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Palatini formulation of $f(R, T)$ gravity theory, and its cosmological implications

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Abstract We consider the Palatini formulation of $f(R, T)$ gravity theory, in which a non-minimal coupling between the Ricci scalar and the trace of the energy-momentum tensor is introduced, by considering the metric and the affine connection as independent field variables. The field equations and the equations of motion for massive test particles are derived, and we show that the independent connection can be expressed as the Levi-Civita connection of an auxiliary, energy-momentum trace dependent metric, related to the physical metric by a conformal transformation. Similar to the metric case, the field equations impose the non-conservation of the energy-momentum tensor. We obtain the explicit form of the equations of motion for massive test particles in the case of a perfect fluid, and the expression of the extra force, which is identical to the one obtained in the metric case. The thermodynamic interpretation of the theory is also briefly discussed. We investigate in detail the cosmological implications of the theory, and we obtain the generalized Friedmann equations of the $f(R, T)$ gravity in the Palatini formulation. Cosmological models with Lagrangians of the type $f = R - \alpha^2/R + g(T)$ and $f = R + \alpha^2 R^2 + g(T)$ are investigated. These models lead to evolution equations whose solutions describe accelerating Universes at late times.

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1 Introduction

The observational discovery of the recent acceleration of
the Universe [1–5] has raised the fundamental theoretical
problem if general relativity, in its standard formulation, can
fully account for all the observed phenomena at both galac-
tic and extra-galactic scales. The simplest theoretical expla-
nation for the observed cosmological dynamics consists in
slightly modifying the Einstein field equations, by adding to
it a cosmological constant Λ [6]. Together with the assump-
tion of the existence of another mysterious component of the
Universe, called dark matter [7,8], assumed to be cold and
pressureless, the Einstein gravitational field equations with
the cosmological constant included, can give an excellent fit
to all observed data, thus leading to the formulation of the
standard cosmological paradigm of our present days, called
the Λ Cold Dark Matter (ΛCDM) model. However, despite
its apparent simplicity and naturalness, the introduction of
the cosmological constant raises a number of important the-
oretical and observational question for which no convincing
answers have been provided so far. The ΛCDM model can fit
the observational data at a high level of precision, it is a very
simple theoretical approach, it is easy to use in practice, but
up to now no fundamental theory can explain it. Why is the
cosmological constant so small? Why is it so fine-tuned? And
why did the Universe begin to accelerate only recently? And,
after all, would a cosmological constant really be necessary
to explain all observations?

From a theoretical point of view two possible answers to
the questions raised by the observation of the recent acceler-
ation of the Universe can be formulated. The first, called the
dark energy approach, assumes that the Universe is filled by
a mysterious and unknown component, called dark energy
[9–12], which is fully responsible for the acceleration of the
Universe, as well as for its mass–energy balance. The cos-
mological constant Λ corresponds to a particular phase of
the dynamical dark energy (the ground state of a potential,
say), and the recent de Sitter phase may prove to be just an
attractor of the dynamical system describing the cosmologi-
cal evolution. A second approach, the *dark gravity approach*,
assumes the alternative possibility that at large scales the
gravitational force may have a very different behavior as
compared to the one suggested by standard general relativity.
In the general relativistic description of gravity, the starting
point is the Hilbert–Einstein action, which can be written

down as $S = \int \left(R/2\kappa^2 + L_{\mathrm{m}} \right) \sqrt{-g}\, \mathrm{d}^4 x$, where R is the
Ricci scalar, κ is the gravitational coupling constant, and L_{m}
is the matter Lagrangian, respectively. Hence in dark grav-
ity type theories for a full understanding of the gravitational
interaction a generalization of the Hilbert–Einstein action is
necessary.

There are (at least) two possibilities to construct dark
gravity theories. The first is based on the modification of
the geometric part of the Hilbert–Einstein Lagrangian only.
An example of such an approach is the $f(R)$ gravity the-
ory, introduced in [13,14], and in which the geometric part
of the action is generalized so that it becomes an arbi-
trary function $f(R)$ of the Ricci scalar. Hence in $f(R)$
gravity the total Hilbert–Einstein action can be written as
$S = \int \left(f(R)/2\kappa^2 + L_{\mathrm{m}} \right) \sqrt{-g}\, \mathrm{d}^4 x$. The recent cosmologi-
cal observations can be satisfactorily explained in the $f(R)$
theory, and a solution of the dark matter problem, interpreted
as a geometric effect in the framework of the theory, can also
be obtained [15]. For reviews and in depth discussions of
$f(R)$ and other modified gravity theories see [16–24].

A second avenue for the construction of the dark gravity
theories consists in looking for maximal extensions of the
Hilbert–Einstein action, in which the matter Lagrangian L_{m}
plays an equally important role as the Ricci scalar. Hence in
this more general approach one modifies both the geomet-
ric and the matter terms in the Hilbert–Einstein action, thus
allowing a coupling between matter and geometry [25,39].
The first possibility for such a coupling is to replace the grav-
itational action by an arbitrary function of the Ricci scalar
and the matter Lagrangian L_{m}, thus obtaining the so-called
$f(R, L_{\mathrm{m}})$ class of modified gravity theories [26]. This class
of theories has the potential of explaining the recent acceler-
ation of the Universe without the need of the cosmological
constant, and can give some new insights into the dark mat-
ter problem, and on the nature of the gravitational motion.
The cosmological and physical implications of this theory
have been intensively investigated [27–37]. For a review of
$f(R, L_{\mathrm{m}})$ type theories see [38].

A second extension of the Hilbert–Einstein action can
be obtained by assuming that the gravitational field cou-
ples to the trace T of the energy-momentum tensor of the
matter. This assumption leads to the $f(R, T)$ class of grav-
itational theories [39]. $f(R, T)$ theory may give some hints
for the existence of an effective classical description of the
quantum properties of gravity. As pointed out in [40], by
using a non-perturbative approach for the quantization of
the metric, proposed [41–43], as a consequence of the quan-
tum fluctuations of the metric, a particular type of $f(R, T)$
gravity naturally emerges, with the Lagrangian given by
$L = \left[(1 - \alpha)R/2\kappa^2 + (L_{\mathrm{m}} - \alpha T/2) \right] \sqrt{-g}$, where α is a
constant. This interesting theoretical result suggests that a
deep connection may exist between the quantum field theo-
retical description of gravity, which naturally involves parti-

cle production in the gravitational field, and the corresponding effective classical description of the $f(R, T)$ gravity theory [44]. The astrophysical and cosmological implications of $f(R, T)$ gravity theory were investigated in [45–66].

Einstein's theory of general relativity can be obtained by starting from two different theoretical approaches, called the metric and the Palatini formalism, respectively, the latter being introduced by Albert Einstein [67–69]. In the Palatini variational approach one takes as independent field variables not only the ten components $g_{\mu\nu}$ of the metric tensor, but also the components of the affine connection $\Gamma^{\alpha}_{\beta\mu}$, without assuming, a priori, the form of the dependence of the connection on the metric tensor, and its derivatives [70,71]. When applied to the Hilbert–Einstein action, these two approaches lead to the same gravitational field equations. Moreover, the Palatini formalism also provides the explicit form of the symmetric connection as determined by the derivatives of the metric tensor. However, in $f(R)$ modified gravity, as well as in other modified theories of gravity, this does not happen anymore. In fact it turns out that the gravitational field equations obtained by using the metric approach are generally different from those obtained by using the Palatini variation [70,71]. An important difference is related to the order of the field equations. The metric formulation usually leads to higher-order derivative field equations, while in the Palatini formalism the obtained gravitational field equations are always second-order partial differential equations. A number of new algebraic relations also do appear in the Palatini variational formulation, which describe the subtle relation between the matter fields and the affine connection, which can be determined from a set of equations that couples it not only to the metric, but also to the matter fields. The astrophysical and cosmological implications of the Palatini formulation of $f(R)$ gravity have also been intensively investigated [72–76].

Based on a hybrid combination of the metric and Palatini mathematical formalisms, an extension of the $f(R)$ gravity theory was proposed in [77] and was used to construct a new type of gravitational Lagrangian [77,78]. A simple example of such an hybrid metric-Palatini theory can be constructed by adopting for the gravitational Lagrangian the expression $R + f\left(R(g, \tilde{\Gamma})\right)$, where $R(g, \tilde{\Gamma})$ is the Palatini scalar curvature. A similar formalism that interpolates between the metric and Palatini regimes was proposed in [79,80] for the study of $f(R)$ type theories. This approach is called C-theory. A generalization of the hybrid metric-Palatini gravity was introduced in [81].

Despite the intensive investigations of the theoretical and observational aspects of the modified gravity theories with geometry–matter coupling, their Palatini formulation and properties have attracted considerably less attention. The Palatini formulation of the linear $f(R, L_{\mathrm{m}})$ gravity was introduced in [82], where the field equations and the equa-

tions of motion for massive test particles were derived. The independent connection can be expressed as the Levi-Civita connection of an auxiliary, matter Lagrangian dependent metric, which is related to the physical metric by means of a conformal transformation. Similar to the metric case, the field equations impose the non-conservation of the energy-momentum tensor. The study of Palatini formulation of $f(R, T)$ gravity was initiated in [83]. Analogously to its metric counterpart, the field equations impose of the $f(R, T)$ gravity in the Palatini formulation implies the non-conservation of the energy-momentum tensor, which leads to non-geodesic motion, and to the appearance of an extra force.

