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PAIRED COMPARISONS FOR PAIRED CHARACTERISTICS[1]

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Summary. The present investigation is concerned with the proposal and study of a class of nonparametric paired comparison tests for the hypothesis of no differences among several objects with respect to a pair of characteristics.

1. Introduction. Let there be $t(\geqq 2)$ *objects* which form $\binom{t}{2}$ *pairs*, and let the ith object have $n_{ij}(\geqq 0)$ encounters with the jth object, for $i < j = 1, \cdots, t$ (conventionally, $n_{ij} = n_{ji}$ for $i > j$). We let

$$N = \sum_{i<j=1}^{t} n_{ij} = \tfrac{1}{2} \sum_{i \neq j=1}^{t} n_{ij} . \tag{1.1}$$

For each of the n_{ij} encounters, it is judged whether the ith object is preferred (or not) to the jth object for each of the two characteristics (α, β), and the order of preference is indicated by $\succ$ (e.g., $\alpha_i \succ \alpha_j \Rightarrow$ the ith object is preferred to the jth object for the characteristic α). Thus, each encounter results in one of the following four mutually exclusive and exhaustive outcomes (the probability of ties being neglected):

$$\begin{aligned} &A_{ij}^{(1)}: \alpha_i \succ \alpha_j, \beta_i \succ \beta_j, \qquad A_{ij}^{(2)}: \alpha_i \succ \alpha_j, \beta_i \prec \beta_j, \\ &A_{ij}^{(3)}: \alpha_i \prec \alpha_j, \beta_i \succ \beta_j, \qquad A_{ij}^{(4)}: \alpha_i \prec \alpha_j, \beta_i \prec \beta_j, \end{aligned} \tag{1.2}$$

for all $i \neq j = 1, \cdots, t$, and we let

$$\pi_{ij\cdot k} = P\{A_{ij}^{(k)}\} \quad \text{for} \quad k = 1, 2, 3, 4 \quad \text{and} \quad i \neq j = 1, \cdots, t. \tag{1.3}$$

Thus, the n_{ij} encounters (assumed to be stochastically independent) result in the following multinomial distribution

$$\{n_{ij}!/\textstyle\prod_{k=1}^{4} n_{ij\cdot k}!\} \prod_{k=1}^{4} \pi_{ij\cdot k}^{n_{ij\cdot k}}, \tag{1.4}$$

where $n_{ij\cdot k}$ is the observed frequency of the event $A_{ij}^{(k)}$, for $k = 1, 2, 3, 4$ and $i < j = 1, \cdots, t$. It may be noted that by definition

$$\pi_{ij\cdot k} = \pi_{ji\cdot 5-k} \quad \text{and} \quad n_{ij\cdot k} = n_{ji\cdot 5-k} \qquad \text{for} \quad k = 1, \cdots, 4, \quad i < j = 1, \cdots, t. \tag{1.5}$$

The null hypothesis to be tested relates to the equality of preference of all t objects with respect to both (α, β). This will imply that

$$\pi_{ij\cdot k} = \pi_k \quad \text{for} \quad k = 1, \cdots, 4 \quad \text{and all} \quad i \neq j = 1, \cdots, t, \tag{1.6}$$

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where by virtue of (1.5), we have

$$\pi_1 + \pi_2 = \pi_1 + \pi_3 = \pi_2 + \pi_4 = \pi_3 + \pi_4 = \tfrac{1}{2}. \tag{1.7}$$

Equivalently, (1.7) can be written as

$$\pi_1 = \pi_4 = \tfrac{1}{4}(1 + \theta), \qquad \pi_2 = \pi_3 = \tfrac{1}{4}(1 - \theta), \tag{1.8}$$

where $\theta(-1 \leqq \theta \leqq 1)$ is a real parameter which we shall term the association parameter. Thus, for convenience of discussion, we shall specify the null hypothesis by

$$\begin{aligned} H_0: \pi_{ij\cdot k} &= \tfrac{1}{4}(1 + \theta), \quad \text{if} \quad k = 1, 4, \\ &= \tfrac{1}{4}(1 - \theta), \quad \text{if} \quad k = 2, 3, \quad \text{for} \quad i < j = 1, \cdots, t. \end{aligned} \tag{1.9}$$

The object of the present investigation is to consider some nonparametric tests for H_0 in (1.9) which may be regarded as natural extensions of some well-known tests for a single criterion only (cf. David [2, pp. 30–31]). Various properties of the proposed tests are also studied.

