Paired Approximation Problems and Incompatible Inapproximabilities

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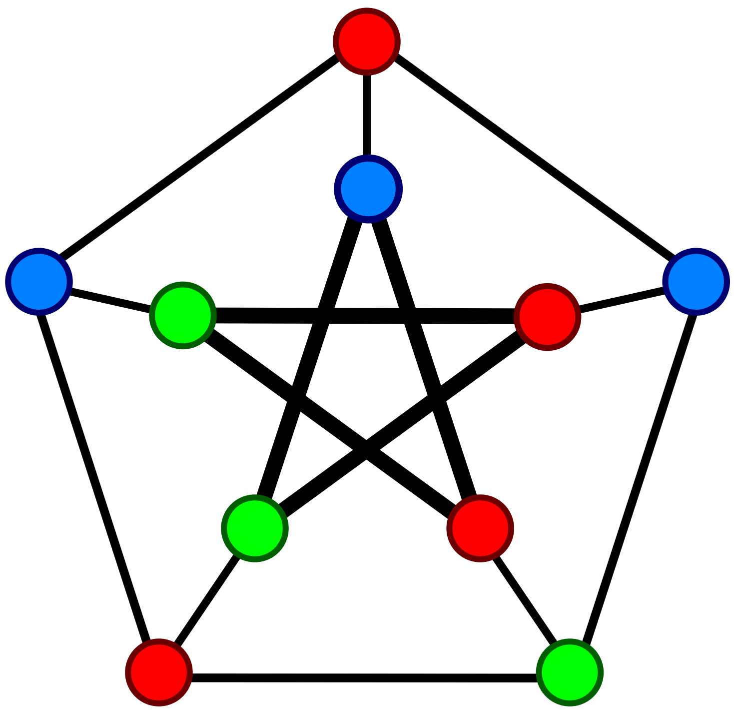
In a nutshell:

Given two approximation
problems A and B

we want an algorithm that
solves one of them
(it gets to choose which one)

What we would expect —
it's as hard as the easier of the two
approximation problems

What actually happens —
it can sometimes be
even easier than that



Graph coloring

Assign colors to vertices of a graph
so each edge has two colors
using as few colors as possible

NP-complete [Karp 1972]

Best known approximation
$O(n/\log^2 n)$ [Boppana&Halldórsson 1992]

Unless P=NP, no polynomial
approximation better than $O(n^{1-\varepsilon})$
[Zuckerman 2006]

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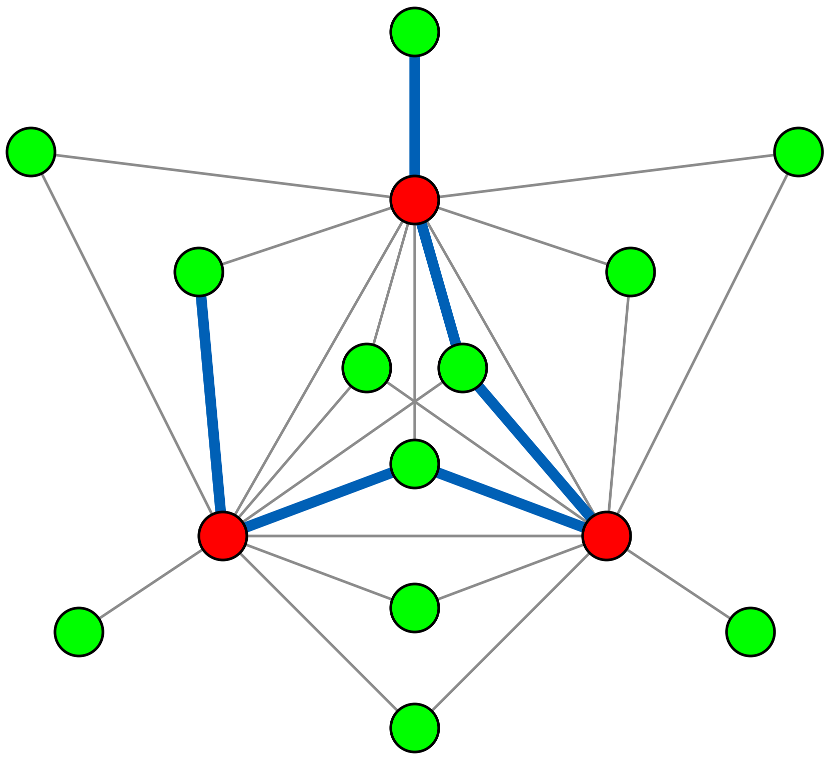
Longest path



