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PAIRWISE SYMMETRY CONDITIONS FOR VOTING EQUILIBRIA

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It is common knowledge that characterizations of majority rule equilibria in multidimensional spaces take the form of pairwise symmetry conditions on utility gradients.[1] Plott [1967], the initial investigator of these conditions, shows that if exactly one utility gradient at an interior point is zero and the number of people is odd, then the point is an equilibrium if and only if the set of nonzero gradients can be partitioned into pairs of exactly opposing vectors. This degree of symmetry seems unlikely to occur. Hence it must be concluded that this type of equilibrium does not usually exist.

However, the condition that all nonzero gradients must be paired is necessary only for equilibria at which only one gradient is zero. One object of this paper is to derive necessary conditions that do not a priori restrict the number of zero gradients. These more general conditions are determined also for the more general case of λ-majority rule, in which a coalition is winning only if it constitutes more than a fraction λ of the voters.[2] The amount of pairwise symmetry required for equilibrium is still restrictive, however, unless many gradients are zero or λ is near one.

Conditions necessary for equilibrium may be less restrictive for equilibria contained in the boundary of a feasible set. Since often the feasible set is a proper subset of the space, such equilibria are certainly worthy of investigation. Plott [1967] makes an initial step in this direction by investigating situations in which the feasible set is a half-space and the equilibrium is contained in the defining hyperplane. His conditions are generalized here by allowing the equilibrium to be contained in the boundary of any convex feasible set, as well as by allowing more than one gradient to "point out" of the feasible set and by considering λ-majority rule. We find that the type of pairwise symmetry required at boundary equilibria is of a lesser degree than that required at interior equilibria. But the symmetry still appears restrictive unless (1) the boundary is highly pointed at the equilibrium, (2) many gradients are zero or "point out" of the feasible set, or (3) λ is near one.

A fundamental characteristic of majority rule is that if two people with diametrically opposed preferences are removed from the set of voters, then any equilibrium remains an equilibrium. The votes of the two individuals merely "cancel each other out." This basic fact is what causes pairwise symmetry conditions to be necessary for equilibrium, as the subsequent proofs are designed to show. All the symmetry conditions are derived as corollaries to theorems stating that various sets of individuals that "disagree" in some sense can be deleted without upsetting equilibrium. This intuitive approach results in relatively concise proofs.

Sufficient conditions involving pairwise symmetries on gradients are important because properties of pairs are relatively easy to verify. The ones derived in section 3 generalize those of

Plott [1967], McKelvey and Wendell [1976], and Slutsky [1978] by allowing the point to be on the boundary of a convex feasible set, by allowing more than one gradient at the point to be zero or to "point out" of the feasible set, and by allowing for λ-majority rule.

1. PRELIMINARIES

The set of feasible alternatives is a convex subset V of a Euclidean space W. Denote by x a particular point of V, not necessarily in the interior. Let the set of voters be denoted by $N = \{1, 2, \ldots, n\}$. Each voter has a differentiable utility function defined on W. The gradient of the utility function of voter i evaluated at x is denoted by $u_i \in W$.[3] We are to investigate pairwise symmetries in the set $\{u_1 \ldots, u_n\}$ of gradients associated with x being a voting equilibrium.

The cone of feasible directions in which x can shift is

$$F = \{v \in W \mid \exists \alpha > 0 \ni x + \alpha v \in V\}.$$

Observe that F is a convex cone that includes the origin. If $x \in \text{interior}(V)$, then $F = W$, whereas $x \in \text{boundary}(V)$ implies that F is contained in a halfspace.

Much of the subsequent discussion concerns the dual of F,

$$F^* = \{y \in W \mid v \cdot y \leq 0 \quad \forall v \in F\} \equiv D.$$

Notice that D is a closed convex cone containing the origin, and that $D = \{0\}$ if and only if $F = W$. If $u_i \in D$ then $v \cdot u_i \leq 0$ for all $v \in F$, so that voter i is "happy" with x in the sense of not marginally benefiting by any feasible shift of x.

Define also a cone

$$E = \{y \in W \mid y \notin D, -y \in D\}.$$

E is a convex cone without the origin that may be empty. In particular, $E = \emptyset$ whenever $D = \{0\}$ or D is a subspace of positive dimension. If $u_i \in E$, then i is "unhappy" with x in the sense that $v \cdot u_i \geq 0$ for any $v \in F$, and there exists $\overline{v} \in F$ such that $\overline{v} \cdot u_i > 0$.

Examples of possible cones F, D, and E are illustrated in figure 1. In the figure and hereafter a cone generated by vectors $y_1, \ldots, y_\ell$ is defined by

$$C(y_1, \ldots, y_\ell) = \{y \in W \mid y = \alpha_1 y_1 + \ldots + \alpha_\ell y_\ell,$$
$$\alpha_i \geq 0, \Sigma\alpha_i > 0\}.$$

Also, if $M = \{i_1, \ldots, i_\ell\} \subset N$, the notation $C(M) = C(u_{i_1}, \ldots, u_{i_\ell})$ will be used for convenience.

Define for any cone C the following derived cones:

$$C^+ = \{y \in W \mid y \cdot c > 0 \quad \forall \quad c \in C\}$$

$$C^- = \{y \in W \mid y \cdot c < 0 \quad \forall \quad c \in C\}$$

$$C^0 = \{y \in W \mid y \cdot c = 0 \quad \forall \quad c \in C\}$$

Without fear of ambiguity, for any $v \in W$ let v^+, v^-, and v^0 denote $C(v)^+$, $C(v)^-$, and $C(v)^0$. Then v^+ and v^- are halfspaces and v^0 is a subspace. Observe that $u_i \in v^+$ implies that $v \cdot u_i > 0$, so that voter i benefits if x shifts in direction v. For any subsets $M \subset N$ and $C \subset W$, let

$$S_M(C) = |\{i \in M \mid u_i \in C\}|,$$

and let $S(C) = S_N(C)$. Hence $S_M(v^+)$ is the number of voters in M who benefit by a shift in direction v. For convenience we also adopt the convention that if an upper case letter denotes a subset of voters, then the corresponding lower case letter denotes their number, e.g., $n = |N|$ and $M \subset N$ implies $m = |M|$.

With these definitions in hand, an equilibrium concept can be defined. Let λ be a fixed fraction $0 \leq \lambda < 1$. Then we want x to be an equilibrium provided no coalition of size greater than λn can marginally benefit by a feasible shift of x. So define x to be <u>quasi-undominated</u> (<u>q.u.d.</u>) provided

$$v \in F \implies S(v^+) \leq \lambda n,$$

and define x to be <u>strictly</u> <u>quasi-undominated</u> (<u>s.q.u.d.</u>) provided

$$v \in F \implies S(v^+) < \lambda n.$$

Notice that x is q.u.d. if x is s.q.u.d. Conversely, x is s.q.u.d. if x is q.u.d. and λn is nonintegral, which is the case when n is odd and $\lambda = 1/2$, the majority rule case studied by Plott [1967].

Two alternative concepts of equilibrum for x are <u>local undominance</u>, which requires the existence of a neighborhood U of x such that no point in $U \cap V$ is unanimously preferred to x by a coalition of size greater than λn, and <u>global</u> <u>undominance</u>, which requires x to be locally undominated in every neighborhood $U \subset W$. When there is a finite number of voters, each with a differentiable utility function, global undominance implies local undominance.

implies quasi-undominance. The reverse implications require utility functions to first be locally pseudoconcave (see appendix B of chapter II) and then pseudoconcave (Kats and Nitzan [1976]). The reader is referred to the cited references for these results, and to Sloss [1973], McKelvey and Wendell [1976], and Slutsky [1978] for further discussions of the relationship between quasi-undominance and other equilibrium concepts. Hence attention here can be focused solely upon quasi-undominance.

It will be convenient for the determination of quasi-undominance to test only directions contained in the relative interior of F. Lemma 2 below justifies this procedure. It also allows us to assume henceforth that F is a closed convex cone, so that $D^* = F^{**} = F$.[4]

Lemma 1: Let $M \subset N$ and $\bar{v} \in W$. Then there exists a neighborhood U of $\bar{v}$ such that $S_M(v^+) \geq S_M(\bar{v}^+)$ for all $v \in U$.

