Luschgy, H.
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PAIRWISE SUFFICIENCY AND INVARIANCE

HARALD LUSCHGY

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Introduction

For a statistical experiment LeCam [3] introduced the concept of a sufficient sublattice (in the M-space of the experiment) thus replacing the usual measure theoretic notion of sufficiency by a vector lattice framework. In this note it is pointed out that the sufficient sublattices are exactly the fixed spaces of the subsemigroups of the semigroup of all measurewise experiment-preserving positive operators on the M-space of the experiment. So reduction of an experiment by sufficiency and reduction by invariance coincide. We derive the same result for pairwise sufficient subfields and sufficient subfields under suitable conditions on the experiment. Some examples which concern the semigroup of operators arising from measurewise experiment-preserving point transformations are given. This is related to Basu's [1] work.

1. Sufficient sublattices and invariance

Let $\mathcal{E}=(X, \mathcal{A}, \mathcal{P})$ be an experiment, i.e. X is a set, $\mathcal{A}$ is a σ-field on X and $\mathcal{P}$ a non-empty set of probability measures on $(X, \mathcal{A})$. The band $L(\mathcal{E})$ generated by $\mathcal{P}$ in the space $\mathrm{ca}(X, \mathcal{A})$ of all bounded signed measures on $(X, \mathcal{A})$ is called the L-space of the experiment $\mathcal{E}$ and its topological dual $L(\mathcal{E})'$, denoted by $M(\mathcal{E})$, is called the M-space of $\mathcal{E}$ ([3]). $M(\mathcal{E})$ has a unit e defined by $\langle e, m\rangle = m(X)$ for every $m\in L(\mathcal{E})$. Let $T(\mathcal{E})$ denote the semigroup of all positive linear operators $V: M(\mathcal{E})\to M(\mathcal{E})$ which satisfy $Ve=e$ and $\langle Vu, P\rangle=\langle u, P\rangle$ for every $u\in M(\mathcal{E})$, $P\in\mathcal{P}$. The semigroup of all measurable maps $g: X\to X$ such that $gP=P$ for every $P\in\mathcal{P}$, denoted by $T_1(\mathcal{E})$, and the group $T_2(\mathcal{E})$ of all bijective and bimeasurable elements of $T_1(\mathcal{E})$ are of special interest. We shall identify $g\in T_1(\mathcal{E})$ with the operator $V\in T(\mathcal{E})$ given by $\langle Vu, m\rangle=\langle u, gm\rangle$, $u\in M(\mathcal{E})$, $m\in L(\mathcal{E})$. Let $\mathcal{L}(M(\mathcal{E})_\sigma)$ denote the space of all $\sigma(M(\mathcal{E}), L(\mathcal{E}))$-continuous linear operators on $M(\mathcal{E})$ equipped with the topology of pointwise convergence. Then $T(\mathcal{E})$ is a subset of $\mathcal{L}(M(\mathcal{E})_\sigma)$ ([3] Lemma 2). One easily verifies the following property of $T(\mathcal{E})$.

Lemma 1. *$T(\mathcal{E})$ is a compact convex semigroup in $\mathcal{L}(M(\mathcal{E})_\sigma)$.*

A semigroup S in $\mathcal{L}(M(\mathcal{E})_\sigma)$ is called mean ergodic if the semigroup $co(S)^-$ has a zero element by which is meant an element Φ such that $\Phi V = V\Phi = \Phi$ for every $V \in co(S)^-$, where $co(S)^-$ denotes the closed convex hull of S. For $K \subset \mathcal{P}$, let $\mathcal{E}_K = (X, \mathcal{A}, K)$ and put $D(\mathcal{E}) = \{K \subset \mathcal{P}: \mathcal{E}_K \text{ is dominated}\}$.