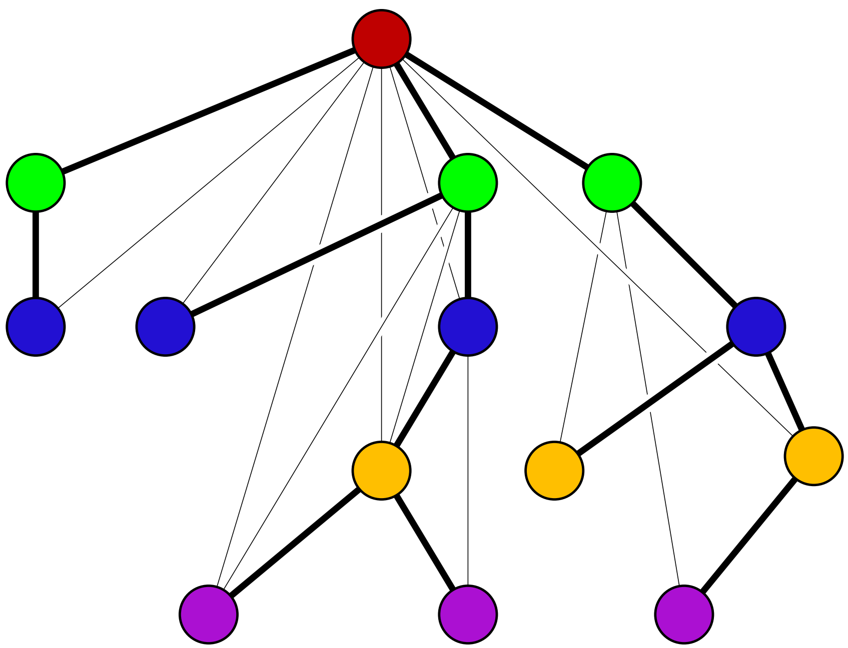
Find a path of as many steps as possible in a given graph, avoiding repeated vertices

NP-complete [Garey & Johnson 1979]

Best known approximation for undirected graphs $O(n(\log \log n \,/\, \log n)^2)$
[Björklund & Husfeldt 2003]
for digraphs $O(n \,/\, \log n)$
[Alon, Yuster & Zwick 1995]

Unless P=NP, no polynomial approximation (for digraphs) better than $O(n^{1-\varepsilon})$
[Bjorklund, Husfeldt & Khanna 2004]

Simultaneous approximation for coloring + long path



Find a depth-first search tree

If it contains a path of length ≥ $\sqrt{n}$, return it

Otherwise, color vertices by their level in the tree

Result is ≤ $\sqrt{n}$ approximation either to optimal coloring or optimal long path