Proof: Follows from the continuity of an inner product and the finiteness of M.

Lemma 2: Let $M \subset N$ and $\beta > 0$. If $S_M(v^+) \leq \beta$ for all v contained in the relative interior of F, then $S_M(v^+) \leq \beta$ for all $v \in$ closure(F).

Proof: Since F is convex, every neighborhood of any $v \in$ closure(F) contains points in the relative interior of F. Hence the result follows from lemma 1.

Henceforth, without loss of generality, we assume F is closed.

The basic feature of majority rule we wish to exploit is that if the number of people who prefer alternative a_1 to a_2 is not a majority, and $Q \subset N$ is a set that can be partitioned into pairs with strictly opposite preferences on $\{a_1, a_2\}$, then when Q is deleted, the number of voters preferring a_1 to a_2 is still not a majority. More generally, if the number of people preferring a_1 is less than λn, then when Q is deleted, the number of people who prefer a_1 is less than $\lambda n - 1/2q$. Now, our general method will be to show that the deletion of coalitions analogous to Q will leave x quasi-undominated, in some sense, in the remaining set of voters. But if $K = N - Q$, the above reasoning indicates that only $S_K(v^+) \leq \lambda n - 1/2(n-k)$ can be guaranteed by $S(v^+) \leq \lambda n$. Hence we shall say that x is q.u.d. in $K \subset N$ provided

$$v \in F \implies S_K(v^+) \leq \lambda n - 1/2(n-k) = \lambda_k k,$$

where λ_k is defined by

$$\lambda_k = \lambda + (\lambda - 1/2)(n/k - 1).$$

Similarly, x is s.q.u.d. in K provided

$$v \in F \implies S_K(v^+) < \lambda_k k.$$

We now prove a simple proposition to illustrate the meaning of quasi-undominance in subsets of N. Say that a pair $\{i,j\} \in N$ strongly disagree provided $u_i \notin D$, $u_j \notin D$, and

$$v \cdot u_i > 0 \iff v \cdot u_j < 0$$

for all $v \in W$. Observe that i and j strongly disagree if and only if there is a ray $r \subset W$ not intersecting D such that $u_i \in r$ and $u_j \in -r$. Thus, if D contains no line, i and j strongly disagree exactly when u_i and u_j are a pair of gradients exactly opposing each other in the sense of Plott [1967]. We show that removing or adding pairs of strongly disagreeing voters preserves quasi-undominance. The following lemma is useful.

Lemma 3: Let $T \subset W$ be a subspace, and let $\overline{v} \in T$, $\overline{v} \neq 0$. Suppose $Q \subset N$ and $u_i \notin T^0$ for each $i \in Q$. If U is a neighborhood of $\overline{v}$, then there exists $v \in U \cap T$ such that $v \cdot u_i \neq 0$ for all $i \in Q$.

Proof: $U' = U \cap T$ is an open set of T. If $u_i \notin T^0$, then $T \not\subset u_i^0$, so that $\dim(T \cap u_i^0) < \dim(T)$. Hence for each $i \in Q$, $T \cap u_i^0$ is a nowhere dense subset of T. Since a countable union of nowhere dense sets cannot contain an open set (Baire's theorem),

$$U \cap T = U' \not\subset \bigcup_{i \in Q} (T \cap u_i^0).$$

Therefore there exists $v \in U \cap T$ such that $v \cdot u_i \neq 0$ for each $i \in Q$.

Proposition 1: Let Q be a subset of N that can be partitioned into strongly disagreeing pairs, and let $K = N - Q$. Then x is (s.)q.u.d. in K iff x is (s.)q.u.d.

Proof: Suppose x is (s.)q.u.d. Let $\overline{v}$ be contained in the relative interior of F. Let T be the smallest subspace containing F. Hence there is a neighborhood U' of $\overline{v}$ such that $U' \cap T \subset F$. By lemma 1 there exists a neighborhood $U \subset U'$ such that $S_K(v^+) \geq S_K(\overline{v}^+)$ for any $v \in U$. Since $u_i \notin D$ for each $i \in Q$,

$u_i \notin T^0$ for each $i \in Q$. Hence lemma 3 implies the existence of $v \in U \cap T \subset F$ such that $v \cdot u_i \neq 0$ for each $i \in Q$. But Q can be partitioned into pairs of strongly disagreeing individuals, so that $S_Q(v^+) = q/2$. Therefore

$$S_K(\overline{v}^+) \leq S_K(v^+) = S(v^+) - q/2$$
$$\leq \lambda n - q/2 = \lambda_k k,$$

with the second inequality strict if x is s.q.u.d. By lemma 2, this proves x is (s.)q.u.d. in K. Now assume x is (s.)q.u.d. in K. Let $v \in F$. Then $S_Q(v^+) \leq q/2 \Rightarrow S(v^+) \leq S_K(v^+) + q/2 \leq \lambda_k k + q/2 = \lambda n$ (second inequality strict if x is s.q.u.d. in K). So x is (s.)q.u.d.

Proposition 1 actually does not lead to strong pairwise symmetry conditions, even for the case of an interior x. In the next section, symmetry conditions for an interior x are obtained easily by a different route. But a result analogous to proposition 2 regarding the deletion of pairs that disagree in a weaker sense is very useful for the case of a boundary x. Hence define a pair $\{i,j\} \subset N$ to weakly disagree provided $u_i \notin D$, $u_j \notin D$, and for any $v \in F$,

$$v \cdot u_i > 0 \Rightarrow v \cdot u_j < 0$$

$$\text{and} \quad v \cdot u_j > 0 \Rightarrow v \cdot u_i < 0 .$$

Let $\mathcal{D}$ be the symmetric binary relation on N denoting weak disagreement, so that $i\mathcal{D}j$ means i and j weakly disagree. If x is an interior point of V, then $F = W$ and weak disagreement implies strong disagreement. Otherwise it is possible that $i\mathcal{D}j$ even though $v \cdot u_i < 0$ and $v \cdot u_j < 0$ for some $v \in F$. But if $i\mathcal{D}j$ and $v \cdot u_i > 0$ and $v \cdot u_j > 0$, then $v \notin F$; weakly disagreeing pairs can agree only on infeasible directions. The next proposition characterizes weakly disagreeing pairs.

Proposition 2: If $u_i \notin D$ and $u_j \notin D$, then $i\mathcal{D}j$ iff $C(u_i,u_j) \cap D \neq \emptyset$.

Proof: D and $C(u_i,u_j) \cup \{0\}$ are closed convex cones. Hence if $C(u_i,u_j) \cap D = \emptyset$, by a separation theorem there exists $v \neq 0$ such that $v \cdot y > 0$ for all $y \in C(u_i,u_j)$ and $v \in D^* = F$. Hence, since $v \cdot u_i > 0$ and $v \cdot u_j > 0$, $i\mathcal{D}j$ is false. Conversely, suppose there exists $y = \alpha_i u_i + \alpha_j u_j \in C(u_i,u_j) \cap D$. Then $\alpha_i > 0$ and $\alpha_j > 0$. Hence, because $v \cdot y \leq 0$ for all $v \in F$, $i\mathcal{D}j$.

Finally, basic necessary conditions are derived via the deletion of individuals who are malcontent in a different way. For any subspace $T \subset W$, say that voter $i \subset N$ is content with T provided $u_i \in T^0$. Let $\mathcal{C}(T) \subset N$ be the subset of N content with T. To interpret $\mathcal{C}(T)$, suppose a subset of public goods is associated with T. Then any $i \in \mathcal{C}(T)$ is content with the allocation of those particular goods at x in the sense of being indifferent to any proposal to change their amounts. Letting $M(T) = N - \mathcal{C}(T)$, each $i \in M(T)$ is discontented with T at x in the sense of preferring a change in allocation of the goods associated with T.

Define a free subspace to be a subspace $T \subset W$ for which $T \subset F$. It is easy to show

Lemma 4: A subspace T is free iff $D \subset T^0$.