Lemma 2. *Each subsemigroup of $T(\mathcal{E})$ is mean ergodic.*

Proof. Let S be a subsemigroup of $T(\mathcal{E})$. The adjoint $V'\colon L(\mathcal{E}) \to L(\mathcal{E})$ of $V \in S$ with respect to the duality $\langle M(\mathcal{E}), L(\mathcal{E})\rangle$ is a transition, that is, V' is positive and $V'm(X) = m(X)$ for every $m \in L(\mathcal{E})$, and $V'P = P$ holds for every $P \in \mathcal{P}$. The adjoint semigroup $S' = \{V'\colon V \in S\}$ is a semigroup in the space $\mathcal{L}(L(\mathcal{E}))$ of all continuous linear operators on $L(\mathcal{E})$ equipped with the topology of pointwise convergence. Let $K \in D(\mathcal{E})$. Since $L(\mathcal{E}_K)$ coincides with the closure of

$$\{m \in ca(X, \mathcal{A})\colon \text{there exists } n \in C_K \text{ such that } |m| \leqslant n\},$$

where C_K denotes the convex cone generated by K, we obtain $V'L(\mathcal{E}_K) \subset L(\mathcal{E}_K)$ for every $V \in S$. For $V \in S$, let V'_K denote the restriction of V' to $L(\mathcal{E}_K)$. Then $S'_K = \{V'_K\colon V \in S\}$ is a semigroup in the space $\mathcal{L}(L(\mathcal{E}_K))$ being equipped with the topology of pointwise convergence. The band $L(\mathcal{E}_K)$ can be generated by a probability measure n_K of the from $n_K = \sum c_P P$, where c is a prior distribution on K with countable support. Since $V'_K n_K = n_K$ for every $V \in S$, the semigroup S'_K is mean ergodic ([8] Korollar 2.3). Let Φ'_K denote the zero element of $co(S'_K)^-$.

In view of the uniqueness of the zero element it is clear that $\Phi'_{K_2} | L(\mathcal{E}_{K_1}) = \Phi'_{K_1}$ if $K_1 \subset K_2 \in D(\mathcal{E})$. Since

$$L(\mathcal{E}) = \bigcup_{K \in D(\mathcal{E})} L(\mathcal{E}_K)$$

([6] Lemme 1), we may define a map $\Phi'\colon L(\mathcal{E}) \to L(\mathcal{E})$ by $\Phi' m = \Phi'_K m$ if $m \in L(\mathcal{E}_K)$, $K \in D(\mathcal{E})$. Then Φ' is a continuous linear projection satisfying $\Phi' V' = V'\Phi' = \Phi'$ for every $V \in S$ and $\Phi' m \in co(S'm)^-$ for every $m \in L(\mathcal{E})$, since the operators Φ'_K, $K \in D(\mathcal{E})$, have the corresponding properties (see [11] III.7.2). This implies that Φ' is a zero element of $co(S')^-$ ([8] Theorem 1.2). Hence, the semigroup S' is mean ergodic and then this also holds for the adjoint semigroup $S'' = S$.

The preceding lemma is the key to the characterization of sufficient sublattices by invariance. According to LeCam [3], a $\sigma(M(\mathcal{E}), L(\mathcal{E}))$-closed vector sublattice H of $M(\mathcal{E})$ containing e is called sufficient for $\mathcal{E}$ if there exists a positive linear projection Π of $M(\mathcal{E})$ onto H such that $\Pi \in T(\mathcal{E})$. The projection Π is uniquely determined by H ([3] Prop. 9). It is called the sufficient projection for H. For $S \subset T(\mathcal{E})$, let

$$M_S = \{u \in M(\mathcal{E})\colon Vu = u \quad \text{for every} \quad V \in S\}$$

be the space of all fixed points of S in $M(\mathcal{E})$ and for $H\subset M(\mathcal{E})$, let

$$T(H) = \{V\in T(\mathcal{E})\colon Vu = u \quad \text{for every} \quad u\in H\}.$$

Then $T(H)$ is a compact convex subsemigroup of $T(\mathcal{E})$ and $M_S=M_{co(S)^-}$. Put $H(\mathcal{E})=M_{T(\mathcal{E})}$.

Theorem. *For a subset H of $M(\mathcal{E})$ the following statements are equivalent:*
(i) *H is a sufficient sublattice for $\mathcal{E}$.*
(ii) *$H=M_S$ for some subsemigroup S of $T(\mathcal{E})$.*
(iii) *$H=M_{T(H)}$.*
In particular, $H(\mathcal{E})$ is the smallest sufficient sublattice for $\mathcal{E}$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Clearly, M_S is a $\sigma(M(\mathcal{E}), L(\mathcal{E}))$-closed vector subspace of $M(\mathcal{E})$ containing e. In order to show that M_S is a sublattice, take $u\in M_S$ and $V\in S$. Then $u^+\geqslant u$ implies $Vu^+\geqslant Vu=u$ and therefore $Vu^+\geqslant u^+$. Since $\langle Vu^+-u^+, P\rangle=0$ for every $P\in\mathcal{P}$ and $L(\mathcal{E})$ coincides with the closure of

