.81}$$

Explicit expression for $R^{(2)}$

The RH problem (4.67) for $k = 2$ reads

$$\begin{cases} R^{(2)} \text{ is analytic in } \mathbb{C} \setminus \left(\partial U \cup \partial \widehat{U} \right), \\ R_+^{(2)}(z) = R_-^{(2)}(z) + W^{(2)}(z) + R_-^{(1)}(z) W^{(1)}(z), & z \in \partial U, \\ R_+^{(2)}(z) = R_-^{(2)}(z) + \widehat{W}^{(2)}(z) + R_-^{(1)}(z) \widehat{W}^{(1)}(z), & z \in \partial \widehat{U}, \\ R^{(2)}(z) = \mathcal{O}(1/z), & z \to \infty. \end{cases} \tag{4.82}$$

The jumps on ∂U and $\partial \widehat{U}$ consist of two terms. Both terms have an analytic extension inside U and $\widehat{U}$ with a simple pole at $\pm\sqrt{8}$. Therefore $R^{(2)}$ can be given in an explicit form in the same way as $R^{(1)}$.

Lemma 4.12. *The solution to* (4.82) *is given by*

$$\begin{aligned} R^{(2)}(z) = \frac{1}{z - \sqrt{8}} & \operatorname*{Res}_{z=\sqrt{8}} \left[W^{(2)}(z) + R^{(1)}(z) W^{(1)}(z) \right] \\ & + \frac{1}{z + \sqrt{8}} \operatorname*{Res}_{z=-\sqrt{8}} \left[\widehat{W}^{(2)}(z) + R^{(1)}(z) \widehat{W}^{(1)}(z) \right], \quad z \in \mathbb{C} \setminus (\overline{U} \cup \overline{\widehat{U}}), \end{aligned} \tag{4.83}$$

while for $z \in U$ ($z \in \widehat{U}$) we have that $R^{(2)}(z)$ is given by (4.83) *minus the jump matrix for $R^{(2)}$ on ∂U, (on $\partial \widehat{U}$).*

We can further evaluate the residues in (4.83) using Lemma 4.9. Indeed, from (4.68) and (4.69) we get

$$\begin{aligned} & \operatorname*{Res}_{z=\sqrt{8}} \left[W^{(2)}(z) + R^{(1)}(z) W^{(1)}(z) \right] \\ & \quad = \frac{1}{2} c_3 \mathcal{H}_\alpha^2(x) I + \frac{1}{2} c_3 y_\alpha(x) \sigma_2 - 2^{1/4} c_3^{1/2} \mathcal{H}_\alpha(x) R_1(\sqrt{8}) (\sigma_3 - \mathrm{i}\sigma_1) \end{aligned} \tag{4.84}$$

Since $(\sigma_3 - \mathrm{i}\sigma_1)^2 = 0$, we find that in evaluating $R_1(\sqrt{8})(\sigma_3 - \mathrm{i}\sigma_1)$ we can ignore terms in $R_1(\sqrt{8})$ involving $\sigma_3 - \mathrm{i}\sigma_1$. Using (4.68), (4.70), (4.72), and

(4.74) in the formula (4.78), we see that the only other terms in $R_1(\sqrt{8})$ involve $\sigma_3 + \mathrm{i}\sigma_1$. Sine $(\sigma_3 + \mathrm{i}\sigma_1)(\sigma_3 - \mathrm{i}\sigma_1) = 2(I + \sigma_2)$, the result is that

$$\begin{aligned} R^{(1)}(\sqrt{8})(\sigma_3 - \mathrm{i}\sigma_1) &= \left(2^{-9/4}c_3^{1/2}\mathcal{H}_\beta(x) + 2^{-9/4}\mathcal{H}_\alpha(x)\lim_{z\to\sqrt{8}}\left(\frac{z-\sqrt{8}}{f(z)}\right)^{1/2}\right)2(I+\sigma_2) \\ &= 2^{-5/4}c_3^{1/2}\left(\mathcal{H}_\beta(x) + \mathcal{H}_\alpha(x)\right)(I+\sigma_2) \end{aligned}$$

