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PAIRS OF RINGS WITH THE SAME PRIME IDEALS, II

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Introduction. Much of [2] was devoted to studying pairs of subrings $A \subset B$ of a field with the property that A and B have the same prime ideals. In this paper, we continue that investigation, but we no longer assume that A and B are comparable. Interestingly, most of the results of [2] carry over to this more general context. Besides such extensions of [2], additional motivation for the more general context comes from the need to explicate some naturally occurring examples (see Examples 2.5, 3.6, and 4.3).

Section 2 begins by showing that we may reduce to the case in which R is a quasilocal domain with nonzero maximal ideal M and quotient field K. Proposition 2.3 establishes that the set $\mathcal{C}(R)$ of all subrings A of K with $\mathrm{Spec}(A) = \mathrm{Spec}(R)$ forms a complete semilattice. Theorem 2.4 shows that $\mathcal{C}(R)$ is naturally isomorphic to the complete semilattice $\mathcal{F}(A)$ of all subfields of the ring $A = (M : M)/M$. Conversely, Theorem 2.6 shows that for any commutative ring A which contains a field, $\mathcal{F}(A)$ may be realized as $\mathcal{C}(R)$ for some quasilocal domain R.

In Section 3, we investigate various common ring-theoretic properties of the rings in $\mathcal{C}(R)$, with special emphasis on the Noetherian property. Specifically, Theorem 3.3 gives several equivalent conditions for each $A \in \mathcal{C}(R)$ to be Noetherian; when these conditions hold, $\mathcal{C}(R)$ is finite. In the final section, we study the semilattice $\mathcal{F}(A)$ and give several examples that illuminate the preceding material.

All rings are assumed to be commutative, with 1. Usually, R will denote a quasilocal domain with nonzero maximal ideal M and quotient field K, and k will be the prime subfield of R/M. As usual, we write

$$(M : M) = \{x \in K | xM \subset M\};$$

the group of units of a ring A will be denoted by $U(A)$; and the finite field with q elements will be denoted by $\mathbf{F}_q$. Any unexplained material is standard, as in [4], [5], or [6].

2. The semilattice $\mathcal{C}(R)$. Let L be a field. Given subrings A and B of L, we write $A \sim B$ if A and B have the same set of prime ideals, that is, if $\mathrm{Spec}(A) = \mathrm{Spec}(B)$. Then $\sim$ is an equivalence relation on the set of subrings of L, and the $\sim$-equivalence class containing L is just the set of subfields of L. In this paper, we are interested in the $\sim$-equivalence classes determined by

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subrings of L which are not fields. First, we give a few observations about such rings which are reminiscent of results from [**2**, Section 3].

PROPOSITION 2.1. *Let A and B be subrings of a field such that A is not a field. Then:*

(a) *If* $\mathrm{Spec}(A) = \mathrm{Spec}(B)$, *then:*

(1) *A and B have the same quotient field.*

(2) *If A is not quasilocal, then $A = B$.*

(3) *If A is quasilocal with maximal ideal M, then*

$$B \subset (M : M).$$

(b) $\mathrm{Spec}(A) = \mathrm{Spec}(B)$ *if and only if* $\mathrm{Max}(A)$ *and* $\mathrm{Max}(B)$ *are comparable.*

Proof. (a) (1) Let I be any nonzero common ideal of A and B. Then the quotient field of A (or B) consists of the elements i/j, where $i \in I$ and $0 \neq j \in I$. (2) Suppose that A and B have two distinct common maximal ideals M and N. Then $A = M + N = B$. (3) This follows since M is also an ideal of B.

(b) The "only if" assertion is clear. For the converse, we may assume $\mathrm{Max}(A) \subset \mathrm{Max}(B)$. By (2) of part (a), we may also assume that A is quasilocal with nonzero maximal ideal M.

Let $C = A \cap B$. Clearly M is a prime ideal of C; we shall show that M is actually a maximal ideal of C. Choose $a \in C - M$. Then there is an $x \in A$ such that $ax = 1$. Since M is also a maximal ideal of B, $ay + m = 1$ for some $y \in B$ and $m \in M$. Then

$$x = xay + xm = y + xm \in y + M \subset B.$$

Hence $x \in C$ and $a \in U(C)$. Thus $M \in \mathrm{Max}(C)$. By [**2**, Theorem 3.10], as applied to $C \subset A$, we have $\mathrm{Spec}(A) = \mathrm{Spec}(C)$. Another application of [**2**, Theorem 3.10] (or [**2**, Proposition 3.8]) yields $\mathrm{Spec}(C) = \mathrm{Spec}(B)$. Hence $\mathrm{Spec}(A) = \mathrm{Spec}(B)$.