$$\{m\in ca(X, \mathcal{A})\colon \text{there exists } n\in C \text{ such that } |m|\leqslant n\},$$

where C denotes the convex cone generated by $\mathcal{P}$, this yields $\langle Vu^+-u^+, m\rangle=0$ for every $m\in L(\mathcal{E})$. Thus $Vu^+=u^+$, and M_S is a sublattice. If Φ denotes the zero element of the semigroup $co(S)^-$ whose existence is established in Lemma 2, then Φ is a linear projection of $M(\mathcal{E})$ onto M_S (see [11] III.7.2) and by Lemma 1, $\Phi\in T(\mathcal{E})$.

(ii) $\Leftrightarrow$ (iii). For every subsemigroup S of $T(\mathcal{E})$ we have $M_S=M_{T(M_S)}$.

Thus it is demonstrated that in this framework the (maximal) sufficiency reduction of an experiment is the same as the (maximal) invariance reduction. Several other characterizations of sufficient sublattices one can find in [4] Chap. 5, Sect. 3.

REMARK 1. The map $S\mapsto M_S$ defines a bijection between the set of subsemigroups of $T(\mathcal{E})$ of the form $T(H)$, $H\subset M(\mathcal{E})$, and the set of sufficient sublattices for $\mathcal{E}$. This follows from the preceding theorem in view of $T(M_{T(H)})=T(H)$. Furthermore, if H is a sufficient sublattice for $\mathcal{E}$ then the sufficient projection for H coincides with the zero element of the semigroup $T(H)$.

REMARK 2. From Lemma 2 and [11] III.7.2 it follows that

$$M(\mathcal{E}) = H(\mathcal{E})\oplus M_0(\mathcal{E}),$$

where $M_0(\mathcal{E})$ denotes the $\sigma(M(\mathcal{E}), L(\mathcal{E}))$-closed linear hull of the set $\{Vu-u\colon u\in M(\mathcal{E}), V\in T(\mathcal{E})\}$. Furthermore, let $L_m(\mathcal{E})$ denote the minimal L-space of $\mathcal{E}$, i.e. $L_m(\mathcal{E})$ is the closed vector sublattice of $L(\mathcal{E})$ generated by $\mathcal{P}$. This space is

of some interest in the theory of experiments (see [4]). Using [11] Cor. 1, p. 120, it is easy to see that $L_m(\mathcal{E})$ coincides with the space of all fixed points in $L(\mathcal{E})$ of the adjoint semigroup of $T(\mathcal{E})$. Arguing as above one obtains

$$L(\mathcal{E}) = L_m(\mathcal{E}) \oplus L_0(\mathcal{E}),$$

where $L_0(\mathcal{E})$ denotes the closed linear hull of the set $\{V'm - m: m \in L(\mathcal{E}), V \in T(\mathcal{E})\}$.

REMARK 3. Let ex $T(\mathcal{E})$ denote the set of extreme points of $T(\mathcal{E})$. According to Lemma 1 and the Krein-Milman theorem, $T(\mathcal{E}) = co(ex\ T(\mathcal{E}))^-$. This yields $H(\mathcal{E}) = M_{ex\ T(\mathcal{E})}$.

2. Pairwise sufficient subfields and invariance

We now use the correspondence between sufficient sublattices and pairwise sufficient subfields. Let $\mathcal{B}$ be a sub-σ-field (subfield for short) of $\mathcal{A}$. For $K \subset \mathcal{P}$, let $\bar{\mathcal{B}}^{(K)}$ denote the subfield generated by $\mathcal{B}$ and the K-null sets in $\mathcal{A}$. Put $\bar{\mathcal{B}} = \bigcap_{K \in D(\mathcal{E})} \bar{\mathcal{B}}^{(K)}$. Then $\mathcal{B}$ is pairwise sufficient if and only if $\bar{\mathcal{B}}$ is pairwise sufficient for $\mathcal{E}$. A pairwise sufficient subfield $\mathcal{B}$ is said to be smallest pairwise sufficient for $\mathcal{E}$ if $\mathcal{B} \subset \bar{\mathcal{C}}$ for any other pairwise sufficient subfield $\mathcal{C}$. Here we stress that "smallest" refers to the partial order $<$ defined by $\mathcal{B} < \mathcal{C}$ iff $\mathcal{B} \subset \bar{\mathcal{C}}$. Let i denote the canonical map of the space $B(X, \mathcal{A})$ of all measurable bounded real valued functions on X into $M(\mathcal{E})$ given by