2. The likelihood ratio test and the spurious degrees of freedom. By (1.4), the unrestricted maximum of the likelihood function comes out as

$$L(\hat{\Omega}) = \prod_{i<j=1}^{t} [\{n_{ij}!/\prod_{k=1}^{4} n_{ij\cdot k}!\} \prod_{k=1}^{4} (n_{ij\cdot k}/n_{ij})^{n_{ij\cdot k}}], \tag{2.1}$$

(where we adopt the convention that $x^0 = 1$ for $x = 0$). Under H_0 in (1.9), the likelihood function is

$$[\prod_{i<j=1}^{t} \{n_{ij}!/\prod_{k=1}^{4} n_{ij\cdot k}!\}]4^{-N}(1 + \theta)^{N_1}(1 - \theta)^{N-N_1}, \tag{2.2}$$

where $N_1 = \sum_{i<j=1}^{t} (n_{ij\cdot 1} + n_{ij\cdot 4})$. Thus, the maximum likelihood estimator of θ is

$$\hat{\theta}_N = (2N_1 - N)/N. \tag{2.3}$$

Hence, the maximum of the likelihood function under H_0 is

$$L(\hat{\omega}) = [\prod_{i<j=1}^{t} \{n_{ij}!/\prod_{k=1}^{4} n_{ij\cdot k}!\}](N_1/N)^{N_1}[(N - N_1)/N]^{N-N_1}. \tag{2.4}$$

Thus, the likelihood ratio (L.R.) criterion is

$$\begin{aligned} \lambda_N &= L(\hat{\omega})/L(\hat{\Omega}) \\ &= N^{-N}\{N_1^{N_1}(N - N_1)^{(N-N_1)}\} \cdot \prod_{i<j=1}^{t} \{n_{ij}^{n_{ij}}/\prod_{k=1}^{4} n_{ij\cdot k}^{n_{ij\cdot k}}\}. \end{aligned} \tag{2.5}$$

Now, under H_0 in (1.9), the distribution of N_1 is given by

$$\binom{N}{N_1} 2^{-N}(1 + \theta)^{N_1}(1 - \theta)^{N-N_1} : 0 \leqq N_1 \leqq N, \tag{2.6}$$

and hence, under H_0, the likelihood function conditional on N_1 or $\hat{\theta}_N$ is

$$\binom{N}{N_1}^{-1} 2^{-N} \prod_{i<j=1}^{t} \{n_{ij}!/\prod_{k=1}^{4} n_{ij\cdot k}!\}. \tag{2.7}$$

Let us now consider the following randomized test function corresponding to significance level $\epsilon (0 < \epsilon < 1)$

$$(2.8) \qquad \begin{aligned} \phi &= 1, & \lambda_N &> \lambda_\epsilon(\hat{\theta}_N), \\ &= a_\epsilon(\hat{\theta}_N), & \lambda_N &= \lambda_\epsilon(\hat{\theta}_N), \\ &= 0, & \lambda_N &< \lambda_\epsilon(\hat{\theta}_N), \end{aligned}$$

where $\lambda_\epsilon(\hat{\theta}_N)$ and $a_\epsilon(\hat{\theta}_N)$ are so chosen that

$$(2.9) \qquad E\{\phi \mid H_0, \hat{\theta}_N\} = \epsilon.$$

Since $E\{\phi \mid H_0\} = E_{\hat{\theta}_N}[E\{\phi \mid H_0, \hat{\theta}_N\}]$, (2.9) implies that ϕ is a size ϵ similar test for H_0. [A non-randomized test (of size $\leqq \epsilon$) can be constructed by letting ϕ be 1 or 0 according as $\lambda_N >$ or $\leqq \lambda_\epsilon(\hat{\theta}_N)$.] For large n_{ij}'s, using the results of Wald [7] we conclude that under H_0 in (1.9), $-2 \log \lambda_N$ has asymptotically a χ^2 distribution with $3\binom{t}{2} - 1$ degrees of freedom. Thus, asymptotically

$$(2.10) \qquad -2 \log \lambda_\epsilon(\hat{\theta}_N) \cong \chi^2_{3\binom{t}{2}-1,\epsilon},$$

where $\chi^2_{r,\epsilon}$ is the upper $100\epsilon\%$ point of a χ^2 distribution with r df.