It is the purpose of the present paper to derive the gravitational field equations of the generalized $f(R, T)$ type gravity models, with non-minimal coupling between matter, described by the trace of the energy-momentum tensor, and geometry, characterized by the Ricci scalar By taking separately two independent variations of the gravitational action with respect to the metric and the connection, respectively, we obtain the field equations and the connection associated to the Ricci tensor, which, due to the coupling between the trace of the energy-momentum tensor and the geometry, is also a function of T. The metric that defines the new independent connection is conformally related to the initial space-time metric, with the conformal factor given by a function of the trace of the energy-momentum tensor, and of the Ricci scalar. After the conformal factor is obtained, the gravitational field equations can be written down easily in both metrics. Similar to the case of the metric $f(R, T)$ gravity, after taking the divergence of the gravitational field equations we obtain the important result that the energy-momentum tensor of the matter is not conserved. Similar to the metric case, the motion of the particles is not geodesic, and due to the matter–geometry coupling, an extra force arises. However, this force has the same expression as in the metric case, and therefore no new physics is expected to arise during the motion of massive test particles in the Palatini formulation of $f(R, T)$ gravity. As the next step in our analysis we investigate in detail the cosmological implications of the Palatini formulation of the $f(R, T)$ gravity theory. We obtain the generalized Friedmann equations, which explicitly contain the extra terms generated by the coupling between the trace of the energy-momentum tensor and geometry. The general properties of the cosmological evolution are obtained, including the behavior of the deceleration parameter, of the effective energy density and pressure, and of the parameter of the equation of state of the dark energy. Cosmological models with Lagrangians of the type $L = R - \alpha^2/R + g(T)$ and $L = R + \alpha^2 R^2 + g(T)$ are considered in detail, and it is shown that these models lead to evolution equations whose solutions tend to a de Sitter type Universe at late times.

The present paper is organized as follows. After a brief review of the metric formalism, the field equations of $f(R, T)$ gravity theory are obtained by using the Palatini formalism of gravitational theories in Sect. 1. The energy and momentum balance equations are obtained, after taking the divergence of the energy-momentum tensor, in Sect. 3. The thermodynamical interpretation of the theory is also briefly discussed. The cosmological implications of the Palatini $f(R, T)$ theory is investigated in Sect. 4. We discuss and conclude our results in Sect. 5. The details of the derivation of the field equations in the metric formalism are given in Appendix A, while the divergence of the matter energy-momentum tensor is derived in Appendix B. The explicit computations of the various geometric quantities for the Friedmann–Robertson–Walker geometry are presented in Appendix C.

2 Palatini formulation of $f(R, T)$ gravity

In the present section, after a brief review of the metric formulation of $f(R, T)$ gravity theory, we derive the field equations of the theory by using the Palatini formalism.

2.1 The metric formalism

The $f(R, T)$ gravity theory is described by the action [39][1]

$$S = \int \left[\frac{\sqrt{-g}}{16\pi} f(R, T) + \sqrt{-g} L_{\mathrm{m}} \right] \mathrm{d}^4 x, \tag{1}$$

where $g \equiv \det(g_{\mu\nu})$, f is an arbitrary function of the Ricci scalar $R = R(g)$ and of the trace $T = g^{\mu\nu} T_{\mu\nu}$ of the matter energy-momentum tensor $T_{\mu\nu}$; the matter Lagrangian L_{m} is assumed to be independent of $\partial_\lambda g^{\mu\nu}$. $T_{\mu\nu}$ is generally obtained as [89]

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \left\{ \frac{\partial \left(\sqrt{-g} L_{\mathrm{m}} \right)}{\partial g^{\mu\nu}} + \partial_\lambda \left[\frac{\partial \left(\sqrt{-g} L_{\mathrm{m}} \right)}{\partial \left(\partial_\lambda g^{\mu\nu} \right)} \right] \right\}$$
$$= \frac{-2}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} L_{\mathrm{m}} \right)}{\partial g^{\mu\nu}} = -2 \frac{\partial L_{\mathrm{m}}}{\partial g^{\mu\nu}} + g_{\mu\nu} L_{\mathrm{m}}. \tag{2}$$

To describe the variation of the energy-momentum tensor with respect to the metric, we also introduce the tensor $\Theta_{\mu\nu}$, defined as

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}. \tag{3}$$

[1] Throughout this article we use the natural system of units with $c = G = 1$. For the metric tensor we adopt the signature convention $(-, +, +, +)$.

For a perfect fluid characterized by its energy density ρ and isotropic pressure P only, the energy-momentum tensor is given by

$$T_\nu^\mu = (\rho + P) u^\mu u_\nu + P \delta_\nu^\mu, \tag{4}$$

where the four-velocity u^μ satisfies the normalization condition $u^\mu u_\mu = -1$. In the comoving frame its components are $u^\mu = (-1, 0, 0, 0)$, and in this frame the components of the energy-momentum tensor become $T_\nu^\mu = (-\rho, P, P, P)$.

The components of the affine connection are defined to be

$$\Gamma_{\mu\nu}^\rho(g) = \frac{g^{\rho\sigma}}{2} \left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu} \right), \tag{5}$$

which are currently regarded as functions of the metric. In the following we assume that the connection is symmetric, that is, $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$. Varying Eq. (1) with respect to $g^{\mu\nu}$, we obtain

$$\delta S = \int \frac{\sqrt{-g}}{16\pi} \Big[f_R \delta R(g) + f_T \delta T - \frac{g_{\mu\nu}}{2} f \delta g^{\mu\nu}$$
$$- 8\pi T_{\mu\nu} \delta g^{\mu\nu} \Big] \mathrm{d}^4 x, \tag{6}$$

where $f_R \equiv \frac{\partial f}{\partial R}$, $f_T \equiv \frac{\partial f}{\partial T}$ and

$$\delta R(g) = R_{\mu\nu}(g) \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}(g). \tag{7}$$

From the condition $\delta S = 0$ we obtain the field equations of $f(R(g), T)$ gravity theory as (for the computational details see Appendix A),

$$\left[R_{\mu\nu}(g) + g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right] f_R + \left(T_{\mu\nu} + \Theta_{\mu\nu} \right) f_T$$
$$- \frac{g_{\mu\nu}}{2} f = 8\pi T_{\mu\nu}. \tag{8}$$

The field equations (8) can be rewritten with the help of the Einstein tensor $G_{\mu\nu}(g) = R_{\mu\nu}(g) - g_{\mu\nu} R(g)/2$ as

$$G_\nu^\mu(g) = \frac{1}{f_R} \Bigg\{ 8\pi T_\nu^\mu - \left(T_\nu^\mu + \Theta_\nu^\mu \right) f_T$$
$$+ \frac{\delta_\nu^\mu}{2} \left[f - R(g) f_R \right] + \left(\nabla^\mu \nabla_\nu - \delta_\nu^\mu \Box \right) f_R \Bigg\}. \tag{9}$$

From Eq. (9) it follows that the matter energy-momentum tensor is not conserved, and its divergence is given by [52] (for the computational details see Appendix B),

$$\nabla_\mu T_\nu^\mu = \frac{f_T}{8\pi - f_T} \Bigg[\left(T_\nu^\mu + \Theta_\nu^\mu \right) \nabla_\mu \ln |f_T| + \nabla_\mu \Theta_\nu^\mu$$
$$- \frac{1}{2} \nabla_\nu T \Bigg] \equiv Q_\nu. \tag{10}$$

2.2 Palatini formulation of $f(R, T)$ gravity

2.2.1 Field equations from metric variation

An alternative formulation of gravitational theories can be obtained within the Palatini formalism, which *consists in taking separately in the gravitational action two independent variations, with respect to the metric and the connection*, respectively. The action is *formally identical to the metric one*, but the *Riemann tensor and the Ricci tensor are constructed with the independent symmetric connection* $\tilde{\Gamma}$. Hence in the Palatini formulation the gravitational action of $f(R, T)$ gravity is given by

$$S = \int \left[\frac{\sqrt{-g}}{16\pi} f\left(R\left(g, \tilde{\Gamma} \right), T \right) + \sqrt{-g} L_{\mathrm{m}}\left(g, \psi \right) \right] \mathrm{d}^4 x. \tag{11}$$

In Eq. (11) the Ricci scalar is defined as

$$R\left(g, \tilde{\Gamma} \right) = g^{\mu\nu} \tilde{R}_{\mu\nu}\left(\tilde{\Gamma} \right), \tag{12}$$

with the Ricci tensor $\tilde{R}_{\mu\nu}\left(\tilde{\Gamma} \right)$ expressed only in terms of the Palatini connection $\tilde{\Gamma}$, with the connection coefficients $\tilde{\Gamma}^\lambda_{\mu\nu}$ *determined self-consistently through the independent variation of the gravitational field action* Eq. (11), and *not constructed directly from the metric by using the usual Levi-Civita definition*. In the following we define $\tilde{R}_{\mu\nu}\left(\tilde{\Gamma} \right)$ with the help of the yet undetermined Palatini connection as

$$\tilde{R}_{\mu\nu}\left(\tilde{\Gamma} \right) = \partial_\lambda \tilde{\Gamma}^\lambda_{\mu\nu} - \partial_\nu \tilde{\Gamma}^\lambda_{\mu\lambda} + \tilde{\Gamma}^\lambda_{\mu\nu} \tilde{\Gamma}^\alpha_{\lambda\alpha} - \tilde{\Gamma}^\alpha_{\mu\lambda} \tilde{\Gamma}^\lambda_{\nu\alpha}. \tag{13}$$

The matter Lagrangian $L_{\mathrm{m}}\left(g, \psi \right)$ is assumed to be a function of the metric tensor g and of the physical fields ψ only.