Remark 2.2. Most of the results of this paper carry over for a commutative quasilocal ring A whose maximal ideal M contains a regular element. However, if M consists entirely of zero divisors, then both (1) and (3) of Proposition 2.1(a) may fail since A may itself be a total quotient ring. For example, consider the dual numbers over the reals: let $A = \mathbf{R}[\epsilon]$ with $\epsilon^2 = 0$. Then A is a quasilocal ring whose maximal ideal $M = \mathbf{R}\epsilon$ consists entirely of zero divisors. Let B be the subring $\mathbf{Q} + M$. Then $\mathrm{Spec}(A) = \mathrm{Spec}(B)(= \{M\}$ since $M^2 = 0)$, while A and B are distinct total quotient rings.

By Proposition 2.1, we reduce to the case in which R is a domain with proper quotient field K, and write

$$C(R) = \{A | A \text{ is a subring of } K \text{ and } \mathrm{Spec}(A) = \mathrm{Spec}(R)\}$$

for the $\sim$-equivalence class of R. Clearly, $\mathcal{C}(R)$ is nonempty since it contains R. If R is not quasilocal, then in fact $\mathcal{C}(R) = \{R\}$ by part (2) of Proposition 2.1(a). For this reason, we shall usually assume that R is a quasilocal domain with maximal ideal M. Then $\mathcal{C}(R)$ is just the set of all (quasilocal) subrings of K which have M as (a) maximal ideal.

$\mathcal{C}(R)$ is a partially ordered set under set-theoretic inclusion. In fact, we next show that $\mathcal{C}(R)$ is a complete (meet) semilattice with respect to intersections (i.e., each nonempty subset of $\mathcal{C}(R)$ has an infimum). In general, $\mathcal{C}(R)$ need not be a lattice (see Example 4.3(b)). However, since $\mathcal{C}(R)$ is closed under unions of chains, we see via Zorn's Lemma that $\mathcal{C}(R)$ always has maximal elements. (Cf. also [**2**, Lemma 3.32].) Moreover, $\mathcal{C}(R)$ is closed under directed unions.

PROPOSITION 2.3. *Let R be a domain which is not a field. Then $\mathcal{C}(R)$ is a complete semilattice with respect to set-theoretic inclusion and intersection. Moreover, $\mathcal{C}(R)$ is a (complete) lattice if and only if $\mathcal{C}(R)$ has a maximum element.*

Proof. We may assume that R is quasilocal with maximal ideal M. To show that $\mathcal{C}(R)$ is a complete semilattice we need only show that $\mathcal{C}(R)$ is closed under arbitrary (nonempty) intersections. Let $\{R_\alpha\}$ be a nonempty family of subrings of K with each $R_\alpha \in \mathcal{C}(R)$. We show that $T = \cap R_\alpha \in \mathcal{C}(R)$. Indeed, T is quasilocal with maximal ideal M. Thus $\mathrm{Spec}(T) = \mathrm{Spec}(R)$ by Proposition 2.1(b), whence $T \in \mathcal{C}(R)$. The "moreover" statement is clear from the above remarks.

For future use, we next define two important subsets of $\mathcal{C}(R)$. Given $T \in \mathcal{C}(R)$, let

$$\mathcal{L}(T) = \{A \in \mathcal{C}(R) | A \subset T\} \quad \text{and}$$
$$\mathcal{U}(T) = \{A \in \mathcal{C}(R) | T \subset A\}.$$

Note that $\mathcal{L}(T)$ and $\mathcal{U}(T)$ are each complete subsemilattices of $\mathcal{C}(R)$ and that $\mathcal{L}(T)$ is actually a lattice. For a fixed domain T, these two sets were studied extensively (without this notation) in [**2**].