A major result of the next section is that quasi-undominance is preserved when M(T) is removed and T is free. Intuitively, if the amounts of the goods associated with T can be increased or decreased freely at x, then the votes of those discontented with the amounts of these goods must "cancel out" for x to be in equilibrium.

2. NECESSARY CONDITIONS

Theorem 1: x is (s.)q.u.d. iff x is (s.)q.u.d. in $\mathcal{C}(T)$ for every free subspace T.

Remark 1: This theorem actually only provides a necessary condition for x to be (s.)q.u.d., since $T = \{0\}$ is always a free subspace and $\mathcal{C}(\{0\}) = N$. Subsequently an example will be presented indicating that a true sufficient condition cannot be obtained by requiring T to be nondegenerate.

Remark 2: The freeness of T is necessary for theorem 1. Consider a case with $W = R^2$, $n = 3$, $\lambda = 1/2$, and with $D = C(0,p)$ with $p = (0,1)$. Let $u_1 = u_2 = p$, and $u_3 = (1,0)$. If T is taken as the line $C(p, -p)$, which is not free, then $\mathcal{C}(T) = \{3\}$. But x is clearly not s.q.u.d. in $\{3\}$, even though x is s.q.u.d. in $\{1,2,3\}$.

Lemma 5: Suppose x is q.u.d. If $v \in F$, $a \in v^0$, and $v + a \in F$, then

$$S(v^+) + S(v^0 \cap a^+) \leq \lambda n,$$

with the inequality strict if x is s.q.u.d.

Proof: By the continuity of the inner product, there exists a neighborhood U of v such that $y \cdot u_i > 0$ for all $y \in U$, $u_i \in v^+$. As F is convex, there exists $0 < \delta \leq 1$ such that $b = v + \delta a \in F \cap U$. Since $b \cdot u_i > 0$ for any $u_i \in v^0 \cap a^+$, and since x is q.u.d., we have

$$S(v^+) + S(v^0 \cap a^+) \leq S(b^+) \leq \lambda n,$$

with the last inequality strict if x is s.q.u.d.

Proof of Theorem 1: (Figure 2 may be helpful.) Suppose x is q.u.d. and $T \neq \{0\}$ is a free subspace. Let $M = C(T)$. Since $i \in N - M \iff u_i \notin T^0$, lemma 3 implies the existence of $v \in T$ such that $v \cdot u_i \neq 0 \iff i \in N - M$. Hence $n = S(v^+) + S(v^-) + m$. We can assume $S(v^-) \leq S(v^+)$, switching v with −v if necessary, so that

$$S(v^+) \geq 1/2(n - m).$$

Let $\overline{v} \in F$. $\overline{v}$ can be expressed as $\overline{v} = a + b$, where $a \in T^0$, $b \in T$. For any $p \in D$, $p \cdot a = p \cdot (\overline{v} - b) = p \cdot \overline{v} \leq 0$, since the freeness of T implies $p \in T^0$. Hence $a \in D^* = F$. T being free also implies $v \in F$, so that $v + a \in F$ by the convexity of F. Applying lemma 5, we now have

$$S(v^+) + S_M(a^+) \leq \lambda n$$

because our choice of v implies $S(v^0 \cap a^+) = S_M(a^+)$. (This inequality is strict if x is s.q.u.d.) Finally, since $i \in M \Rightarrow u_i \in T^0 \Rightarrow \overline{v} \cdot u_i = a \cdot u_i$, we obtain

$$S_M(\overline{v}^+) = S_M(a^+).$$

Putting the pieces together leads to

$$S_M(\overline{v}^+) \leq \lambda n - S(v^+) \leq \lambda n - 1/2(n - m) = \lambda_m m,$$

with the first inequality strict if x is s.q.u.d. The theorem is proved.

Corollary 1: Let T be a free subspace and $M = C(T)$. If x is q.u.d. and $v \in F$, then

$$S_M(v^+) - S_M(v^-) \leq S_M(v^0) + (2\lambda - 1)n,$$

with the inequality strict if x is s.q.u.d.

Proof: Theorem 1 implies $S_M(v^+) \leq \lambda n - 1/2(n - m)$, so the inequality follows from substituting $S_M(v^+) + S_M(v^-) + S_M(v^-)$ for m.

Corollary 2 (Generalized Plott Theorem 1):

Suppose x is an interior point of V and r is a ray without the origin. If x is q.u.d. then

(i) $|S(r) - S(-r)| \leq S(0) + (2\lambda - 1)n$

(ii) $S(0) \geq (1 - 2\lambda)n$,

with both inequalities strict if x is s.q.u.d. If Q is a maximal subset of N that can be partitioned into disagreeing pairs, then $n = q + S(0)$ whenever either one of the following holds:

(iii) x is q.u.d. and $S(0) < 1 - (2\lambda - 1)n$

(iv) x is s.q.u.d. and $S(0) \leq 1 - (2\lambda - 1)n$.

Proof: $T = r^0$ is a free subspace, since $F = W$. Letting $M = C(T)$, $i \in M \iff u_i \in -r \cup \{0\} \cup r$. Hence for any $v \in r$, $S_M(v^+) = S(r)$, $S_M(v^-) = S(-r)$, and $S_M(v^0) = S(0)$. Applying corollary 1 first to v and then to −v now results in (i). Expression (i) implies (ii) when r is chosen so that no gradients are contained in r or -r. If either (iii) or (iv) hold, then (i) implies $|S(r) - S(-r)| = 0$ for all rays r. This implies $n = q + S(0)$, since $n - q - S(0) = \sum_{i \in I} |S(r_i) - S(-r_i)|$, where I indexes the lines $\ell_i = -r_i \cup \{0\} \cup r_i$ that contain nonzero gradients.

Remark 3: Corollary 2 states the complete pairwise symmetry required of the set of utility gradients at interior equilibria. The simple example of figure 3, which has $W = R^2$, $n = 5$ and $\lambda = 1/2$, indicates that (i) and (ii) are only necessary conditions, since $S(v^+) = 3$. The example also serves to show that x being s.q.u.d. in $C(T)$ for every free, nondegenerate T does not imply that x is q.u.d., as x is s.q.u.d. in all the subsets content with nondegenerate subspaces: {1,2}, {1,2,3}, {1,2,4}, {1,2,5}.

Remark 4: A converse of corollary 2 is true. Specifically, if $Q \subset N$ can be partitioned into weakly disagreeing pairs and $n = q + S(0)$, then x is q.u.d. if $S(0) \geq (1 - 2\lambda)n$ and x is s.q.u.d. if $S(0) > (1 - 2\lambda)n$. This follows easily from the observation that $S(v^+) = S_Q(v^+) \leq q/2$ for any feasible direction $v \in F$. This converse is true of any D and is generalized in section 3.

Theorem 1 is only the first step in proving symmetry conditions hold at boundary equilibria. However, it does imply necessary lower bounds on $S(D) - S(E)$ in important cases. This is

not unexpected, since the vote of an individual in D "cancels" the vote of an individual in E for any feasible direction, just as the votes of individuals whose gradients are contained in opposing rays cancel. Hence one expects an analog of (i) in corollary 2 to bound $S(D) - S(E)$. But an example will be presented subsequently showing this is not always true. First, the following corollary provides a sufficient condition for $S(D) - S(E)$ to be bounded below.

Corollary 3: Suppose T is a free subspace such that $C(T) = \{i \in N \mid u_i \in D \cup E\}$. If x is q.u.d., then

$$S(D) - S(E) \geq (1 - 2\lambda)n,$$

with the inequality strict if x is s.q.u.d.

Proof: Let $M = C(T)$. Let $\bar{v} \in$ relative interior(F), which exists because F is convex. Hence if T_F is the smallest subspace containing F, there is a neighborhood U of $\bar{v}$ such that $U \cap T_F \subset F$. Let $Q = \{i \in N \mid u_i \in E\}$. For each $i \in Q$ there exists $v \in F$ such that $v \cdot u_i > 0$, so that $u_i \notin T_F^0$. Hence lemma 3 implies the existence of $v \in U \cap T_F \subset F$ such that $v \cdot u_i > 0$ for all $i \in Q$. Therefore $S(E) = S_M(v^+)$ and $S(D) = S_M(v^-) + S_M(v^0)$, implying $S(D) - S(E) \geq (1 - 2\lambda)n$ by corollary 1.