We vary now the gravitational action (11) with respect to the metric tensor $g^{\mu\nu}$, *under the assumption* $\delta \tilde{R}_{\mu\nu}\left(\tilde{\Gamma} \right) = 0$, that is, *by keeping the connection constant*. As a result we immediately obtain the field equations

$$\tilde{R}_{\mu\nu}\left(\tilde{\Gamma} \right) f_R = 8\pi T_{\mu\nu} - \left(T_{\mu\nu} + \Theta_{\mu\nu} \right) f_T + \frac{g_{\mu\nu}}{2} f. \tag{14}$$

By contracting the above equation with $g^{\mu\nu}$ we obtain for the Ricci scalar (12) the expression

$$R\left(g, \tilde{\Gamma} \right) f_R = 8\pi T - (T + \Theta) f_T + 2f, \tag{15}$$

where $\Theta = \Theta^\mu_\mu$. In terms of the Einstein tensor

$$G_{\mu\nu}\left(g, \tilde{\Gamma} \right) = \tilde{R}_{\mu\nu}\left(\tilde{\Gamma} \right) - \frac{1}{2} g_{\mu\nu} R\left(g, \tilde{\Gamma} \right), \tag{16}$$

the Palatini field equations can be written as

$$G_{\mu\nu}\left(g, \tilde{\Gamma} \right) = \frac{1}{f_R} \left\{ 8\pi \left(T_{\mu\nu} - \frac{g_{\mu\nu}}{2} T \right) \right.$$
$$\left. - \left[\left(T_{\mu\nu} + \Theta_{\mu\nu} \right) - \frac{g_{\mu\nu}}{2} \left(T + \Theta \right) \right] f_T - \frac{g_{\mu\nu}}{2} f \right\}. \tag{17}$$

2.2.2 The Palatini connection

We vary now the gravitational action (11) with respect to the connection $\tilde{\Gamma}$, *by keeping the metric constant*, so that

$$\delta R\left(g, \tilde{\Gamma} \right) = g^{\mu\nu} \delta \tilde{R}_{\mu\nu}\left(\tilde{\Gamma} \right). \tag{18}$$

According to the Palatini identity [82], we have

$$\delta \tilde{R}_{\mu\nu}\left(\tilde{\Gamma} \right) = \tilde{\nabla}_\lambda \left(\delta \tilde{\Gamma}^\lambda_{\mu\nu} \right) - \tilde{\nabla}_\nu \left(\delta \tilde{\Gamma}^\lambda_{\mu\lambda} \right), \tag{19}$$

where $\tilde{\nabla}_\lambda$ is the covariant derivative associated with $\tilde{\Gamma}$. Hence the variation of the action (11) with respect to $\tilde{\Gamma}$ leads to

$$\delta S = \int \frac{\sqrt{-g}}{16\pi} A^{\mu\nu} \left[\tilde{\nabla}_\lambda \left(\delta \tilde{\Gamma}^\lambda_{\mu\nu} \right) - \tilde{\nabla}_\nu \left(\delta \tilde{\Gamma}^\lambda_{\mu\lambda} \right) \right] \mathrm{d}^4 x, \tag{20}$$

where we have denoted

$$A^{\mu\nu} = f_R g^{\mu\nu}. \tag{21}$$

We integrate now by parts to obtain

$$16\pi \delta S = \int \tilde{\nabla}_\lambda \left[\sqrt{-g} \left(A^{\mu\nu} \delta \tilde{\Gamma}^\lambda_{\mu\nu} - A^{\mu\lambda} \delta \tilde{\Gamma}^\alpha_{\mu\alpha} \right) \right] \mathrm{d}^4 x$$
$$- \int \tilde{\nabla}_\lambda \left[\sqrt{-g} \left(A^{\mu\nu} \delta^\lambda_\alpha - A^{\mu\lambda} \delta^\nu_\alpha \right) \right] \delta \tilde{\Gamma}^\alpha_{\mu\nu} \, \mathrm{d}^4 x. \tag{22}$$

The first term in Eq. (22) is a total derivative, and thus after transforming it into a surface integral it vanishes. Therefore the variation of the action with respect to the connection $\tilde{\Gamma}$ becomes

$$\tilde{\nabla}_\lambda \left[\sqrt{-g} \left(A^{\mu\nu} \delta^\lambda_\alpha - A^{\mu\lambda} \delta^\nu_\alpha \right) \right] = 0. \tag{23}$$

Equation (23) can be significantly simplified by taking into account that for $\alpha = \lambda$ the equation is identically zero. Hence for the case $\lambda \neq \alpha$, we find

$$\tilde{\nabla}_\lambda \left(\sqrt{-g} f_R g^{\mu\nu} \right) = 0. \tag{24}$$

Equation (24) shows that the connection $\tilde{\Gamma}$ *is compatible with a conformal metric* $\tilde{g}_{\mu\nu}$, conformally related to the initial metric $g_{\mu\nu}$ by means of the relations

$$\tilde{g}_{\mu\nu} \equiv f_R g_{\mu\nu} = F g_{\mu\nu}, \tag{25}$$

with the conformal factor F defined as

$$F \equiv f_R \left(R \left(g, \tilde{\Gamma} \right), T \right). \tag{26}$$

Moreover, we have

$$\sqrt{-\tilde{g}}\, \tilde{g}^{\mu\nu} = \sqrt{-g}\, f_R g^{\mu\nu} = \sqrt{-F^4 g} \left(F^{-1} g^{\mu\nu} \right), \tag{27}$$

where $\tilde{g} \equiv \det \left(\tilde{g}_{\mu\nu} \right)$. Thus Eq. (24) gives *the geometric interpretation of the Palatini connection* $\tilde{\Gamma}$ as the Levi-Civita connection corresponding to the metric $\tilde{g}_{\mu\nu}$, conformally related to $g_{\mu\nu}$,

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} \tilde{g}^{\lambda\rho} \left(\partial_\nu \tilde{g}_{\rho\mu} + \partial_\mu \tilde{g}_{\rho\nu} - \partial_\rho \tilde{g}_{\mu\nu} \right). \tag{28}$$

2.2.3 Conformal geometry and g frame field equations

Since the metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are conformally related, the connection $\tilde{\Gamma}$ can be expressed in terms of the Levi-Civita connection Γ, whose components are given in Eq. (5), as

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2F} \left(\delta^\lambda_\mu \partial_\nu + \delta^\lambda_\nu \partial_\mu - g_{\mu\nu} \partial^\lambda \right) F. \tag{29}$$

In terms of the tensor $R_{\mu\nu}(g)$, constructed from the metric by using the Levi-Civita connection (5), the Ricci tensor $\tilde{R}_{\mu\nu}$ in the conformally transformed metric is given by [84,85]

$$\tilde{R}_{\mu\nu} = R_{\mu\nu}(g) + \frac{1}{F} \left[\frac{3}{2F} \nabla_\mu F \nabla_\nu F - \left(\nabla_\mu \nabla_\nu + \frac{g_{\mu\nu}}{2} \Box \right) F \right]. \tag{30}$$

Since $\tilde{R} \equiv \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu} = F^{-1} R \left(g, \tilde{\Gamma} \right)$ and $\tilde{G}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \tilde{g}_{\mu\nu} \tilde{R}/2 = G_{\mu\nu} \left(g, \tilde{\Gamma} \right)$, the Ricci scalar (12) and the Einstein tensor in Eq. (17) can be obtained in the conformally related frames [84,85]:

$$R \left(g, \tilde{\Gamma} \right) = F \tilde{R} = R(g) + \frac{3}{F} \left[\frac{1}{2F} (\nabla F)^2 - \Box F \right] \tag{31}$$

and

$$G_{\mu\nu} \left(g, \tilde{\Gamma} \right) = \tilde{G}_{\mu\nu} = G_{\mu\nu}(g) + \frac{1}{F} \left\{ \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) F \right. \\ \left. + \frac{3}{2F} \left[\nabla_\mu F \, \nabla_\nu F - \frac{g_{\mu\nu}}{2} (\nabla F)^2 \right] \right\}, \tag{32}$$

respectively, *with all the covariant derivatives and algebraic operations performed with the help of the metric* $g_{\mu\nu}$.

By using the expression of the Einstein tensor as given by Eq. (32) in Eq. (17), we obtain finally the $f(R, T)$ field equations of the Palatini formulation *expressed solely in the g frame*:

$$G_{\mu\nu}(g) + \frac{1}{F} \left\{ \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) F + \frac{3}{2F} \left[\nabla_\mu F \, \nabla_\nu F \right. \right. \\ \left. \left. - \frac{g_{\mu\nu}}{2} (\nabla F)^2 \right] \right\} = \frac{1}{F} \left\{ 8\pi \left(T_{\mu\nu} - \frac{g_{\mu\nu}}{2} T \right) \right. \\ \left. - \left[(T_{\mu\nu} + \Theta_{\mu\nu}) - \frac{g_{\mu\nu}}{2} (T + \Theta) \right] f_T - \frac{g_{\mu\nu}}{2} f \right\}. \tag{33}$$

The Ricci scalar can be obtained:

$$R(g) + \frac{3}{F} \left[\frac{1}{2F} (\nabla F)^2 - \Box F \right] \\ = \frac{1}{F} \left[8\pi T - (T + \Theta) f_T + 2f \right]. \tag{34}$$

2.2.4 Field equations in the $\tilde{g}$ frame

In the previous subsection we have obtained the gravitational field equations in the Palatini formulation of $f(R, T)$ gravity as expressed in terms of the metric tensor $g_{\mu\nu}$. This approach usually involves higher-order differential equations for the physical and geometrical quantities. An alternative approach, which keeps the order of the differential equations of the model not higher than two would be to solve first the gravitational field equations in the conformal metric $\tilde{g}_{\mu\nu}$, and to recover the metric $g_{\mu\nu}$ with the use of the conformal transformation (25). To obtain the field equations in the conformal frame we follow the procedure introduced in [83].