where we used (4.76) to obtain the last equality. Subsituting this in (4.84), we find

$$\begin{aligned} &\operatorname*{Res}_{z=\sqrt{8}}\left[W^{(2)}(z) + R^{(1)}(z)W^{(1)}(z)\right] \\ &\quad = \frac{1}{2}c_3\left(\mathcal{H}_\alpha^2(x)I + y_\alpha(x)\sigma_2 - \mathcal{H}_\alpha(x)\left(\mathcal{H}_\alpha(x) + \mathcal{H}_\beta(x)\right)(I+\sigma_2)\right) \\ &\quad = \frac{1}{2}c_3\left(y_\alpha(x)\sigma_2 - \mathcal{H}_\alpha\mathcal{H}_\beta I - \mathcal{H}_\alpha(\mathcal{H}_\alpha(x) + \mathcal{H}_\beta(x))\sigma_2\right). \end{aligned} \tag{4.85}$$

Similarly, we have

$$\begin{aligned} &\operatorname*{Res}_{z=-\sqrt{8}}\left[\hat{W}^{(2)}(z) + R^{(1)}(z)\hat{W}^{(1)}(z)\right] \\ &\quad = \frac{1}{2}c_3\left(y_\beta(x)\sigma_2 + \mathcal{H}_\alpha\mathcal{H}_\beta I - \mathcal{H}_\beta(\mathcal{H}_\alpha(x) + \mathcal{H}_\beta(x))\sigma_2\right). \end{aligned} \tag{4.86}$$

If we use (4.85) and (4.86) in (4.83) and expand around $z = \infty$, we obtain the following corollary.

Corollary 4.13. *We have that*

$$R^{(2)}(z) = \frac{R_1^{(2)}}{z} + \frac{R_2^{(2)}}{z^2} + \mathcal{O}(z^{-3}), \qquad z \to \infty \tag{4.87}$$

where

$$R_1^{(2)} = \frac{1}{2}c_3\left(y_\alpha(x) + y_\beta(x) - (\mathcal{H}_\alpha(x) + \mathcal{H}_\beta(x))^2\right)\sigma_2 \tag{4.88}$$

and

$$R_2^{(2)} = \sqrt{2}c_3\left((y_\alpha(x) - y_\beta(x))\sigma_2 - 2\mathcal{H}_\alpha(x)\mathcal{H}_\beta(x)I - (\mathcal{H}_\alpha^2(x) - \mathcal{H}_\beta^2(x))\sigma_2\right). \tag{4.89}$$

4.10 Recurrence coefficients in terms of R

We start from the formulas (2.9) and (2.10) that express $a_{n,n}(t)$ and $b_{n,n}(t)$ in terms of the solution Y of the RH problem (2.3). Following the effects of the transformations $Y \mapsto T \mapsto S \mapsto R$ we obtain the following expressions for $a_{n,n}(t)$ and $b_{n,n}(t)$ in terms of the coefficients R_1 and R_2 in the expansion

$$R(z) = I + \frac{R_1}{z} + \frac{R_2}{z^2} + \mathcal{O}(z^{-3}), \qquad z \to \infty. \tag{4.90}$$

Lemma 4.14. *We have that*

$$a_{n,n}(t) = \left((R_1)_{21} - \mathrm{i}\sqrt{2}\right)\left((R_1)_{12} + \mathrm{i}\sqrt{2}\right) \tag{4.91}$$

and

$$b_{n,n}(t) = \frac{(R_1)_{11} - 2^{-1/2}\mathrm{i}(R_2)_{12}}{1 - 2^{-1/2}\mathrm{i}(R_1)_{12}} - (R_1)_{22}. \tag{4.92}$$