The L.R. test considered above is really a test for the identity of $\binom{t}{2}$ multinomial laws in (1.4) under the further specifications in (1.9), and hence carries $3\binom{t}{2} - 1$ df. In actual practice, we are often not interested in such a broad class of alternatives but rather in a comprehensive test for an analysis of variance problem posed below. Let us define

$$(2.11) \qquad \pi_{ij}(\alpha) = P\{\alpha_i > \alpha_j\} = \pi_{ij\cdot 1} + \pi_{ij\cdot 2},$$

$$(2.12) \qquad \pi_{ij}(\beta) = P\{\beta_i > \beta_j\} = \pi_{ij\cdot 1} + \pi_{ij\cdot 3}, \qquad i \neq j = 1, \cdots, t.$$

Under H_0 in (1.9), $\pi_{ij}(\alpha) = \pi_{ij}(\beta) = \frac{1}{2}$ for all $i \neq j = 1, \cdots, t$. Thus, if we want to test for the homogeneity of $\pi_{ij}(\alpha)$'s and of $\pi_{ij}(\beta)$'s, the number of df cannot exceed $2[\binom{t}{2} - 1]$. Further, preference behavior is frequently stochastically transitive, i.e. if $\pi_{ij}(\alpha) \geqq \frac{1}{2}$ and $\pi_{jk}(\alpha) \geqq \frac{1}{2}$, then $\pi_{ik}(\alpha) \geqq \frac{1}{2}$. Indeed, stronger restrictions on the preference probabilities are often reasonable and the $\binom{t}{2}$ quantities $\pi_{ij}(\alpha)$ may be expressible in terms of only t parameters, say $\pi_i^*(\alpha)$, $i = 1, \cdots, t$. In a similar manner, we may proceed with $\pi_{ij}(\beta)$'s. Thus, for the homogeneity of the parameters in (2.11) and in (2.12), we can have alternative tests carrying only $2(t - 1)$ df. To sum up, the L.R. test will have a much broader scope, but, for the specific purpose of testing homogeneity of the parameters in (2.11) and (2.12), it carries many spurious df. [It is well-known (cf.[4]) that in χ^2 tests the effect of increasing the degrees of freedom is to reduce the power of the test unless the noncentrality parameter increases at a sufficiently fast rate to compensate; in the present situation, the spurious df may not contribute much to the noncentrality parameter.]

The alternative test to be proposed is a natural generalization of the paired comparison test for a single characteristic (cf. David [3, pp. 30–31]) and will be appropriate if the characteristics (α, β) can be related to an underlying bivariate trait or random variable (X, Y) in whose locations we are interested. Fortunately this is possible in a wide range of problems involving paired comparisons. In such a setting, the feasibility and optimality of an appropriate L.R. test requires knowledge of the underlying bivariate distribution. This in turn limits the scope of the inferences and usually renders quite complicated the solution of the likelihood equations necessary for the evaluation of the L.R. criterion. On the other hand, our proposed methods appear to be quite simple and reasonably efficient

3. A permutationally distribution free test. We rewrite (2.7) as

$$\left[\binom{N}{N_1}^{-1} \prod_{i<j=1}^{t} \binom{n_{ij}}{n_{ij}^{(1)}}\right] \prod_{i<j=1}^{t} \left\{\left[\binom{n_{ij}^{(1)}}{n_{ij\cdot 1}} \left(\tfrac{1}{2}\right)^{n_{ij}^{(1)}}\right]\left[\binom{n_{ij}^{(2)}}{n_{ij\cdot 2}} \left(\tfrac{1}{2}\right)^{n_{ij}^{(2)}}\right]\right\}, \tag{3.1}$$

where $n_{ij}^{(1)} = n_{ij\cdot 1} + n_{ij\cdot 4}$, $n_{ij}^{(2)} = n_{ij\cdot 2} + n_{ij\cdot 3}$, for $i < j = 1, \cdots, t$. Thus, the first factor of (3.1) is a (generalized) hypergeometric distribution, while the second factor is the product of $t(t - 1)$ independent binomial distributions; all these distributions are simple and well-tabulated. Let us now define

$$u_{ij} = [n_{ij\cdot 1} + n_{ij\cdot 2} - n_{ij\cdot 3} - n_{ij\cdot 4}]/n_{ij}^{\frac{1}{2}}, \quad \text{if} \quad n_{ij} > 0, \tag{3.2}$$
$$= 0, \quad \text{otherwise};$$

$$v_{ij} = [n_{ij\cdot 1} - n_{ij\cdot 2} + n_{ij\cdot 3} - n_{ij\cdot 4}]/n_{ij}^{\frac{1}{2}}, \quad \text{if} \quad n_{ij} > 0, \tag{3.3}$$
$$= 0, \quad \text{otherwise}, \quad \text{for} \quad i \neq j = 1, \cdots, t.$$

Also let

$$T_{N,i}^{(1)} = \textstyle\sum_{j=1,\neq i}^{t} u_{ij}, \; T_{N,i}^{(2)} = \sum_{j=1,\neq i}^{t} v_{ij}, \quad \text{for} \quad i = 1, \cdots, t. \tag{3.4}$$