First we multiply the Palatini field equation (14) with $\tilde{g}^{\lambda\mu} = f_R^{-1} g^{\lambda\mu}$, thus obtaining

$$\tilde{R}^\lambda_\nu = \frac{1}{F^2} \left[8\pi T^\lambda_\nu - \left(T^\lambda_\nu + \Theta^\lambda_\nu \right) f_T + \frac{\delta^\lambda_\nu}{2} f \right], \tag{35}$$

where $\tilde{R}^\lambda_\nu$ is now a function of the metric $\tilde{g}$ only. As for the energy-momentum tensor, we have, respectively,

$$T_{\mu\nu} \to \tilde{T}_{\mu\nu} = \frac{-2}{\sqrt{-\tilde{g}}} \frac{\partial \left(\sqrt{-\tilde{g}} L_m \right)}{\partial \tilde{g}^{\mu\nu}} = -2 \frac{\partial L_m}{\partial \tilde{g}^{\mu\nu}} + \tilde{g}_{\mu\nu} L_m \tag{36}$$

and

$$T^\mu_\nu \to \tilde{T}^\mu_\nu = -2 \tilde{g}^{\mu\lambda} \frac{\partial L_m}{\partial \tilde{g}^{\lambda\nu}} + \delta^\mu_\nu L_m = T^\mu_\nu. \tag{37}$$

That is, the mixed components T^μ_ν of the energy-momentum tensor are conformally invariant. Additionally, we have

$$\Theta^\mu_\nu \to \tilde{\Theta}^\mu_\nu = \Theta^\mu_\nu, \quad T \to \tilde{T} = T, \quad \Theta \to \tilde{\Theta} = \Theta. \tag{38}$$

By contracting Eq. (35) by taking $\nu = \lambda$ we obtain the expression of the Ricci scalar in the conformal frame, thus

$$\tilde{R} = \frac{1}{F^2} \left[8\pi T - (T + \Theta) f_T + 2f \right]. \tag{39}$$

Therefore in the conformal frame the full set of the Einstein equations can be written with $f = f \left(F\tilde{R} = R \left(g, \tilde{\Gamma} \right), T \right)$ as

$$\tilde{G}^\mu_\nu = \tilde{R}^\mu_\nu - \frac{\delta^\mu_\nu}{2} \tilde{R} = \frac{1}{F^2} \left\{ 8\pi \left(T^\mu_\nu - \frac{\delta^\mu_\nu}{2} T \right) \right.$$
$$\left. - \left[\left(T^\mu_\nu + \Theta^\mu_\nu \right) - \frac{\delta^\mu_\nu}{2} (T + \Theta) \right] f_T - \frac{\delta^\mu_\nu}{2} f \right\}. \tag{40}$$

These equations determine the conformal metric $\tilde{g}$ as a function of the thermodynamic parameters that enter in the definition of the matter energy-momentum tensor.

2.3 The Newtonian limit

To investigate the Palatini $f(R, T)$ gravity under the weak field, slow motion and static approximation, namely the Newtonian limit, we assume the metric to be a Minkowski metric plus a perturbation, given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \quad \gamma_{\mu\nu} \ll 1. \tag{41}$$

Hence $g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu}$ and the relation $g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta$ still holds. In this context, we also assume the conformal metric (25) to be nearly flat, so that

$$\tilde{g}_{\mu\nu} = F g_{\mu\nu} = \eta_{\mu\nu} + \tilde{\gamma}_{\mu\nu}, \tag{42}$$

where $\tilde{\gamma}_{\mu\nu} \ll 1$ is of the same order as $\gamma_{\mu\nu}$. Hence $F \approx 1$. From Eqs. (41) and (42) we obtain

$$(F - 1)\eta_{\mu\nu} = \tilde{\gamma}_{\mu\nu} - F\gamma_{\mu\nu}. \tag{43}$$

If we take $F = e^{2W}$ and expand it to $F = 1 + 2W$, then the above equation shows that $W = (\tilde{\gamma} - \gamma)/2(4 + \gamma)$, where $\tilde{\gamma} \equiv \eta^{\mu\nu} \tilde{\gamma}_{\mu\nu}$ and $\gamma \equiv \eta^{\mu\nu} \gamma_{\mu\nu}$. Thus $W \sim O(\gamma) \sim O(\gamma_{\mu\nu})$.

Let us now consider the Palatini field equation (14) under the Newtonian limit. First we obtain the g-frame Ricci tensor:

$$R_{\mu\nu}(g) = \frac{1}{2} \left(\partial^\lambda \partial_\mu \gamma_{\nu\lambda} + \partial^\lambda \partial_\nu \gamma_{\mu\lambda} - \partial^2 \gamma_{\mu\nu} - \partial_{\mu\nu} \gamma \right). \tag{44}$$

Omitting all higher-order terms with respect to $O(\gamma_{\mu\nu})$ and taking into account the gauge

$$\partial^\mu \left(\gamma_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \gamma \right) = 0, \tag{45}$$

then some algebra gives the expression of the Ricci tensor (30) as

$$\tilde{R}_{\mu\nu} = -\frac{1}{2} \partial^2 \left(\gamma_{\mu\nu} + 2\eta_{\mu\nu} w \right) - 2\partial_{\mu\nu} w. \tag{46}$$

With the use of the above equations, and preserving only the first-order terms, the field equation (14) becomes

$$-\frac{1}{2} \partial^2 \tilde{\gamma}_{\mu\nu} - 2\partial_{\mu\nu} w - \frac{\eta_{\mu\nu}}{2} f = 8\pi T_{\mu\nu} - \left(T_{\mu\nu} + \Theta_{\mu\nu} \right) f_T. \tag{47}$$

For perfect fluids, $T^\mu_\nu = \mathrm{diag}(-\rho, P, P, P)$ and $\Theta^\mu_\nu = \delta^\mu_\nu P - 2T^\mu_\nu$ (see Eq. (61)); besides, in the Newtonian limit, $P \to 0$ and $\partial_0 \to 0$. Hence we can obtain immediately the generalized Poisson equation in the Palatini formulation of $f(R, T)$ gravity:

$$-\frac{1}{2} \vec{\nabla}^2 \tilde{\gamma}_{00} = (8\pi + f_T) \rho - \frac{f}{2}. \tag{48}$$

The same result can also be found in [83].

2.4 Violation of the equivalence principle

An interesting feature of the modified gravities, including their Palatini extensions, is the violation of the equivalence principle. In Palatini $f(R)$ gravity this problem was discussed in [86]. In the following we will generalize some results from the case of $f(R)$ theory to the Palatini $f(R, T)$ gravity. In the conformal frame the field equations of the Palatini $f(R, T)$ gravity are given by Eq. (40), where $f = f \left(F\tilde{R}, T \right) = f \left(R \left(g, \tilde{\Gamma} \right), T \right)$.

In the weak field limit we can represent the gravitational Lagrangian as

$$f \left(R \left(g, \tilde{\Gamma} \right), T \right) = R(g, \tilde{\Gamma}) + \epsilon K \left(R \left(g, \tilde{\Gamma} \right) \right) + \epsilon' g(T), \tag{49}$$

where ϵ, ϵ' are constants, $K \left(R(g, \tilde{\Gamma}) \right)$ is an arbitrary function of the argument $R(g, \tilde{\Gamma})$, while g is an arbitrary function of the trace of the matter energy-momentum tensor. In the limit of small ϵ, ϵ', with $\epsilon, \epsilon' \to 0$, it follows that $f - R(g, \tilde{\Gamma})F \approx 0$. In addition, by neglecting the matter energy-momentum tensor T^μ_ν for weak sources, it follows that Eq. (40) becomes $\tilde{G}^\mu_\nu \approx 0$, which leads to $\tilde{g}_{\mu\nu} \approx \eta_{\mu\nu}$, or, equivalently,

$$g_{\mu\nu} \approx F^{-1} \eta_{\mu\nu} \approx \left(1 - \epsilon \frac{\partial K}{\partial R} \right) \eta_{\mu\nu}. \tag{50}$$

The above equation tells us that, similar to the metric and the Palatini formulation of $f(R)$ gravity [86], *in the Palatini formulation of $f(R,T)$ gravity it is impossible to recover the flat Minkowski metric even in local frames with external gravitational fields screened.* This result violates the basic postulate of general relativity according to which *in locally free falling frames the non-gravitational laws of physics are those of special relativity* [86]. Since this postulate assumes that the Einstein equivalence principle holds, it follows that, similar to Palatini $f(R)$ theory, in the Palatini formulation of $f(R,T)$ gravity the equivalence principle does not hold exactly. The deviation of the current metric with respect to the Minkowski metric is $\eta_{\mu\nu} - g_{\mu\nu} \approx \epsilon (\partial K/\partial R) \eta_{\mu\nu}$.

In order to give a quantitative estimate for the deviation of the $f(R,T)$ metric from the Minkowski metric we will consider the cosmological case, to be discussed in detail in Sect. 4. We assume that the gravitational action takes the form $f(R(g,\tilde{\Gamma}), T) = R(g,\tilde{\Gamma}) + \frac{\alpha}{16\pi} R^2(g,\tilde{\Gamma}) + 8\pi\beta T$, where $\alpha, \beta \to 0$, as an example. From Eq. (141), to be derived in Sect. 4, it follows that $F = 1 + \beta_0 \alpha\rho = 1 + \beta_0 \alpha\rho_0 a^{-\beta_1}$, where ρ_0 is the present-day matter density, a is the scale factor, $\beta_0 = 1 - 3w + \beta(3-5w)$, $\beta_1 = 3(1+\beta)(1+w)/[1+\beta(3-w)/2]$, and w is the parameter of the matter equation of state. By estimating all quantities at the present time $t = t_0$, then when $\alpha, \beta \to 0$, and $w = 0$, $a(t_0) = 1$, the deviation $\epsilon (\partial K/\partial R)$ from the Minkowki metric can be obtained:

$$1 - F^{-1} = 1 - [1 + \beta_0|_{t=t_0} \alpha\rho_0 a_0^{-\beta_1|_{t=t_0}}]^{-1}$$
$$\approx (1 + 3\beta) \alpha\rho_0 \approx \alpha\rho_0 \approx \alpha \frac{3H_0^2}{8\pi}, \quad (51)$$

where H_0 is the present-day value of the Hubble function, and we have assumed for the present-day density of the Universe the critical value.

