J. Austral. Math. Soc. (*Series A*) **26** (1978), 241–250

PAIRWISE CONCEPTS IN BITOPOLOGICAL SPACES

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(Received 24 December 1976; revised 23 May 1977)

Communicated by N. Smythe

Abstract

Several pairwise concepts have been studied for bitopological spaces. In this note an attempt has been made to see how much 'bitopological' these pairwise concepts are. For example pairwise T_1 is purely a topological concept whereas pairwise normality is very much 'bitopological'. Several questions pertaining to this theme are dealt with.

Subject classification (*Amer. Math. Soc.* (*MOS*) 1970): 54E55, 54G20.

1. Introduction

The idea of bitopological spaces was introduced by Kelly (1963). Since then several topological notions have been generalised to the setting of bitopological spaces under the name 'pairwise P'. These concepts have been defined in such a fashion that for the 'trivial' bitopological space they become the corresponding topological concepts. In an attempt to have this requirement several concepts are defined in such a way that they become just the topological concepts. For example a bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is pairwise T_1 (Reilly, 1972) if both the topological spaces $(X, \mathscr{P})$ and $(X, \mathscr{Q})$ are T_1. Similarly pairwise compactness (Swart, 1971) of $(X, \mathscr{P}, \mathscr{Q})$ is nothing but the compactness of the space $(X, \mathscr{P} + \mathscr{Q})$, where $\mathscr{P} + \mathscr{Q}$ is the coarsest topology finer than both $\mathscr{P}$ and $\mathscr{Q}$. Contrary to this, pairwise normality of $(X, \mathscr{P}, \mathscr{Q})$ is independent of the normality of $(X, \mathscr{P} + \mathscr{Q})$ or the normality of $(X, \mathscr{P})$ and $(X, \mathscr{Q})$. Thus pairwise normality is very much 'bitopological'. In this note we are interested in investigating the 'topologicalness' of various bitopological concepts. In brief we are interested in the following questions: Given a pairwise P bitopological space $(X, \mathscr{P}, \mathscr{Q})$ what can be said about the situations (i) both $(X, \mathscr{P})$ and $(X, \mathscr{Q})$ have property P, (ii) $(X, \mathscr{P} + \mathscr{Q})$ has property P and vice-versa?

2. Preliminaries

Let $(X, \mathscr{P}, \mathscr{Q})$ be a bitopological space. Then there are associated

(a) a pair of topological spaces $(X, \mathscr{P})$ and $(X, \mathscr{Q})$;

(b) a topological space $(X, \mathscr{P}+\mathscr{Q})$, where $\mathscr{P}+\mathscr{Q}$ denotes the coarsest topology finer than both $\mathscr{P}$ and $\mathscr{Q}$. Given a topological property P we say that $(X, \mathscr{P}, \mathscr{Q})$ is

(i) *p-P* if $(X, \mathscr{P}, \mathscr{Q})$ is pairwise P;
(ii) *bi-P* if both $(X, \mathscr{P})$ and $(X, \mathscr{Q})$ have property P; and
(iii) *sup-P* if $(X, \mathscr{P}+\mathscr{Q})$ has property P.

In this note we are interested in the following possible implications

$$\text{sup-}P \rightleftarrows \text{p-}P \rightleftarrows \text{bi-}P.$$

We study these implications for various topological properties. The implications which hold are shown by an arrow $\rightarrow$ and those which do not hold by $\not\rightarrow$. The implications about which we are not certain are not shown in the figures. Proofs of most of the implications are simple and therefore omitted. As such the note is mainly concerned with counter examples. These are described in Section 6. Throughout this note 'space' means a bitopological space and implication means one of the four implications listed above.

$\mathbf{R}$ denotes the set of reals, $\mathscr{L}$ the topology generated by $\{]-\infty, a[: a \in \mathbf{R}\}$ and $\mathscr{U}$ the topology generated by $\{]a, +\infty[: a \in \mathbf{R}\}$. For $A \subseteq X$, $\mathscr{P}\text{-cl}\, A$ denotes the closure of A in $(X, \mathscr{P})$.