Proof. Since ν_t is a measure with an even density, we have

$$\begin{aligned} g_t(z) &= \int \log(z-x)\, \mathrm{d}\nu_t(x) \\ &= \log(z) - \frac{1}{z}\int x\, \mathrm{d}\nu_t(x) - \frac{1}{2z^2}\int x^2\, \mathrm{d}\nu_t(x) + \mathcal{O}(z^{-3}) \\ &= \log(z) - cz^{-2} + \mathcal{O}(z^{-3}), \qquad z \to \infty, \end{aligned}$$

for some constant c. Then

$$\mathrm{e}^{ng_t(z)} = z^n\left(1 - cnz^{-2} + \mathcal{O}(z^{-3})\right),$$

so that we get from (4.14) and (2.8) that

$$\begin{aligned} T(z) &= \mathrm{e}^{-nl_t\sigma_3/2}\left(I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \mathcal{O}(z^{-3})\right)\left(I + \frac{cn\sigma_3}{z^2} + \mathcal{O}(z^{-3})\right)\mathrm{e}^{nl_t\sigma_3/2} \\ &= I + \frac{T_1}{z} + \frac{T_2}{z^2} + \mathcal{O}(z^{-3}), \qquad z \to \infty, \end{aligned} \tag{4.93}$$

with $T_1 = \mathrm{e}^{-nl_t\sigma_3/2}Y_1\mathrm{e}^{nl_t\sigma_3/2}$ and $T_2 = \mathrm{e}^{-nl_t\sigma_3/2}\left(Y_2 + cn\sigma_3\right)\mathrm{e}^{nl_t\sigma_3/2}$. Then we find after some straightforward calculations

$$a_{n,n}(t) = (Y_1)_{21}(Y_1)_{12} = (T_1)_{21}(T_1)_{12} \tag{4.94}$$

and

$$b_{n,n}(t) = \frac{(Y_2)_{12}}{(Y_1)_{12}} - (Y_1)_{22} = \frac{(T_2)_{12}}{(T_1)_{12}} - (T_1)_{22}. \tag{4.95}$$

Since by (4.17) and (4.55) we have that $T = S = RM$ outside the disks and the lens, and M has the expansion

$$M(z) = I + \frac{M_1}{z} + \frac{M_2}{z^2} + \mathcal{O}(z^{-3}) = I - \frac{\sqrt{2}}{z}\sigma_2 + \frac{1}{z^2}I + \mathcal{O}(z^{-3}), \qquad z \to \infty.$$

we find from (4.90) and (4.93) that

$$T_1 = R_1 + M_1 = R_1 - \sqrt{2}\sigma_2, \qquad T_2 = R_2 + R_1M_1 + M_2 = R_2 - \sqrt{2}R_1\sigma_2 + I. \tag{4.96}$$

Inserting (4.96) into (4.94) and (4.95) we arrive at (4.91) and (4.92). □

4.11 Proof of Theorem 1.2

Finally, we are ready for the proof of Theorem 1.2.

Proof. For each $k \geq 1$ we have that $R^{(k)}$ has a Laurent expansion at infinity

$$R^{(k)}(z) = \frac{R_1^{(k)}}{z} + \frac{R_2^{(k)}}{z^2} + \mathcal{O}(z^{-3}), \qquad z \to \infty.$$

Because of (4.65) and (4.90) we then get that R_1 and R_2 have asymptotic expansions in powers of $n^{-1/5}$,

$$R_j \sim \sum_{k=1}^{\infty} R_j^{(k)} n^{-k/5}, \qquad j = 1, 2. \tag{4.97}$$

Using (4.97) in (4.91) and (4.92) we find that $a_{nn}(t)$ and $b_{n,n}(t)$ have asymptotic expansions in powers of $n^{-1/5}$.