It may be noted that by definition

$$\textstyle\sum_{i=1}^{t} T_{N,i}^{(k)} = 0, \quad \text{for} \quad k = 1, 2. \tag{3.5}$$

[If $n_{ij} = n$ for all $i < j = 1, \cdots, t$, we may simplify (3.4) a little further. On characteristic $\alpha(\beta)$, denote the score of the ith object in its n encounters with the jth object by $a_{ij}(b_{ij})$, so that $a_{ij} = n_{ij\cdot 1} + n_{ij\cdot 2}(b_{ij} = n_{ij\cdot 1} + n_{ij\cdot 3})$, for $i \neq j = 1, \cdots, t$. Let

$$a_i = \textstyle\sum_{j=1,\neq i}^{t} a_{ij} \quad \text{and} \quad b_i = \sum_{j=1,\neq i}^{t} b_{ij}, \qquad i = 1, \cdots, t \tag{3.6}$$

be the total scores. Then

$$T_{N,i}^{(1)} = n^{-\frac{1}{2}}[2a_i - n(t - 1)], \qquad T_{N,i}^{(2)} = n^{-\frac{1}{2}}[2b_i - n(t - 1)], \quad \text{for} \quad i = 1, \cdots, t. \tag{3.7}$$

Thus, the $T_{N,i}$'s are related to standardized deviations of the total scores from their expectations].

To formulate the test in a convenient way, we further define

$$(3.8)\qquad Z_{N,i}^{(1)} = \tfrac{1}{2}(T_{N,i}^{(1)} + T_{N,i}^{(2)}) = \sum_{j=1,\neq i}^{t} n_{ij}^{-\frac{1}{2}}(n_{ij\cdot 1} - n_{ij\cdot 4}),$$

$$(3.9)\qquad Z_{N,i}^{(2)} = \tfrac{1}{2}(T_{N,i}^{(1)} - T_{N,i}^{(2)}) = \sum_{j=1,\neq i}^{t} n_{ij}^{-\frac{1}{2}}(n_{ij\cdot 2} - n_{ij\cdot 3}),$$

for $i = 1, \cdots, t$. From (3.1), we get after some simple manipulations that

$$(3.10)\qquad E\{Z_{N,i}^{(k)} \mid H_0, \hat{\theta}_N\} = 0, \quad \text{for} \quad k = 1, 2; \qquad i = 1, \cdots, t;$$

$$(3.11)\qquad E\{Z_{N,i}^{(k)} \cdot Z_{N,j}^{(q)} \mid H_0, \hat{\theta}_N\} = \delta_{kq}(\delta_{ij}t - 1)N_k/N,$$

for $k, q = 1, 2; i, j = 1, \cdots, t$, where δ_{uv} is the Kronecker delta and $N_2 = N - N_1$. Thus considering the linearly independent set of random variables $\{Z_{N,i}^{(k)}, k = 1, 2; i = 1, \cdots, t - 1\}$, taking the reciprocal of the covariance matrix as a suitable discriminant of their quadratic form, and finally symmetrizing, we obtain an appropriate test statistic as

$$(3.12)\qquad \begin{aligned} D_N &= t^{-1} \sum_{k=1}^{2} \sum_{i=1}^{t} (N/N_k)[Z_{N,i}^{(k)}]^2 \\ &= (t(1 - \hat{\theta}_N^{\,2}))^{-1} \{\sum_{i=1}^{t} [(T_{N,i}^{(1)})^2 - 2\hat{\theta}_N T_{N,i}^{(1)} T_{N,i}^{(2)} + (T_{N,i}^{(2)})^2]\}, \end{aligned}$$

where $\hat{\theta}_N = (N_1 - N_2)/N$. Since D_N is a positive semidefinite quadratic form, it will increase stochastically with increasing heterogeneity among the $T_{N,i}^{(1)}$ and/or $T_{N,i}^{(2)} (i = 1, \cdots, t)$. Hence, it seems natural to consider the following test function

$$(3.13)\qquad \begin{aligned} \Psi &= 1, & D_N &> D_\epsilon(\hat{\theta}_N), \\ &= a_\epsilon^{\,*}(\hat{\theta}_N), & D_N &= D_\epsilon(\hat{\theta}_N), \\ &= 0, & D_N &< D_\epsilon(\hat{\theta}_N), \end{aligned}$$

where $D_\epsilon(\hat{\theta}_N)$ and $a_\epsilon^{\,*}(\hat{\theta}_N)$ are so chosen that $E\{\Psi \mid H_0, \hat{\theta}_N\} = \epsilon$. As for the test ϕ in (2.8) and (2.9), Ψ will have unconditional significance level ϵ. For small values of n_{ij}'s, we may use (3.1) directly to construct Ψ in (3.13), while we formulate the following approach for large n_{ij}'s.

