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PAIRED COMPARISONS FOR MULTIPLE CHARACTERISTICS: AN ANOCOVA APPROACH

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Abstract. An analysis of covariance model is developed for paired comparisons to situations in which responses (on a preference order) to paired comparisons are obtained on some primary as well as concomitant traits. Along with the general rationality of the proposed test, its asymptotic properties are studied.

1. Introduction

The method of paired comparisons as originally developed by psychologists relates to a number of objects which are presented in pairs to a set of judges who verdict (independently) a relative preference of one over the other within each pair. This dichotomous response allows subjective judgement to a greater extent than in the so called method of m–ranking where each judge has to rank simultaneously all the objects. Paired comparisons designs are thus incomplete block designs with blocks of two plots and the dichotomous response relate to the ordering of the intra–block plot yields. As such, in paired comparisons designs circular triads may arise in a natural way, and this may lead to intrasitiveness of statistical procedures in a decision theoretic formulation. This intransitiveness is also shared by the Pitman (1937) measure of closeness (PMC) where competing estimators belonging to a class are only compared by pairs at a time. While the plausible intransitiveness may concern some decision theorists, in many real life problems this may crop up in a natural way, and therefore there is a genuine need to incorporate such a phenomenon as a vital component of the basic problem. Such

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examples abound in paired comparisons models and elsewhere [viz., Keating, Mason and Sen (1993)], and David (1988) addressing this issue in a superb manner commented in a different perspective: "It is a valuable feature of the method of paired comparisons that allows such contradictions to show themselves..." Nevertheless, in paired comparisons designs, suitable models have been considered by a host of workers which take into account such intransitivenss in an appropriate manner.

Paired comparisons for multiple characteristics have received due attention in the past [viz., Sen and David (1968), Davidson and Bradley (1969, 1970) and David (1988), among others]. The basic point is that with respect to p (≥ 1) characteristics or traits, for each pair of objects, the outcome response is a p–vector $\underset{\sim}{s} = (s_1,...,s_p)'$ where each s_j is either $<$ or $>$ (ordering), so that there are 2^p possible realizations. [If ties are admissible, this number may jump to 3^p.] Therefore with t objects forming $\binom{t}{2}$ pairs, the totality of response vectors in $2^p\binom{t}{2}$, so that intransitivenss may crop up even more noticably when p is > 1. This puts an additional constraint on multivariate paired comparisons procedures. Yet most of these procedures developed in the spirit of multi–variate analysis of variance (MANOVA) models address this intransitiveness issue in a rational way. Often, some of the characteristics may conveniently be regarded as primary traits while the others as concomitant ones. Analysis of covariance (ANOCOVA) models are therefore relevant to such paired comparisons models and will be studied here.

In Section 2, this ANOCOVA paired comparisons model is illustrated for the simplest bivariate case, with motivations from Chatterjee (1966) and Sen and David (1968). A basic assumption needed in this context is clarified; this also applies to the work of Davidson and Bradley (1969, 1970). A representation of probability laws for p (≥ 1) dichotomous attributes, similar to Bahadur (1961), is exploited in Section 3 to stress this fundamental assumption needed for multivariate paired comparisons. Section 4 deals with general MANOCOVA paired comparisons models. Some general remarks are appended in the concluding section.

2. ANOCOVA paired comparisons for paired characteristics

Consider t (≥ 2) objects which are considered in pairs (i,j), $1 \leq i < j \leq t$, and judged with respect to two characteristics, say 1 and 2. Thus, for the pair (i,j), the response $\underset{\sim}{X}_{ij} = (X_{ij}^{(1)}, X_{ij}^{(2)})'$ is a stochastic vector where $X_{ij}^{(\ell)}$ takes on the value +1 or −1 according as the ith object is judged better (or not) than the jth one on the ℓth trait, for $\ell = 1,2$; $1 \leq i < j \leq t$. Let $\pi_{++}^{(ij)}$, $\pi_{+-}^{(ij)}$, $\pi_{-+}^{(ij)}$ and $\pi_{--}^{(ij)}$ be respectively the probability that $\underset{\sim}{X}_{ij}$ is $(+1, +1)'$, $(+1, -1)'$, $(-1, +1)'$ and $(-1,-1)'$. This leads to a 2×2 table

$$\begin{array}{cc|c} \pi_{++}^{(ij)} & \pi_{-+}^{(ij)} & \pi_{\cdot+}^{(ij)} \\ \pi_{+-}^{(ij)} & \pi_{--}^{(ij)} & \pi_{\cdot-}^{(ij)} \\ \hline \pi_{+\cdot}^{(ij)} & \pi_{-\cdot}^{(ij)} & 1 \end{array} \tag{2.1}$$

where

$$\pi_{\cdot-}^{(ij)} = 1 - \pi_{\cdot+}^{(ij)} \quad \text{and} \quad \pi_{-\cdot}^{(ij)} = 1 - \pi_{+\cdot}^{(ij)} . \tag{2.2}$$

As in Chatterjee (1966), we introduce an association parameter

$$\theta_{ij} = \pi_{++}^{(ij)} / (\pi_{+\cdot}^{(ij)} \pi_{\cdot+}^{(ij)}) - 1, \qquad 1 \leq i < j \leq t. \tag{2.3}$$

Then, we may rewrite the $\pi_{\pm\pm}^{(ij)}$ equivalently as

$$\pi_{\pm\pm}^{(ij)} = \pi_{\pm\cdot}^{(ij)} \pi_{\cdot\pm}^{(ij)} + (\pm + \pm)\theta_{ij} \pi_{+\cdot}^{(ij)} \pi_{\cdot+}^{(ij)}, \tag{2.4}$$

for $1 \leq i < j \leq t$, and for $j < i$, $\pi_{\pm\pm}^{(ij)} = \pi_{\mp\mp}^{(ji)}$ can be represented in a similar way. Note that for each (i,j), we reparameterize the vector $\underset{\sim}{\pi}^{(ij)} = (\pi_{++}^{(ij)}, \pi_{+-}^{(ij)}, \pi_{-+}^{(ij)}, \pi_{--}^{(ij)})'$ as $(\pi_{+\cdot}^{(ij)}, \pi_{\cdot+}^{(ij)}, \theta_{ij})$. In this manner, for the $\pi_{+\cdot}^{(ij)}$ (or $\pi_{\cdot+}^{(ij)}$) we may use the univariate paired comparisons models, while the association parameter θ_{ij} is a nuisance parameter. Let us formulate the two null hypotheses relating to the marginals as

$$H_0^{(1)} : \pi_{+\cdot}^{(ij)} = 1/2, \ \forall\ 1 \leq i < j \leq t, \tag{2.5}$$

$$H_0^{(2)} : \pi_{\cdot+}^{(i,j)} = 1/2, \forall\ 1 \leq i < j \leq t, \tag{2.6}$$

and note that under $H_0 = H_0^{(1)} \cap H_0^{(2)}$ we may not still have the homogeneity of the $\binom{t}{2}$ tables in (2.1). This brings us to the following (fundamental assumption):

[A] $$\theta_{ij} = \theta, \ \forall \ 1 \le i < j \le t. \tag{2.7}$$

In Sen and David (1968), (2.7) was explicitly formulated through the Chatterjee (1966) characterization, while in Davidson and Bradley (1969), it was made without any explanation. With paired characteristics, a distribution–free (even conditionally) test rests on the homogeneity of the $\underset{\sim}{\pi}^{(ij)}$, $1 \le i < j \le t$, and hence, if one wants to test for H_0, pertaining to the marginals, [A] is a vital presumption. Since the $\underset{\sim}{\pi}^{(ij)}$ do not conform to a location–scale model, in general, the homogeneity of the θ_{ij} may not directly follow from the marginal homogenity of the $\pi_{+\cdot}^{(ij)}$ or $\pi_{\cdot+}^{(ij)}$.