3. Separation Axioms

DEFINITION (Murdeshwar and Naimpally, 1966). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be $p\text{-}T_0$ if for each pair of distinct points of X, there is a $\mathscr{P}$-open set or a $\mathscr{Q}$-open set containing one of the points, but not the other.

THEOREM 1. *$(X, \mathscr{P}, \mathscr{Q})$ is* p-T_0 *if and only if it is* sup-T_0. *Also if it is* bi-T_0 *(even if one of $\mathscr{P}$ and $\mathscr{Q}$ is T_0) then $(X, \mathscr{P}, \mathscr{Q})$ is* p-T_0. *The fourth implication does not hold.*

PROOF. It is shown in Singal and Jain (unpublished) that $(X, \mathscr{P}, \mathscr{Q})$ is p-T_0 if and only if it is sup-T_0. Bi-T_0 implies p-T_0 is obvious. For the fourth implication 1(a) is a counter example. Thus for $P = T_0$,

$$\text{sup-}P \rightleftarrows \text{p-}P \not\rightleftarrows \text{bi-}P$$

DEFINITION (Reilly, 1972). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be $p\text{-}T_1$ if for each pair of distinct points x, y, there exist $U \in \mathscr{P}$, $V \in \mathscr{Q}$ such that $x \in U, y \notin V$ and $x \notin U, y \in V$.

THEOREM 2. *$(X, \mathscr{P}, \mathscr{Q})$ is* p-T_1 *if and only if it is* bi-T_1 *and only if it is* sup-T_1. *The fourth implication does not hold.*

PROOF. A simple proof of the first part is given in Reilly (1972). The second part is trivial. 2(a) is a counter example.

$$\text{sup-}P \;\substack{\not\rightarrow \\ \leftarrow}\; \text{p-}P \rightleftarrows \text{bi-}P$$

DEFINITION (Kelly, 1963). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be *p-T_2* if given distinct points $x, y \in X$ there exist $U \in \mathscr{P}$, $V \in \mathscr{Q}$ such that $x \in U$, $y \in V$, $U \cap V = \varnothing$.

THEOREM 3. *If $(X, \mathscr{P}, \mathscr{Q})$ is* p-T_2, *then it is* sup-T_2. *No other implication holds.*

PROOF. This is easy to establish. For counter examples see 3, 4 and 5(a).

$$\text{sup-}P \;\substack{\not\rightarrow \\ \leftarrow}\; \text{p-}P \;\substack{\not\rightarrow \\ \not\leftarrow}\; \text{bi-}P$$

DEFINITION (Kelly, 1963). In a space $(X, \mathscr{P}, \mathscr{Q})$, $\mathscr{P}$ is said to be regular with respect to $\mathscr{Q}$ if for each $x \in X$ and a $\mathscr{P}$-closed set F such that $x \in F$ there exist a $\mathscr{P}$-open set U and a $\mathscr{Q}$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$. $(X, \mathscr{P}, \mathscr{Q})$ is said to be *p-regular* if $\mathscr{P}$-is regular with respect to $\mathscr{Q}$ and $\mathscr{Q}$ is regular with respect to $\mathscr{P}$.

DEFINITION (Lane, 1967). In a space $(X, \mathscr{P}, \mathscr{Q})$, $\mathscr{P}$ is said to be completely regular with respect to $\mathscr{Q}$ if for each $x \in X$ and a $\mathscr{P}$-closed F such that $x \notin F$ there exists a $\mathscr{P}$-upper semi continuous and $\mathscr{Q}$-lower semi continuous function $f: X \to [0, 1]$ such that $f(x) = 0$, $f(F) = \{1\}$. $(X, \mathscr{P}, \mathscr{Q})$ is said to be *p-completely regular* if $\mathscr{P}$ is completely regular with respect to $\mathscr{Q}$ and $\mathscr{Q}$ is completely regular with respect to $\mathscr{P}$.