For the asymptotic theory we assume that

$$(3.14)\qquad \lim_{N\to\infty} N^{-1} n_{ij} = \rho_{ij} : 0 < \rho_{ij} < 1, \quad \text{for all} \quad i < j = 1, \cdots, t;$$

$$(3.15)\qquad |\theta| < 1, \qquad \text{where} \quad \theta \quad \text{is defined by (1.8).}$$

First, we consider the limiting distribution of the following random variables. Let $V_N = N^{\frac{1}{2}}(\hat{\theta}_N - \theta) = N^{-\frac{1}{2}}[(N_1 - N_2) - N\theta]$ and

$$(3.16)\qquad \mathbf{U}_N^{(k)} = (U_{N,1}^{(k)}, \cdots, U_{N,t-1}^{(k)}),\ U_{N,i}^{(k)} = (N/N_k)^{\frac{1}{2}} Z_{N,i}^{(k)}, \qquad i = 1, \cdots, t - 1; k = 1, 2,$$

where $Z_{N,i}^{(k)}$ is defined by (3.8) and (3.9). Now, under (1.9), (3.14) and (3.15), $N_k/N \to_P \frac{1}{2}(1 + (-1)^{k-1}\theta)$, $k = 1, 2$, and $1 \pm \theta \neq 0$. Let $\mathbf{Z}_N^{(k)} = (Z_{N,1}^{(k)}, \cdots, Z_{N,t-1}^{(k)})$, $k = 1, 2$. Then $(\mathbf{U}_N^{(1)}, \mathbf{U}_N^{(2)})$ and $([\frac{1}{2}(1 + \theta)]^{-\frac{1}{2}}\mathbf{Z}_N^{(1)}, [\frac{1}{2}(1 - \theta)]^{\frac{1}{2}}\mathbf{Z}_N^{(2)})$ have the same limiting distribution, if any. Now,

$Z_{N,i}^{(k)}$, $i = 1, \cdots, t-1$, $k = 1, 2$, and V_N are all linear functions of $\binom{t}{2}$ independent sets of multinomial variables. Hence, under (1.9), (3.14) and (3.15), as $N \to \infty$, $([\frac{1}{2}(1+\theta)]^{-\frac{1}{2}}\mathbf{Z}_N^{(1)}, [\frac{1}{2}(1-\theta)]^{-\frac{1}{2}}\mathbf{Z}_N^{(2)}, V_N)$ converges in law to a multivariate normal distribution with a null mean vector and a dispersion matrix

$$\text{(3.17)} \qquad \begin{pmatrix} t\mathbf{I} - \mathbf{11}' & \mathbf{0} & 0 \\ \mathbf{0} & t\mathbf{I} - \mathbf{11}' & 0 \\ 0 & 0 & (1-\theta^2) \end{pmatrix}; \mathbf{I} = ((\delta_{ij}))_{i,j=1,\cdots,t-1}, \quad \mathbf{1}' = (1, \cdots, 1).$$

Consequently, $[\mathbf{U}_N^{(1)}, \mathbf{U}_N^{(2)}, V_N]$ also converges (under (1.9), (3.14) and (3.15)) in law to a multinormal distribution with a null mean vector and the dispersion matrix in (3.17). This implies that V_N is bounded in probability. Combining these results with that of Theorem 2.4 of Steck [6], we arrive at the following.

THEOREM 3.1. *Under* (1.9), (3.14) *and* (3.15), (i) *the conditional distribution of* $(\mathbf{U}_N^{(1)}, \mathbf{U}_N^{(2)})$, *given* V_N *converges in probability to a multivariate normal distribution with a null mean vector and dispersion matrix given by the cofactor of* $(1-\theta^2)$ *in* (3.17), *and* (ii) *the unconditional distribution of* $(\mathbf{U}_N^{(1)}, \mathbf{U}_N^{(2)})$ *also converges to the same multinormal law.*

Noting that the reciprocal matrix of $t\mathbf{I} - \mathbf{11}'$ is $t^{-1}(\mathbf{I} + \mathbf{11}')$, and rewriting D_N in (3.12) as

$$\text{(3.18)} \qquad D_N = \sum_{k=1}^{2} \mathbf{U}_N^{(k)}[t^{-1}(\mathbf{I} + \mathbf{11}')]\mathbf{U}_N^{(k)\prime},$$

we obtain from Theorem 3.1, (3.18) and some simplifications, the following.