Granted [A] in (2.7), for testing (2.5) and (2.6), we may readily appeal to the classical paired comparisons model with single characteristics, and hence, as in the Bradley–Terry (1952) model, we may set

$$\pi_{+\cdot}^{(ij)} = \frac{\alpha_i}{\alpha_i + \alpha_j}, \quad \pi_{\cdot+}^{(ij)} = \frac{\beta_i}{\beta_i + \beta_j}, \quad 1 \le i < j \le t, \tag{2.8}$$

where the α_i and β_i are nonnegative quantities and

$$\Sigma_{i=1}^{t} \alpha_i = 1 = \Sigma_{j=1}^{t} \beta_j. \tag{2.9}$$

Note that under [A], the number of free parameters appearing in (2.4) (for $1 \le i < j \le t$) is $2\binom{t}{2} + 1$, where the modeling in (2.8)–(2.9) reduces it to $2(t-1)+1 = 2t-1$. This reduction of the parameter space was made in Sen and David (1968) through a permutation (sign–invariance) argument of Chatterjee (1966), while Davidson and Bradley (1969) motivated this primarily through the classical likelihood ratio test (in a parametric setup), and thereby their test is essentially a large sample one.

In a conventional linear ANOCOVA model, the basic assumptions are the following [see Scheffé (1959)]:

(i) The concomitant vector $\underset{\sim}{Z}$ has a distribution unaffected by plausible treatment differences,

(ii) The regression of the primary variate (Y) on $\underset{\sim}{Z}$ is linear with a conditional variance $\sigma^2_{Y\cdot\underset{\sim}{Z}}$ independent of $\underset{\sim}{Z}$ (the homoscedasticity condition), and

(iii) treatment effects are additive, and the errors are normally distributed.

Therefore conditionally on $\underset{\sim}{Z} = \underset{\sim}{z}$, Y has a linear regression function depending on $\underset{\sim}{z}$ by a term $\underset{\sim}{\gamma}'\underset{\sim}{z}$ and the design variables, its conditional variance, given $\underset{\sim}{Z} = \underset{\sim}{z}$, $\sigma^2_{Y\cdot\underset{\sim}{z}}$ is $\leq \sigma^2_Y$, its marginal variance, and the equality sign holds only when the multiple correlation of Y on $\underset{\sim}{Z}$ is equal to 0. Thus, in an ANOCOVA model, one ends up with a smaller mean square due to errors, albeit the degrees of freedom (DF) due to error is smaller than in the ANOVA model; the reduction of DF is equal to rank $(\underset{\sim}{\gamma})$. This reduction of the error mean square in ANOCOVA accounts for its greater efficacy than ANOVA (unless the error DF is sufficiently small to compensate this gain).

Our formulation of ANOCOVA paired comparisons model is adapted from Sen and David (1968). The conventional linear ANOCOVA model is not useful in the current context mainly because (i) the primary as well as concomitant response variables are binary, and (ii) due to possible intransitiveness, other constraints on the model may be necessary. Borrowing the first assumption in the linear ANOCOVA model, we may assume that for the ANOCOVA paired comparisons model

$$\pi^{(ij)}_{\cdot+} = \tfrac{1}{2},\ \forall\ 1 \leq i < j \leq t \Rightarrow \beta_1 = \cdots = \beta_t = t^{-1} \tag{2.10}$$

where the β_j are defined by (2.8). We may also assume that the $\pi^{(ij)}_{+\cdot}$ satisfy (2.8) with nonnegative $\alpha_1,\ldots,\alpha_t : \Sigma^t_{j=1}\ \alpha_j = 1$. The third assumption in the linear ANOCOVA model is not tenable for paired comparisons, while to justify the homoscedasticity condition (in (ii)), we may note that by (2.4), (2.7) and (2.8),

$$\pi^{(ij)}_{++}/\pi^{(ij)}_{\cdot+} = \pi^{(ij)}_{+\cdot}\,(1+\theta),$$

$$\pi^{(ij)}_{+-}/\pi^{(ij)}_{\cdot-} = \pi^{(ij)}_{+\cdot}\,(1-\theta),\ \ 1 \leq i < j \leq t, \tag{2.11}$$

so that under $H^{(1)}_0$ in (2.5), $\pi^{(ij)}_{++}/\pi^{(ij)}_{\cdot+} = \pi^{(ij)}_{--}/\pi^{(ij)}_{\cdot-} = \frac{1}{2}\,(1+\theta)$ and $\pi^{(ij)}_{+-}/\pi^{(ij)}_{\cdot-} = \pi^{(ij)}_{-+}/\pi^{(ij)}_{\cdot+} = \frac{1}{2}\,(1-\theta)$, and the homoscedasticity condition holds (as the common variance is equal to $\frac{1}{4}(1-\theta^2)$.) On the other hand, if [A] in (2.7) may not hold or if $H_0^{(1)}$ does not hold this homoscedasticity may not be true. This explains the basic role of [A] in (2.7) for formulating an ANOCOVA paired comparisons model, and also suggests that a somewhat different approach is needed here.

Note that under (2.7), (2.8) and (2.10), for the ANOCOVA model, we have in all t parameters $(\alpha_1,\ldots,\alpha_t : \Sigma_{j=1}^{t}\alpha_j = 1, \theta)$, where under $H_0^{(1)}$, θ is the only unknown parameter. Hence, for the ANOCOVA model, a test for $H_0^{(1)}$ should relate to t–1 DF (in some sense). For the MANOVA model, the test proposed by Sen and David (1968) corresponds to 2(t–1)DF. Thus, we need to eliminate the component due to the concomitant trait and get a test statistic having a comparable t–1 DF. For this we proceed as follows.