We have already observed that $\mathcal{C}(R)$ need not be a lattice. It is well known that any partially ordered set may be completed, in the sense of Dedekind-MacNeille, to a complete lattice (cf. [**6**, Proposition 5, page 44]). For $\mathcal{C}(R)$ (or any complete semilattice), this completion is particularly simple: we just add a maximum element. Specifically, for any complete (meet) semilattice $(S, \leqq, \wedge)$ adjoin a new element ∞ to S to get $S^* = S \cup \{\infty\}$, and extend the ordering on S to S^* by decreeing $x < \infty$ for all $x \in S$. For any $x, y \in S^*$, define

$$x \vee y = \bigwedge \{z \in S^* | x \leqq z \text{ and } y \leqq z\}.$$

It is easily verified that S^* is a complete lattice. Moreover, any (nontrivial) complete lattice arises from a complete semilattice (which is not a lattice) in this manner.

For any commutative ring A, we let $\mathcal{F}(A)$ denote the set of subrings of A which are fields. If A is a field L, then $\mathcal{F}(L)$ is just the (complete) lattice of subfields of L. However, $\mathcal{F}(A)$ may be empty (for instance, if $A = \mathbf{Z}$). In fact, $\mathcal{F}(A)$ is nonempty if and only if either A has prime characteristic or A is a $\mathbf{Q}$-algebra. Like $\mathcal{C}(R)$, $\mathcal{F}(A)$ is a complete (meet) semilattice with respect to inclusion and intersection. Moreover, $\mathcal{F}(A)$ is a (complete) lattice if and only if $\mathcal{F}(A)$ has a maximum element. The semilattice $\mathcal{F}(A)$ will be studied in more detail in Section 4.

Our next theorem establishes an order-isomorphism between $\mathcal{C}(R)$ and $\mathcal{F}(A)$, for a suitable ring A defined in terms of R. It may often be used to reduce ring-theoretic questions to field-theoretic questions. It also generalizes the bijection given in [**2**, Theorem 3.25].

THEOREM 2.4. *Let R be a quasilocal domain with nonzero maximal ideal M and let $A = (M : M)/M$. Then the correspondence $T \leftrightarrow T/M$ gives an order-isomorphism from $\mathcal{C}(R)$ onto $\mathcal{F}(A)$.*

Proof. Let $\pi : (M : M) \longrightarrow A$ be the natural surjection. By part (3) of Proposition 2.1(a), each $T \in \mathcal{C}(R)$ is contained in $(M : M)$. It is easy to see that the function $\Psi : \mathcal{C}(R) \longrightarrow \mathcal{F}(A)$, given by $\Psi(T) = \pi(T) = T/M$, is a well-defined injection that preserves and reflects order.

Let $F \in \mathcal{F}(A)$. To show that Ψ is surjective, we need only show that $D = \pi^{-1}(F) \in \mathcal{C}(R)$; for then $F = D/M = \Psi(D)$. Note that D has M as a maximal ideal. Proposition 2.1(b) then yields $\text{Spec}(D) = \text{Spec}(R)$. Hence $D \in \mathcal{C}(R)$, as desired.

In the above bijection between $\mathcal{C}(R)$ and $\mathcal{F}(A)$, the minimum element of $\mathcal{C}(R)$ corresponds to the prime subfield of R/M. Moreover for any $T \in \mathcal{C}(R)$, $\mathcal{L}(T)$ corresponds to $\mathcal{F}(T/M)$, and $\mathcal{U}(T)$ corresponds to the subsemilattice of $\mathcal{F}(A)$ of all fields which are contained in A and contain T/M.

Example 2.5. Let L be any field and $R = L[[X]] = L + M$, where $M = XR$ is the maximal ideal of R. In this case, $(M : M) = R$ and $R/M \cong L$. Theorem 2.4 therefore gives a bijection between $\mathcal{C}(R)$ and $\mathcal{F}(L)$, namely $k + M \leftrightarrow k$ for each subfield k of L. If we choose L to be either $\mathbf{F}_p$ or $\mathbf{Q}$, then $\mathcal{C}(R) = \{R\}$. Thus $\mathcal{C}(R)$ may be a singleton even when R is quasilocal (cf. (2) of Proposition 2.1(a)). In Example 4.3(a), we shall give an example of a quasilocal domain R for which $\mathcal{C}(R) = \{R\}$, but with $(M : M)$ a proper overring of R.

The above reasoning leads to the following conclusion. Let R be a domain with nonzero maximal ideal M. Then $\mathcal{C}(R) = \{R\}$ if and only if either (a) R is not quasilocal or (b) R is quasilocal and

$$\mathcal{F}((M : M)/M) = \{R/M\}.$$

Moreover, if (b) holds, then R/M is canonically either $\mathbf{F}_p$ or $\mathbf{Q}$.