Recently the first results obtained by the MICROSCOPE satellite, whose aims are to constrain the weak equivalence principle in outer space by determining the Eötvös parameter η, have been published [87]. The Eötvös parameter is defined as the normalized difference of accelerations between two bodies i and j, located in the same gravitational field. The MICROSCOPE determinations give for η the value $\eta = (-1 \pm 27) \times 10^{-15}$ at a 2σ confidence level [87]. These results allow one to constrain possible sources of violation of the weak equivalence principle, like, for example, the existence of light or massive scalar fields with coupling to matter weaker than the gravitational coupling [88]. For a massive scalar field of mass smaller than 10^{-12} eV, the coupling is constrained as $|\alpha_C| < 10^{-11}$, if the scalar field couples to the baryon number, and to $|\alpha_C| < 10^{-12}$ if the scalar field couples to the difference between the baryon and the lepton numbers, respectively. We expect a similar order of magnitude for the coupling between matter and geometry in both the metric and the Palatini formulations of $f(R,T)$ gravity.

3 Energy and momentum balance equations

An interesting and important consequence of modified gravity theories with geometry–matter coupling is the non-conservation of the matter energy-momentum tensor. This property of the theory has a number of far reaching physical implications, and it may represent the main link between the interpretation of $f(R,T)$ theory as an effective classical description of the quantum theory of gravity. In the present section we obtain the general expression of the divergence of the energy-momentum tensor in $f(R,T)$ gravity theory, and, by using it, we obtain the energy-momentum balance equations, which describe the energy transfer processes from geometry to matter, and the deviations from the geodesic motion, respectively.

3.1 The divergence of the matter energy-momentum tensor

We begin our analysis by calculating first the divergence of the Einstein tensor in the Palatini frame. Since $\nabla^\mu G_{\mu\nu}(g) = 0$, Eq. (32) yields

$$\nabla^\mu \tilde{G}_{\mu\nu} = \nabla^\mu \left[\tilde{G}_{\mu\nu} - G_{\mu\nu}(g) \right]$$
$$= 2 (\nabla_\nu \Box - \Box \nabla_\nu) w + 2\Box w \nabla_\nu w + 2\nabla^\mu w \nabla_\mu \nabla_\nu w$$
$$+ \nabla_\nu (\nabla w)^2$$
$$= -2R_{\mu\nu}(g)\nabla^\mu w + 2\Box w \nabla_\nu w + 4\nabla^\mu w \nabla_\mu \nabla_\nu w, \quad (52)$$

where to simplify the calculation we have taken $F = e^{2w}$, and we have used the mathematical identities [84]

$$(\nabla_\nu \Box - \Box \nabla_\nu) \phi = g^{\alpha\beta} \left(\nabla_\nu \nabla_\alpha \nabla_\beta - \nabla_\alpha \nabla_\beta \nabla_\nu \right) \phi$$
$$= g^{\alpha\beta} (\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu) \nabla_\beta \phi$$
$$= g^{\alpha\beta} R^\mu_{\beta\alpha\nu} \nabla_\mu \phi$$
$$= -R_{\mu\nu} \nabla^\mu \phi \quad (53)$$

and

$$\nabla_\nu (\nabla\phi)^2 = \nabla_\nu \left(\nabla^\mu \phi \nabla_\mu \phi \right) = \nabla_\mu \phi \nabla_\nu \nabla^\mu \phi$$
$$+ \nabla^\mu \phi \nabla_\nu \nabla_\mu \phi = 2\nabla_\mu \phi \nabla_\nu \nabla^\mu \phi$$
$$= 2\nabla^\mu \phi \nabla_\nu \nabla_\mu \phi, \quad (54)$$

respectively, where $\phi(x^\mu)$ is an arbitrary scalar. We have also used the relation $g^{\alpha\beta} R_{\mu\beta\alpha\nu} = -g^{\alpha\beta} R_{\beta\mu\alpha\nu} = -R_{\mu\nu}$ in the last step of Eq. (53). Note that the above two identities are valid in both the metric and the Palatini formulations [84].

Substituting Eq. (30) into Eq. (52), we find

$$\nabla^\mu \tilde{G}_{\mu\nu} = -2R_{\mu\nu}(g)\nabla^\mu w + 2\Box w \nabla_\nu w + 4\nabla^\mu w \nabla_\mu \nabla_\nu w$$
$$= -2\tilde{R}_{\mu\nu}\nabla^\mu w = -\frac{\nabla^\mu F}{F} \tilde{R}_{\mu\nu}. \quad (55)$$

The covariant divergence of the field equation (17) yields (note that $G_{\mu\nu}\left(g,\tilde{\Gamma}\right) = \tilde{G}_{\mu\nu}$ and $g_{\mu\nu}R\left(g,\tilde{\Gamma}\right) = \tilde{g}_{\mu\nu}\tilde{R}$)

$$
\nabla^\mu \left[F G_{\mu\nu}\left(g,\tilde{\Gamma}\right)\right] + \frac{g_{\mu\nu}}{2}R\left(g,\tilde{\Gamma}\right)\nabla^\mu F
$$
$$
= \left[\tilde{G}_{\mu\nu} - \tilde{R}_{\mu\nu} + \frac{\tilde{g}_{\mu\nu}}{2}\tilde{R}\right]\nabla^\mu F = 0
$$
$$
= \nabla_\mu \left[8\pi T^\mu_\nu - \left(T^\mu_\nu + \Theta^\mu_\nu\right)f_T\right]
$$
$$
+ \frac{\delta^\mu_\nu}{2}\left[\nabla_\mu f - F\nabla_\mu R\left(g,\tilde{\Gamma}\right)\right]
$$
$$
= \nabla_\mu \left[8\pi T^\mu_\nu - \left(T^\mu_\nu + \Theta^\mu_\nu\right)f_T\right] + \frac{f_T}{2}\nabla_\nu T, \quad (56)
$$

where we have used Eq. (55).

One can check now by comparison with Eq. (10) that the above expression gives the same result as in the metric formulation, except for the functional form of the function f,

$$
\nabla_\mu T^\mu_\nu = \frac{f_T}{8\pi - f_T}\left[\left(T^\mu_\nu + \Theta^\mu_\nu\right)\nabla_\mu \ln|f_T| + \nabla_\mu \Theta^\mu_\nu \right.
$$
$$
\left. - \frac{1}{2}\nabla_\nu T\right], \quad f = f\left(R\left(g,\tilde{\Gamma}\right),T\right). \quad (57)
$$

From its definition with the use of Eq. (2), the tensor $\Theta_{\mu\nu}$ for perfect fluids can be obtained:

$$
\Theta_{\mu\nu} = g^{\alpha\beta}\left[\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}}L_m + g_{\alpha\beta}\frac{\partial L_m}{\partial g^{\mu\nu}} - 2\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}\right]
$$
$$
= g_{\mu\nu}L_m - 2T_{\mu\nu} - 2g^{\alpha\beta}\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}, \quad (58)
$$

where the relation

$$
\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} = -g_{\alpha\mu}g_{\beta\nu} \quad (59)
$$

can be derived from the relations

$$
g^{\alpha\beta}\left(\delta g_{\alpha\beta} + g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}\right) = g^{\alpha\beta}\delta g_{\alpha\beta} + g_{\mu\nu}\delta g^{\mu\nu} = 0,
$$
$$
\Rightarrow \delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}. \quad (60)
$$

For perfect fluids we fix L_m to be P [52], while T^μ_ν takes the form of (4), then

$$
\Theta_{\mu\nu} = g_{\mu\nu}P - 2T_{\mu\nu}, \quad \Theta^\mu_\nu = \delta^\mu_\nu P - 2T^\mu_\nu; \quad (61)
$$
$$
T = \delta^\nu_\mu T^\mu_\nu = -\rho + 3P, \quad \Theta \equiv \delta^\nu_\mu \Theta^\mu_\nu = 4P - 2T. \quad (62)
$$

Hence

$$
T_{\mu\nu} + \Theta_{\mu\nu} = g_{\mu\nu}P - T_{\mu\nu} \quad (63)
$$

and

$$
T + \Theta = 4P - T = \rho + P. \quad (64)
$$

Multiplying Eq. (57) by u^ν [46] we obtain the $f(R,T)$ perfect-fluid energy balance equation,

$$
\dot\rho + 3(\rho + P)H = \frac{-f_T}{8\pi + f_T}\left[(\rho + P)u^\mu \nabla_\mu \ln|f_T|\right.
$$
$$
\left. + u^\mu \nabla_\mu \left(\frac{\rho - P}{2}\right)\right], \quad f = f\left(R\left(g,\tilde{\Gamma}\right),T\right), \quad (65)
$$

where we have denoted $H = \left(\nabla_\mu u^\mu\right)/3$, and $\dot{} = d/ds = u^\mu \nabla_\mu$.