By virtue of [A] in (2.7), the null hypothesis H_0 may be rephrased as

$$H_0 : \pi_{++}^{(ij)} = \pi_{--}^{(ij)} = \tfrac{1}{4}(1+\theta),\ \pi_{+-}^{(ij)} = \pi_{-+}^{(ij)} = \tfrac{1}{4}(1-\theta), \tag{2.12}$$

for every $1 \le i < j \le t$; so that θ is a nuisance parameter. Let n_{ij} be the number of independent responses on the (i,j) pair and denote the cell frequencies for the four cells (±, ±) by n_{ij}^{++}, n_{ij}^{+-}, n_{ij}^{-+}, n_{ij}^{--} respectively, for $1 \le i < j \le t$. Conventionally, for $j < i$, we let $n_{ij} = n_{ji}$ and $n_{ij}^{++} = n_{ji}^{--}$, n_{ij}^{+-}, $n_{ji}^{-+} = n_{ij}^{-+} = n_{ji}^{+-}$ and $n_{ij}^{--} = n_{ji}^{++}$. Further let

$$C_{ij} = n_{ij}^{++} + n_{ij}^{--} \quad \text{and} \quad D_{ij} = n_{ij} - C_{ij} = n_{ij}^{+-} + n_{ij}^{-+} \tag{2.13}$$

be the number of concordant and discordant responses for the pair (i,j), $1 \le i < j \le t$. Then, under H_0, the likelihood function (for θ) is

$$\prod_{1\le i<j\le t}\left\{\frac{(n_{ij})!\ (1+\theta)^{C_{ij}}\ (1-\theta)^{D_{ij}}}{(n_{ij}^{++})!\,(n_{ij}^{+-})!\ (n_{ij}^{-+})!\ (n_{ij}^{--})!}\,4^{-n_{ij}}\right\} \tag{2.14}$$

The maximum likelihood estimator (MLE) of θ derived from (2.14) is therefore given by

$$\begin{aligned}\hat{\theta}_n &= \Sigma_{1\le i<j\le t}(C_{ij} - D_{ij})/n \\ &= 2n^{-1}C - 1,\end{aligned} \tag{2.15}$$

where $n = \Sigma_{1\le i<j\le t}\, n_{ij}$ and $C = \Sigma_{1\le i<j\le t}\, C_{ij}$ is the total number of concordant responses.

We also let

$$n_{i\,j}^{+\cdot} = n_{i\,j}^{++} + n_{i\,j}^{+-} \text{ and } n_{i\,j}^{\cdot +} = n_{i\,j}^{++} + n_{i\,j}^{-+},\ 1 \le i < j \le t, \tag{2.16}$$

and for $j < i$, let $n_{i\,j}^{+\cdot} = n_{ij} - n_{j\,i}^{+\cdot}$, $n_{i\,j}^{\cdot +} = n_{ij} - n_{j\,i}^{\cdot +}$. Let then

$$T_{n,i}^{(1)} = \Sigma_{j=1,\neq i}^{t}\, n_{ij}^{-\frac{1}{2}}\,(2n_{i\,j}^{+\cdot} - n_{ij}),$$

$$T_{n,i}^{(2)} = \Sigma_{j=1,\neq i}^{t}\, n_{ij}^{-\frac{1}{2}}\,(2n_{i\,j}^{\cdot +} - n_{ij}),\ 1 \le i \le t. \tag{2.17}$$

The MANOVA paired comparisons test proposed by Sen and David (1968) is based on the statistic

$$\mathscr{L}_n = t^{-1}(1-\hat{\theta}_n^2)^{-1} \sum_{i=1}^{t} \left\{ (T_{n,i}^{(1)})^2 + (T_{n,i}^{(2)})^2 - 2\hat{\theta}_n\, T_{n,i}^{(1)}\, T_{n,i}^{(2)} \right\} \tag{2.18}$$

For small sample sizes $\{n_{ij},\ 1 \le i < j \le t\}$, under H_0, conditional on the C_{ij}, the $n_{i\,j}^{\pm\pm}$ have the product binomial law:

$$\prod_{1 \le i < j \le t} \left\{ \binom{C_{ij}}{n_{ij}^{++}} 2^{-C_{ij}} \binom{D_{ij}}{n_{ij}^{+-}} 2^{-D_{ij}} \right\}, \tag{2.19}$$

so that the (conditional) distribution of $\mathscr{L}_n$ under H_0 (given the C_{ij}, $1 \le i < j \le t$) can be obtained by using (2.19). This task becomes prohibitively laborious as the n_{ij} increases. Nevertheless, if we assume that as $n \to \infty$,

$$n_{ij}/n \to \rho_{ij} : 0 < \rho_{ij} < 1;\ \Sigma_{i<j}\, \rho_{ij} = 1, \tag{2.20}$$

then under [A] in (2.7) and H_0,

$$\mathscr{L}_n \xrightarrow{\mathscr{D}} \chi^2_{2(t-1),0} \tag{2.21}$$

where $\chi^2_{p,\delta}$ stands for a random variable having noncentral chi square distribution with p DF and noncentrality parameter δ (≥ 0); for $\delta = 0$, this reduces to a central chi square distribution. Thus, corresponding to a given significance level $\epsilon : 0 < \epsilon < 1$, the critical level of $\mathscr{L}_n$ may be closely approximated by $\chi^2_{2(t-1),0}(\epsilon)$, the upper $100\epsilon\%$ point of $\chi^2_{2(t-1),0}$. We refer to Sen and David (1968) for details.

Next, if we ignore totally the concomitant trait (2), then for the primary trait (1), the model reduces to the classical single characteristic paired comparisons model, for which an appropriate (ANOVA) test statistic is

$$\mathscr{L}_{n,1} = t^{-1} \sum_{i=1}^{t} (T_{n,i}^{(1)})^2 ; \tag{2.22}$$

$$\mathscr{L}_{n,1} \xrightarrow{\mathscr{D}} \chi^2_{t-1,0} \text{ under } H_0^{(1)}. \tag{2.23}$$

Side by side, ignoring the primary trait (1) and solely based on the concomitant trait (2), we construct an (ANOVA) paired comparisons test for $H_0^{(2)}$ based on the test statistic

$$\mathscr{L}_{n,2} = t^{-1} \sum_{i=1}^{t} (T_{n,i}^{(2)})^2. \tag{2.24}$$

Here also, under $H_0^{(2)}$, $\mathscr{L}_{n,2}$ is distribution–free and as n increases,

$$\mathscr{L}_{n,2} \xrightarrow{\mathscr{D}} \chi^2_{t-1,0}, \text{ under } H_0^{(2)}. \tag{2.25}$$