Remark 5: If $D \cup E$ is a subspace, then the hypothesis of corollary 3 is satisfied for $T = (D \cup E)^0$. One case is $D = \{0\}$, $E = \emptyset$, for which the result is merely (ii) of corollary 2. Another case is $D + C(0,p)$, $E = C(-p)$, which occurs when V is uniquely supported by a hyperplane at x. If $D \cup E$ is not a subspace, the hypothesis may not be satisfied, and the bound on $S(D) - S(E)$ can be violated if $\dim(W) > 2$. An example with $\dim(W) = 3$, $n = 9$, and $\lambda = 1/2$ is shown in figure 4. There, none of $\{u_1,\ldots,u_6\}$ are in $E \cup D$, $\{u_7,u_8\} \subset E$, and $u_9 \in D$. x is s.q.u.d., since directions v in the corners of F get $S(v^+) = 4 < 9/2$ votes and directions in the middle of F get only 2 votes. But $S(D) - S(E) = -1 \not\geq 0$.

Pairwise symmetries at boundary equilibria will be implied by the following theorem. It refers to situations in which x is (s.)q.u.d. in a coalition whose members' gradients are contained in a two dimensional subspace. This occurs when x is (s.)q.u.d. and a subspace T of dimension $\dim(W) - 2 \geq 0$ is free, for then the gradients of members of $C(T)$ are in the two dimensional subspace T^0 and x is (s.)q.u.d. in $C(T)$ by theorem 1. Hence, for example, theorem 2 will be shown to imply necessary pairwise symmetries when x is contained in the boundary of V and V is uniquely supported at x by a hyperplane, since in this case many subspaces of dimension $\dim(W) - 2$ are free.

<u>Theorem 2</u>: Let T be a two dimensional subspace and $M = \{i \in N \mid u_i \in T\}$. Let Q be a maximal subset of M that can be partitioned into weakly disagreeing pairs, and let $K = M - Q$. Then x is (s.)q.u.d. in K if x is (s.)q.u.d. in M.

<u>Remark 6</u>: This theorem differs from the analogous proposition 1 concerning strongly disagreeing pairs by referring to only a two-dimensional subspace and by requiring Q to be maximal. Neither additional hypothesis can be eliminated. Figure 5(a) depicts a situation with $n = 5$, $\lambda = 1/2$, $W = R^2$, $D = C(0, p)$, and x s.q.u.d. By proposition 2, $2\mathcal{D}5$, $3\mathcal{D}5$, and $2\mathcal{D}4$. If $Q = \{3,5\} \cup \{2,4\}$ is deleted, x is s.q.u.d. in $\{1\}$, but $Q = \{2,5\}$ cannot be deleted because x is not s.q.u.d. in $\{1,3,4\}$. This shows Q must be taken maximal. In figure 5(b), $n = 7$, $\lambda = 1/2$, $W = R^3$, and $D = C(0,p)$. All gradients except u_4 and u_5 are in the plane of the figure, with u_5 receding behind and u_4 coming up off the page. The gradients u_3, u_4 and u_5 are all slightly lower than the plane p^0 seen in cross-section as H. Hence $C = u_3^+ \cap u_4^+ \cap u_5^+$ is a narrow cone containing $-p$. The only disagreeing pair is $\{6,7\}$. If $\{6,7\}$ is deleted, then $S_{\{1,\ldots,5\}}(-p^+) = 3$ and x is not q.u.d. in $\{1,\ldots,5\}$. But, as $u_6^+ \cap C = u_7^+ \cap C \neq \emptyset$, x is s.q.u.d. in $\{1,\ldots 7\}$. Hence, figure 5(b) shows T must be assumed two dimensional in theorem 2.

<u>Lemma 6</u>: Let T, M, Q and K be defined as in theorem 2. Suppose $T \cap D \neq \{0\}$ and $T \cap D$ contains no line. Then there exists $\hat{Q} \subset Q$ such that $\hat{q} = q/2$ and $C(\hat{K} \cup \hat{Q}) \cap D = \emptyset$, where $\hat{K} = \{i \in K \mid u_i \notin D\}$.

<u>Proof</u>: Let $\bar{r} \in T \cap D$ be a nondegenerate ray containing the origin. For any nonzero $v \in T$ let $\alpha(v)$ be the angle measured counterclockwise from $\bar{r}$ to v, with the convention $0 \leq \alpha(v) < 2\pi$. Number the members of Q as $1,2,\ldots,q$ so that $i < j$ implies $\alpha(u_i) \leq \alpha(u_j)$, as in figure 6. Because Q can be partitioned into weakly disagreeing pairs, a tedious

but straightforward argument that we omit establishes that $i\mathcal{D}(i + q/2)$ for each $1 \le i \le q/2$. Let $\sigma(\cdot)$ be defined by $\sigma(i) = i + q/2$, so that $i\mathcal{D}\sigma(i)$ for $i \le i \le q/2$.

Let $A \subset Q \cup \hat{K}$. Because $T \cap D$ contains no line and $u_i \notin D$ for any $i \in Q \cup \hat{K}$, it can be shown that $\dim(T) = 2$ implies $C(A) \cap D = \emptyset \Longleftrightarrow C(A) \cap \overline{r} = \emptyset$. Thus we need only establish the existence of $\hat{Q} \subset Q$ such that $\hat{q} = q/2$ and $C(\hat{K} \cup \hat{Q}) \cap \overline{r} = \emptyset$.

Now consider $C(\hat{K})$. Let $a \in \hat{K}$ satisfy $\alpha(u_a) \le \alpha(u_i)$ for all $i \in \hat{K}$ and let $b \in \hat{K}$ satisfy $\alpha(u_b) \ge \alpha(u_i)$ for all $i \in \hat{K}$. Then $C(\hat{K}) \cap \overline{r} \ne \emptyset \Longleftrightarrow \alpha(u_b) - \alpha(u_a) \ge \pi \Longleftrightarrow C(u_a,u_b) \cap \overline{r} \ne \emptyset$. But then $C(\hat{K}) \cap \overline{r} \ne \emptyset$ implies $a\mathcal{D}b$, contrary to the maximality of Q. Hence $C(\hat{K}) \cap \overline{r} = \emptyset$ and $C(\hat{K}) = C(u_a,u_b)$.

Suppose $C(\hat{K} \cup \{1\}) \cap \overline{r} = \emptyset$. Then, since $\alpha(u_b) - \alpha(u_1) < \pi$, $\alpha(u_{q/2}) - \alpha(u_1) < \pi$, and $\alpha(u_b) - \alpha(u_a) < \pi$, we have

$$\max\{\alpha(u_b),\ \alpha(u_{q/2})\} - \min\{\alpha(u_a),\ \alpha(u_1)\} < \pi.$$

Therefore $C(\hat{K} \cup \{1,\dots,q/2\}) \cap \overline{r} = \emptyset$, and the lemma is proved. Similarly, the lemma is proved if $C(\hat{K} \cup \{q\}) \cap \overline{r} = \emptyset$. Furthermore, letting $Q_i = \{i,\ i+1,\dots,\sigma(i-1)\}$, the lemma is proved if $C(\hat{K} \cup Q_i) \cap \overline{r} = \emptyset$ for any $1 < i \le q/2$. Hence it remains to consider the case where $C(\hat{K} \cup \{1\})$, $C(\hat{K} \cup \{q\})$, and $C(\hat{K} \cup Q_i)$ for $1 < i \le q/2$ all intersect $\overline{r}$.

Now, $C(\hat{K} \cup \{1\}) \cap \overline{r} \ne \emptyset$ implies $1\mathcal{D}b$ and $C(\hat{K} \cup \{q\}) \cap \overline{r} = \emptyset$ implies $a\mathcal{D}q$. For $1 < i \le q/2$, $C(\hat{K} \cup Q_i) \cap \overline{r} \ne \emptyset$ implies $i\mathcal{D}b$ or $a\mathcal{D}(\sigma(i-1))$ or $i\mathcal{D}(\sigma(i-1))$. Let i_0 be the maximal $1 \le i \le q/2$ such that $i\mathcal{D}b$. Let j_0 be the minimal $i_0 < j \le q/2 + 1$ such that $a\mathcal{D}(\sigma(j-1))$.