$$\langle i(u), m \rangle = \int u\, dm, \quad m \in L(\mathcal{E}).$$

The $\sigma(M(\mathcal{E}), L(\mathcal{E}))$-closure of $i(B(X, \mathcal{B}))$, denoted by $H_{\mathcal{B}}$, is a vector sublattice of $M(\mathcal{E})$ containing e. If on the other hand H is a $\sigma(M(\mathcal{E}), L(\mathcal{E}))$-closed vector sublattice containing e then $\mathcal{A}(H) = \{A \in \mathcal{A}: i(I_A) \in H\}$ is a subfield of $\mathcal{A}$. Note that $\mathcal{A}(H) = \overline{\mathcal{A}(H)}$. For $S \subset T(\mathcal{E})$, put $\mathcal{A}_S = \mathcal{A}(M_S)$. In case $S \subset T_1(\mathcal{E})$ we have

$$\mathcal{A}_S = \{A \in \mathcal{A}: P(A \triangle g^{-1}A) = 0 \quad \text{for every} \quad g \in S, P \in \mathcal{P}\}.$$

In fact, if $g \in S$ and $V \in T(\mathcal{E})$ denotes the induced operator, then

$$\begin{aligned} \langle V i(I_A), m \rangle &= \langle i(I_A), gm \rangle = gm(A) \\ &= m(g^{-1}A) = \langle i(I_{g^{-1}A}), m \rangle \end{aligned}$$

for every $m \in L(\mathcal{E})$ which yields $V i(I_A) = i(I_{g^{-1}A})$, $A \in \mathcal{A}$. Hence

$$\mathcal{A}_S = \{A \in \mathcal{A}: i(I_{g^{-1}A}) = i(I_A) \quad \text{for every} \quad g \in S\}.$$

Moreover, for $u, v \in B(X, \mathcal{A})$, $i(u) = i(v)$ holds if and only if $u = v$ P-a.e. for every $P \in \mathcal{P}$. This gives the above characterization of $\mathcal{A}_S$.

Corollary 1. (a) *If $\mathcal{B}$ is a pairwise sufficient subfield for $\mathcal{E}$ then $\bar{\mathcal{B}}=\mathcal{A}_S$ for some subsemigroup S of $T(\mathcal{E})$.*

(b) *The following statements are equivalent:*

(i) *There exists a smallest pairwise sufficient subfield for $\mathcal{E}$.*

(ii) *$\mathcal{A}_S$ is pairwise sufficient for every subsemigroup S of $T(\mathcal{E})$.*

(iii) *$\mathcal{A}_{T(\mathcal{E})}$ is pairwise sufficient for $\mathcal{E}$.*

If (iii) *is valid, then $\mathcal{A}_{T(\mathcal{E})}$ is the smallest pairwise sufficient subfield for $\mathcal{E}$.*

Proof. (a) $H_{\mathcal{B}}$ is a suffiicent sublattice for $\mathcal{E}$ ([5] Prop. II.2.8). Hence by the theorem, $H_{\mathcal{B}}=M_S$ for some subsemigroup S of $T(\mathcal{E})$. This implies $\bar{\mathcal{B}}=\mathcal{A}(H_{\mathcal{B}})=\mathcal{A}_S$.

(b) (i) $\Rightarrow$ (iii). Let $\mathcal{B}$ be the smallest pairwise sufficient subfield for $\mathcal{E}$. By the theorem and [12] p. 240, $H_{\mathcal{B}}=H(\mathcal{E})$ and thus $\bar{\mathcal{B}}=\mathcal{A}(H(\mathcal{E}))=\mathcal{A}_{T(\mathcal{E})}$. This yields the assertion.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious.