To propose the ANOCOVA test for H_0, we first note that

$$\mathscr{L}_n^0 = \mathscr{L}_n - \mathscr{L}_{n,2} = t^{-1}(1-\hat{\theta}_n^2)^{-1} \sum_{i=1}^{t} \left\{T_{n,i}^{(1)} - \hat{\theta}_n T_{n,i}^{(2)}\right\}^2, \tag{2.27}$$

for i=1,...,t. This shows that under H_0, given the C_{ij}, $1 \le i < j \le t$, $\mathscr{L}_n^0$ is conditionally distribution–free, and its null (conditional) distribution can be enumerated by using (2.19). Since $\mathscr{L}_n^0$ is nonnegative and a quadratic form in the $T_{n,i}^{(\ell)}$, $\ell = 1,2$, $1 \le i \le t$, proceeding as in Section 4 of Sen and David (1968) and using an asymptotic version of the celebrated Cochran Theorem on quadratic forms [viz., Sen and Singer (1993, p. 137)], we conclude that under H_0, (2.7) and (2.20),

$$\mathscr{L}_n^0 \xrightarrow{\mathscr{D}} \chi^2_{t-1,0}. \tag{2.28}$$

Thus, for both the ANOVA test statistic $\mathscr{L}_{n,1}$ and the ANOCOVA test statistic $\mathscr{L}_n^0$, an asymptotic ϵ–level critical value is $\chi^2_{t-1,0}(\epsilon)$. Further note that in $\mathscr{L}_{n,1}$, the information on the concomitant trait has been totally ignored, while for $\mathscr{L}_n^0$, the residuals in (2.27) incorporate the same to a certain extent. Since the variance–covariance matrix of the residuals in (2.27) under the conditional law in (2.19) is $(1-\hat{\theta}_n^2)\{\underset{\sim}{I} - t^{-1} \underset{\sim}{1}\underset{\sim}{1}'\}$, while under $H_0^{(1)}$, the dispersion matrix of the $T_{n,i}^{(1)}$ is $\underset{\sim}{I} - t^{-1}\underset{\sim}{1}\underset{\sim}{1}'$, we have adjusted for this difference in (2.26) by the scalar factor $(1 - \hat{\theta}_n^2)^{-1}$.

Let us now compare $\mathcal{L}_{n,1}$ and $\mathcal{L}_n^o$ in the light of the Pitman asymptotic relative efficiency (PARE) measure as has been adapted for paired comparisons model in Sen and David (1968) and elsewhere. Note that under $H_0^{(1)}$ and (2.10), we have $\underset{\sim}{\alpha} = t^{-1}\underset{\sim}{1} = \underset{\sim}{\beta}$, while the ANOCOVA alternatives relate only to departures of $\underset{\sim}{\alpha}$ from $t^{-1}\underset{\sim}{1}$, but $\underset{\sim}{\beta}$ is allowed to be equal to $t^{-1}\underset{\sim}{1}$. Hence, we conceive of a sequence $\{K_n^{(1)}\}$ of alternative hypotheses:

$$K_n^{(1)} : \underset{\sim}{\alpha} = t^{-1}\underset{\sim}{1} + n^{-\frac{1}{2}}\underset{\sim}{\xi},\ \underset{\sim}{\beta} = t^{-1}\underset{\sim}{1},\ \underset{\sim}{\xi} \in \mathbb{R}^t, \tag{2.29}$$

and, we set $\underset{\sim}{\xi}'\underset{\sim}{1} = 0$. Let then

$$\mu_i = \Sigma_{j=1,\neq i}^{t}\ \rho_{ij}^{\frac{1}{2}}\ \xi_i/(\xi_i + \xi_j),\ \ 1 \le i \le t; \tag{2.30}$$

$$\Delta = t^{-1}(1-\theta^2)^{-1}\sum_{i=1}^{t}\mu_i^2,\ \ \Delta_0 = \Delta(1-\theta^2). \tag{2.31}$$

Proceeding as in Section 4 of Sen and David (1968), we obtain that under $\{K_n^{(1)}\}$ in (2.29), (2.7) and (2.20),

$$\mathcal{L}_n \xrightarrow{\mathcal{D}} \chi^2_{2(t-1),\Delta},\ \ \mathcal{L}_{n,1} \xrightarrow{\mathcal{D}} \chi^2_{t-1,\Delta_0}; \tag{2.32}$$

$$\mathcal{L}_{n,2} \xrightarrow{\mathcal{D}} \chi^2_{t-1,0} \tag{2.33}$$

Therefore, by (2.26), (2.32) and (2.23), we conclude that under $\{K_n^{(1)}\}$, (2.7) and (2.20),

$$\mathcal{L}_n^o \xrightarrow{\mathcal{D}} \chi^2_{t-1,\Delta}\ . \tag{2.34}$$

From (2.32) and (2.34), we obtain that

$$\begin{aligned} \text{PARE}(\mathcal{L}_{n,1} \mid \mathcal{L}_n^o) &= \Delta_0/\Delta = 1-\theta^2 \\ &\le 1,\ \ \forall\ \underset{\sim}{\xi},\ \theta, \end{aligned} \tag{2.35}$$

where the equality sign holds only when $\theta = 0$, i.e., the primary and concomitant traits are independent. This establishes the (asymptotic) superiority of the ANOCOVA test $\mathcal{L}_n^o$ to the ANOVA test $\mathcal{L}_{n,1}$. Finally, we may remark that under $\{K_n^{(1)}\}$, both $\mathcal{L}_n$ and $\mathcal{L}_n^o$ have the same noncentrality parameter Δ, while $DF(\mathcal{L}_n) = 2(t-1) = 2 \cdot DF(\mathcal{L}_n^o)$. Hence, $\mathcal{L}_n^o$ is more efficient than the MANOVA test $\mathcal{L}_n$ when, in fact, the ANOCOVA alternative $\{K_n^{(1)}\}$ holds. This provides the justification for the ANOCOVA paired comparisons procedures (instead of either the MANOVA or ANOVA procedures) when the trait 2 satisfies the conditions of a concomitant trait as have been laid down earlier.

3. Probability laws for multiple dichotomous attributes

Consider p (≥ 1) dichotomores attributes and let

$$\underset{\sim}{i} = (i_1,\ldots,i_p)' \text{ where } i_j = 0,1; 1 \leq j \leq p. \tag{3.1}$$

There are thus in all 2^p realizations of $\underset{\sim}{i}$, and let $\underset{\sim}{X} = (X_1,\ldots,X_p)'$ be a random p–vector, such that

$$P\{\underset{\sim}{X} = \underset{\sim}{i}\} = \pi(\underset{\sim}{i}),\ \underset{\sim}{i} \in \mathcal{I}, \tag{3.2}$$

where $\mathcal{I}$ stands for the set of all possible $\underset{\sim}{i}$ in (3.1). The probability law $\pi(\underset{\sim}{i})$ is defined on a 2^p–simplex

$$\pi(\underset{\sim}{i}) \geq 0,\ \forall\ \underset{\sim}{i} \in \mathcal{I},\ \Sigma_{\underset{\sim}{i} \in \mathcal{I}}\, \pi(\underset{\sim}{i}) = 1, \tag{3.3}$$

so that there are 2^p-1 linearly independent parameters. Keeping in mind the multivariate paired comparisons models (as well as the case of $p = 2$, treated in Section 2), we consider the following modification of the Bahadur (1961) basic result. Let

$$\pi_{*(i)}^{(j)} = P\{X_j = i\},\ i = 0,1;\ 1 \leq j \leq p. \tag{3.4}$$

Also, for every $\ell : 2 \leq \ell \leq p;\ 1 \leq i_1 < \cdots < i_\ell \leq p$, define a pth order association parameter

$$\theta_{i_1 \cdots i_\ell},\ \text{where there are } \binom{p}{\ell} \text{ such parameters.} \tag{3.5}$$