Then substitution of

$$\{i_0,b\} \cup \{i_0+1,\ \sigma(i_0)\} \cup \dots \cup \{j_0-1,\ \sigma(j_0-2)\} \cup \{a,\ \sigma(j_0-1)\}$$

for $\{i_0,\ \sigma(i_0)\} \cup \{i_0+1,\ \sigma(i_0+1)\} \cup \dots \cup \{j_0-1,\ \sigma(j_0-1)\}$ in the partition $Q = \{1,\ \sigma(1)\} \cup \dots \cup \{q/2,\ \sigma(q/2)\}$ yields a partition of $Q \cup \{a,b\}$ into weakly disagreeing pairs. This contradiction of Q maximal finishes the proof.

<u>Proof of Theorem 2</u>: <u>Case 1</u>: $T \cap D = \{0\}$. In this case each weakly disagreeing pair in Q is strongly disagreeing and the theorem follows by proposition 1. <u>Case 2</u>: $T \cap D = T$. Then $Q = \emptyset$ and the theorem is trivial. <u>Case 3</u>: $T \cap D \ne T$ contains a line ℓ. Because $\dim(T) = 2$, there exists nonzero $v \in T$ such that $\ell = v^0$. Since D is convex, $T \cap D = v^0$ or $T \cap D = v^0 \cup v^+$ (switching v and -v if necessary). If $T \cap D = v^0 \cup v^+$ and $u_i,u_j \notin D$ for some $i,j \in M$, then $C(u_i,u_j) \cap D = \emptyset$. Hence $Q = \emptyset$ and the theorem is trivial if $T \cap D = v^0 \cup v^+$. If $T \cap D = v^0$, then for any $i,j \in M$, $i\mathcal{D}j \Longleftrightarrow C(u_i,u_j) \cap v^0 \ne \emptyset$. Hence all of $\{u_i \mid i \in K\}$ and half of $\{u_i \mid i \in Q\}$ are contained in one halfspace (v^+ or v^-). Therefore there exists $\hat{Q} \subset Q$ such that $\hat{q} = q/2$ and $C(\hat{K} \cup \hat{Q}) \cap D = \emptyset$. By lemma 6, such a $\hat{Q}$ also exists for the remaining <u>Case 4</u>: $T \cap D \ne \emptyset$ and $T \cap D$ contains no line. Therefore we must prove the theorem for cases 3 and 4 assuming such a $\hat{Q}$ exists. But then $C(K \cup \hat{Q})$ is a closed, convex and pointed cone not intersecting the convex closed cone D, so a separation theorem implies the existence

of $\overline{v} \in D^* = F$ such that $C(\hat{K} \cup \hat{Q}) \subset \overline{v}^+$. Hence, since x is q.u.d. in M,

$$\hat{k} + \hat{q} \leq \lambda_m m = \lambda n - 1/2(n-q-k).$$

This implies, as $\hat{q} = q/2$, that $\hat{k} \leq \lambda n - 1/2(n-k) = \lambda_k k$, with the inequality strict if x is s.q.u.d. in M. Since $S_K(v^+) \leq \hat{k}$ for all $v \in F$, x is (s.)q.u.d. in K.

<u>Corollary 4 (Generalized Plott Theorem 2)</u>: Suppose $D = C(0,p_1,p_2)$, with p_1 and p_2 nonzero but not necessarily distinct. Let T be a two dimensional subspace containing D, $M = \{i \in N \mid u_i \in T\}$, Q a maximal subset of M that can be partitioned into weakly disagreeing pairs, and $\hat{K} = \{i \in M - Q \mid u_i \notin D\}$. Then if x is q.u.d.,

(i) $\hat{k} \leq S(D) + (2\lambda - 1)n$

(ii) $m - S(D) - S(E) \geq q \geq m - 2S(D) - (2\lambda - 1)n$,

with the inequality in (i) and the second inequality in (ii) strict if x is s.q.u.d. Furthermore, if $\overline{Q}$ is the maximal subset of N that can be partitioned into weakly disagreeing pairs, and $p_1 = \pm p_2$, then $n = \overline{q} + S(D) + S(E)$ if

(iii) x is q.u.d. and $S(D) - S(E) < 1 - (2\lambda-1)n$

or

(iv) x is s.q.u.d. and $S(D) - S(E) \leq 1 - (2\lambda-1)n$.

<u>Proof</u>: Since T contains D, $M = C(T^0)$ and T^0 is a free subspace. By theorem 1, x is (s.)q.u.d. in M. Hence by theorem 2, x is (s.)q.u.d. in M - Q. Also, for $D = C(0,p_1,p_2)$, cases 3 or 4 of the proof of theorem 2 apply, so that $\hat{k} \leq \lambda n - 1/2(n - k)$, where $k = \hat{k} + S(D)$. Hence (i) follows. The second inequality in (ii) follows from (i) by substituting $m - q - S(D)$ for $\hat{k}$ in (i). The first inequality in (ii) holds because $E \cup D \subset T$ and no $i \in M$ with $u_i \in E \cup D$ can weakly disagree with anybody. By (ii), $q = m - S(D) - S(E)$ if either (iii) or (iv) hold. If also $p_1 = \pm p_2$, then all n gradients are contained in the union of a finite number of two dimensional subspaces that each contain D. Summing over these subspaces consequently yields $n = \overline{q} + S(D) + S(E)$ if (iii) or (iv) holds and $p_1 = \pm p_2$.

<u>Remark 7</u>: Observe the analogy between corollaries 2 and 4. Expression (i) in corollary 2 puts a bound on the minimal set of people whose gradients are in a one dimensional subspace containing $D = \{0\}$ that does not contain a disagreeing pair. Expression (i) in corollary 4 puts a bound on the minimal set of people, whose gradients are in a two dimensional subspace containing a $D \neq \{0\}$, that does not contain a weakly disagreeing pair. Expressions (iii) and (iv) in the two corollaries are obviously similar.

<u>Remark 8</u>: Corollary 4(ii) indicates the pairwise symmetry that must hold at boundary equilibria if D is at most two dimensional, since then $i\mathcal{D}j$ iff u_i and u_j occupy symmetrical positions about D. D is at most two dimensional if V is uniquely supported at x by a hyperplane, or if F can be defined as the intersection of only two half-spaces with boundaries containing x. The pairwise symmetry of all gradients is implied by (iii) or (iv) only if V is uniquely supported

at x by a hyperplane. Clearly, less symmetry is required if V is so "pointed" at x that D is more than two dimensional; it seems that corollaries 2 and 4 indicate the only situations in which required symmetries involve pairs of gradients.

Remark 9: Notice that because D is two dimensional, (ii) of corollary 4 implies the validity of $S(D) - S(E) \geq (1-2\lambda)n$ without requiring the condition that $D \cup E$ be contained in a subspace containing only gradients in $D \cup E$, which was needed in corollary 3.

Remark 10: A converse of corollary 4 is also true: If $\overline{Q} \subset N$ can be partitioned into weakly disagreeing pairs and $n = \overline{q} + S(D) + S(E)$, then x is q.u.d. if $S(D) - S(E) \gtrsim (1 - 2\lambda)n$ and x is s.q.u.d. if $S(D) - S(E) > (1 - 2\lambda)n$. This follows easily from the observation that $S(v^+) \leq S_{\overline{Q}}(v^+) + S(E) \lesssim \overline{q}/2 + S(E)$ for any feasible $v \in F$. This converse is true for any D and is generalized in section 3.

3. SUFFICIENT CONDITIONS

Most conditions sufficient for quasi-undomination are not as general as the necessary ones and, unfortunately, require more notation for their derivation. However, there is one general result providing a necessary as well as a sufficient condition, although it is not often useful if F is "large".