Inserting

$$R_1 = R_1^{(1)} n^{-1/5} + R_1^{(2)} n^{-2/5} + \mathcal{O}(n^{-3/5}), \tag{4.98}$$

into (4.91) yields

$$\begin{aligned} a_{n,n}(t) = 2 &+ \sqrt{2}\mathrm{i}\left(\left(R_1^{(1)}\right)_{21} - \left(R_1^{(1)}\right)_{21}\right) n^{-1/5} \\ &+ \left(\left(R_1^{(1)}\right)_{12}\left(R_1^{(1)}\right)_{21} + \sqrt{2}\mathrm{i}\left(\left(R_1^{(2)}\right)_{21} - \left(R_1^{(2)}\right)_{12}\right)\right) n^{-2/5} + \mathcal{O}(n^{-3/5}) \end{aligned} \tag{4.99}$$

From the explicit expression for $R_1^{(1)}$ in (4.80) it follows that

$$\left(R_1^{(1)}\right)_{12} = \left(R_1^{(1)}\right)_{21} = 2^{1/4}c_3^{1/2}\mathrm{i}\left(\mathcal{H}_\alpha(x) + \mathcal{H}_\beta(x)\right) \tag{4.100}$$

and so the order $n^{-1/5}$ in (4.99) vanishes. From the explicit expression for $R_1^{(2)}$ in (4.88) it follows that

$$\left(R_1^{(2)}\right)_{21} = -\left(R_1^{(2)}\right)_{12} = \frac{1}{2}c_3\mathrm{i}\left(y_\alpha(x) + y_\beta(x) - (\mathcal{H}_\alpha(x) + \mathcal{H}_\beta(x))^2\right). \tag{4.101}$$

Therefore by (4.100) and (4.101) the coefficient in the term of order $n^{-2/5}$ in (4.99) is

$$\begin{aligned}
-2^{1/2}c_3\left(\mathcal{H}_\alpha(x) + \mathcal{H}_\beta(x)\right)^2 - 2^{1/2}c_3\left(y_\alpha(x) + y_\beta(x) - (\mathcal{H}_\alpha(x) + \mathcal{H}_\beta(x))^2\right) \\
= -2^{1/2}c_3(y_\alpha(x) + y_\beta(x)).
\end{aligned}$$

This proves equation (1.28) in Theorem 1.2 since $c_2 = 2^{1/2}c_3$.

The calculations for $b_{n,n}(t)$ are slightly more involved. Besides (4.98) we also use

$$R_2 = R_2^{(1)}n^{-1/5} + R_2^{(2)}n^{-2/5} + \mathcal{O}(n^{-3/5}). \tag{4.102}$$

and the explicit expressions (4.80)-(4.81) and (4.88)-(4.89) for $R_j^{(k)}$, $j,k = 1,2$. This yields

$$\begin{aligned}
(R_1)_{11} - 2^{-1/2}\mathrm{i}(R_2)_{12} = 2^{1/4}c_3^{1/2}(\mathcal{H}_\alpha(x) - \mathcal{H}_\beta(x))n^{-1/5} \\
+ c_3(-y_\alpha(x) + y_\beta(x) + \mathcal{H}_\alpha^2(x) - \mathcal{H}_\beta^2(x))n^{-2/5} + \mathcal{O}(n^{-3/5}),
\end{aligned} \tag{4.103}$$

$$1 - 2^{-1/2}\mathrm{i}(R_1)_{12} = 1 + 2^{-1/4}c_3^{1/2}(\mathcal{H}_\alpha(x) + \mathcal{H}_\beta(x))n^{-1/5} + \mathcal{O}(n^{-2/5}) \tag{4.104}$$

and

$$(R_1)_{22} = 2^{1/4}c_3^{1/2}(\mathcal{H}_\alpha(x) - \mathcal{H}_\beta(x))n^{-1/5} + \mathcal{O}(n^{-3/5}). \tag{4.105}$$

Inserting (4.103)-(4.105) into (4.92), we obtain (1.29).

This completes the proof of Theorem 1.2. □

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