Let then

$$\Theta = \{\theta_{i_1 \cdots i_\ell} :\ 1 \leq i_1 < \cdots < i_\ell \leq p;\ \ell = 2,\ldots,p\} \tag{3.6}$$

and, conventionally, for $p = 2$, we let $\Theta = \theta$. Then noting that $\pi_{*(0)}^{(j)} + \pi_{*(1)}^{(j)} = 1,\ \forall\ 1 \leq j \leq p$, we conclude that there p unknown parameters in (3.4) and in (3.6), there are in all $\binom{p}{2} + \cdots + \binom{p}{p} = 2^p-p-1$ unknown parameters, so that in all there are 2^p-1 unknown parameters. We write $\underset{\sim}{\pi}_* = (\pi_{*(0)}^{(1)},\cdots,\pi_{*(0)}^{(p)})'$, and consider the reparameterization:

$$\{\pi(\underset{\sim}{i}),\ \underset{\sim}{i} \in \mathcal{I}\} \longrightarrow \{\underset{\sim}{\pi}_*,\ \underset{\sim}{\Theta}\}, \tag{3.7}$$

where we set for every $\underset{\sim}{i} \in \mathscr{I}$,

$$\pi(\underset{\sim}{i}) = \prod_{j=1}^{p} \pi_{*(ij)}^{(j)} + \sum_{1 \le j_1 < j_2 \le p} (-1)^{i_{j_1}+i_{j_2}} \theta_{j_1 j_2} \prod_{r=1}^{2} \pi_{*(0)}^{(j_r)} \prod_{\substack{s=1 \\ \neq j_1, j_2}}^{p} \pi_{*(i_s)}^{(s)}$$

$$+ \sum_{1 \le j_1 < j_2 < j_3 \le p} (-1)^{i_{j_1}+i_{j_2}+i_{j_3}} \theta_{j_1 j_2 j_3} \prod_{r=1}^{3} \pi_{*(0)}^{(i_j)} \prod_{\substack{s=1 \\ \neq j_1, j_2, j_3}}^{p} \pi_{*(i_s)}^{(s)}$$

$$+ \cdots + (-1)^{i_1+\cdots+i_p} \theta_{1\cdots p} \prod_{r=1}^{p} \pi_{*(0)}^{(r)} . \tag{3.8}$$

For p = 2, (3.8) reduces to (2.4). Moreover, (3.8) implies, for example, that

$$\pi(i_1,\ldots,i_{p-1},i_p) + \pi(i_1,\ldots,i_{p-1}, 1-i_p)$$
$$= \pi(i_1,\ldots,i_{p-1};*) = P\{(x_1,\ldots,x_{p-1}) = (i_1,\ldots,i_{p-1})\},$$
$$\text{for every } (i_1,\ldots,i_p). \tag{3.9}$$

A similar consistency result holds for any lower dimensional joint probability for $(X_{i_1},\ldots,X_{i_\ell})$, $1 \le \ell \le p$, $1 \le i_1 < \cdots < i_\ell \le p$, and (3.8) can be established by induction from (2.4). This representation underlies the multivariate paired comparisons procedure in Sen and David (1968), and in the next section we examine its role in general MANOCOVA paired comparisons procedure.

4. MANOCOVA paired comparisons models and analyses

As in Section 2, we consider t objects, forming $\binom{t}{2}$ possible pairs, and for the pair (i,j) : $1 \le i < j \le t$, we denote the response vector by $\underset{\sim}{X}_{ij} = (X_{ij}^{(1)},\cdots,X_{ij}^{(p)})'$ (where we have p dichotomous attributes). The probability law of $\underset{\sim}{X}_{ij}$ over the 2^p–simplex is denoted by $\underset{\sim}{\pi}_{ij} = \{\pi_{ij}(\underset{\sim}{i}) : \underset{\sim}{i} \in \mathscr{I}\}$, for $1 \le i < j \le t$.

As a first step, we consider the same reparameterization in (3.7) and write

$$\{\underset{\sim}{\pi}_{ij}(\underset{\sim}{i}),\ \underset{\sim}{i} \in \mathscr{I}\} \longrightarrow \{\underset{\sim}{\pi}_{*ij}, \Theta_{ij}\}, \tag{4.1}$$

for $1 \le i < j \le t$. Then, parallel to (2.7), the fundamental assumption here is the following:

[A] $$\underset{\sim}{\Theta}_{ij} = \underset{\sim}{\Theta}, \ \forall \ 1 \le i < j \le t. \tag{4.2}$$

Also, note that in (4.1), we have

$$\underset{\sim}{\pi}_{*ij} = (\pi^{(1)}_{*ij(0)}, \ldots, \pi^{(p)}_{*ij(0)})', \ 1 \le i < j \le t, \tag{4.3}$$

so that parallel to (2.8), we may set

$$\pi^{(r)}_{*ij(0)} = \alpha_{ri}/(\alpha_{ri} + \alpha_{rj}), \ 1 \le i < j \le t; \ 1 \le r \le p, \tag{4.4}$$

where for each r (=1,...,p), the α_{ri} are nonnegative and

$$\Sigma^{p}_{j=1} \alpha_{rj} = 1, \ \text{for } r = 1,\ldots,p. \tag{4.5}$$

Thus, there are in all p(t–1) linearly independent parameters in the set (4.3). The multivariate ANOVA as well as ANOCOVA paired comparisons models relate to the α_{ri}, $1 \le i \le t$, $1 \le r \le p$, under [A] in (4.2) and treating $\underset{\sim}{\Theta}$ as a nuisance parameter vector. In this formulation, unlike the case of Davidson and Bradley (1969), we do not need to assume that $\underset{\sim}{\Theta} = (\theta_{12}, \cdots, \theta_{p-1p}, \underset{\sim}{0})'$. In Sen and David (1968), we also did not require the above restriction on $\underset{\sim}{\Theta}$, and we shall make this point clear at a later stage. First, we introduce the component null hypotheses:

$$H^{(r)}_0 : \alpha_{r1} = \cdots = \alpha_{rp} = t^{-1}, \ \text{for } r = 1,\ldots,p, \tag{4.6}$$

so that in a MANOVA model we have the over all null hypothesis H_0 given by

$$H_0 = \bigcap_{1 \le r \le p} H^{(r)}_0. \tag{4.7}$$

In a MANOCOVA model, we let $p = p_1 + p_2$, $p_i \ge 1$, $i = 1,2$, and set the last p_2 traits as concomitant ones, while the first p_1 as the primary ones. Note that the discussion made in Section 2 on the concomitant trait pertains to this general case of $p_2 \ge 1$ as well. We rewrite $H_0 = H_{01} \cap H_{02}$ where $H_{01} = \cap_{r \le p_1} H^{(r)}_0$ and $H_{02} = \cap_{p_1 < r \le p_2} H^{(r)}_0$. Then the MANOCOVA null hypothesis is framed as

$$H^*_0 = H_0 | H_{02} \equiv H_{01} | H_{02}. \tag{4.8}$$

For the MANOVA model, Davidson and Bradley (1969) assumed that $\theta_{j_1 \cdots j_\ell} = 0$, $\forall \ \ell \ge 3$, $1 \le j_1 < \cdots < j_\ell \le p$, and then incorporated the usual likelihood function (of the α_{ri} $1 \le i \le t$ and

θ_{ij}, $1 \le i < j \le p$) for deriving the MLE of these parameters. They have correctly noted that for $p > 2$ and/or $t > 2$, an explicit solution to the likelihood equations can not be found, and thus considerations must be given to solving the equations iteratively for the MLE. In fact, they have elaborated such an iteration procedure. We like to pressure a point of distinction here, and by an appeal to a conditional argument similar to that in Sen and David (1968), we may as well use some partial MLE (PMLE) of these parameters to construct a conditionally distribution–free test for H_0 in (4.7). This test (mentioned in Sen and David (1968)) has the same asymptotic properties as the large sample test proposed by Davidson and Bradley (1969), and moreover, for small sample sizes, it is conditionally distribution–free whereas the other test does not have this property.