Our next theorem may be viewed as a converse to Theorem 2.4. We show that for any ring A which contains a field, there is a quasilocal domain R with nonzero maximal ideal M such that $A \cong (M : M)/M$. Thus by Theorem 2.4, the semilattice $\mathcal{F}(A)$ may be realized as $\mathcal{C}(R)$.

THEOREM 2.6. *A commutative ring A has the form $(M : M)/M$ for some quasilocal domain R with nonzero maximal ideal M if and only if A contains a field k; equivalently, if and only if either A has prime characteristic or A is a $\mathbf{Q}$-algebra. In this case, we may choose R so that $R/M \cong k$. Moreover, R may be chosen to be Noetherian if A is finite-dimensional over k.*

Proof. If $A = (M : M)/M$ for some quasilocal domain R with nonzero maximal ideal M, then A contains the field $k = R/M$.

Conversely, suppose that A contains a field k. Then $A \cong k[\{X_\alpha\}]/I$ for some set $\{X_\alpha\}$ of indeterminates and nonzero ideal I. Let $T = k[\{X_\alpha\}]$ and let $\pi : T \longrightarrow A$ be the natural surjection with $\ker \pi = I$. Define

$$S = \{u \in T | \pi(u) \in U(A)\}.$$

Then S is a saturated, multiplicatively closed subset of T. Also, π induces a surjective homomorphism

$$\pi^* : T_S \longrightarrow A,$$

given by

$$\pi^*(t/s) = \pi(t)\pi(s)^{-1},$$

with $\ker \pi^* = I_S$. Moreover, $x \in U(T_S)$ if and only if $\pi^*(x) \in U(A)$. This follows easily from the fact that S is saturated, as does the assertion that

$$U(T_S) = \{s_1/s_2 | s_1, s_2 \in S\}.$$

Since $\pi^*(1 + I_S) = 1, 1 + I_S \subset U(T_S)$. Thus I_S is a nonzero ideal contained in $\text{rad}(T_S)$. We claim that $R = k + I_S$ is a quasilocal subring of T_S with nonzero maximal ideal $M = I_S$. To see this, it is enough to show that $\alpha + i/s \in U(R)$ for each $0 \neq \alpha \in k$ and $i/s \in I_S$. Since $I_S \subset \text{rad}(T_S)$, $\alpha + i/s \in U(T_S)$, and hence $(\alpha + i/s)x = 1$ for some $x \in T_S$. Thus

$$x = \alpha^{-1} - \alpha^{-1}(i/s)x \in k + I_S = R,$$

proving the claim. Since $M = I_S$ is a nonzero ideal of the completely integrally closed (Krull) domain T_S, we have $(M : M) = T_S$ [**4**, (34.3) Theorem]. Thus

$$(M : M)/M = T_S/I_S \cong A.$$

The "equivalently" statement has been noted earlier.

Next, we prove the "moreover" assertion. Suppose that A is finite-dimensional over k. Then by the above proof, T, and hence T_S, may be chosen to be Noetherian. In addition, T_S is finitely generated as an R-module since $A(\cong T_S/M)$ is finitely generated as a $k(= R/M)$-vector space. Hence R is Noetherian, by Eakin's Theorem.

Remark 2.7. An easier proof of Theorem 2.6 is available if A is a domain which contains a field k. In this case, let $T = A[[X]]$ and $M = XT \subset \text{rad}(T)$. Then $R = k + M$ is quasilocal with maximal ideal M and $(M : M) = T$. Thus

$$(M : M)/M = A[[X]]/XA[[X]] \cong A.$$

3. $\mathcal{C}(R)$ for R Noetherian. We next investigate what common ring-theoretic properties are shared by the elements of $\mathcal{C}(R)$. In [**2**], we investigated the ascent and descent of various ring-theoretic properties between comparable pairs of rings with the same prime ideals. Those techniques can sometimes be used for incomparable elements of $\mathcal{C}(R)$. Let $A, B \in \mathcal{C}(R)$ and let C be the subring $A \cap B \in \mathcal{C}(R)$. If a certain property holds in A and is preserved by both descent and ascent to rings with the same prime ideals, then it holds in C, and hence also in B. (This technique has already been used in the proof of Proposition 2.1(b).) Another such extension applies to [**2**, Proposition 3.5]: if $\text{Spec}(A) = \text{Spec}(B)$ for domains which are not fields, then $A_P = B_P$ for each nonmaximal prime ideal $P \in \text{Spec}(A)$, and $\text{Spec}(A)$ and $\text{Spec}(B)$ are homeomorphic as topological spaces with the Zariski topology. Many other such extensions of [**2**] may be found in this way: consider [**2**, Propositions 2.2, 3.15, and B.1], for instance.