Theorem 3: Let $\{T_\alpha\}$ be a collection of subspaces such that $F \subset \bigcup_\alpha T_\alpha$. Then x is (s.)q.u.d. if and only if for every subspace T_α that intersects F, x is (s.)q.u.d. when every person's gradient is projected onto T_α.

Proof: Given a subspace T, write $u_i = a_0^i + a_1^i$, where $a_0^i \in T^0$, $a_1^i \in T$. The set $\{a_1^i\}$ is the set of gradients projected onto T, and the result follows from the fact that $v \cdot u_i > 0$ if and only if $v \cdot a_1^i > 0$ when $v \in F \cap T$.

The usefulness of the criterion provided by theorem 3 is severely limited by the tradeoff between checking many subspaces of low dimension and checking fewer subspaces of higher dimension. To obtain more tractable conditions, we introduce new notation. Let $\overline{M} = \{i \in N \mid u_i \in E \cup D\}$. For any $\overline{M} \subset M \subset N$ and for any $v \in F$, define

$$n_M(v) = S_{M-\overline{M}}(v^+) - S_{M-\overline{M}}(v^- \cup v^0)$$

and

$$n_M = \max_{v \in F} n_M(v).$$

Now we have what will prove to be a very useful result.

Theorem 4: Let $M_1, \ldots, M_h$ be a collection of subsets of N satisfying $N = M_1 \cup \ldots \cup M_h$ and $M_i \cap M_j = \overline{M}$ for $i \neq j$. Then x is q.u.d. if

$$\sum_{i=1}^{h} n_{M_i} \leq S(D) - S(E) + (2\lambda-1)n,$$

and x is s.q.u.d. if the inequality is strict.

Proof: Let $v \varepsilon F$. Then

$$S(v^+) \leq S(E) + S_{N-\overline{M}}(v^+)$$

$$= S(E) + \sum_{i=1}^{h} S_{M_i-\overline{M}}(v^+)$$

$$\leq S(E) + \sum_{i=1}^{h} n_{M_i} + \sum_{i=1}^{h} S_{M_i-\overline{M}}(v^- \cup v^0)$$

$$\leq S(D) + \sum_{i=1}^{h} S_{M_i-\overline{M}}(v^- \cup v^0) + (2\lambda-1)n$$

$$\leq S(v^- \cup v^0) + (2\lambda-1)n.$$

Now $S(V^+) \leq \lambda n$ follows by substituting $n - S(v^+)$ for $S(v^- \cup v^0)$. The proof that x is s.q.u.d. if strict inequality holds is identical.

Corollary 5: Suppose $x \varepsilon$ interior(V). Let Q be a maximal subset of N that can be partitioned into disagreeing pairs. Then x is q.u.d. if $n - q \leq 2S(0) + (2\lambda-1)n$, and x is s.q.u.d. if $n - q < 2S(0) + (2\lambda-1)n$.

Remark 11: Observe that

$$n - q - S(0) = \sum_{i\varepsilon I} | S(r_i) - S(-r_i)|,$$

where I indexes the lines $\ell_i = -r_i \cup \{0\} \cup r_i$ that contain nonzero gradients. Hence the sufficient condition for x to be q.u.d. is that

$$\sum_{i\varepsilon I} | S(r_i) - S(-r_i) | \leq S(0) + (2\lambda-1)n.$$

Notice the relationship to (i) in corollary 2.

Proof of Corollary 5: In theorem 4, take $M_i = \{i\varepsilon N | u_i \varepsilon \ell_i)$ for each $i \varepsilon I$. Since $D = \{0\}$, these M_i satisfy the hypothesis of theorem 4.

Also, $n_{M_i} = | S(r_i) - S(-r_i) |$. Hence, by remark 11, $n - q \leq 2S(0) + (2\lambda-1)n$ implies $\sum_{i\varepsilon I} n_{M_i} \leq S(0) + (2\lambda-1)n = S(D) - S(E) + (2\lambda-1)n$. Therefore the result follows from theorem 4.

Remark 12: The condition of corollary 5 is not necessary for x to be q.u.d., as figure 7 illustrates. There, $n = 9$, $\lambda = 1/2$, $D = \{0\}$, x is s.q.u.d. since max $S(v^+) = 4$, but

$$\Sigma n_{M_i} = \sum_{i=1}^{3} |S(r_i) - S(-r_i)|$$

$$= 3 \not\leq 2 = S(0).$$

Remark 13: The simple sufficient condition mentioned in remark 4 is a special case of corollary 5.

Corollary 6: Suppose $x \in$ boundary(V) with $D = C(0,p)$ $(p \neq 0)$. Let Q be a maximal subset of N that can be partitioned into weakly disagreeing pairs. Then x is q.u.d. if $n - q \leq 2S(D) + (2\lambda-1)n$, and x is s.q.u.d. if $n - q < 2S(D) + (2\lambda-1)n$.

Remark 14: Notice the relationship of this inequality to the second inequality in (ii) of corollary 4.

Proof of corollary 6: Let $T_1,\ldots,T_h$ be a set of two dimensional subspaces that collectively contain all nonzero gradients and that satisfy $D \subset T_i$. Let $M_i = \{i \varepsilon N \mid u_i \varepsilon T_i\}$, and notice $M_1,\ldots M_h$ satisfy the hypothesis of theorem 4. Let Q_i be a maximal subset of M_i that can be partitioned into weakly disagreeing pairs. Then

$q = \sum_{i=1}^{h} q_i$. Let $\hat{K}_i = \{j \in M_i - Q_i \mid u_j \notin D\}$. Then as in cases 3 and 4 of the proof of theorem 2, there exists $v_i \in F$ satisfying

$$S_{M_i}(v_i^+) = \hat{k}_i + q_i/2 = S_{M_i-\overline{M}}(v_i^+) + S(E)$$

and

$$S_{M_i-\overline{M}}(v_i^- \cup v_i^0) = q_i/2.$$

This v_i yields the greatest $n_{M_i}(v^+)$, so that $n_{M_i} = \hat{k}_i - S(E)_i$.

Noticing that $n - q = \sum_{i=1}^{h} (\hat{k}_i - S(E)) + S(D) + S(E)$, we have

$$\sum_{i=1}^{h} n_{M_i} = n - q - S(D) - S(E)$$

$$\leq S(D) - S(E) + (2\lambda-1)n.$$

Hence theorem 4 implies corollary 6.

We conclude with a useful theorem that can be easily applied if $D = \{0\}$ or $D = C(0,p)$.

Theorem 5 (Partial converse to theorem 1):

Let $T_1,\ldots,T_h$ be any collection of free subspaces such that $C(T_1) \cup \ldots \cup C(T_h) = N$ and $C(T_i) \cap C(T_j) = \overline{M}$ for $i \neq j$. Then x is q.u.d. if

(i) $S(D) - S(E) < 1-(2\lambda-1)n$ and x is q.u.d. in each $C(T_i)$,

and x is s.q.u.d. if

(ii) $S(D) - S(E) \leq 1-(2\lambda-1)n$ and x is s.q.u.d. in each $C(T_i)$.

Lemma 7: For any $M \subset N$ that contains $\overline{M}$, x is q.u.d. in M iff

$$n_M \leq S(D) - S(E) + (2\lambda-1)n,$$

and x is s.q.u.d. in M iff the inequality is strict.

Proof: By lemma 2, there exists $\overline{v} \in$ relative interior (F) such that $S_M(\overline{v}^+) \geq S_M(v^+)$ for all $v \in F$. By suitable applications of lemmas 1 and 3, $\overline{v} \in$ relative interior (F) can be shown to imply that $\overline{v} \cdot u_i > 0$ for each $u_i \in E$. Hence, since $\overline{M} \subset M$ and $\overline{v} \cdot u_i \leq 0$ for all $u_i \in D$,

$$S_M(\overline{v}^+) = S_{M-\overline{M}}(\overline{v}^+) + S(E).$$

Similarly, there exists $\hat{v} \in$ relative interior (F) such that $S_{M-\overline{M}}(\hat{v}^+) \geq S_{M-\overline{M}}(v^+)$ for any $v \in F$ and

$$S_M(\hat{v}^+) = S_{M-\overline{M}}(\hat{v}^+) + S(E).$$

Hence $S_{M-\overline{M}}(\overline{v}^+) = S_M(\overline{v}^+) - S(E) \geq S_M(\hat{v}^+) - S(E) = S_{M-\overline{M}}(\hat{v}^+)$ implies $S_{M-\overline{M}}(v^+)$ is maximized on F at $\overline{v}$.