Multiplying (57) by the projection operator h^ν_λ, defined as $h^\nu_\lambda \equiv \delta^\nu_\lambda + u^\nu u_\lambda$ [46], with the properties $u_\nu h^\nu_\lambda = 0$, $h^\nu_\lambda \nabla_\mu u_\nu = \nabla_\mu u_\lambda$, and $h^{\nu\lambda}\nabla_\nu = \left(g^{\nu\lambda} + u^\nu u^\lambda\right)\nabla_\nu = \nabla^\lambda + u^\lambda u^\nu \nabla_\nu$, respectively, we obtain the (non-geodesic) equation of motion of massive test particles:

$$
u^\nu \nabla_\nu u^\lambda = \frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}u^\mu u^\nu = \frac{-h^{\nu\lambda}\nabla_\nu P + h^{\nu\lambda}Q_\nu}{\rho + P}, \quad (66)
$$

where

$$
h^{\nu\lambda}Q_\nu = \frac{f_T/2}{8\pi + f_T}h^{\nu\lambda}\nabla_\nu(\rho - P). \quad (67)
$$

3.2 Balance equations in the conformal frame

In the following for notational simplicity we define first a mixed-component vector field,

$$
V^\mu_\nu \equiv 8\pi\left(T^\mu_\nu - \frac{\delta^\mu_\nu}{2}T\right) - \frac{\delta^\mu_\nu}{2}f
$$
$$
- \left[\left(T^\mu_\nu + \Theta^\mu_\nu\right) - \frac{\delta^\mu_\nu}{2}(T + \Theta)\right]f_T. \quad (68)
$$

Since $\tilde\nabla_\mu \tilde{G}^\mu_\nu \equiv 0$, from Eq. (40) we at once get the conservation equations in the conformal frame:

$$
\tilde\nabla_\mu V^\mu_\nu - 2V^\mu_\nu \tilde\nabla_\mu \ln|F|
$$
$$
= 2F\tilde{G}^\mu_\nu \tilde\nabla_\mu F - 2V^\mu_\nu \tilde\nabla_\mu \ln|F| = 0
$$
$$
= \nabla_\mu V^\mu_\nu + \left(\tilde\Gamma^\mu_{\lambda\mu} - \Gamma^\mu_{\lambda\mu}\right)V^\lambda_\nu - \left(\tilde\Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\nu\mu}\right)V^\mu_\lambda
$$
$$
- 2V^\mu_\nu \tilde\nabla_\mu \ln|F|
$$
$$
= \nabla_\mu V^\mu_\nu + 2V^\mu_\nu \left(\partial_\mu - \tilde\nabla_\mu\right)\ln|F| - \frac{V}{2}\partial_\nu \ln|F|, \quad (69)
$$

where $V \equiv \delta^\nu_\mu V^\mu_\nu = -\left[8\pi T - (T + \Theta)f_T + 2f\right] = -FR\left(g,\tilde{\Gamma}\right)$ according to Eq. (15), and we have used Eq. (28). Taking into account that F is a scalar and $\tilde\nabla_\mu F = \nabla_\mu F = \partial_\mu F$, then

$$
\nabla_\mu V^\mu_\nu = \frac{V}{2}\partial_\nu \ln|F|, \quad (70)
$$

or equivalently

$$\nabla_\mu \left[8\pi T_\nu^\mu - \left(T_\nu^\mu + \Theta_\nu^\mu \right) f_T \right] - \frac{\nabla_\nu}{2} \left[FR \left(g, \tilde{\Gamma} \right) - f \right]$$

$$= -\frac{R \left(g, \tilde{\Gamma} \right) \partial_\nu F}{2}, \tag{71}$$

where we have used the relation $8\pi T - (T + \Theta) f_T + f = FR \left(g, \tilde{\Gamma} \right) - f$ on the left hand side. Hence one can easily see that the equations above are exactly the same as the energy balance equation (57).

3.3 Thermodynamic interpretation of $f(R, T)$ gravity theories

For the sake of completeness we briefly present the thermodynamic interpretation of $f(R, T)$ gravity theories, as discussed in [46]. The non-conservation of the matter energy-momentum tensor strongly suggests that, due to the matter–geometry coupling, a particle creation processes may take place during the cosmological evolution. This phenomenon is also specific to quantum field theories in curved space-times, as pointed out in [90–92], and it is a consequence of a time varying gravitational field. Hence, $f(R, T)$ theory, which also involves particle creation, may lead to the possibility of a semiclassical description of quantum field theoretical processes in gravitational fields.

3.3.1 Particle and entropy fluxes, and the creation pressure

Particle creation implies that the covariant divergence of the basic equilibrium quantities, including the particle and entropy fluxes, as well as of the energy-momentum tensor, are now different from zero. Consequently, all the balance equilibrium equations must be modified to include particle creation [93–95]. In the presence of gravitationally generated matter, the balance equation for the particle flux $N^\mu \equiv nu^\mu$, where n is the particle number density, becomes

$$\nabla_\mu N^\mu = \dot{n} + 3Hn = n\Psi, \tag{72}$$

where Ψ is the particle production rate, which can be neglected in the case that $\Psi \ll H$. The entropy flux vector is defined to be $S^\mu \equiv su^\mu = n\sigma u^\mu$, where s is the entropy density, and σ is the entropy per particle. The divergence of the entropy flux gives

$$\nabla_\mu S^\mu = n\dot{\sigma} + n\sigma\Psi \geq 0. \tag{73}$$

If we consider a specific σ which is constant, then

$$\nabla_\mu S^\mu = n\sigma\Psi = s\Psi \geq 0, \tag{74}$$

that is, the variation of the entropy is entirely due to (adiabatic gravitational) particle creation processes. Since $s > 0$, from the above equation it follows that the particle creation rate must satisfy the condition $\Psi \geq 0$, that is, gravitational fields can generate particles, but the inverse process is prohibited. The energy-momentum tensor of a fluid in the presence of particle creation must also be modified to take into account the second law of thermodynamics, so that [96]

$$T^{\mu\nu} = T_{\text{eq}}^{\mu\nu} + \Delta T^{\mu\nu}, \tag{75}$$

where $T_{\text{eq}}^{\mu\nu}$ denotes the equilibrium component (4), and $\Delta T^{\mu\nu}$ is the correction due to particle creation. Due to the isotropy and homogeneity of space-time, the extra contribution to the equilibrium energy-momentum tensor must be represented by a scalar process. Generally one can write

$$\Delta T_0^0 = 0, \quad \Delta T_i^j = -P_c \delta_i^j, \tag{76}$$

where P_c is the dynamic *creation pressure* that describes phenomenologically the thermodynamic effect of particle creation in a macroscopic system. In a covariant representation we have [96]

$$\Delta T^{\mu\nu} = -P_c h^{\mu\nu} = -P_c \left(g^{\mu\nu} + u^\mu u^\nu \right), \tag{77}$$

which immediately gives $u_\mu \nabla_\nu \Delta T^{\mu\nu} = 3H P_c$. Therefore in the presence of particle creation the total energy balance equation $u_\mu \nabla_\nu T^{\mu\nu} = 0$, which follows from Eq. (75), immediately gives

$$\dot{\rho} + 3H \left(\rho + P + P_c \right) = 0. \tag{78}$$

The thermodynamic quantities must also satisfy the Gibbs law, which can be formulated as [94]

$$n\mathcal{T} d \left(\frac{s}{n} \right) = n\mathcal{T} d\sigma = d\rho - \frac{\rho + p}{n} dn, \tag{79}$$

where $\mathcal{T}$ is the thermodynamic temperature of the system.

3.3.2 Thermodynamic quantities in $f(R, T)$ gravity

After some simple algebraic manipulations the energy balance Eq. (93) can be reformulated as

$$\dot{\rho} + 3H \left(\rho + P + P_c \right) = 0, \tag{80}$$

where the creation pressure P_c is defined as

$$P_c = \frac{-f_T (1+w) \rho}{8\pi + f_T} \left\{ \ln |f_T| + \frac{1-w}{2(1+w)} - \frac{1}{3H} \right.$$
$$\left. \times \left[\nabla_\mu \left(u^\mu \ln |f_T| \right) + \frac{1}{(1+w)\rho} \nabla_\mu \left(u^\mu \frac{(1-w)\rho}{2} \right) \right] \right\},$$
$$(81)$$

where we have denoted $w = P/\rho$. Then the generalized balance equation (93) can be derived from the divergence of the total energy momentum tensor $T^{\mu\nu}$, defined as

$$T^{\mu\nu} = (\rho + P + P_c) u^\mu u^\nu + (P + P_c) g^{\mu\nu}. \qquad (82)$$

On the other hand under the assumption of adiabatic particle production, with $\dot{\sigma} = 0$, the Gibbs law gives

$$\dot{\rho} = (\rho + P) \frac{\dot{n}}{n} = (\rho + P)(\Psi - 3H), \qquad (83)$$

which together with the energy balance equation gives immediately the relation between the particle creation rate and the creation pressure as

$$\Psi = \frac{-3H P_c}{\rho(1+w)}. \qquad (84)$$

In the framework of $f(R, T)$ gravity theory we obtain for the particle creation rate the general expression

$$\Psi = \frac{f_T}{8\pi + f_T} \left\{ 3H \left[\ln |f_T| + \frac{1-w}{2(1+w)} \right] \right.$$
$$\left. - \left[\nabla_\mu \left(u^\mu \ln |f_T| \right) + \frac{1}{(1+w)\rho} \nabla_\mu \left(u^\mu \frac{(1-w)\rho}{2} \right) \right] \right\}. \qquad (85)$$

Hence the condition $\Psi \geq 0$ imposes a strong constraint on the physical parameters of the theory. In the case of pressureless dust, with $P = 0$, $w = 0$, under the assumption $f_T > 0$, we obtain the following general cosmological constraint that must be satisfied by the function f_T for all times:

$$3H \left(\ln |f_T| + \frac{1}{2} \right) \geq \nabla_\mu \left(u^\mu \ln |f_T| \right) + \frac{1}{2\rho} \nabla_\mu \left(\rho u^\mu \right). \quad (86)$$

The divergence of the entropy flux vector can be reformulated in terms of the creation pressure as

$$\nabla_\mu S^\mu = \frac{-3n\sigma H P_c}{\rho(1+w)}. \qquad (87)$$

Finally, we consider the temperature evolution in a system with particle creation. In order to fully determine the time behavior of a relativistic fluid we must add two equations of state for the density and pressure, which have the general

form $\rho = \rho(n, \mathcal{T})$ and $p = p(n, \mathcal{T})$, respectively. Then we obtain

$$\dot{\rho} = \left(\frac{\partial \rho}{\partial n} \right)_\mathcal{T} \dot{n} + \left(\frac{\partial \rho}{\partial \mathcal{T}} \right)_n \dot{\mathcal{T}}. \qquad (88)$$

By using the energy and particle balance equations we find

$$-3H (\rho + P + P_c) = \left(\frac{\partial \rho}{\partial n} \right)_\mathcal{T} n (\Psi - 3H) + \left(\frac{\partial \rho}{\partial \mathcal{T}} \right)_n \dot{\mathcal{T}}. \qquad (89)$$

With the use of the thermodynamic identity [96]

$$\mathcal{T} \left(\frac{\partial P}{\partial \mathcal{T}} \right)_n = \rho + P - n \left(\frac{\partial \rho}{\partial n} \right)_\mathcal{T}, \qquad (90)$$

Equation (89) yields the temperature evolution of a relativistic fluid in the presence of matter creation:

$$\frac{\dot{\mathcal{T}}}{\mathcal{T}} = \left(\frac{\partial P}{\partial \rho} \right)_n \frac{\dot{n}}{n}. \qquad (91)$$

In the particular case $(\partial P/\partial \rho)_n = w = $ constant, we obtain for the temperature–particle number dependence the simple expression $\mathcal{T} \sim n^w$.