We may partition $\mathcal{J}$ into 2^{p-1} buckets, such that within each bucket, we have a product binomial law as in (2.14), so that taking the product over such 2^{p-1} buckets, we get the following (conditional) likelihood function of the $C^{(1)}_{ij,s}$, $C^{(1)}_{ij,s}$, $s \in S$, $\#S = 2^{p-1}$:

$$\prod_{1 \le i < j \le t} \prod_{s \in S} \left\{ 2^{-(C^{(0)}_{ij,s} + C^{(1)}_{ij,s})} \binom{C^{(0)}_{ij,s} + C^{(1)}_{ij,s}}{C^{(0)}_{ij,s}} \right\}, \tag{4.9}$$

which would characterize the conditional distribution–freeness of the proposed tests. If we consider the bivariate marginal probabilities, say, for example, $\pi(i_1,i_2,*,*,\cdots *)$, then

$$\theta_{12} = \pi(0,0,*,\cdots,*)/\pi^{(1)}_{*(0)}\, \pi^{(2)}_{*(0)} - 1, \tag{4.10}$$

and a similar expression holds for each other θ_{rs}, $1 \le r < s \le p$. Thus, considering the bivariate joint distribution of the (r,s)th trait responses for each of the $\binom{t}{2}$ pairs, we may obtain a likelihood function as in (2.14), and the PMLE of θ_{rs}, denoted by $\tilde{\theta}_{rs}$, is then obtained as in (2.15), for every $1 \le r < s \le p$. We let $\tilde{\theta}_{sr} = \tilde{\theta}_{rs}$, for $s < r$, and consider the following $p \times p$ matrix.

$$\hat{\underset{\sim}{\Gamma}}_n = ((\hat{\gamma}_{nrs})) = ((\tilde{\theta}_{rs})); \quad \tilde{\theta}_{rr} = 1,\ 1 \le r \le p. \tag{4.11}$$

As in (2.16), we then define for each pair (i,j)

$$n^{(r)}_{ij*}(i_r) = \Sigma^{p}_{s=1,\ne r} \Sigma^{1}_{i_s=0}\ n_{ij}(\underset{\sim}{i}),\quad 1 \le i < j \le t, \tag{4.12}$$

and r = 1,...,p, and as in (2.17) we define

$$T_{n,i}^{(r)} = \Sigma_{j=1,\neq i}^{t} n_{ij}^{-\frac{1}{2}} \{n_{ij*}^{(r)}(0) - n_{ij*}^{(r)}(1)\} \tag{4.13}$$

for i = 1,...,t, r = 1,...,p. Here also for j < i, we define the $n_{ij*}^{(r)}(i_r)$ etc., as in Section 2, by $n_{ji*}^{(r)}(1-i_r)$ etc. Let then

$$\underset{\sim}{T}_{n,i} = (T_{n,i}^{(1)},\cdots,T_{n,i}^{(p)})'. \tag{4.14}$$

As in Sen and David (1968), we consider then the MANOVA test for H_0 in (4.7) based on the statistic

$$\mathscr{L}_n = t^{-1} \Sigma_{i=1}^{t} \underset{\sim}{T}_{n,i}' \hat{\underset{\sim}{\Gamma}}_n^{-1} \underset{\sim}{T}_{n,i} \tag{4.15}$$

Again under [A] in (4.2), the null hypothesis distribution of $\mathscr{L}_n$, given the $(C_{ij,s}^{(0)} + C_{ij,s}^{(1)})$, $s \in S$, $1 \leq i < j \leq t$, can be enumerated by using (4.9), and as $n \to \infty$,

$$\mathscr{L}_n \xrightarrow{\mathscr{D}} \chi^2_{p(t-1),0}, \text{ under } H_0. \tag{4.16}$$

In a similar manner, to test for H_{01} alone, we may ignore the last p_2 traits, and contracted to the first p_1 traits, we have $\underset{\sim}{T}_{n,i}^{(1)}$ as the sub–vector of $\underset{\sim}{T}_{n,i}$ consisting only of the first p_1 elements in (4.14), i.e.,

$$\underset{\sim}{T}_{n,i}' = (\underset{\sim}{T}_{n,i}^{(1)\prime}, \underset{\sim}{T}_{n,i}^{(2)\prime}), \ i = 1,\dots,t \tag{4.17}$$

Similarly, we partition

$$\hat{\underset{\sim}{\Gamma}}_n = \begin{bmatrix} \hat{\underset{\sim}{\Gamma}}_{n11} & \hat{\underset{\sim}{\Gamma}}_{n12} \\ \hat{\underset{\sim}{\Gamma}}_{n21} & \hat{\underset{\sim}{\Gamma}}_{n22} \end{bmatrix}; \quad \hat{\underset{\sim}{\Gamma}}_{nij} \text{ is } p_i \times p_j \quad i,j = 1,2. \tag{4.18}$$

Then, the test statistic for testing H_{01} is

$$\mathscr{L}_{n1} = t^{-1} \Sigma_{i=1}^{t} (\underset{\sim}{T}_{n,i}^{(1)})' \hat{\underset{\sim}{\Gamma}}_{n11}^{-1} (\underset{\sim}{T}_{n,i}^{(1)}), \tag{4.19}$$

and parallel to (4.16), we have

$$\mathscr{L}_{n1} \xrightarrow{\mathscr{D}} \chi^2_{p_1(t-1),0}, \text{ under } H_{01}. \tag{4.20}$$

Finally, to pose the MANOCOVA test for H_0^*, we let

$$\underset{\sim}{T}^*_{n,i} = \underset{\sim}{T}^{(1)}_{n,i} - \hat{\underset{\sim}{\Gamma}}_{n12}\, \hat{\underset{\sim}{\Gamma}}^{-1}_{n22}\, \underset{\sim}{T}^{(2)}_{n,i},\ 1 \le i \le t, \tag{4.21}$$

$$\hat{\underset{\sim}{\Gamma}}_{n11\cdot 2} = \hat{\underset{\sim}{\Gamma}}_{n11} - \hat{\underset{\sim}{\Gamma}}_{n12}\, \hat{\underset{\sim}{\Gamma}}^{-1}_{n22}\, \hat{\underset{\sim}{\Gamma}}_{n21} \tag{4.22}$$