Therefore, if x is q.u.d. in M then

$$\begin{aligned}
n_M &= \max_{v\in F} \left\{S_{M-\overline{M}}(v^+) - S_{M-\overline{M}}(v^- \cup v^0)\right\} \\
&= \max_{v\in F} \left\{S_{M-\overline{M}}(v^+) - [m - S_{M-\overline{M}}(v^+) - S(D) - S(E)]\right\} \\
&= S(D) + S(E) - m + 2 \max_{v\in F} S_{M-\overline{M}}(v^+), \\
&= S(D) - S(E) - m + 2S_M(\overline{v}^+) \\
&\leq S(D) - S(E) - m + 2\lambda_m m \\
&= S(D) - S(E) + (2\lambda-1)n,
\end{aligned}$$

with the inequality strict if x is s.q.u.d. in M. The other direction of proof is straightforward and very similar to the proof used in theorem 4.

Proof of Theorem 5: Let $M_i = C(T_i)$ and observe that $M_1,\ldots,M_h$ satisfy the hypothesis of theorem 4. Suppose (i) holds. Then by lemma 7,

$$n_{M_i} \le S(D) - S(E) + (2\lambda-1)n < 1.$$

Hence, as each n_{M_i} is nonpositive, $\sum_{i=1}^{h} n_{M_i} \le n_{M_h} \le S(D) - S(E) + (2\lambda-1)n$.

Therefore x is q.u.d. by theorem 4. If (ii) holds, then by lemma 7,

$$n_{M_i} < S(D) - S(E) + (2\lambda-1)n \le 1.$$

Therefore $\sum_{i=1}^{h} n_{M_i} \le n_{M_h} < S(D) - S(E) + (2\lambda-1)n$ and x is s.q.u.d. by theorem 4.