3.3.3 The case $w = -1$

Based on the homogeneous and isotropic Friedmann–Robertson–Walker metric, and on the energy conservation equation $\dot{\rho} + 3H(1+w)\rho = 0$, in [97], general cosmological thermodynamic properties with an *arbitrary, varying equation-of-state parameter* $w(a)$, where a is the scale factor, were discussed. The $w = -1$-crossing problem of w was explicitly pointed out, and the behaviors of the quantities $(\rho(a), \mu(a), \mathcal{T}(a), $ etc.$)$ at/near $w = -1$ were discussed. As a result of this study it was concluded that *all cosmological quantities must be regular and well defined for all values of w(a), and indeed they are* [97]. In the present thermodynamical approach we have assumed that matter is created in an ordinary form, and therefore all our previous results are valid for $w \geq 0$. However, the thermodynamic approach and the interpretation of $f(R, T)$ gravity can be extended to the case $w < 0$. In the following we will consider this problem, and we show that our results are valid, in the sense of regularity and of being well defined, even in the case of $w = -1$. In particular, we concentrate on the temperature evolution equation,

$$\frac{\dot{\mathcal{T}}}{\mathcal{T}} = \left(\frac{\partial P}{\partial \rho} \right)_n \frac{\dot{n}}{n}, \qquad (92)$$

which still holds even if $w = P/\rho = -1$. The demonstration is as follows. First we consider the perfect-fluid energy-momentum balance equation,

$$\dot{\rho} + 3(\rho + P)H = \frac{-f_T}{8\pi + f_T}\left[(\rho + P)u^\mu \nabla_\mu \ln|f_T|\right.$$
$$\left. + u^\mu \nabla_\mu \left(\frac{\rho - P}{2}\right)\right], \quad f = f(R(g,\tilde{\Gamma}),T). \quad (93)$$

When $w = -1$, Eq. (93) becomes

$$\dot{\rho} = \frac{-f_T}{8\pi + f_T}u^\mu \nabla_\mu \rho \equiv -3HP_c, \quad (94)$$

where P_c is the matter creation pressure. Under the assumption of adiabatic particle production, with $\dot{\sigma} = 0$, the Gibbs law gives

$$\dot{\rho} = (\rho + P)\frac{\dot{n}}{n} = 0, \quad (95)$$

where σ is the entropy per particle. That is, from the above two equations, we obtain

$$\dot{\rho} = P_c = 0. \quad (96)$$

Since $\rho = \rho(n, T)$, we have

$$\dot{\rho} = \left(\frac{\partial \rho}{\partial n}\right)_T \dot{n} + \left(\frac{\partial \rho}{\partial T}\right)_n \dot{T} = 0. \quad (97)$$

Combining the above equation and the thermodynamic identity [96]

$$T\left(\frac{\partial P}{\partial T}\right)_n = \rho + P - n\left(\frac{\partial \rho}{\partial n}\right)_T = -n\left(\frac{\partial \rho}{\partial n}\right)_T, \quad (98)$$

it immediately follows that Eq. (92) still holds even for negative values of $w = -1$. If $w = -1 = $ constant, we can find from Eq. (92) that nT is a constant, or $T \sim 1/n$. This relation shows that for very low density "dark energy" particles their thermodynamic temperature is extremely high, while high particle number (density) systems have a very low temperature. In the limit $n \to \infty$, the temperature of the "dark energy" made system tends to zero.

4 Cosmology of Palatini $f(R, T)$ gravity

In the present section we investigate the cosmological implications of the Palatini formulation of $f(R, T)$ gravity. We assume that the Universe is flat, homogeneous and isotropic, with the metric given in comoving coordinates by the Friedmann–Robertson–Walker metric,

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \quad (99)$$

where $a(t)$ is the scale factor. We also introduce the Hubble function, defined as $H = \dot{a}/a$. We assume that the matter content of the Universe consists of a perfect fluid that can be characterized by two thermodynamic parameters, the energy density ρ, and the pressure P, respectively. As for the relations of the geometric quantities in the g and $\tilde{g}$ frames, their detailed computation is presented in Appendix C.

4.1 Generalized Friedmann equations in Palatini $f(R, T)$ gravity

Substituting the expression of the perfect-fluid energy-momentum tensor as well as Eqs. (61), (62) and (C11) into the Palatini field Eq. (33), from the 00 component we obtain the first modified Friedmann equation as

$$3H^2 = \frac{1}{2F}\left[8\pi(\rho + 3P) + (\rho + P)f_T\right.$$
$$\left. + f - \frac{3\dot{F}^2}{2F} - 6H\dot{F}\right]. \quad (100)$$

Similarly, with the help of Eqs. (61), (62) and (C12), we can obtain the second modified Friedmann equation from the 'ii' components of Eq. (33),

$$2\dot{H} + 3H^2 = \frac{-1}{2F}\left[8\pi(\rho - P) + (\rho + P)f_T\right.$$
$$\left. - f - \frac{3\dot{F}^2}{2F} + 2\ddot{F} + 4H\dot{F}\right]. \quad (101)$$

Note that the first modified Friedmann equation (100) can be written more compactly as

$$\left(H + \frac{\dot{F}}{2F}\right)^2 = \frac{8\pi(\rho + 3P) + (\rho + P)f_T + f}{6F}. \quad (102)$$

Finally, we substitute Eqs. (61) and (C10) into Eq. (34) and obtain the trace equation

$$FR\left(g,\tilde{\Gamma}\right) - 2f = 8\pi T - (T + \Theta)f_T$$
$$= 6F\left(\dot{H} + 2H^2\right)$$
$$- 3\left(\frac{\dot{F}^2}{2F} - \ddot{F} - 3H\dot{F}\right) - 2f. \quad (103)$$

By eliminating the term $3H^2$ between the two generalized Friedmann equations we obtain the evolution equation for the Hubble function as given by

$$\dot{H} = \frac{-1}{2F}\left[(8\pi + f_T)(\rho + P) - \frac{3\dot{F}^2}{2F} + \ddot{F} - H\dot{F}\right].$$
(104)

The $f\left(R\left(g, \tilde{\Gamma}\right), T\right) \to f\left(R\left(g, \tilde{\Gamma}\right)\right)$ limits of the two modified Friedmann equations can be given by

$$6H^2 F - f = 8\pi(\rho + 3P) - \frac{3\dot{F}^2}{2F} - 6H\dot{F}$$
(105)

and

$$-2F\left(2\dot{H} + 3H^2\right) + f = 8\pi(\rho - P) - \frac{3\dot{F}^2}{2F} + 2\ddot{F} + 4H\dot{F}.$$
(106)

If we go one step further and take $f\left(R\left(g, \tilde{\Gamma}\right)\right) \to R\left(g, \tilde{\Gamma}\right)$, then $f = R(g) = 6\left(\dot{H} + 2H^2\right)$ and the above two equations reduce to

$$6H^2 - 6\left(\dot{H} + 2H^2\right) = -6\left(\dot{H} + H^2\right) = 8\pi(\rho + 3P)$$
(107)

and

$$-2\left(2\dot{H} + 3H^2\right) + 6\left(\dot{H} + 2H^2\right)$$
$$= 2\dot{H} + 6H^2 = 8\pi(\rho - P).$$
(108)

That is,

$$\dot{H} + H^2 = -\frac{4\pi}{3}(\rho + 3P)$$
(109)

and

$$H^2 = \frac{8\pi}{3}\rho.$$
(110)

Hence the first modified Palatini $f(R, T)$ Friedmann equation (100) reduces to the ordinary second Friedmann equation (109) when $f(R, T) \to R$.