Then we have the proposed test statistic

$$\mathscr{L}^*_n = t^{-1} \Sigma^t_{i=1} (\underset{\sim}{T}^*_{n,i})'\, \hat{\underset{\sim}{\Gamma}}^{-1}_{n11\cdot 2}\, (\underset{\sim}{T}^*_{n,i}). \tag{4.23}$$

Under H_0 and the law (4.9), the (conditional) distribution of $\mathscr{L}^*_n$ can be enumerated by using (4.9). For large n, under [A] in (4.2) and H^*_0,

$$\mathscr{L}^*_n \xrightarrow{\mathscr{D}} \chi^2_{p_1(t-1),0}\,. \tag{4.24}$$

We are in a position to compare the MANOVA test $\mathscr{L}_{n_1}$ and the MANOCOVA test $\mathscr{L}^*_n$ for local Pitman type alternatives pertaining to the set of p_1 primary traits. Such alternatives can be formulated as in (2.29) for each $\underset{\sim}{\alpha}^{(r)}$, $r = 1,\dots,p_1$ and the $\mu^{(r)}_i$ can then be defined as in (2.30). We denote these p_1–vectors by $\underset{\sim}{\mu}_i = (\mu^{(1)}_i,\dots,\mu^{(p_1)}_i)'$, $i = 1,\dots,t$. Also, let $\underset{\sim}{\Gamma} = ((\gamma_{ij})) = ((\theta_{ij}))_{i,j=1,\dots,p}$ be defined as in (4.11) and partition $\underset{\sim}{\Gamma}$ into $\underset{\sim}{\Gamma}_{rs}$, r, s = 1,2, as in (4.18). Let $\underset{\sim}{\Gamma}_{11\cdot 2} = \underset{\sim}{\Gamma}_{11} - \underset{\sim}{\Gamma}_{12}\, \underset{\sim}{\Gamma}^{-1}_{22}\, \underset{\sim}{\Gamma}_{21}$ and

$$\Delta = t^{-1} \Sigma^t_{i=1}\, \underset{\sim}{\mu}'_i\, \underset{\sim}{\Gamma}^{-1}_{11\cdot 2}\, \underset{\sim}{\mu}_i\,, \tag{4.25}$$

$$\Delta_0 = t^{-1} \Sigma^t_{i=1}\, \underset{\sim}{\mu}'_i\, \underset{\sim}{\Gamma}^{-1}_{11}\, \underset{\sim}{\mu}_i. \tag{4.26}$$

Then as in (2.32)–(2.34), we obtain that under such a sequence $\{K^{(1)}_n\}$ of local alternatives, as $n \to \infty$,

$$\mathscr{L}_n \xrightarrow{\mathscr{D}} \chi^2_{p(t-1),\Delta},\quad \mathscr{L}_{n1} \xrightarrow{\mathscr{D}} \chi^2_{p_1(t-1),\Delta_0}; \tag{4.27}$$

$$\mathscr{L}^*_n \xrightarrow{\mathscr{D}} \chi^2_{p_1(t-1),\Delta}. \tag{4.28}$$

Next, by an application of the Courant Theorem [see Sen & Singer (1993, p. 28)]

$$\inf\{\Delta/\Delta_0 : (\underset{\sim}{\mu}_1,\dots,\underset{\sim}{\mu}_t) \in \mathbb{R}^{pt_1}\}$$
$$= ch_{\min}\{\underset{\sim}{\Gamma}_{11}\, \underset{\sim}{\Gamma}^{-1}_{11\cdot 2}\}$$

$$= \mathrm{ch}_{\min}\{(\underset{\sim}{I} - \underset{\sim}{\Gamma}_{12}\underset{\sim}{\Gamma}_{22}^{-1}\underset{\sim}{\Gamma}_{21}\underset{\sim}{\Gamma}_{11}^{-1})^{-1}\}$$

$$= (\mathrm{ch}_{\max}\{\underset{\sim}{I} - \underset{\sim}{\Gamma}_{12}\underset{\sim}{\Gamma}_{22}^{-1}\underset{\sim}{\Gamma}_{21}\underset{\sim}{\Gamma}_{11}^{-1}\})^{-1}$$

$$\geq 1 \tag{4.29}$$

where the strict equality sign holds when the positive semidefinite matrix $\underset{\sim}{\Gamma}_{11}^{-\frac{1}{2}}\underset{\sim}{\Gamma}_{12}\underset{\sim}{\Gamma}_{22}^{-1}\underset{\sim}{\Gamma}_{21}\underset{\sim}{\Gamma}_{11}^{-\frac{1}{2}}$ ($= \underset{\sim}{\Gamma}^*$, say) has a null minimum eigen value. On the other hand, if at least one of the eigen values of $\underset{\sim}{\Gamma}^*$ is strictly positive, it follows from (4.29) and a more structured version of the Courant Theorem that Δ/Δ_0 is > 1, for some part of parameter space (under the alternative). From (4.27), (4.28) and (4.29) we conclude that (i) under $\{K_n^{(1)}\}$, $\mathscr{L}_n^*$ is asymptotically more powerful than $\mathscr{L}_n$ (as $p > p_1$), and (ii) $\mathscr{L}_n$ is asymptotically at least as powerful as $\mathscr{L}_{n1}$, and at least one a part of the alternative hypothesis parameter space, $\mathscr{L}_n^*$ is more powerful than $\mathscr{L}_{n1}$. Thus, for the MANOCOVA paired comparisons model, $\mathscr{L}_n^*$ is a better choice.

5. Concluding remarks

The main thrust of the current study is on the development of the analysis of covariance approach to the classical paired comparisons model through a multivariate approach, mainly adapted from Sen and David (1968). The relevance of the same (conditional) distribution–freeness as in Sen and David (1968) has been established, and the ANOCOVA test statistic $\mathscr{L}_n^*$ has also been singled out as a better alternative than the other. In this context, there are certain issues that need some discussion and are presented below:

(i) In principle, the Davidson and Bradley (1969) procedure (for MANOVA paired comparisons) may as well be extended to the MANOCOVA problem. For this setup, assumption [A] in (4.2) remains in tact, with the additional assumption that the $\theta_{j_1 \cdots j_\ell}$ for $\ell \geq 3$ are all null. This latter condition is not needed in the current approach. Secondly, with respect to (4.3)–(4.4), in the MANOCOVA model, we have $\pi_{*ij(0)}^{(\ell)} = 1/2$, for $\ell = p_1 + 1,\ldots,p$, p, $1 \leq i < j \leq t$, so that $\alpha_{ri} = t^{-1}$, for every $r = p_1+1,\ldots,p$ and $i = 1,\ldots,t$. Therefore in the Davidson–Bradley (1969) likelihood function, we will have only the set of parameters α_{ri}, $1 \leq r \leq p_1$, $1 \leq i \leq t$ and the $\theta_{j\ell}$, $1 \leq j < \ell \leq p$. This would lead to a computationally simpler likelihood equation. Nevertheless, an iterative solution is generally needed to solve for the MLE of these α_{ri} and $\theta_{j\ell}$, as would be needed for their proposed likelihood ratio test. Because here we allow the $\theta_{j_1 \cdots j_\ell}$, $\ell \geq 3$ to be arbitrary, model wise we have a less restricted model and hence, the proposed procedure is more model–robust. This conclusion also applies to the MANOVA paired comparisons test of Sen and David (1968) in relation to the parallel one due to Davidson and Bradley (1969) which has purely a large sample flavor.