THEOREM 4. *If $(X, \mathscr{P}, \mathscr{Q})$ is p-(completely) regular, then it is* sup-*(completely) regular. None of the other implications holds.*

PROOF. Let A be $\mathscr{P} + \mathscr{Q}$-closed and $x \notin A$. If A is $\mathscr{P}$-closed or $\mathscr{Q}$-closed, then we are done. Otherwise $A = F \cup G$, where F is $\mathscr{P}$-closed and G is $\mathscr{Q}$-closed. If $(X, \mathscr{P}, \mathscr{Q})$ is p-regular, $x \notin F$ imples the existence of disjoint sets $U_1 \in \mathscr{P}$, $V_1 \in \mathscr{Q}$ such that $x \in U_1$, $F \subseteq V_1$. Similarly $x \notin G$ implies the existence of disjoint sets $U_2 \in \mathscr{P}$, $V_2 \in \mathscr{Q}$ such that $G \subseteq U_2$, $x \in V_2$. Now $U = U_1 \cap V_2$ and $V = U_2 \cup V_1$ are $\mathscr{P} + \mathscr{Q}$ open sets such that $x \in U$, $A \subseteq V$ and $U \cap V = \varnothing$.

If $(X, \mathscr{P}, \mathscr{Q})$ is p-completely regular, then there exists a $\mathscr{P} + \mathscr{Q}$-continuous function $f: X \to [0,1]$ such that $f(x) = 1, f(F) = \{0\}$. Similarly there exists $\mathscr{P} + \mathscr{Q}$-continuous function $g: X \to [0,1]$ such that $g(x) = 1$, $g(G) = \{0\}$. Clearly $h = \min\{f, g\}$ is a $\mathscr{P} + \mathscr{Q}$-continuous function separating x and A.

Examples 2(b), 1(c) and 1(b) are the required counter examples. Thus for P = regular, completely regular, we have the diagram

$$\text{sup-}P \underset{\leftarrow}{\not\rightarrow} \text{p-}P \underset{\not\leftarrow}{\rightarrow} \text{bi-}P$$

Definition (Kelly, 1963). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be *p-normal* if given a $\mathscr{P}$-closed set F and a $\mathscr{Q}$-closed G with $F \cap G = \varnothing$, there exist a $\mathscr{P}$-open set U and a $\mathscr{Q}$-open set V such that $F \subseteq V$, $G \subseteq U$, $U \cap V = \varnothing$.

Definition (Patty, 1967). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be *p-completely normal* if whenever A and B are subsets of X such that $\mathscr{P}\text{-cl}\, A \cap B = \varnothing = \mathscr{Q}\text{-cl}\, B \cap A$ then there exist $\mathscr{Q}$-open set U and $\mathscr{P}$-open set V such that $A \subseteq U$, $B \subseteq V$, $U \cap V = \varnothing$.

Normality and complete normality have entirely different behaviour. In fact for these two properties none of the four implications holds as is shown by examples 2(c), 6, 7 and 11. Thus for P normal, completely normal we have

$$\text{sup-}P \underset{\not\leftarrow}{\not\rightarrow} \text{p-}P \underset{\not\leftarrow}{\not\rightarrow} \text{bi-}P$$

Remarks. Several authors introduced the pairwise concepts in their own way keeping different aspects in mind. For example p-T_1 and p-T_0 have been defined in Murdeshwar and Naimpally (1966) and in Reilly (1972) also. Here we took one definition from Murdeshwar and Naimpally (1966) and one from Reilly (1972). Similar investigations may be made adopting the definitions of Murdeshwar and Naimpally (1966) and Reilly (1972). Next, several other separation axioms like T_D, E_0, E_1, R_0, R_D, almost regular, almost completely regular, mildly normal and so on have been generalized to the setting of bitopological spaces. It should be interesting to study the above implications for these properties also.

4. Compactness like properties

Definition (Swart, 1971). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be *p-compact* if each cover $\mathscr{U} \subseteq \mathscr{P} \cup \mathscr{Q}$ has a finite subcover.