We next concentrate on what can be said about $\mathcal{C}(R)$ when R is Noetherian. The following result will be useful both for studying $\mathcal{C}(R)$ and for constructing examples in the next section.

PROPOSITION 3.1. *Let $\{R_\alpha | \alpha \in \Lambda\}$ be a nonempty family of commutative rings and $R = \Pi R_\alpha$. Fix an element $\beta \in \Lambda$. Suppose that for each $\alpha \in \Lambda$ we have an $F_\alpha \in \mathcal{F}(R_\alpha)$ and an isomorphism $\varphi_\alpha : F_\beta \longrightarrow F_\alpha$ (with $\varphi_\beta = 1$). Then*

$$F = \{(\varphi_\alpha(x))_{\alpha\in\Lambda} | x \in F_\beta\} \in \mathcal{F}(R).$$

Conversely, any $F \in \mathcal{F}(R)$ arises in such a manner from suitable $(F_\alpha), (\varphi_\alpha)$.

Proof. The first assertion admits a routine verification, and so we omit the details. For the converse, consider $F \in \mathcal{F}(R)$. Then $F_\alpha = p_\alpha(F) \in \mathcal{F}(R_\alpha)$, where $p_\alpha : R \longrightarrow R_\alpha$ is the natural projection. Fix $\beta \in \Lambda$. Consider $x_\beta \in F_\beta$. Since the restriction of p_β to F gives an isomorphism between F and F_β, there is a unique $x \in F$ such that $p_\beta(x) = x_\beta$. For each $\alpha \in \Lambda$, define $\varphi_\alpha : F_\beta \longrightarrow F_\alpha$ by $\varphi_\alpha(x_\beta) = p_\alpha(x)$. Then each φ_α is an isomorphism and

$$F = \{(\varphi_\alpha(x))_{\alpha\in\Lambda} | x \in F_\beta\}.$$

COROLLARY 3.2. *Let* $\{R_\alpha\}$ *and* R *be as in Proposition* 3.1. *Then:*

(a) $\mathcal{F}(R)$ *is nonempty if and only if either each* R_α *has the same prime characteristic or each* R_α *is a* $\mathbf{Q}$*-algebra.*

(b) *If* Λ *and each* $\mathcal{F}(R_\alpha)$ *are finite, then* $\mathcal{F}(R)$ *is finite.*

THEOREM 3.3. *Let* R *be a quasilocal domain with nonzero maximal ideal* M, *and let* k *the prime subfield of* R/M. *Then the following statements are equivalent:*

(a) *Each* $A \in \mathcal{C}(R)$ *is Noetherian;*

(b) *The minimum element* B *of* $\mathcal{C}(R)$ *is Noetherian;*

(c) *Some* $D \in \mathcal{C}(R)$ *is Noetherian and* $[D/M : k] < \infty$.

Moreover, if any of the above equivalent statements holds, then $\mathcal{C}(R)$ *is finite and* $[A/M : k] < \infty$ *for each* $A \in \mathcal{C}(R)$.

Proof. It is clear that (a) $\Rightarrow$ (b). Moverover, (b) $\Rightarrow$ (c) since it follows from the comments after Theorem 2.4 that $B/M = k$. We shall prove (c) $\Rightarrow$ (a).

Take D as in (c) and once again let B be the minimum element of $\mathcal{C}(R)$. We have $B/M = k$ and $B \subset D$. By [**2**, Corollary 3.29], B is Noetherian if and only if both D is Noetherian and $[D/M : B/M] < \infty$. Hence, by (c), B is Noetherian. Another application of [**2**, Corollary 3.29] now yields that each $A \in \mathcal{C}(R)$ is Noetherian (and $[A/M : k] < \infty$).