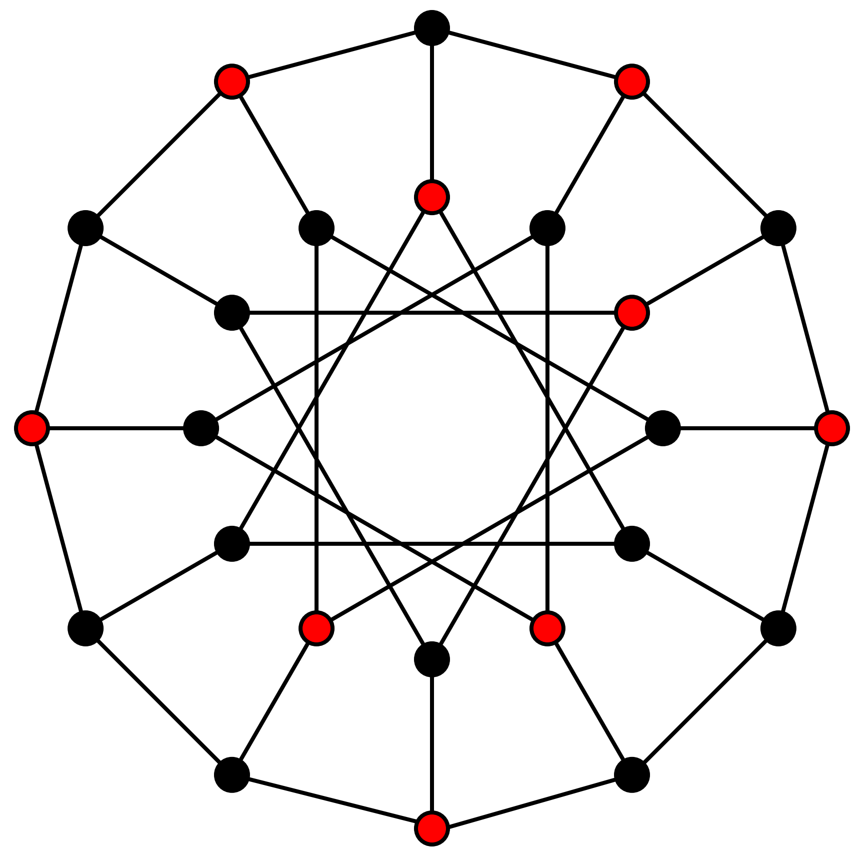
Maximum independent set

Subset of vertices of a graph
no two adjacent
using as many vertices as possible

NP-complete [Cook 1971]

Best known approximation
$O(n(\log \log n)^2/\log^3 n)$
[Feige 2004]

Unless P=NP, no polynomial
approximation better than $O(n^{1-\varepsilon})$
[Zuckerman 2006]



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Traveling salesman

Cyclic ordering of the points
in a metric space
minimizing sum of distances
of adjacent pairs

NP-complete
even when all distances in {1,2}
[Garey & Johnson 1979]

Best known approximation 3/2
[Christofides 1976]
or 8/7 when all distances in {1,2}
[Berman & Karpinski 2006]

Unless P=NP, no polynomial
approximation better than 741/740
for distances in {1,2}
[Engebretsen & Karpinski 2001]

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Simultaneous approximation for independent set & TSP

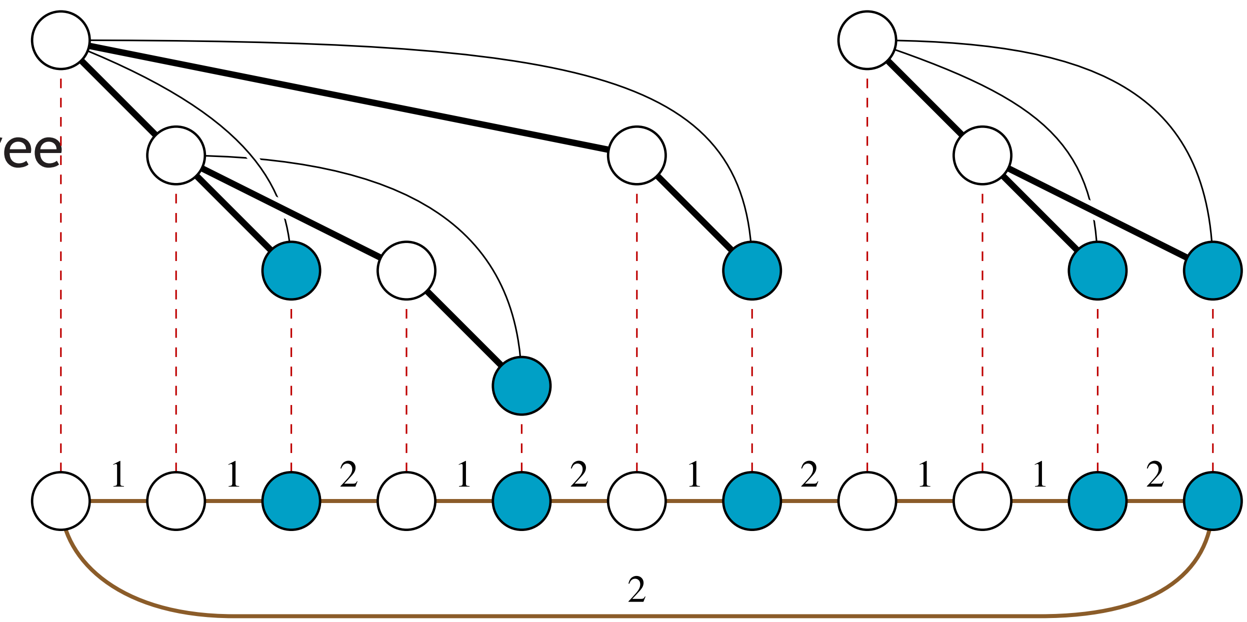
Given graph representing distance-1 pairs