If the underlying experiment $\mathcal{E}$ is coherent by which is meant $i(B(X, \mathcal{A}))=M(\mathcal{E})$ one can obtain an analogous result for sufficient subfields. By [6] Lemma 4, this notion coincides with coherence in the sense of [2]; for further equivalent conditions see e.g. [7]. Especially, dominated experiments and discrete experiments (Basu-Ghosh-structures) are coherent. Note that for these experiments $\bar{\mathcal{B}}$ is sufficient provided $\mathcal{B}$ is a pairwise sufficient subfield. In particular, for coherent experiments pairwise sufficiency of $\mathcal{A}_S$ implies sufficiency of $\mathcal{A}_S$, because $\mathcal{A}_S=\bar{\mathcal{A}}_S$. Furthermore, $\bar{\mathcal{B}}=\hat{\mathcal{B}}$ for sufficient subfields, where $\hat{\mathcal{B}}=\bar{\mathcal{B}}(\mathcal{P})$. Since it is well known that for coherent experiments there exists a smallest pairwise sufficient subfield for $\mathcal{E}$ (see [2] and notice that the smallest sufficient subfield whose existence is established in [2] is smallest pairwise sufficient; see also [6], [12]), the following corollary is an immediate consequence of Corollary 1.

Corollary 2. *Suppose that $\mathcal{E}$ is a coherent experiment. Then a subfield $\mathcal{B}$ of $\mathcal{A}$ is sufficient for $\mathcal{E}$ if and only if $\hat{\mathcal{B}}=\mathcal{A}_S$ for some subsemigroup S of $T(\mathcal{E})$. In particular, $\mathcal{A}_{T(\mathcal{E})}$ is the smallest sufficient subfield for $\mathcal{E}$.*

Basu [1] has shown that $\mathcal{A}_{T_2(\mathcal{E})}$ is sufficient for dominated experiments and Trenkler [13] has proven the same fact for discrete experiments. Both results are contained in the preceding corollary. In general, $\mathcal{A}_{T_2(\mathcal{E})}$ (and $\mathcal{A}_{T_1(\mathcal{E})}$) is not the smallest sufficient subfield for dominated experiments (see [1]).

The following examples may serve as illustrations of the properties of $\mathcal{A}_{T_2(\mathcal{E})}$ and $M_{T_2(\mathcal{E})}$. Note that in all examples $M_{T_1(\mathcal{E})}=M_{T_2(\mathcal{E})}$ holds. The first example provides an experiment $\mathcal{E}$ such that $H(\mathcal{E})\neq M_{T_2(\mathcal{E})}$ and $\mathcal{A}_{T_2(\mathcal{E})}$ is not pairwise sufficient for $\mathcal{E}$. We remark that the latter fact disproves a result of Petit [9] (second part of Théorème 3).

EXAMPLE 1. Let $X=(-1, 0)\cup(0, 1)$ and $\mathcal{A}$ be the Borel-field of X. Let

$\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$ with $\mathcal{P}_1=\{P_1, P_2\}$ and $\mathcal{P}_2=\{(\varepsilon_x+\varepsilon_{-x})/2: x\in(0,1)\}$, where P_1 and P_2 denote the Lebesgue measures on $(0,1)$ and $(-1,0)$, respectively. Since $L(\mathcal{E})=L(\mathcal{E}_{\mathcal{P}_1})\oplus L(\mathcal{E}_{\mathcal{P}_2})$ holds and $L(\mathcal{E}_{\mathcal{P}_1})$ can be identified with $L^1(Q)$ and $L(\mathcal{E}_{\mathcal{P}_2})$ with the band of discrete signed measures on the power set $\mathcal{P}(X)$ of X, we obtain

$$M(\mathcal{E}) = L^\infty(Q)\times B(X)$$

with $B(X)=B(X, \mathcal{P}(X))$ and $Q=(P_1+P_2)/2$. We claim that

$$T_1(\mathcal{E}) = T_2(\mathcal{E})=\{id_X\, I_A-id_X\, I_{A^c}: A\in\mathcal{A}, A=-A, Q(A^c)=0\}\,.$$