We next prove the "moreover" statement. Suppose that the minimum element B of $\mathcal{C}(R)$ is Noetherian. Then $T = (M : M)$ is a finitely generated B-module. Hence $S = T/M$ is a finitely generated $k(= B/M)$-module. If $k = \mathbf{F}_p$, then S is finite and hence $\mathcal{F}(S)$ is also finite; Theorem 2.4 then yields $\mathcal{C}(R) \cong \mathcal{F}(S)$ is finite. Thus, we may assume that $k = \mathbf{Q}$.

Since S is Artinian, we have $S = S_1 \times \ldots \times S_n$, where each S_i is a (complete) local Artinian ring with maximal ideal M_i and residue field K_i. As S_i is finite dimensional over k, we have that $[K_i : k] < \infty$. Moreover, by Cohen structure theory [**7**, (31.10) Corollary], each S_i has a unique coefficient field, which is isomorphic to K_i. Thus each $\mathcal{F}(S_i)$ is finite, and hence $\mathcal{F}(S)$ is finite by Corollary 3.2(b). By Theorem 2.4, $\mathcal{C}(R)$ is then also finite. The second part of the "moreover" statement was noted in the above proof that (c) $\Rightarrow$ (a).

COROLLARY 3.4. *Let* R *be a quasilocal Noetherian domain which is not a field. Then the following statements are equivalent:*

(a) $\mathcal{C}(R)$ *is finite;*

(b) $\mathcal{L}(R)$ *is finite;*

(c) *Each* $A \in \mathcal{C}(R)$ *is Noetherian.*

Proof. (a) $\Rightarrow$ (b) is trivial; and (c) $\Rightarrow$ (a) is included in the "moreover" assertion in Theorem 3.3. We next prove (b) $\Rightarrow$ (c). Assume that $\mathcal{L}(R)$ is finite. Then $\mathcal{F}(R/M)$ is finite, by the comments following Theorem 2.4. Thus $[R/M : k] < \infty$, where k is the prime subfield of R/M. Hence each $A \in \mathcal{C}(R)$ is Noetherian, by the "(c) $\Rightarrow$ (a)" part of Theorem 3.3.

COROLLARY 3.5. *Let R be a quasilocal domain which is not a field, such that $\mathcal{C}(R)$ is finite. Then the following statements are equivalent:*
(a) *R is Noetherian;*
(b) *Each $A \in \mathcal{C}(R)$ is Noetherian;*
(c) *Some $A \in \mathcal{C}(R)$ is Noetherian.*

Proof. (a) $\Rightarrow$ (b) is just the "(a) $\Rightarrow$ (c)" part of Corollary 3.4; and (b) $\Rightarrow$ (c) is clear. Finally, (c) $\Rightarrow$ (a) follows from the "(a) $\Rightarrow$ (c)" part of Corollary 3.4 since $\mathcal{C}(A) = \mathcal{C}(R)$.

Of course, $\mathcal{C}(R)$ may be finite and $[A/M : k] < \infty$ for each $A \in \mathcal{C}(R)$ even when R is not Noetherian (see Example 4.3(a)).

In Example 4.3(c), we shall give an example of a (necessarily nonNoetherian) quasilocal domain R with maximal ideal M such that $[A/M : k] < \infty$ for each $A \in \mathcal{C}(R)$, but $\mathcal{C}(R)$ is infinite. In contrast to Corollary 3.4, we next give examples to show that for a quasilocal Noetherian domain R, we may have either $\mathcal{U}(R)$ finite and $\mathcal{C}(R)$ infinite, or both $\mathcal{L}(R)$ and $\mathcal{U}(R)$ infinite.

Example 3.6. (a) Let

$$R = \mathbf{F}_2[\{X_n | 1 \leqq n < \infty\}]_N,$$

where $\{X_n\}$ is a denumerable set of indeterminates and

$$N = (\{X_n | 1 \leqq n < \infty\}).$$

Then $\mathcal{C}(R) = \{R\}$, but R is not Noetherian.

(b) Let $R = \mathbf{R}[[X]]$ as in Example 2.5. Then R is Noetherian, $\mathcal{U}(R) = \{R\}$, and $\mathcal{L}(R)(= \mathcal{C}(R) \cong \mathcal{F}(\mathbf{R}))$ is infinite.