Find a depth-first search tree

If it has ≥ n/ε leaves, return the leaves as an independent set

Otherwise, use a preorder traversal as a TSP tour

Result is 1/ε-approximation to independent set or (1+ε)-approximation to TSP



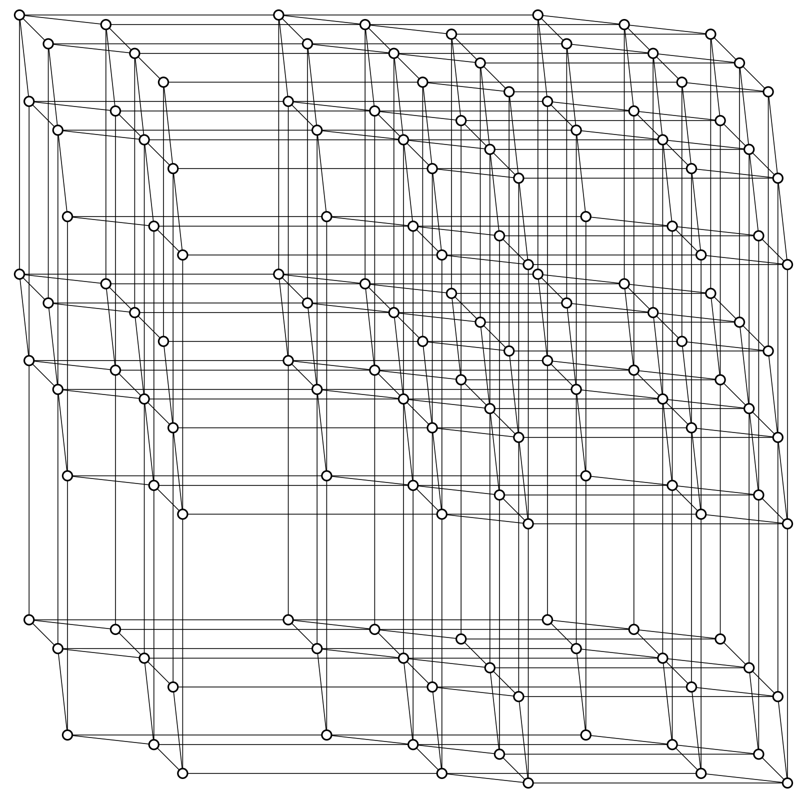
Motivation

Find minimum dimension embedding of a graph into a Fibonacci cube [Cabello, E. & Klavzar 2009]

Translate into finding TSP of (1,2) metric on auxiliary graph

If approximation algorithm finds a good TSP, done

If it finds a big independent set then auxiliary graph is tiny enough to apply exponential algorithms on it



10-dimensional Fibonacci cube

Related combinatorial inequalities

Chromatic number ≤ clique minor size

Hadwiger's conjecture [H. 1943], still unproven

Min leaves in spanning tree ≤ max independent set size

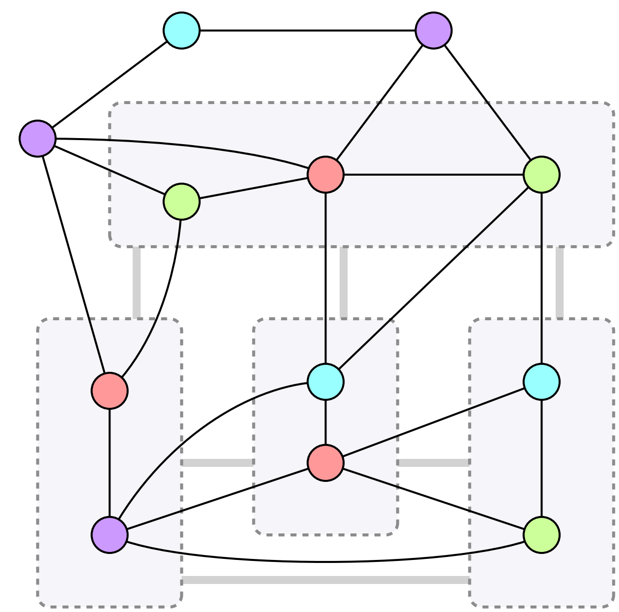
Use DFS tree again [Gargano et al. 2004, Sun 2007]