Theorem 5. *$(X, \mathscr{P}, \mathscr{Q})$ is* p-*compact if and only if it is* sup-*compact and only if it is* bi-*compact. The fourth implication does not hold.*

Proof. The first part is shown in Swart (1971). The second part is trivial. For the fourth implication example 5(b) works.

$$\text{sup-}P \rightleftarrows \text{p-}P \underset{\not\leftarrow}{\rightarrow} \text{bi-}P$$

DEFINITION (Singal and Singal, 1970). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be p-*countably compact* if each proper $\mathscr{Q}$-closed set is $\mathscr{P}$-countably compact and each proper $\mathscr{P}$-closed set is $\mathscr{Q}$-countably compact.

THEOREM 6. *$(X, \mathscr{P}, \mathscr{Q})$ is* p-*countably compact if it is* sup-*countably compact. No other implication holds.* See examples 1(d) and 5(c).

$$\text{sup-}P \mathrel{\substack{\rightarrow \\ \not\leftarrow}} \text{p-}P \mathrel{\substack{\not\rightarrow \\ \not\leftarrow}} \text{bi-}P$$

DEFINITION (Reilly, 1973). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be p-*Lindelöf* if each pairwise open cover $\mathscr{U}$ (i.e., $\mathscr{U} \subseteq \mathscr{P} \cup \mathscr{Q}$, $\mathscr{U} \cap \mathscr{P} \neq \emptyset$, $\mathscr{U} \cap \mathscr{Q} \neq \emptyset$) has a countable subcover.

THEOREM 7. *If $(X, \mathscr{P}, \mathscr{Q})$ is* sup-*Lindelöf, then it is* p-*Lindelöf. Other implications do not hold.*

For the implications which do not hold see examples 1(e) and 5(d). Thus for P = Lindelöf

$$\text{sup-}P \mathrel{\substack{\rightarrow \\ \not\leftarrow}} \text{p-}P \mathrel{\substack{\not\rightarrow \\ \not\leftarrow}} \text{bi-}P$$

DEFINITION (Saegrove, 1973). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be p-*pseudo compact* if each pairwise continuous map $f:(X, \mathscr{P}, \mathscr{Q}) \to (\mathbf{R}, \mathscr{U}, \mathscr{L})$ is bounded.

THEOREM 8. *If $(X, \mathscr{P}, \mathscr{Q})$ is* sup-*pseudo compact then it is* p-*pseudo compact.*

PROOF. Every pairwise continuous $f:(X, \mathscr{P}, \mathscr{Q}) \to (\mathbf{R}, \mathscr{U}, \mathscr{L})$ is continuous from $(X, \mathscr{P} + \mathscr{Q}) \to (\mathbf{R}, \mathscr{U} + \mathscr{L})$ and $\mathscr{U} + \mathscr{L}$ is the usual topology on $\mathbf{R}$.

Example 1(f) shows that a p-pseudo compact space need not be bi-pseudo compact or sup-pseudo compact. We do not know about the fourth implication, though we strongly suspect that this too does not hold. As such for P = pseudo-compact we have the following diagram

$$\text{sup-}P \mathrel{\substack{\rightarrow \\ \not\leftarrow}} \text{p-}P \not\rightarrow \text{bi-}P$$

REMARKS. There are as many as five definitions of p-compactness in bitopological spaces given by Birsan (1969), Fletcher, Hoyle and Patty (1969), Kim (1968) Saegrove (1973) and Swart (1971). For P = compactness we consider Swart's definition, whereas for P = countable compactness we take a generalization of Kim's definition (Singal and Singal, 1970) and for P = Lindelöf we take FHP-compactness (Reilly, 1973) as a model. However a close look at the results and examples shows that if we define p-countable compactness and p-Lindelöf following Swarts definition, then for P = countable compactness or Lindelöf we get the same implication diagram as for P = compactness. Next if for p-compactness we take Kim's definition and define p-Lindelöf likewise, then for these two properties we get the same implication diagrams as for P = countable compactness. Similarly if compactness and countable compactness are defined following (Fletcher, Hoyle and Patty, 1969), and are substituted for Lindelöf in the implication diagram following Theorem 7 the diagram remains unchanged. It may, however, be noted here that in view of a result of Cooke and Reilly (1975), the two ways of defining p-countable compactness and p-Lindelöf are essentially the same. Here pseudo-compactness is a part of the definition of compactness due to Saegrove (1973).