THEOREM 3.2. *Under* (1.9), (3.14) *and* (3.15), *the conditional distribution of* D_N, *given* $\hat{\theta}_N$ *converges (as* $N \to \infty$*) in probability to a chi-square distribution with* $2(t-1)$ *degrees of freedom. Also, the unconditional distribution of* D_N *(under* H_0 *in* (1.9)*) asymptotically reduces to a chi-square distribution with* $2(t-1)$ *degrees of freedom.*

REMARK. The condition (3.14) can be replaced by the following. We denote the incidence matrix by $((n_{ij}))_{i,j=1,\cdots,t}$, where $n_{ii} = 0$ for $i = 1, \cdots, t$, and $n_{ij} = n_{ji}$ for $i < j = 1, \cdots, t$. A sufficient condition for Theorems 3.1 and 3.2 to hold is that each row (or column) of this matrix contains at least one non-zero n_{ij} and only the set of non-zero n_{ij}'s satisfies (3.14). The proof of this follows on the same lines. This extension covers some incomplete block designs where not all possible pairings are made.

It follows from Theorem 3.2 that under (3.14) and (3.15),

$$\text{(3.19)} \qquad D_\epsilon(\hat{\theta}_N) \to \chi^2_{2(t-1),\epsilon} \quad \text{and} \quad a_\epsilon^*(\hat{\theta}_N) \to 0, \quad \text{in probability, as } N \to \infty.$$

Also, it follows that if we consider an asymptotically distribution-free size ϵ test function

$$\text{(3.20)} \qquad \begin{aligned} \Psi^* &= 1, \quad \text{if } D_N \geqq \chi^2_{2(t-1),\epsilon} \\ &= 0, \quad \text{if } D_N < \chi^2_{2(t-1),\epsilon} \end{aligned}$$

then under H_0 in (1.9), and granted (3.14) and (3.15), Ψ in (3.13) and Ψ^* are asymptotically equivalent.

4. Performance characteristics of the tests. We shall now prove the consistency of the tests against appropriate alternatives and also study their power properties. When the $\binom{t}{2}$ multinomial laws of the type (1.4) are not all identical, let us define

(4.1) $$\bar{\pi}_{k,N} = (1/2N) \sum_{i\neq j=1}^{t} n_{ij}\pi_{ij\cdot k}, \quad \text{for} \quad k = 1, 2, 3, 4,$$

where $\pi_{ij\cdot k}$'s are defined in (1.3) and (1.5). (1.5) and (4.1) imply that

(4.2) $$\bar{\pi}_{1,N} = \bar{\pi}_{4,N}, \bar{\pi}_{2,N} = \bar{\pi}_{3,N}, \quad \text{and} \quad \bar{\pi}_{1,N} + \bar{\pi}_{2,N} = \bar{\pi}_{1,N} + \bar{\pi}_{3,N} = \tfrac{1}{2},$$

whatever be the $\binom{t}{2}$ multinomial laws. Let us therefore write

(4.3) $$\bar{\pi}_{1,N} = \bar{\pi}_{4,N} = \tfrac{1}{4}(1 + \bar{\theta}_N), \qquad \bar{\pi}_{2,N} = \bar{\pi}_{3,N} = \tfrac{1}{4}(1 - \bar{\theta}_N).$$

Like θ in (1.8), $\bar{\theta}_N$ also lies in the interval $(-1, 1)$.

THEOREM 4.1. *Whatever be the* $\pi_{ij\cdot k}$'s $(i < j = 1, \cdots, t, k = 1, 2, 3, 4)$, $\hat{\theta}_N$ *in* (2.3) *is the MVU estimator of* $\bar{\theta}_N$ *in* (4.3).

PROOF. Writing $\hat{\theta}_N$ in (2.3) equivalently as

(4.4) $$(1/2N) \sum_{i\neq j=1}^{t} (n_{ij\cdot 1} - n_{ij\cdot 2} - n_{ij\cdot 3} + n_{ij\cdot 4}),$$

the unbiasedness of $\hat{\theta}_N$ as an estimator of $\bar{\theta}_N$, follows readily. Also, for multinomial distributions of the type (1.4), $(n_{ij\cdot 1} - n_{ij\cdot 2} - n_{ij\cdot 3} + n_{ij\cdot 4})/n_{ij}$ is the MVU estimator of $(\pi_{ij\cdot 1} - \pi_{ij\cdot 2} - \pi_{ij\cdot 3} + \pi_{ij\cdot 4})$, for all $i < j = 1, \cdots, t$, and these estimators are all independent. The rest of the proof is simple and is omitted. Hence, the theorem.