Treewidth ≤ longest path length

Use DFS tree again [Bodlaender 1993]

Chromatic number ≤ longest path length

(1,2)-TSP ≤ n + max independent set size



A 4-vertex clique minor
in a 4-chromatic graph

Formalization

Given a language L of problem instances, and objective functions f_0 and f_1 from instances to real numbers:

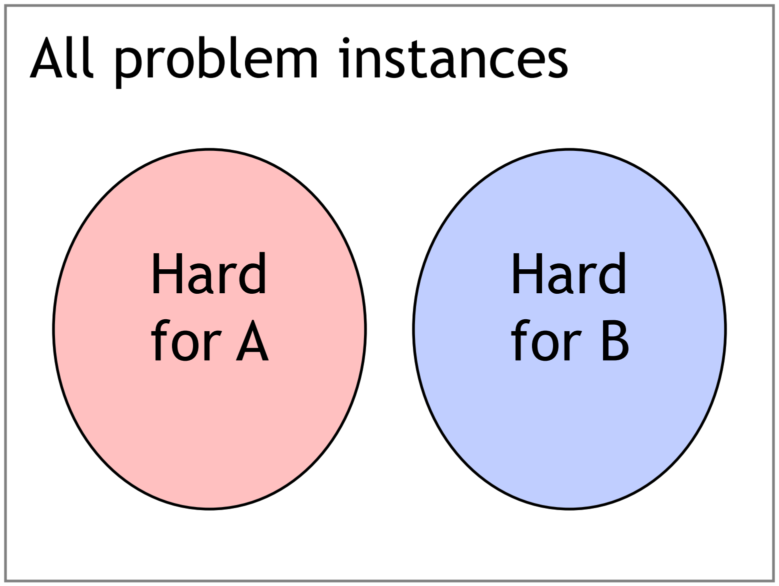
Algorithm A is a "simultaneous approximation algorithm" if for any x it returns a pair (i,y) with i in {0,1} and $y \geq f_i(x)$

It is a "(g_0,g_1)-simultaneous approximation algorithm" if, for all instances x of length n, $y \leq g_i(n)\, f_i(x)$

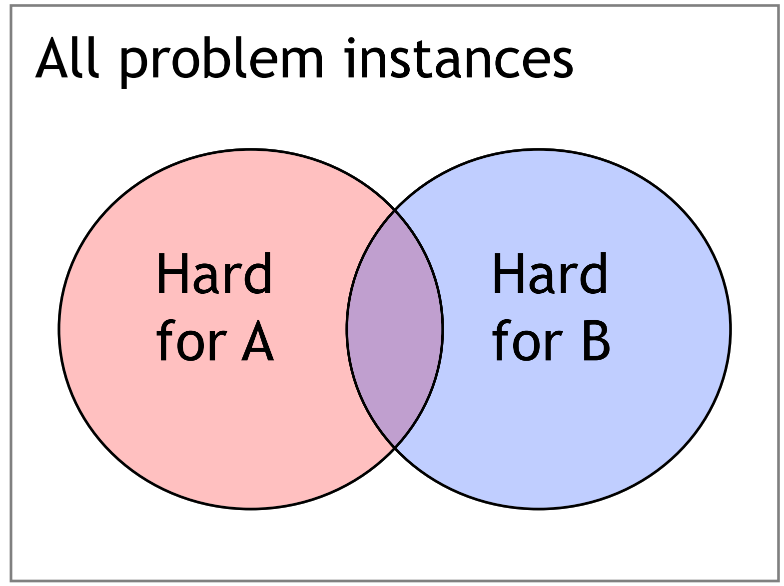
(Coloring, Longest Path) has a $(\sqrt{n}, \sqrt{n})$-simultaneous approximation

(Indep.Set, (1,2)-TSP) has a $(1/\varepsilon, 1 + \varepsilon)$-simultaneous approximation

Informalization



Has simultaneous approximation better than approximation ratio for separate problems