(c) Let $K = \mathbf{F}_p(s, t)$ for indeterminates s and t, and let F be the subfield $\mathbf{F}_p(s^p, t^p)$. Let $R = K[[X]] = K + M$, and put $A = F + M$. Then A is Noetherian by [**2**, Corollary 3.29], since $[K : k] < \infty$. However, both $\mathcal{L}(A)$ and $\mathcal{U}(A)$ are infinite since there are infinitely many subfields between F and $\mathbf{F}_p$, and infinitely many subfields between K and F (cf. [**5**, Exercise 15, page 289]).

We close this section by studying whether three other classical properties are stable under ascent/descent in $\mathcal{C}(R)$.

Remark 3.7. Let R and S be domains which are not fields such that $R \sim S$, that is, such that Spec(R) = Spec(S). Then R satisfies accp (ascending chain condition on principal ideals) if and only if S satisfies accp. Indeed, since R and S share the same nonunits, a chain of proper principal ideals $Ra_1 \subset Ra_2 \subset \ldots$ in R is strictly ascending if and only if the corresponding chain $Sa_1 \subset Sa_2 \subset \ldots$ in S is strictly ascending.

Besides "Noetherian," the most natural sufficient condition for accp is "UFD" (unique factorization domain). However, for this property, the above type of descent fails. To see this, consider $R = \mathbf{C}[[X]]$ and $S = \mathbf{R} + X\mathbf{C}[[X]]$. Then

$R \sim S$, R is a UFD, but S is not a UFD since S is not completely integrally closed. Indeed, if $R \sim S$ for any domains which are not fields, then R and S have the same complete integral closure (cf, [**2**, Proposition 3.15]).

Thirdly, if $R \sim S$, then R satisfies PIT ($\text{ht}(P) = 1$ for each prime P of R which is minimal over a nonzero principal ideal of R) if and only if S satisfies PIT (cf. [**3**, Corollary 3.2(b)]).

4. The semilattice $\mathcal{F}(A)$. In this section, we make a few remarks about the semilattice $\mathcal{F}(A)$ and give some examples. First, let's consider the case in which A is a field, L. In this case, $\mathcal{F}(L)$ is the complete lattice of all subfields of L. When L is a finite algebraic extension of its prime subfield k, Galois theory gives an order-reversing bijection between $\mathcal{F}(L)$ and a certain lattice of subgroups.

Specifically, when $k = \mathbf{F}_p$ and $[L : k] = n < \infty$, $\mathcal{F}(L)$ is anti-isomorphic to the lattice of subgroups of $\mathbf{Z}/n\mathbf{Z}$, or equivalently, the lattice of positive divisors of n. If $k = \mathbf{Q}$ and $[L : \mathbf{Q}] < \infty$, let E be the normal closure of L over $\mathbf{Q}$, $G = \text{Aut}(E/\mathbf{Q})$, and $H = \text{Aut}(L/\mathbf{Q})$; then $\mathcal{F}(L)$ is anti-isomorphic to the lattice of subgroups of G which contain H.

When L is an arbitrary infinite extension of k, the structure of $\mathcal{F}(L)$ is much more complicated and does not seem to have been studied extensively. Recently, the lattice of intermediate fields between $F(X)$ and F has been investigated in [**1**].

Proposition 3.1 may be used to give a satisfactory description of $\mathcal{F}(L_1 \times \ldots \times L_n)$ when each L_i is a finite field. Out next result follows easily from Proposition 3.1 and the following two well known facts about finite fields:

$$\mathbf{F}_p m \subset \mathbf{F}_p n \Leftrightarrow m|n; \text{ and}$$

$$\text{Aut}(\mathbf{F}_p n) = \langle \sigma \rangle (\cong \mathbf{Z}/n\mathbf{Z}),$$

where $\sigma(x) = x^p$.

Proposition 4.1. *Let $A = \mathbf{F}_{p^{n_1}} \times \ldots \times \mathbf{F}_{p^{n_t}}$, with $t \geqq 2$, and $e = \gcd(n_1, \ldots, n_t)$ Then each $F \in \mathcal{F}(A)$ has the form*

$$F = \{(a, \sigma_1(a), \ldots, \sigma_{t-1}(a)) | a \in \mathbf{F}_p d\}$$

for a fixed integer $d \geqq 1$ with $d|e$ and fixed $\sigma_i \in \text{Aut}(\mathbf{F}_p d)$. In particular,

$$|\mathcal{F}(A)| = \sum_{d|e} d^{t-1}.$$

Corollary 4.2. *Let $A = \mathbf{F}_{p^{n_1}} \times \ldots \times \mathbf{F}_{p^{n_t}}$, with $t \geqq 2$. Then $\mathcal{F}(A)$ is a lattice (i.e., A has a maximum subfield) if and only if $\gcd(n_1, \ldots, n_t) = 1$. Moreover, in this case, $\mathcal{F}(A) = \{\mathbf{F}_p\}$.*

We next give several examples promised earlier in the paper.