5. Disconnectedness-like properties

DEFINITION (Previn, 1967). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be *p-disconnected* if there exist subsets A, B such that $A \cup B = X$ and $(\mathscr{P}\text{-cl}\, A \cap B) \cup (A \cap \mathscr{Q}\text{-cl}\, B) = \varnothing$.

THEOREM 9. *If* $(X, \mathscr{P}, \mathscr{Q})$ *is* p-*disconnected, then it is* sup-*disconnected. None of the other implications holds.*

See examples 1(g), 2(d) and 8.

DEFINITION (Swart, 1971). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be p-*totally disconnected* if given $x \neq y$ there exist A, B with $x \in A, y \in B$ (or $x \in B, y \in A$) such that $A \cup B = X$, $(\mathscr{P}\text{-cl}\, A \cap B) \cup (A \cap \mathscr{Q}\text{-cl}\, B) = \varnothing$.

DEFINITION (Dutta, 1971). A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be *p-zero dimensional* if $\mathscr{P}$ has a base of $\mathscr{Q}$-closed sets and $\mathscr{Q}$ has a base of $\mathscr{P}$-closed sets.

THEOREM 10. *If* $(X, \mathscr{P}, \mathscr{Q})$ *is* p-*totally desconnected* (*p-zero dimensional*), *then it is* sup-*totally disconnected* (sup-*zero dimensional*).

PROOF. For total disconnectedness the proof is trivial. For zero dimensional if $\mathscr{B}$ is a base for $\mathscr{P}$ consisting of $\mathscr{Q}$-closed sets and $\mathscr{C}$ is a base for $\mathscr{Q}$ consisting of $\mathscr{P}$-closed sets, then $\mathscr{B} \cap \mathscr{C} = \{B \cap C : B \in \mathscr{B}, C \in \mathscr{C}\}$ is base for $\mathscr{P} + \mathscr{Q}$ consisting of $\mathscr{P} + \mathscr{Q}$-closed sets.

For any of these two properties none of the three remaining implications holds, as is shown by examples 1(g), 2(d), 10, 12 and 13. Summarizing for P = totally disconnected or zero dimensional we have

$$\text{sup-}P \rightleftarrows \text{p-}P \not\rightleftarrows \text{bi-}P$$

DEFINITION. A space $(X, \mathscr{P}, \mathscr{Q})$ is said to be p-*extremally disconnected* if $\mathscr{P}$-closure of each $\mathscr{Q}$-open set is $\mathscr{Q}$-open and $\mathscr{Q}$-closure of each $\mathscr{P}$-open set is $\mathscr{P}$-open or equivalently given $\mathscr{P}$-open set U and $\mathscr{Q}$-open set V with $U \cap V = \varnothing$ we have $\mathscr{Q}\text{-cl}\, U \cap \mathscr{P}\text{-cl}\, V = \varnothing$.

With respect to the three implications

$$\text{sup-}P \rightarrow \text{p-}P \rightleftarrows \text{bi-}P$$

this property behaves in a similar manner as the other three properties in this section. See examples 2(e) and 9. However about the remaining implication we are not certain. May be it is true and its behaviour is same as that of the preceding properties, but we suspect that this implication is also false and this property behaves in a different manner from the other properties discussed in this section.

REMARKS. As we see above out of the four implications some hold while others do not. It would be interesting to find conditions under which the implications which do not hold in general hold. For example for $P =$ extremally disconnected bi-P + p-P implies sup-P. We do not know whether bi-P is a necessary condition for p-P to imply sup-P. It would also be interesting to see whether these conditions are necessary too. Next, in some of the examples above in order to show that the space is not bi-P we show that one of the spaces is not P (e.g., $P = T_0$). It whould be of interest to find examples for such situations where none of the topologies is P.

6. Examples

1. Let X be an infinite set, $\mathcal{I}$ be the indiscrete topology and $\mathcal{D}$ be the discrete topology. Consider the space $(X, \mathcal{I}, \mathcal{D})$.

(a) $(X, \mathcal{I}, \mathcal{D})$ is p-T_0, but not bi-T_0.