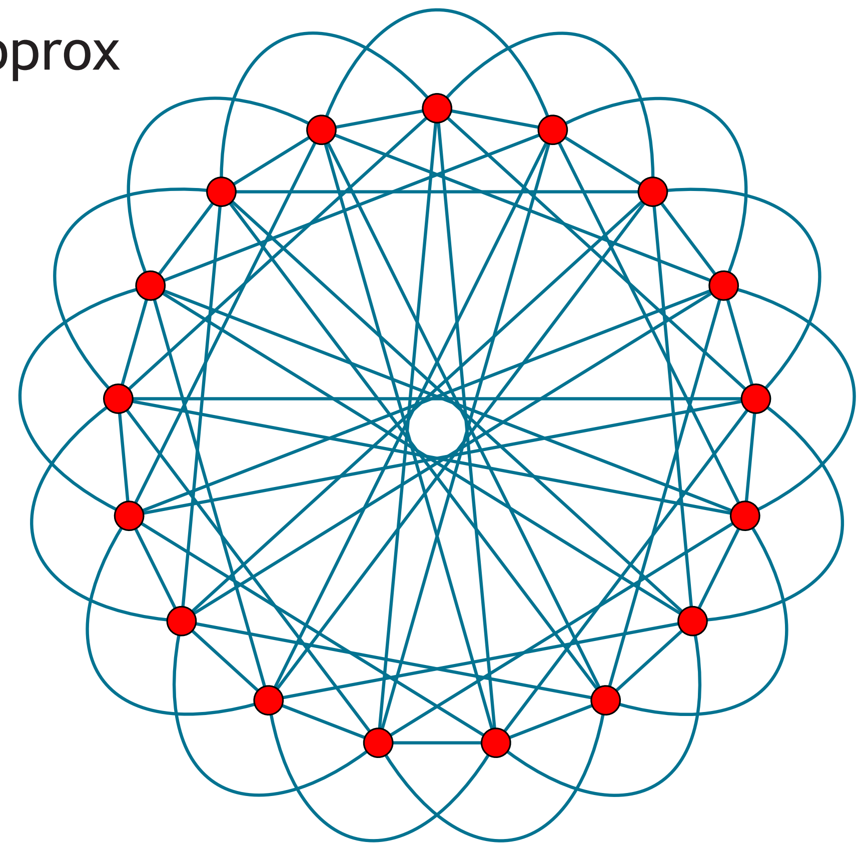
Simultaneous approximation is not better than approximation ratio for separate problems

Simultaneous approx of clique and independent set?

Not quite the same as classical approx of largest homogeneous subgraph

(we allow approximating the smaller of the two as long as the approximation is accurate)

Ramsey's theorem trades off clique size vs independent set but only weakly (logarithmically)



(4,4)-Ramsey graph

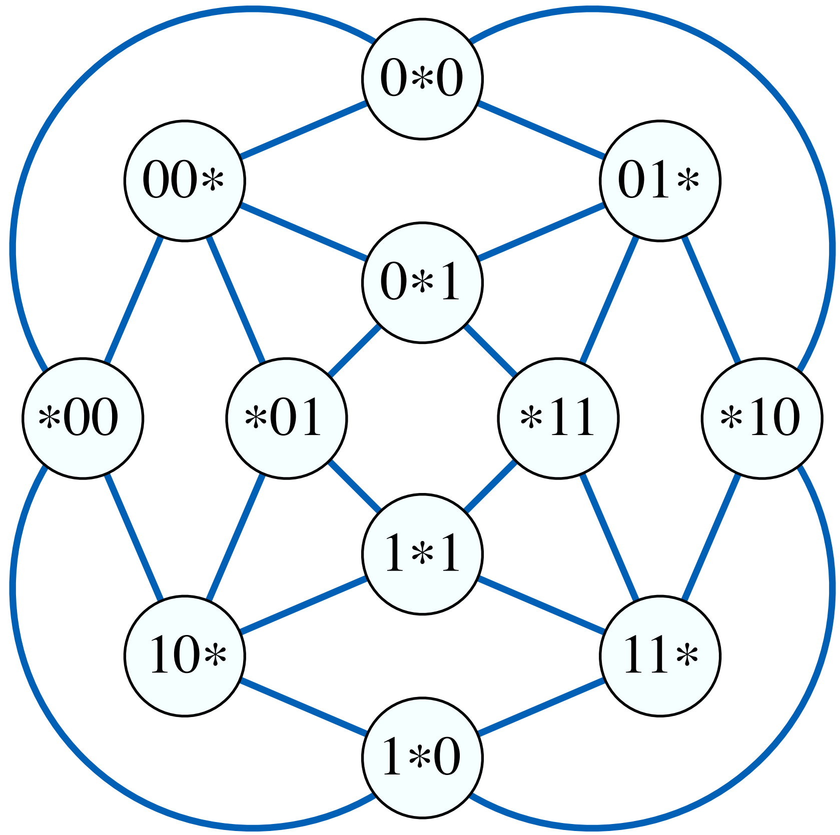
Hard instances for clique

Form graph in which vertices represent k-bit samples from a longer q-bit "proof string"