Since the multinomial law in (1.4) carries 3 df, we may write

(4.5) $$\pi_{ij\cdot 1} = \tfrac{1}{4}[(1 + \Delta_{ij})(1 + \epsilon_{ij}) + (\bar{\theta}_N + \eta_{ij})],$$

(4.6) $$\pi_{ij\cdot 2} = \tfrac{1}{4}[(1 + \Delta_{ij})(1 - \epsilon_{ij}) - (\bar{\theta}_N + \eta_{ij})],$$

(4.7) $$\pi_{ij\cdot 3} = \tfrac{1}{4}[(1 - \Delta_{ij})(1 + \epsilon_{ij}) - (\bar{\theta}_N + \eta_{ij})],$$

(4.8) $$\pi_{ij\cdot 4} = \tfrac{1}{4}[(1 - \Delta_{ij})(1 - \epsilon_{ij}) + (\bar{\theta}_N + \eta_{ij})], \qquad \text{for} \quad i \neq j = 1, \cdots, t,$$

where $(\Delta_{ij}, \epsilon_{ij}, \eta_{ij})$ are all real parameters. The Δ's and ϵ's account for heterogeneity of locations and the η's for heterogeneity of association. Thus, from (1.5) and (4.3), we have $\Delta_{ji} = -\Delta_{ij}$, $\epsilon_{ji} = -\epsilon_{ij}$, $\eta_{ij} = \eta_{ji}$, so that

(4.9) $$(1/2N) \sum_{i\neq j=1}^{t} n_{ij}\Delta_{ij} = (1/2N) \sum_{i\neq j=1}^{t} n_{ij}\epsilon_{ij} = (1/2N) \sum_{i\neq j=1}^{t} n_{ij}\eta_{ij} = 0.$$

Using (3.17), let us then define

(4.10) $$\Delta_{i\cdot} = \sum_{j=1,\neq i}^{t} \rho_{ij}^{\frac{1}{2}}\Delta_{ij}, \qquad \epsilon_{i\cdot} = \sum_{j=1,\neq i}^{t} \rho_{ij}^{\frac{1}{2}}\epsilon_{ij}, \qquad \text{for} \quad i = 1, \cdots, t,$$

and let

(4.11) $$\Delta^2 = \sum_{i=1}^{t} \Delta_{i\cdot}^2, \qquad \epsilon^2 = \sum_{i=1}^{t} \epsilon_{i\cdot}^2.$$

THEOREM 4.2. *The test based on D_N is consistent against the set of alternatives* $\Delta^2 + \epsilon^2 > 0$.

PROOF. We rewrite D_N in (3.12) in the alternative form

$$(4.12)\quad D_N = t^{-1} \{\textstyle\sum_{i=1}^t [T_{N,i}^{(k)}]^2 + (1 - \hat{\theta}_N{}^2)^{-1} \sum_{i=1}^t [T_{N,i}^{(k)} - \hat{\theta}_N T_{N,i}^{(q)}]^2\},$$

where (k, q) is any permutation of $(1, 2)$. Now, using (3.2), (3.3), (3.17), (4.5) through (4.11), it can be shown that

$$(4.13)\quad N^{-1} \textstyle\sum_{i=1}^t [T_{N,i}^{(1)}]^2 \to_P \Delta^2,\ N^{-1} \sum_{i=1}^t [T_{N,i}^{(2)}]^2 \to_P \epsilon^2.$$

Thus, if $\Delta^2 + \epsilon^2 > 0$ it follows from (4.12) and (4.13) that D_N can be made stochastically indefinitely large as $N \to \infty$. Hence, the theorem follows from (3.13) and (3.19).

In view of the consistency of the test, we shall consider alternatives infinitely close to the null hypothesis to study its asymptotic power. By an adaptation to the categorical situation of Pitman's [4] types of alternatives, we let in (4.5) through (4.8), $\bar{\theta}_N = \theta$ and

$$(4.14)\quad H_N{:}\Delta_{ij} = N^{-\frac{1}{2}}\mu_{ij}, \qquad \epsilon_{ij} = N^{-\frac{1}{2}}\nu_{ij} \quad \text{and} \quad \eta_{ij} = N^{-\frac{1}{2}}\xi_{ij},$$

where μ_{ij}, ν_{ij} and ξ_{ij} $(i \neq j = 1, \cdots, t)$ are all real and finite. Further, as in (4.10), we write

$$(4.15)\quad \mu_{i\cdot} = \textstyle\sum_{j=1,\neq i}^t \rho_{ij}^{\frac{1}{2}}\mu_{ij}, \qquad \nu_{i\cdot} = \sum_{j=1,\neq i}^t \rho_{ij}^{\frac{1}{2}}\nu_{ij}, \qquad \text{for} \quad i = 1, \cdots, t.$$

Then we have the following.