Example 4.3. (a) Let $A = \mathbf{F}_p \times \mathbf{F}_p$. By Theorems 2.6 and 2.4, there is a local Noetherian domain R with nonzero maximal ideal M such that $\mathcal{C}(R)$ is order-isomorphic to $\mathcal{F}(A)$. By Proposition 4.1, $|\mathcal{C}(R)| = |\mathcal{F}(A)| = 1$. Thus $\mathcal{C}(R) = \{R\}$, but $R/M(\cong \mathbf{F}_p)$ is a proper subring of $(M : M)/M(\cong A)$.

(b) Let $A = \mathbf{F}_4 \times \mathbf{F}_4$. By Theorems 2.6 and 2.4, there is a local Noetherian domain R with nonzero maximal ideal M such that $\mathcal{C}(R)$ is order-isomorphic to $\mathcal{F}(A)$. By Proposition 4.1, $|\mathcal{C}(R)| = |\mathcal{F}(A)| = 3$. By Corollary 4.2 and Theorem 2.4, $\mathcal{C}(R)(\cong \mathcal{F}(A))$ is not a lattice, even though R is Noetherian and $\mathcal{C}(R)$ is finite.

A concrete example for the domain R in both Example (a) and (b) may be constructed as follows. Let $T = \mathbf{F}_q[X]_S$, where

$$S = \mathbf{F}_q[X] - ((X) \cup (X + 1)).$$

Next, let $R = \mathbf{F}_q + M$, where $M = X(X + 1)T$. Then R is a local Noetherian domain with nonzero maximal ideal M (cf. [**7**, (E2.1), page 204]) such that

$$(M : M)/M \cong T/M \cong \mathbf{F}_q \times \mathbf{F}_q.$$

(c) Let A be a direct product of denumberably many copies of $\mathbf{F}_4$. By Proposition 3.1, the only subfields of A are its prime subfield $(= \mathbf{F}_2)$ and an infinite number of fields each isomorphic to $\mathbf{F}_4$. By Theorem 2.6, there is a quasilocal domain R with nonzero maximal ideal M such that $(M : M)/M \cong A$. By Theorem 2.4, $\mathcal{C}(R)(\cong \mathcal{F}(A))$ is infinite, but $[D/M : \mathbf{F}_2] < \infty$ for each $D \in \mathcal{C}(R)$ (and hence R is not Noetherian by Theorem 3.3).

We close this paper with one special case in which $\mathcal{F}(A)$ (and hence $\mathcal{C}(R)$) is a lattice.

PROPOSITION 4.4. (a) *Let A be a domain such that $\mathcal{F}(A)$ is nonempty and finite. Then $\mathcal{F}(A)$ is a lattice.*

(b) *Let R be a quasilocal domain with nonzero maximal ideal M such that $(M : M)/M$ is a domain and $\mathcal{C}(R)$ is finite. Then $\mathcal{C}(R)$ is a lattice.*

Proof. (a) In this case, each $F \in \mathcal{F}(A)$ is a finite algebraic extension of its prime subfield k. Since A is a domain,

$$K = \{a \in A | a \text{ is algebraic over } k\}$$

is a field, and hence K is the maximum subfield of A. Thus the semilattice $\mathcal{F}(A)$ has a maximum element (K), and so $\mathcal{F}(A)$ is a lattice.

(b) This follows readily via Theorem 2.4 and (a).

Our final example shows that the assertion in Proposition 4.4(a) fails if we remove the hypothesis that $\mathcal{F}(A)$ is finite.

Example 4.5. Let $A_1 = \mathbf{Q}(X)$ and $A_2 = \mathbf{Q}(Y)$, with X and Y indeterminates. Let A be the subring of $\mathbf{Q}(X, Y)$ generated by A_1 and A_2. It is easily verified that the domain A is not a field. Hence, $\mathcal{F}(A)$ is not a lattice since A_1 and A_2 are distinct maximal elements of $\mathcal{F}(A)$. Note, however, that $\mathcal{F}(A)$ is infinite.

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