Cliques represent subsets of samples from the same string

Keep only vertices representing computations of a PCP checker that cause it to accept a proof

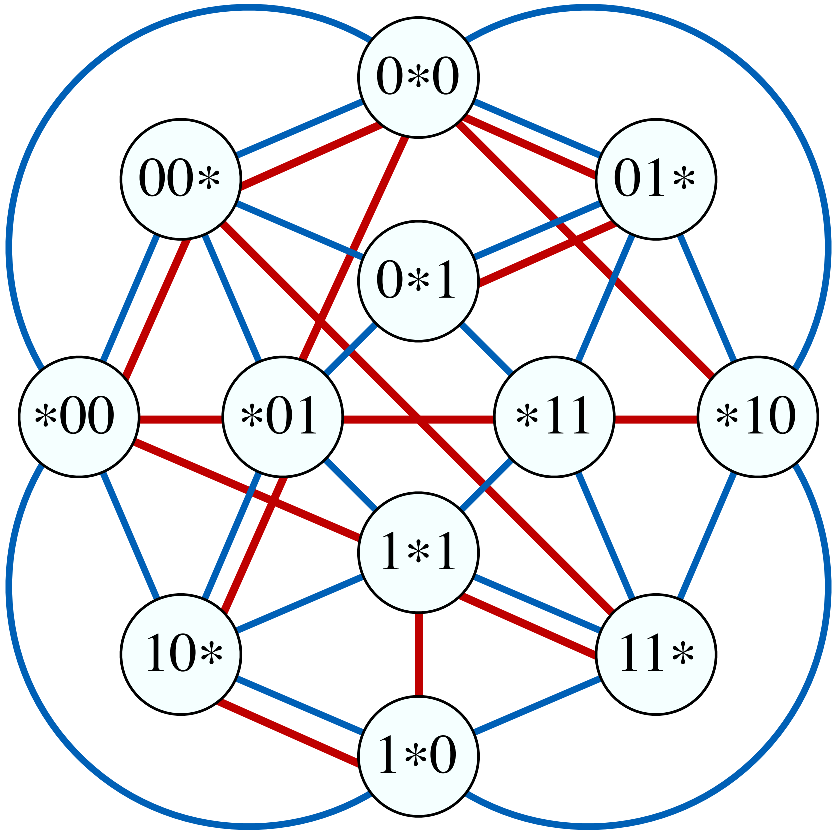
Remaining graph has a big clique iff some proof string is accepted by many PCP checkers, iff initial SAT instance is satisfiable



Compatibility graph
of 2-bit samples from 3-bit strings

Controlling independence in PCP graphs

PCP graph



Ramsey graph
(no large bicliques
or independent sets)