(b) $(X, \mathcal{I}, \mathcal{D})$ is bi-completely regular, but not even p-regular.

(c) $(X, \mathcal{I}, \mathcal{D})$ is sup-completely regular, but not even p-regular.

(d) $(X, \mathcal{I}, \mathcal{D})$ is p-countably compact, but neither bi-countably compact nor sup-countably compact.

(e) If X is uncountable then $(X, \mathcal{I}, \mathcal{D})$ is p-Lindelöf, but neither bi-Lindelöf nor sup-Lindelöf.

(f) If X is countable, then $(X, \mathcal{I}, \mathcal{D})$ is p-pseudo compact, but is neither sup-pseudo compact nor bi-pseudo compact.

(g) $(X, \mathcal{I}, \mathcal{D})$ is sup-totally disconnected, but not p-disconnected.

2. Let X be an infinite set and let p be a fixed point of X. Let $\mathcal{P} = \{U \subseteq X : p \in U\} \cup \{\emptyset\}$ and $\mathcal{Q} = \{V \subseteq X : p \notin V\} \cup \{X\}$.

(a) $(X, \mathcal{P}, \mathcal{Q})$ is sup-T_1, but not p-T_1.

(b) $(X, \mathcal{P}, \mathcal{Q})$ is p-completely regular, but not even bi-regular.

(c) $(X, \mathcal{P}, \mathcal{Q})$ is p-completely normal, but not even bi-normal.

(d) $(X, \mathcal{P}, \mathcal{Q})$ is p-totally disconnected and p-zero dimensional, but neither bi-disconnected nor bi-zero dimensional.

https://doi.org/10.1017/S1446788700011733 Published online by Cambridge University Press

(e) $(X, \mathscr{P}, \mathscr{Q})$ is p-extremally disconnected, since each proper $\mathscr{P}$-open set is $\mathscr{Q}$-closed and vice-versa. However it is not bi-extremally disconnected, because $\mathscr{Q}$ does not have two disjoint closed sets though it has plenty of disjoint $\mathscr{Q}$-open sets.

3. $(\mathbf{R}, \mathscr{U}, \mathscr{L})$ is sup-T_2, but not p-T_2.

4. Let X be an infinite set. If $\mathscr{P}$ be the cofinite topology on X and $\mathscr{Q}$ be the discrete topology, then $(X, \mathscr{P}, \mathscr{Q})$ is p-T_2, but not bi-T_2.

5. Let X be an infinite set and let $p \neq q$ be two points of X. Let $\mathscr{P} = \mathscr{T}(p)$ and $\mathscr{Q} = \mathscr{T}(q)$ where $\mathscr{T}(x)$ is the topology in which each $y \neq x$ is open and an open neighbourhood of x is a set containing x and all but finitely many points of X.

(a) $(X, \mathscr{P}, \mathscr{Q})$ is bi-T_2, but not p-T_2.

(b) $(X, \mathscr{P}, \mathscr{Q})$ is bi-compact, but not p-compact, since $\mathscr{P} + \mathscr{Q}$ is the discrete topology.

(c) $(X, \mathscr{P}, \mathscr{Q})$ is bi-countably compact, but not p-countably compact. $X - \{q\}$ is $\mathscr{P}$-closed of which $\{\{x\} : x \neq q\}$ is a $\mathscr{Q}$-open cover having no subcover.

(d) $(X, \mathscr{P}, \mathscr{Q})$ is bi-Lindelöf, but not p-Lindelöf. $\{\{x\} : x \in X\}$ is a pairwise open cover which has no proper subcover.

6. Let $X = [-1, 1]$. Let $\mathscr{P} = \{U : 0 \notin U \text{ or }]-1, 1[\subseteq U\}$ and $\mathscr{Q} = \{U : X - U \text{ is finite or } -1 \notin U\}$. Then $(X, \mathscr{P}, \mathscr{Q})$ is a bi-completely normal space, which is not p-normal. $A = \{-1\}$ and $B = \{0, 1\}$ are $\mathscr{P}$-closed and $\mathscr{Q}$-closed sets respectively such that $A \cap B = \varnothing$. Since the smallest $\mathscr{P}$-open set containing B is $]-1, 1]$, the only non-empty set disjoint from $]-1, 1]$ is $\{-1\}$ which is not $\mathscr{Q}$-open.