FIGURE 1

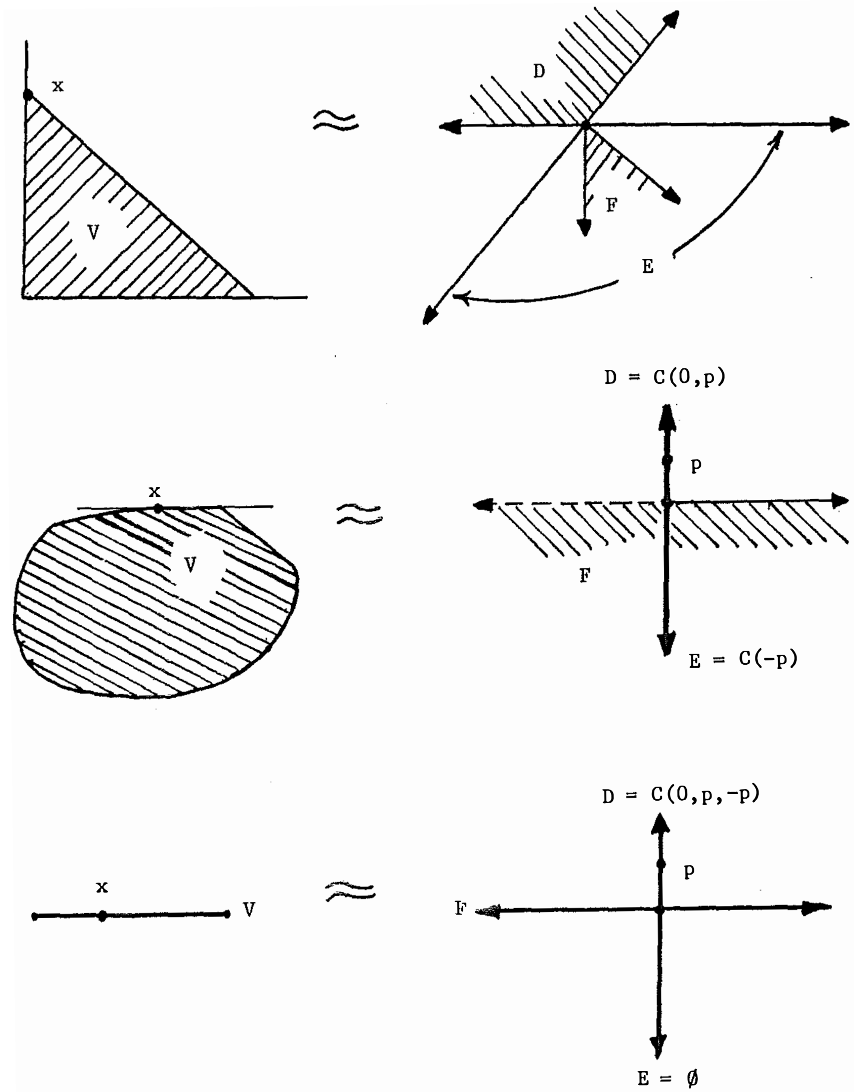


FIGURE 2

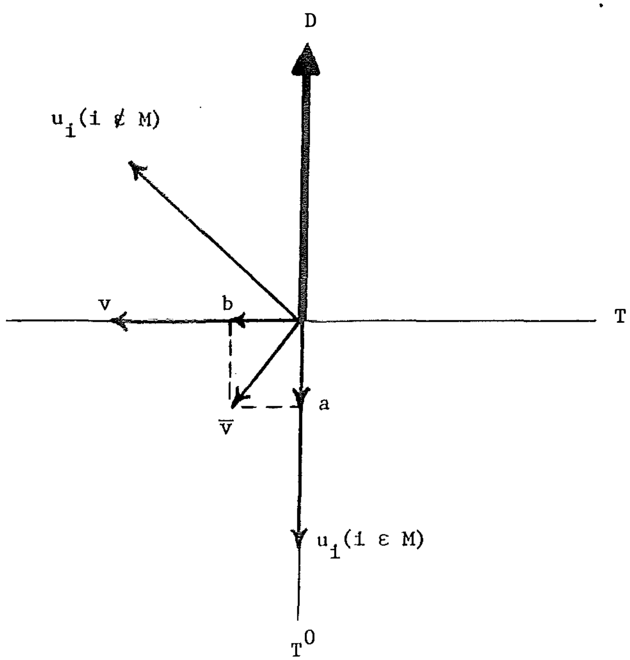


FIGURE 3

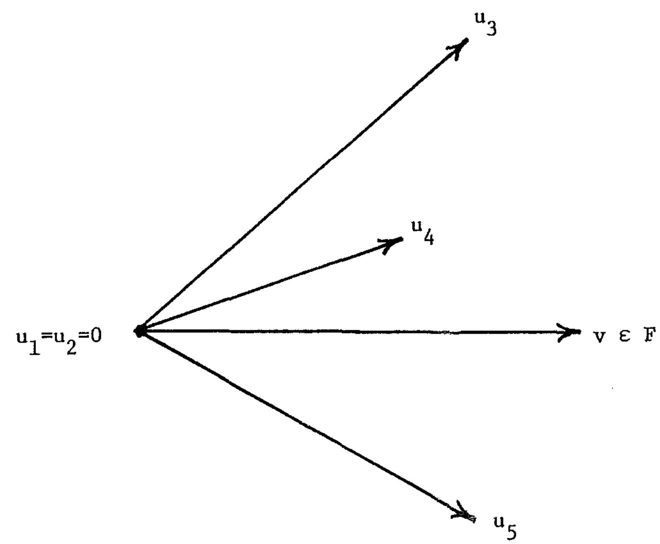


FIGURE 4

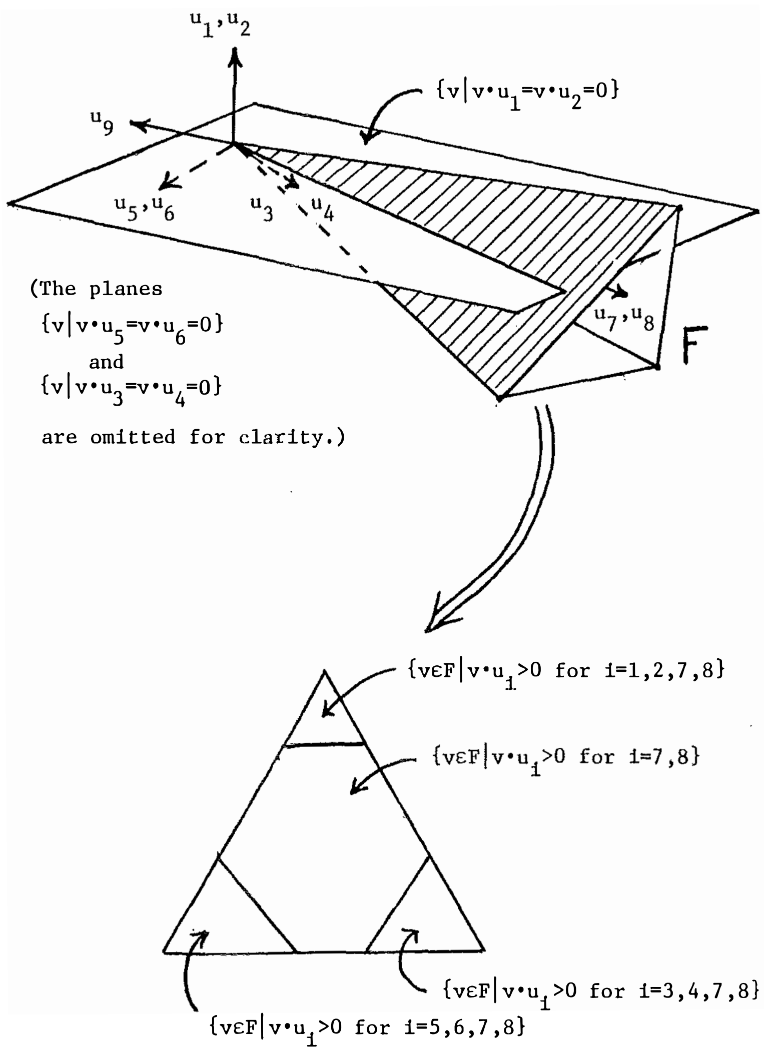


FIGURE 5

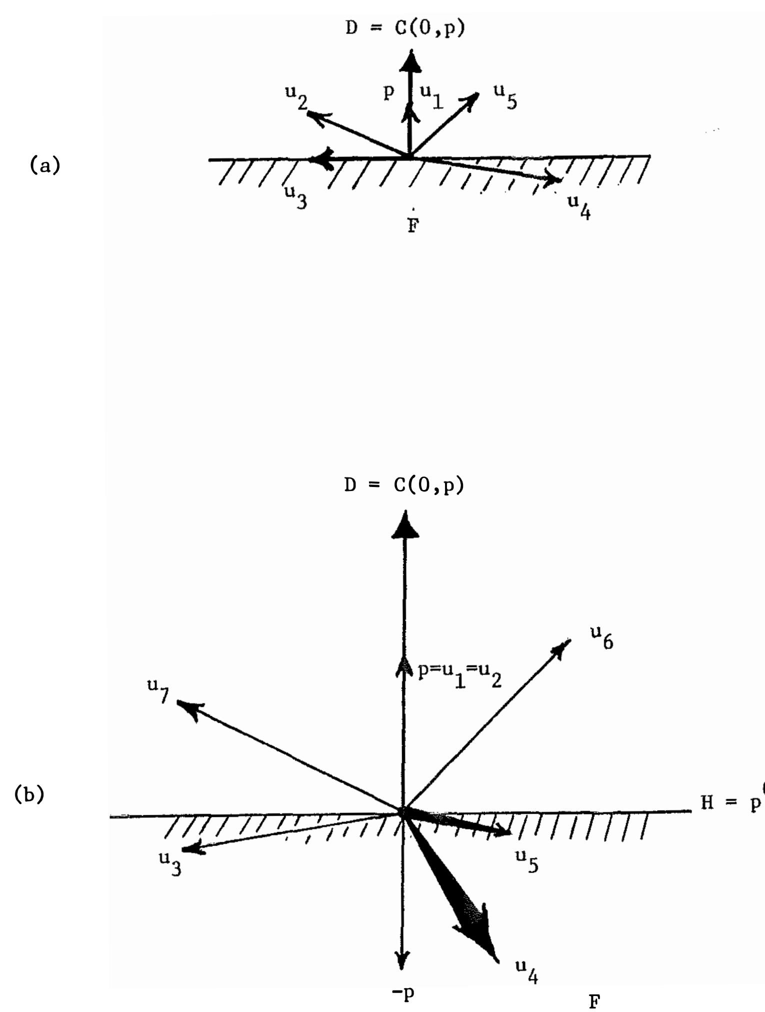


FIGURE 6

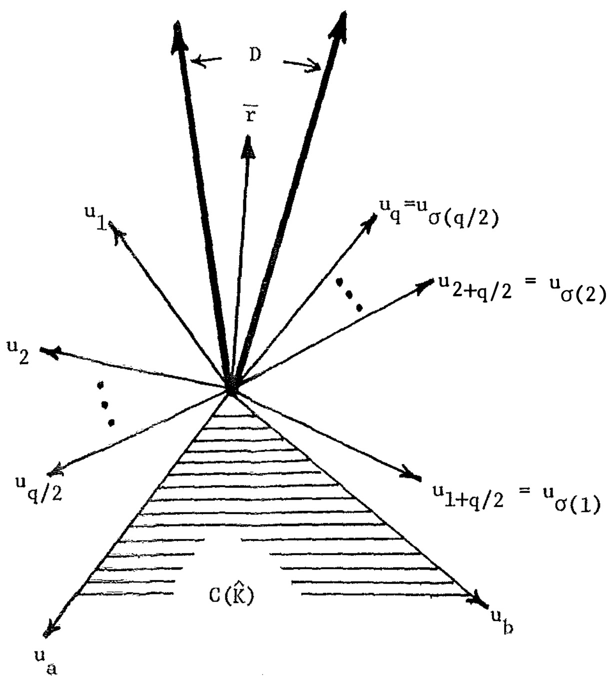
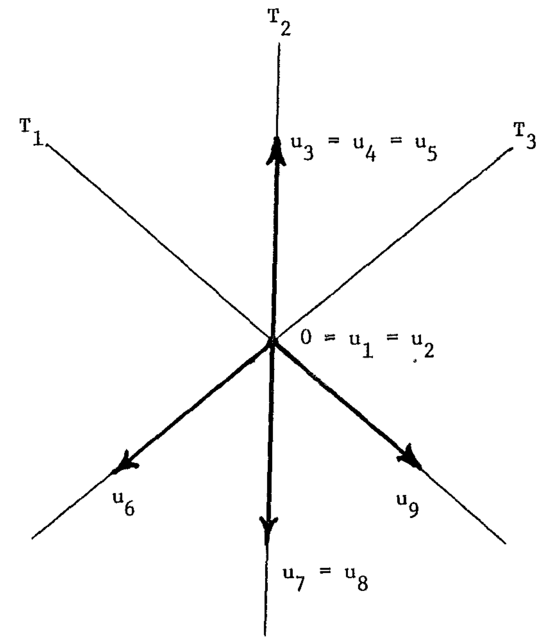


FIGURE 7



FOOTNOTES

1. Although to my knowledge symmetry conditions for <u>pairs</u> of utility gradients have only been studied previously in three papers: Plott [1967], McKelvey and Wendell [1976], and Slutsky [1978].

2. For interior equilibria, Slutsky [1978] has independently derived pairwise symmetry conditions for λ-majority rule equilibria. His conditions are similar to some of those derived here.

3. A simple generalization would be to allow W to be a differentiable manifold, F a convex cone in the tangent space TW_x of W at x, and u_i an element of the dual of TW_x.

4. For this and other results mentioned below concerning convex cones, refer to any standard source such as Fenchel [1953] or Rockafellar [1970].

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