THEOREM 4.3. *Under $\{H_N\}$ in (4.14), D_N has asymptotically a noncentral χ^2 distribution with $2(t-1)$ df and the noncentrality parameter*

$$(4.16)\quad (t(1-\theta^2))^{-1}\{\textstyle\sum_{i=1}^t [\mu_{i\cdot}^2 - 2\theta\mu_{i\cdot}\nu_{i\cdot} + \nu_{i\cdot}^2]\}.$$

PROOF. Under $\{H_N\}$ in (4.14), the joint distribution of (u_{ij}, v_{ij}), defined by (3.2) and (3.3), can be shown to be asymptotically bivariate normal with means $(\rho_{ij}^{\frac{1}{2}}\mu_{ij}, \rho_{ij}^{\frac{1}{2}}\nu_{ij})$, variances unity, and correlation coefficient equal to θ, for all $i < j = 1, \cdots, t$. Thus $(T_{N,i}^{(1)}, T_{N,i}^{(2)})$ with $i = 1, \cdots, t-1$ (defined by (3.4)), will have asymptotically a $2(t-1)$ variate normal distribution with means $(\mu_{i\cdot}, \nu_{i\cdot})$, defined by (4.15), variances equal to $(t-1)$, covariances between $T_{N,i}^{(1)}, T_{N,j}^{(2)}$ equal to $(\delta_{ij}t - 1)\theta$, for $i, j = 1, \cdots, t-1$, and finally, covariances between $T_{N,i}^{(k)}, T_{N,j}^{(k)}$ equal -1 for all $i \neq j = 1, \cdots, t-1$, $k = 1, 2$, where δ_{ij} is the usual Kronecker delta. Consequently, by routine methods we see that $D_N{}^* = (t(1-\theta^2))^{-1}\{\sum_{i=1}^t [(T_{N,i}^{(1)})^2 - 2\theta T_{N,i}^{(1)}T_{N,i}^{(2)} + (T_{N,i}^{(2)})^2]\}$ has asymptotically a noncentral χ^2 distribution with $2(t-1)$ df and the noncentrality parameter in (4.16). Finally, by Theorem 4.1, $\hat{\theta}_N$ converges to θ (under $\{H_N\}$), in probability, and hence $D_N \sim_P D_N{}^*$. Hence the theorem follows.

Finally, as compared to the parametrically optimum paired comparison test (for bivariate normal distribution) based on Hotelling's T^2-statistic, the efficiency of the proposed paired comparison test will be the same as that of the bivariate median test (cf. Chatterjee [1] and Chatterjee and Sen [2]). For brevity, these results are not reproduced again.

5. A further remark. We have so far considered the case of paired characteristics only. The general case of p-tuple characteristics $(\alpha^{(1)}, \cdots, \alpha^{(p)})$ for some $p \geqq 1$, can be tackled in a similar way. In this case, there will be 2^p possible outcomes (as compared to 4 in (1.2)), and there will be $\binom{p}{2}$ two-way marginal tables. For $(\alpha^{(k)}, \alpha^{(q)})$ the same structure holds as in (1.8), (1.9); the corresponding θ is denoted by θ_{kq}, $k \neq q = 1, \cdots, p$, and the estimators by $\hat{\theta}_{N,kq}$ (defined as in (2.3)), for $k \neq q = 1, \cdots, p$. We write

$$(5.1) \qquad \mathbf{\Theta} = ((\theta_{kq}))_{k,q=1,\cdots,p}, \qquad \hat{\mathbf{\Theta}}_N = ((\hat{\theta}_{N,kq}))_{k,q=1,\cdots,p},$$

where conventionally $\theta_{kk} = \hat{\theta}_{N,kk} = 1$ for all $k = 1, \cdots, p$. Let $\hat{\mathbf{\Theta}}_N^{-1}$ be the reciprocal matrix of $\hat{\mathbf{\Theta}}_N$ and define $T_{N,i}^{(k)}$ as in (3.4), for all $k = 1, \cdots, p$, $i = 1, \cdots, t$. Then the test statistic will be

$$(5.2) \qquad D_{N(p)} = t^{-1} \textstyle\sum_{k=1}^{p} \sum_{q=1}^{p} \hat{\theta}_N^{kq} \sum_{i=1}^{t} T_{N,i}^{(k)} T_{N,i}^{(q)}.$$

It can be shown that under H_0 of homogeneity of the t objects, $D_{N(p)}$ will have a known permutation distribution which asymptotically reduces to χ^2 distribution with $p(t-1)$ df. Hence, a test procedure essentially similar to (3.13) can be proposed. Because of the similarity of approach, the details are omitted.

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