7. Let X be an infinite set and let $p \in X$. Let $\mathscr{P} = \{U : p \in U\} \cup \{\varnothing\}$ and $\mathscr{Q} = \{X, \varnothing, \{p\}\}$. Then $(X, \mathscr{P}, \mathscr{Q})$ is not sup-normal. However it is p-completely normal vacuously. That there do not exist $A, B \subseteq X$ with (i) $A \cap \mathscr{Q}\text{-cl}\, B = \varnothing$ and (ii) $\mathscr{P}\text{-cl}\, A \cap B = \varnothing$ can be seen as follows: If $p \in B$, then $\mathscr{Q}\text{-cl}\, B = X$ and so (i) is impossible. If $p \notin B$, then $\mathscr{Q}\text{-cl}\, B = X - \{p\}$ so that (i) implies $A = \{p\}$. Now (ii) is impossible ($\mathscr{P}\text{-cl}\, A = X$).

8. Let $X \neq \varnothing$ and let $\varnothing \neq A \neq B \neq \varnothing$ be two subsets of X. If $\mathscr{P} = \{X, A, X - A, \varnothing\}$ and $\mathscr{Q} = \{X, B, X - B, \varnothing\}$, then $(X, \mathscr{P}, \mathscr{Q})$ is a bi-disconnected space which is not p-disconnected.

9. Let X be an infinite set and p a fixed point of X. Let $\mathscr{P} = \{U : p \in U\} \cup \{\varnothing\}$ and $\mathscr{Q}$ be the discrete topology. The space $(X, \mathscr{P}, \mathscr{Q})$ is bi-extremally disconnected. It is not p-extremally disconnected, because if $q \neq p$, then $\{q\} \cap X - \{q\} = \varnothing$, but $\mathscr{Q}\text{-cl}\,\{q\} = \{q\}$ and $\mathscr{P}\text{-cl}\,(X - \{q\}) = X$ and hence not disjoint. This space is also sup-extremally disconnected, but not p-extremally disconnected.

10. Let X be a non-empty set and let $p \neq q$ be in X. If $\mathscr{P} = \{U : p \in U\} \cup \{\emptyset\}$ and $\mathscr{Q} = \{V : q \in V\} \cup \{\emptyset\}$, then $(X, \mathscr{P}, \mathscr{Q})$ is sup-zero dimensional. However it is not p-zero dimensional, because the only base for $\mathscr{P}$ is $\{\{x, p\} : x \in X\}$ of which $\{q, p\}$ is not $\mathscr{Q}$-closed and similarly for the base $\{\{x, q\} : x \in X\}$ for $\mathscr{Q}$.

11. Consider $(X, \mathscr{P}, \mathscr{Q})$, where $X = \{a, b, c, d,\}$, $\mathscr{P} = \{X, \emptyset, \{a, b\}, \{c, d\}\}$ and $\mathscr{Q} = \{X, \emptyset, \{a, c\}, \{b, d\}, \{a, b, c\}, \{b\}\}$. Here $(X, \mathscr{P}, \mathscr{Q})$ is not p-normal, though it is easily seen to be sup-completely normal.

12. Let X be the set of real numbers, $\mathscr{P}$ be the right half-open interval topology which has as a base the family of all sets of the form $[a, b[$, where $a, b \in X$ and $\mathscr{Q}$ be the left half-open interval topology (base the family $]a, b]$, $a, b \in X$). Here $(X, \mathscr{P}, \mathscr{Q})$ is bi-totally disconnected, but not p-totally disconnected.

13. Let X and $\mathscr{P}$ be as in the example 12 above. If $\mathscr{D}$ is the discrete topology on X, then $(X, \mathscr{P}, \mathscr{D})$ is bi-zero dimensional, but not p-zero dimensional.

Acknowledgements

The author is thankful to the refree for providing examples 12 and 13 and thus modifying a result in Section 5. He is also thankful to Prof. M. K. Singal for his help and guidance during the preparation of this note.

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