Let T_0 denote the right side of the second equality and let $g=id_X\, I_A-id_X\, I_{A^c}\in T_0$. Then $g^2=id_X$, hence g is bijective. Moreover,

$$g^{-1}B = (A\cap B)\cup(A^c\cap(-B)) = (A\cap B)\cup(-(A^c\cap B))$$

holds for every $B\in\mathcal{A}$, hence $T_0\subset T_2(\mathcal{E})$. To prove the inclusion $T_1(\mathcal{E})\subset T_0$, note first that for $g\in T_1(\mathcal{E})$, we have $(\varepsilon_x+\varepsilon_{-x})/2=(\varepsilon_{gx}+\varepsilon_{g(-x)})/2$ which yields $\{gx, g(-x)\}=\{x,-x\}$ for every $x\in X$. Setting $A=\{x\in X: gx=x\}$, this implies $A=-A$ and $g=id_X\, I_A-id_X\, I_{A^c}$. Since

$$A = (\{g<0\}\cap(-1,0))\cup(\{g>0\}\cap(0,1))\,,$$

$A\in\mathcal{A}$ holds, and $gP_i=P_i$, $i=1,2$, implies $Q(A^c)=0$. Thus $g\in T_0$ and our claim is proved. If $g\in T_2(\mathcal{E})$ and $V\in T(\mathcal{E})$ denotes the induced operator, then for every $f\in L^\infty(Q)$, $u\in B(X)$

$$\begin{aligned}\langle V(f,u), m_1+m_2\rangle &= \langle (f,u), gm_1+gm_2\rangle\\ &= \langle f, gm_1\rangle+\langle u, gm_2\rangle = \int f\,dm_1+\int u\circ g\,dm_2\\ &= \langle (f, u\circ g), m_1+m_2\rangle \quad \text{for every} \quad m_1\in L(\mathcal{E}_{\mathcal{P}_1}), m_2\in L(\mathcal{E}_{\mathcal{P}_2})\,,\end{aligned}$$

hence $V(f,u)=(f, u\circ g)$. (Note that we did not make notational distinction between $m_2\in L(\mathcal{E}_{\mathcal{P}_2})$ and its uniquely determined extension to $\mathcal{P}(X)$.) Furthermore, $u\circ g=u$ for every $g\in T_2(\mathcal{E})$ holds if and only if $u=u_s$, where $u_s(x)=(u(x)+u(-x))/2$ for every $x\in X$. This yields

$$M_{T_2(\mathcal{E})} = L^\infty(Q)\times\{u\in B(X): u=u_s\}\,,$$

hence

$$\mathcal{A}_{T_2(\mathcal{E})} = \{A\in\mathcal{A}: A=-A\}\,,$$

because $i(I_A)=(I_A, I_A)$ for every $A\in\mathcal{A}$. By Exemple 2 of [6], $\mathcal{A}_{T_2(\mathcal{E})}$ is not pairwise sufficient for $\mathcal{E}$. Define a map $\Phi: M(\mathcal{E})\to M(\mathcal{E})$ by

$$\Phi(f,u) = \left(\int f\,dP_1\, I_{(0,1)}+\int f\,dP_2\, I_{(-1,0)}, u_s\right).$$

Clearly, $\Phi \in T(\mathcal{E})$. Furthermore, for $V=(V_1, V_2)\in T(\mathcal{E})$, we have for every $f\in L^\infty(Q), u\in B(X)$

$$\int V_1(f, u)\, dP_i = \langle V(f, u), P_i\rangle = \langle (f, u), P_i\rangle = \int f\, dP_i, i = 1, 2$$

and

$$\begin{aligned}(V_2(f, u))_s(x) &= \langle V(f, u), (\varepsilon_x+\varepsilon_{-x})/2\rangle \\ &= \langle (f, u), (\varepsilon_x+\varepsilon_{-x})/2\rangle = u_s(x)\end{aligned}$$

for every $x\in X$, hence $(V_2(f, u))_s=u_s$. This yields $\Phi V=\Phi$ for every $V\in T(\mathcal{E})$, that is, Φ is a left zero of $T(\mathcal{E})$. Since $T(\mathcal{E})$ is mean ergodic by Lemma 2, it follows that Φ is the zero element of $T(\mathcal{E})$. Thus it follows from Remark 1 that

$$H(\mathcal{E}) = M_{\{\Phi\}} = L^\infty(Q\,|\,\mathcal{B})\times \{u\in B(X): u = u_s\}$$

with $\mathcal{B}=\{\phi, X, (0, 1), (-1, 0)\}$.

A slight modification of the preceding example yields an experiment $\mathcal{E}$ such that $H(\mathcal{E})=M_{T_2(\mathcal{E})}$ and $\mathcal{A}_{T_2(\mathcal{E})}$ is not pairwise sufficient for $\mathcal{E}$.