(ii) There is a rational interpretation of the proposed testing procedure in the light of the classical Mantel–Haenszel (1959) procedure as studied in a general multi–dimensional case by Sen (1988). We may remark that for the $\binom{t}{2}$ pairs (i,j), $1 \leq i < j \leq t$, the total number of independent cell probabilities ($\underset{\sim}{\pi}_{ij}$) is equal to $\binom{t}{2}(2^p - 1)$. A test of homogeneity of these $\binom{t}{2}$ multi–dichotomous tables could have been made (with a DF $[\binom{t}{2} - 1][2^p - 1]$). For $t > 2$ and/or $p \geq 2$, this DF is large compared to $p(t-1)$, in the Davidson–Bradley (1969) or Sen–David (1968) procedures. Whereas the ingenuity of the Davidson–Bradley approach was to incorporate (4.4), and impose the restraints that the $\theta_{j_1 \cdots j_\ell}$, $\ell \geq 3$ are all 0, to reduce the number of free parameters to $p(t-1) + \binom{p}{2}$, and through the likelihood principle justify their procedure as being at least asymptotically optimal, there remains the question of model–robustness. In particular, the likelihood ratio test is generally non–robust even in simpler models [Huber (1965)] and with the increase in the number of parameters under testing as well as the nuisance parameters, the degree of non–robustness may accelerate. The Mantel–Haenszel (1959) technique offers a more robust alternative. It simply relates to the choice of a specific number of contrasts in the cell probabilities which are directly relevant to the hypotheses under testing and exploits a suitable conditional argument to render distribution–freeness for finite sample sizes. The Chatterjee (1966) concordance–discordance conditionality argument is an extension of the Mantel–Haenszel principle to the multidimensional contingency tables, and following Sen (1988), we may characterize the proposed testing procedure as a further extension of the Mantel–Haenszel conditional procedure to multidimensional dichotomores tables arising in MANOCOVA paired comparisons models. This way we allow more flexibility with respect to the vector Θ in (4.2).

(iii) In the simple one parameter model, a likelihood ratio test may have some optimality properties even for small sample sizes. In the classical MANOVA (linear model) tests, even asymptotically the likelihood ratio test may not be universally optimal. The Wald (1943) characterization of the asymptotic optimality properties of likelihood ratio tests leaves the door open for other procedures as well. In fact, the Lawley–Hotelling trace statistics and the Wilks likelihood ratio test statistics are known to share such asymptotic optimality properties for MANOVA/MANOCOVA problems. Since our proposed test is more in the spirit of the Lawley–Hotelling trace statistics (with adaptations from the Mantel–Haenszel procedure), it was motivating to note that this procedure should also share the same asymptotic properties with the likelihood ratio tests proposed by Davidson and Bradley (1969). This intuition is indeed true as may easily be verified by comparing our $\mathscr{L}_n^*$ with a parallel version as can be obtained by using the likelihood ratio principle on the Davidson–Bradley model.

(iv) In the classical MANOCOVA model, the asymptotic power–equivalence of the Lawley–Hotelling trace statistics and likelihood ratio statistics is largely due to the "parametric orthogonality" of the regression parameters and the dispersion matrix. In the current situation, we have a non–linear model, and hence this asymptotic equivalence result (discussed in (iii)) casts more light on the model parameters. In our proposed test, we have tactly used $p(t-1) + (2^p-p-1)$ parameters [see (4.2) and (4.4)], treating $p(t-1)$ of them as the ones under testing while the remaining (2^p-p-1) (i.e., $\underset{\sim}{\theta}$) as nuisance parameters. In the Davidson–Bradley (1969) model too, they could have worked with their likelihood ratio principle with all these $p(t-1) + (2^p-p-1)$ parameters. The asymptotic properties of such a likelihood ratio test (for H_0) would have been the same as their original one based on $p(t-1) + \binom{p}{2}$ parameters, although computationally that would have been even more cumbersome; the iteration procedure for this full parameter space model would have been highly involved and complex. But the outcome of this asymptotic equivalence is that even the likelihood ratio test is asymptotically insensitive to the parameters $\theta_{j_1 \cdots j_\ell}$, $\ell \geq 3$ (when $p \geq 3$), ensuring an asymptotic parametric orthogonality with higher order $\theta_{j_1 \cdots j_\ell}$, $\ell \geq 3$. This is not surprising at all. Because of the asymptotic (joint) normality of the $n_{ij}^{-\frac{1}{2}}\{n_{ij*}^{(r)}(0) - n_{ij}\,\pi_{*(0)}^{(r)}\}$, $1 \le r \le p$, $1 \le i < j \le t$, only the $\theta_{j\ell}$, $1 \le j \ne \ell \le p$, enter into their covariance matrix while the higher order θ's cease to have any impact. Moreover, for the unrestricted model, the $n_{ij*}^{(r)}(0)$ are asymptotically BAN estimators, and hence, the higher order θ's, even dropped from the model, do not lead to any asymptotic loss of information. However, from model specification and finite sample analysis considerations, there is no need to assume that the $\theta_{j_1 \cdots j_\ell}$, $\ell \ge 3$, are all 0.

We conclude this section with a note that in the literature there are other procedures relating to paired comparisons designs where for each pair (i,j), $1 \le i < j \le t$, quantitative responses are available on the individual objects, or at least, on their difference. Thus, we may assume that there are n_{ij} observations $\underset{\sim}{X}_{ij,k}$, $k = 1,\ldots,n_{ij}$, for each (i,j), $1 \le i < j \le t$, where the $\underset{\sim}{X}_{ij,k}$ are i.i.d.r. vectors with a continuous distribution function F_{ij}, defined on $\mathbb{R}^p$, and it may be assumed that F_{ij} is (diagonally) symmetric about the location parameter $\underset{\sim}{\mu}_{ij}$. In the same spirit as in Davidson and Bradley (1969), it can be taken for granted that

$$\underset{\sim}{\mu}_{ij} = \underset{\sim}{\xi}_i - \underset{\sim}{\xi}_j, \text{ for } 1 \le i < j \le t, \tag{5.1}$$

so that the null hypothesis relates to the homogeneity of the $\underset{\sim}{\xi}_i$. This relates to a (nonparametric) linear model for which the techniques discussed in detail by Puri and Sen (1985, Sec. 8.3) can readily be adopted to study suitable MANOCOVA tests. Therefore, there is no need to study such tests in detail. Rather, following the general philosophy of David (1988), we are somewhat reserved in characterizing such procedures as genuine paired comparisons procedures, and hence, we refrain ourselves from further deliberations of such MANOCOVA paired comparisons. More robust ANOCOVA procedures considered by Sen (1993) are more appealing in this respect.

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