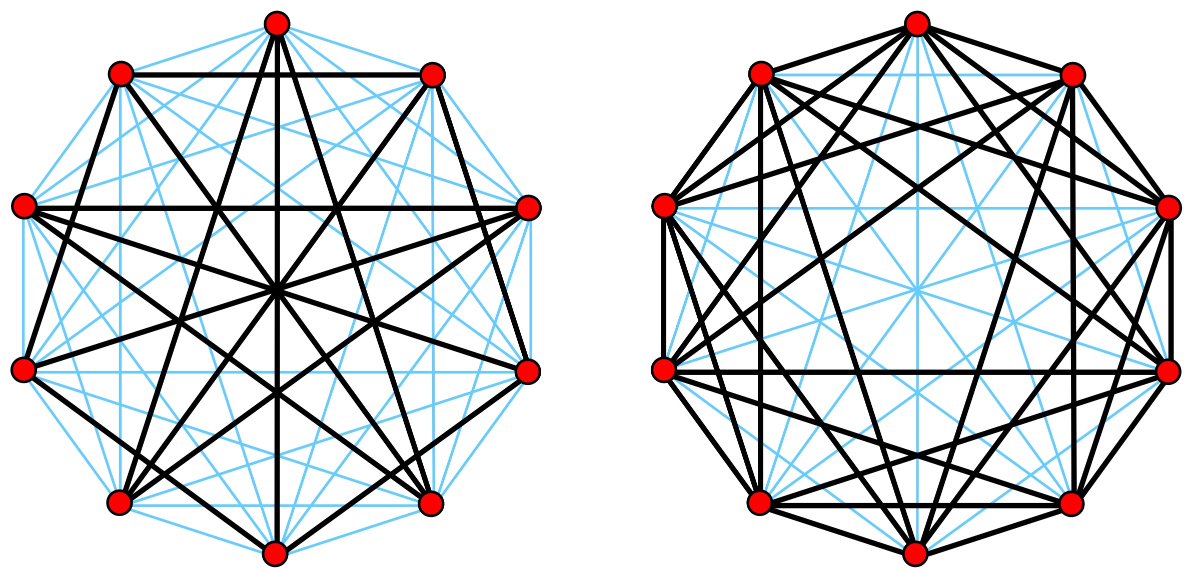
Ramsey graph kills all large independent sets in the union

Due to PCP graph structure, union has no new large cliques

clique members vote on proof string bits
⇒ most clique members disagree with consensus string somewhere
⇒ single position where a large minority disagrees
⇒ large biclique in Ramsey graph

Hardness of simultaneous approximation

Satisfiability ⇒ PCP ⇒ Hard graph for clique
⇒ Union with Ramsey graph
⇒ Disjoint union with self-complement



Resulting graph has large clique (in uncomplemented part)
and a large independent set (in complemented part)
iff input problem is satisfiable

Unless P = NP, no simultaneous approximation
for clique and independent set better than $O(n^{1 - \varepsilon})$

Results

Has simultaneous approximation:

- Coloring + longest path
- Independent set + (1,2)-TSP
- Coloring + clique minor
- Induced acyclic subgraph + directed longest path

Hard to approximate:

- Clique + independent set
- Set cover + hitting set
- (1,2)-TSP + (1,2)-MaxTSP

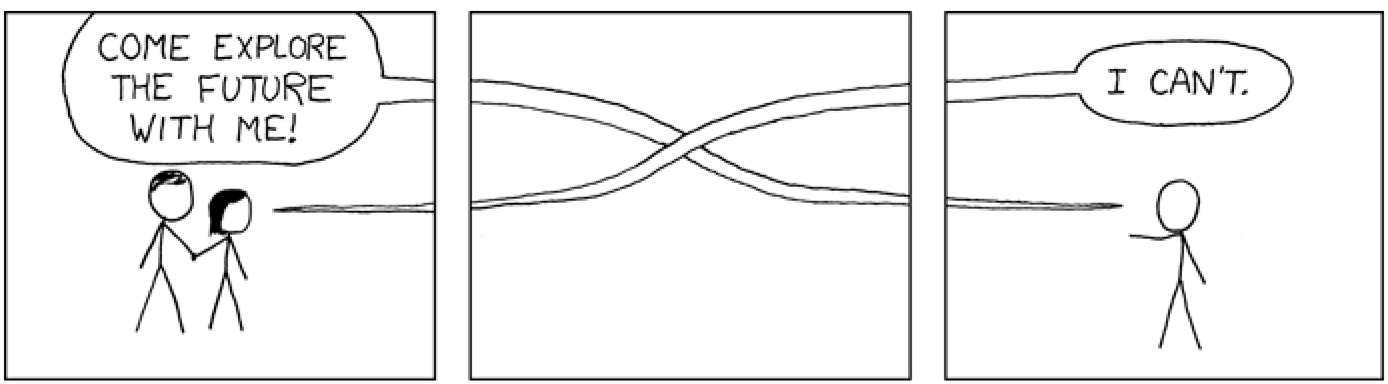
Conclusions and open problems

New and interesting class of approximation problems

Shows limitations of inapproximability proofs
(hardness is not preserved when combining problems)

Upper and lower bounds for more pairs of problems?

Group problems into sets that behave similarly
wrt existence of simultaneous approximations,
avoiding combinatorial explosion in # pairs?



http://xkcd.com/338/