EXAMPLE 2. Replace $\mathcal{P}_1$ in the preceding example by the set of all probability measures which are absolutely continuous with respect to Q. Then $M(\mathcal{E})$ and $T_2(\mathcal{E})$ remain unchanged. But now the sufficient projection Π for $M_{T_2(\mathcal{E})}$ given by $\Pi(f, u)=(f, u_s)$ is the zero element of $T(\mathcal{E})$. Indeed, for $V=(V_1, V_2)\in T(\mathcal{E})$, we have for every $f\in L^\infty(Q)$, $u\in B(X)$

$$\int V_1(f, u)\, dP = \int f\, dP$$

for every probability measure P which is absolutely continuous with respect to Q, hence $V_1(f, u)=f$. This implies that Π is a left zero of $T(\mathcal{E})$ and thus, again by Lemma 2, it is the zero element of $T(\mathcal{E})$. From Remark 1 follows $H(\mathcal{E})=M_{T_2(\mathcal{E})}$.

Next we give an experiment $\mathcal{E}$ such that $H(\mathcal{E})=M_{T_2(\mathcal{E})}$, $\mathcal{A}_{T_2(\mathcal{E})}$ is pairwise sufficient, but it is not sufficient for $\mathcal{E}$. In particular, $\mathcal{A}_{T_2(\mathcal{E})}$ is the smallest pairwise sufficient subfield for $\mathcal{E}$ by Corollary 1.

EXAMPLE 3. Let $X=[0, 1]$ and $\mathcal{A}$ be the Borel-field of X. Let B be a non-Borel subset of $(1/2, 1]$ with the cardinality of the reals and α: $[0, 1/2]\to B$ a bijective map. Let

$$\mathcal{P} = \{(\varepsilon_x+\varepsilon_{\alpha(x)})/2: x\in[0, 1/2]\} \cup \{\varepsilon_x: x\in(1/2, 1]\backslash B\}\,.$$

Then

$$M(\mathcal{E}) = B(X)\,.$$

For $x\in[0, 1/2]$, define g_x: $X\to X$ by

$$g_x = id_X I_{\{x,\alpha(x)\}^c} + \alpha I_{\{x\}} + \alpha^{-1} I_{\{\alpha(x)\}}$$

and let S denote the group generated by $\{g_x: x \in [0, 1/2]\}$. Then S is a subgroup of $T_2(\mathcal{E})$ and $u \in B(X)$ satisfies $u \circ g = u$ for every $g \in S$ if and only if $u(\alpha(x)) = u(x)$ for every $x \in [0, 1/2]$. This gives

$$M_S = \{u \in B(X): u(\alpha(x)) = u(x) \quad \text{for every} \quad x \in [0, 1/2]\}$$

and

$$\mathcal{A}_S = \bigcap_{x \in [0, 1/2]} \{A \in \mathcal{A}: \{x, \alpha(x)\} \subset A \text{ or } \{x, \alpha(x)\} \subset A^c\} .$$

Since there exists a smallest pairwise sufficient subfield for $\mathcal{E}$ (see [12]), $\mathcal{A}_S$ is pairwise sufficient by Corollary 1 (it is not difficult to check this property directly), but $\mathcal{A}_S$ is not sufficient ([10]). The sufficient projection Π for M_S given by

$$\Pi u = u_\alpha I_{[0, 1/2]} + u I_{(1/2, 1] \setminus B} + u_{\alpha^{-1}} I_B ,$$

where $u_\alpha(x) = (u(\alpha(x)) + u(x))/2$ for every $x \in [0, 1/2]$ and $u_{\alpha^{-1}}(x) = (u(\alpha^{-1}(x)) + u(x))/2$ for every $x \in B$, is a left zero of $T(\mathcal{E})$. This follows from $(Vu)_\alpha = u_\alpha$ on $[0, 1/2]$, $Vu = u$ on $(1/2, 1] \setminus B$, and $(Vu)_{\alpha^{-1}} = u_{\alpha^{-1}}$ on B for every $V \in T(\mathcal{E})$, $u \in B(X)$. Hence, Π is the zero element of $T(\mathcal{E})$ and by Remark 1

$$H(\mathcal{E}) = M_{\{\Pi\}} = M_S .$$

In particular, $H(\mathcal{E}) = M_{T_1(\mathcal{E})} = M_{T_2(\mathcal{E})}$ holds.

Finally, we give an experiment $\mathcal{E}$ such that $\mathcal{A}_{T(\mathcal{E})}$ is pairwise sufficient and hence the smallest pairwise sufficient subfield for $\mathcal{E}$, $\mathcal{A}_{T_2(\mathcal{E})}$ is the smallest sufficient subfield for $\mathcal{E}$, but $\mathcal{A}_{T(\mathcal{E})} \neq \mathcal{A}_{T_2(\mathcal{E})}$.

EXAMPLE 4. Let $X = [0, 1]$, $\mathcal{A}$ be the Borel-field, and $\mathcal{P} = \{\varepsilon_x: x \in X\} \cup \{P\}$, where P denotes the Lebesgue measure on X. Then $T_1(\mathcal{E}) = T_2(\mathcal{E}) = \{id_X\}$ and $\mathcal{A} = \mathcal{A}_{T_2(\mathcal{E})}$ is the smallest sufficient subfield. Furthermore, $\mathcal{B} = \{A \in \mathcal{A}: A \text{ or } A^c \text{ is countable}\}$ is the smallest pairwise sufficient subfield for $\mathcal{E}$. To see this, let $A \in \mathcal{A}$ and $x, x_1, x_2 \in X$. Then $I_{A \cap \{x_1, x_2\}}$ is a version of $E_{\varepsilon_{x_i}}(I_A | \mathcal{B})$, $i = 1, 2$, and $I_{A \cap \{x\}} + P(A) I_{\{x\}^c}$ is a version of $E_{\varepsilon_x}(I_A | \mathcal{B})$ and of $E_P(I_A | \mathcal{B})$ which yields pairwise sufficiency of $\mathcal{B}$. To prove that $\mathcal{B}$ is smallest pairwise sufficient, it is enough to show that $A \in \bar{C}^{(K)}$ for every countable set $A \in \mathcal{B}$ and $K = \{\varepsilon_{x_i}: i \in \boldsymbol{N}\} \cup \{P\}$, where C is a given pairwise sufficient subfield. Let $B = A \cap \{x_i: i \in \boldsymbol{N}\}$ and let f denote a version of $E_{\varepsilon_{x_i}}(I_B | C)$ for every $i \in \boldsymbol{N}$ and of $E_P(I_B | C)$. Put $C = \{f = 1\}$. Then $B \subset C$ and $\{x_i: i \in \boldsymbol{N}\} \setminus B \subset C^c$, hence $\varepsilon_{x_i}(A \triangle C) = 0$ for every $i \in \boldsymbol{N}$. Since $f = 0$ P-a.e., we have $P(A \triangle C) = 0$ and hence $A \in \bar{C}^{(K)}$. It follows from Corollary 1 that $\mathcal{A}_{T(\mathcal{E})} = \bar{\mathcal{B}}$, but e.g. $[0, 1/2] \notin \bar{\mathcal{B}}$.

REMARK 4. It is known that a subfield $\mathcal{B}$ is pairwise sufficient for $\mathcal{E}$ if and

only if the deficiency of $\mathcal{E}|\mathcal{B}$ relative to $\mathcal{E}$ (introduced in [3]) is equal to zero by which is meant that there is a transition U: $L(\mathcal{E}|\mathcal{B})\to L(\mathcal{E})$ such that $U(P|\mathcal{B})=P$ for every $P\in\mathcal{P}$, where $\mathcal{E}|\mathcal{B}=(X, \mathcal{B}, \{P|\mathcal{B}: P\in\mathcal{P}\})$ ([5] Théorème 3). To prove the "if" part one can argue as follows. Let R: $L(\mathcal{E})\to L(\mathcal{E}|\mathcal{B})$ denote the restriction to $\mathcal{B}$ and let S be the subsemigroup of $T(\mathcal{E})$ generated by $R'U'$. By the theorem, M_S is a sufficient sublattice. Since $H_{\mathcal{B}}=R'(M(\mathcal{E}|\mathcal{B}))$, we see that $M_S\subset H_{\mathcal{B}}$. This implies that $H_{\mathcal{B}}$ is a sufficient sublattice and thus, $\mathcal{B}$ is pairwise sufficient for $\mathcal{E}$ ([5] Prop. II.2.8). For a more detailed discussion of the above relation see [14], [15].

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University of Münster
Department of Mathematical Statistics

Einsteinstrasse 62
D-4400 Münster
Federal Republic of Germany