4.2 The energy balance equation

With the help of Eq. (C2), we can directly work out the covariant divergence of the energy-momentum tensor (4) as

$$\nabla_\mu T_i^\mu = \partial_\mu T_i^\mu + \Gamma_{\mu\nu}^\mu T_i^\nu - \Gamma_{\mu i}^\nu T_\nu^\mu$$
$$= \partial_i T_i^i + \Gamma_{\mu i}^\mu T_i^i - \Gamma_{0i}^0 T_0^0 - 3\Gamma_{11}^1 T_1^1$$
$$= 0, \quad i = 1, 2, 3,$$
(111)

and

$$\nabla_\mu T_0^\mu = \partial_\mu T_0^\mu + \Gamma_{\mu\nu}^\mu T_0^\nu - \Gamma_{\mu 0}^\nu T_\nu^\mu$$

$$= \partial_0 T_0^0 + \Gamma_{\mu 0}^\mu T_0^0 - \Gamma_{00}^0 T_0^0 - 3\Gamma_{10}^1 T_1^1$$
$$= -\dot{\rho} - 3H(\rho + P).$$
(112)

Substituting the above two equations and Eqs. (61) and (62) into the already known Palatini $f(R, T)$ energy balance equation (93),

$$\nabla_\mu T_\nu^\mu = \frac{f_T}{8\pi - f_T}\left[(T_\nu^\mu + \Theta_\nu^\mu)\nabla_\mu \ln|f_T|\right.$$
$$\left. + \nabla_\mu \Theta_\nu^\mu - \frac{1}{2}\nabla_\nu T\right],$$
(113)

we obtain

$$\dot{\rho} + 3H(\rho + P) = \frac{f_T}{8\pi}\left[\frac{\dot{P} - 3\dot{\rho}}{2} - \left(3H + \frac{\dot{f}_T}{f_T}\right)(\rho + P)\right].$$
(114)

When $f(R, T) \to f(R)$, the energy balance equation reduces to the ordinary conservation equation,

$$\dot{\rho} + 3H(\rho + P) = 0.$$
(115)

4.3 Deceleration parameter and equation of state of the Universe

An important cosmological parameter, indicating the accelerating/decelerating nature of the cosmological dynamics is the deceleration parameter q, defined to be

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = -\frac{\dot{H}}{H^2} - 1.$$
(116)

Using Eqs. (100) and (104), we immediately obtain

$$q = 3\frac{(8\pi + f_T)(\rho + P) - \frac{3\dot{F}^2}{2F} + \ddot{F} - H\dot{F}}{8\pi(\rho + 3P) + (\rho + P)f_T + f - \frac{3\dot{F}^2}{2F} - 6H\dot{F}} - 1.$$
(117)

For a vacuum Universe with $\rho = P = 0$, the condition for accelerated expansion, $q < 0$, reduces to

$$\frac{\ddot{F} - 3\dot{F}^2/2F - H\dot{F}}{f - 3\dot{F}^2/2F - 6H\dot{F}} < \frac{1}{3}.$$
(118)

The deceleration parameter can also be defined, by analogy with the standard general relativistic cosmology, in terms of the effective parameter w_{eff} of the equation of state of the Universe as

$$q = \frac{1 + 3w_{\text{eff}}}{2},$$
(119)

giving

$$w_{\text{eff}} = \frac{2q - 1}{3}. \tag{120}$$

By using the above definition we obtain for the effective parameter of the equation of state of the Universe the expression

$$w_{\text{eff}} = 2\frac{(8\pi + f_T)(\rho + P) - \frac{3\dot{F}^2}{2F} + \ddot{F} - H\dot{F}}{8\pi(\rho + 3P) + (\rho + P)f_T + f - \frac{3\dot{F}^2}{2F} - 6H\dot{F}} - 1. \tag{121}$$

4.4 The de Sitter solution

Next, we investigate the possibility of the existence of a de Sitter type solution in the framework of the Palatini formulation of $f(R, T)$ gravity. The de Sitter solution corresponds to $H = H_0 = \text{constant}$, and $\dot{H} = 0$, respectively. Assuming that the Universe is filled with a pressureless dust, we have $P = 0$, and $T = -\rho$. Moreover, we adopt for the function f the functional form $f(R, T) = k(R) + g(T)$, where for simplicity we take $g(T) = 8\pi\beta T$, with β a constant. Then the energy balance equation (114) takes the form

$$\left(1 + \frac{3\beta}{2}\right)\dot{\rho} = -3H_0(1 + \beta)\rho, \tag{122}$$

with the general solution given by

$$\rho(t) = \rho_0 e^{-\tilde{\alpha}t}, \tag{123}$$

where ρ_0 is an integration constant, and we have denoted

$$\tilde{\alpha} = \frac{3H_0(1 + \beta)}{1 + 3\beta/2}. \tag{124}$$

Equation (104) becomes

$$\ddot{F} - \frac{3\dot{F}^2}{2F} - H_0\dot{F} + 8\pi(1 + \beta)\rho_0 e^{-\tilde{\alpha}t} = 0. \tag{125}$$

In the limit of large times the last, exponential term in the above equation becomes negligibly small, and hence we can approximate the solution of Eq. (125) by

$$F(t) \approx \frac{F_0}{\left(e^{H_0 t} - 1\right)^2}, \tag{126}$$

where F_0 is an arbitrary constant of integration, and without any loss of generality we have taken the second integration

constant as zero. Then the first generalized Friedmann equation (100) gives the Lagrangian function of the model as

$$f(t) = 6H_0^2 F + \frac{3\dot{F}^2}{2F} + 6H_0\dot{F} + 8\pi(1 + \beta)T. \tag{127}$$

On the other hand the trace of Eq. (103) gives (note that in the following $R = R(g)$)

$$FR + 3\left(\frac{\dot{F}^2}{F} + 4H_0\dot{F}\right) = 8\pi(1 + \beta)T + 2f, \tag{128}$$

or

$$R(t) = 12H_0^2 + 24\pi(1 + \beta)\frac{T}{F}. \tag{129}$$

Equations (127) and (129) give a parametric representation of f as a function of R, with t taken as a parameter. Once the function $t = t(R)$ is obtained from Eq. (129), by substituting it in Eq. (127) we can find the explicit dependence of f on R.

4.5 Comparison with the metric $f(R, T)$ cosmology

Using the Friedmann–Robertson–Walker metric, and the field equation (9), we can similarly derive the two Friedmann equations in the metric formulation. The cosmological equations in metric $f(R, T)$ gravity are different from their counterparts in the Palatini formalism, due to the presence of some dynamical terms related to f_R, and they are given by

$$3H^2 = \frac{1}{2F}\left[8\pi(\rho + 3P) + (\rho + P)f_T + f + 3\ddot{f}_R + 3H\dot{f}_R\right], \tag{130}$$

$$2\dot{H} + 3H^2 = \frac{-1}{2F}\left[8\pi(\rho - P) + (\rho + P)f_T - f - \ddot{f}_R - 5H\dot{f}_R\right]. \tag{131}$$

Besides, the deceleration parameter $q = -\dot{H}/H^2 - 1$ can be obtained:

$$q = 3\frac{(8\pi + f_T)(\rho + P) + \ddot{f}_R - H\dot{f}_R}{8\pi(\rho + 3P) + (\rho + P)f_T + f + 3\ddot{f}_R + 3H\dot{f}_R} - 1. \tag{132}$$

For $\rho = P = 0$, an accelerated Universe with $q < 0$ requires that

$$\frac{\ddot{f}_R - H\dot{f}_R}{f + 3\ddot{f}_R + 3H\dot{f}_R} < \frac{1}{3}. \tag{133}$$

The condition for accelerated expansion in the metric $f(R, T)$ gravity is very different from the similar condition given by Eq. (118), in the Palatini formulation of the theory. The presence of the extra term, $3\dot{F}^2/2F$, in Eq. (118) may have a significant effect on the transition from the decelerating to the accelerating phase. In the Palatini formulation the moment of the transition to the accelerated expansion with $q \leq 0$ is determined by the equation

$$\ddot{f}_R - \dot{f}_R^2/f_R + H\dot{f}_R - f/3 = 0, \tag{134}$$

with $f_R = f_R\left(R\left(g, \tilde{\Gamma}\right), T\right)$, while in the metric $f(R, T)$ gravity the same condition is given by the much simpler expression

$$6H\dot{f}_R + f = 0, \tag{135}$$

with $f = f(R(g), T)$. It is interesting to note that the conditions for the transition to an accelerated expansion in the vacuum case are independent in both approaches on f_T. However, the Palatini formulation of $f(R, T)$ gravity allows for a much richer cosmological dynamics, as compared to the metric formulation.

As for the energy-momentum balance equation and the non-geodesic equation of motion of massive test particles: since they are all derived from the divergence of the energy-momentum tensor, and since $\nabla_\mu T^\mu_\nu$ is the same in the two formalisms, independently of the functional form of f, the energy balance equations and the equations of motion have the same functional form in both approaches.

4.6 Specific cosmological models in the Palatini $f(R, T)$ gravity

In the present section we will investigate two specific cosmological models in the framework of the Palatini formulation of $f(R, T)$ gravity. We will assume that the action of the gravitational field has the general form

$$f(R, T) = R\left(g, \tilde{\Gamma}\right) + k\left(R\left(g, \tilde{\Gamma}\right)\right) + 8\pi\beta g(T), \tag{136}$$

where β is a constant, and $k\left(R\left(g, \tilde{\Gamma}\right)\right)$ and $g(T)$ are arbitrary functions of the Ricci scalar and the trace of the matter energy-momentum tensor. For the function $g(T)$ for simplicity we will assume a simple linear dependence on T, so that $g(T) = T$. As for the function $k\left(R\left(g, \tilde{\Gamma}\right)\right)$, we will consider two cases, corresponding to the Starobinsky model $k(R) \sim R^2\left(g, \tilde{\Gamma}\right)$ [98], and the $1/R\left(g, \tilde{\Gamma}\right)$ case, respectively.

4.6.1 $f(R, T) = R\left(g, \tilde{\Gamma}\right) + \frac{\alpha}{16\pi}R^2\left(g, \tilde{\Gamma}\right) + 8\pi\beta g(T)$

We consider a Palatini $f(R, T)$ model, specified by a functional form of $f(R, T)$ given as

$$f(R, T) = R\left(g, \tilde{\Gamma}\right) + \frac{\alpha}{16\pi}R^2\left(g, \tilde{\Gamma}\right) + 8\pi\beta g(T), \tag{137}$$

where α, β are constants, $g(T)$ is a function that depends on T solely, and for simplicity we set $g(T) = T$. For this Lagrangian we immediately find $F = f_R = 1 + (\alpha/8\pi)R\left(g, \tilde{\Gamma}\right)$, $f_T = 8\pi\beta g_T$, $g_T \equiv \partial g(T)/\partial T = 8\pi\beta$. Moreover, we assume that the cosmological fluid satisfies a linear barotropic equation of state of the form $P = w\rho$, $w = $ constant.

Consequently, from the trace of (15) of the Palatini field equations we first obtain

$$\left(1 + \frac{\alpha R}{8\pi}\right)R = 8\pi T - 8\pi\beta(T + \Theta) + 2f, \tag{138}$$

thus

$$R = 8\pi[(1 - 3w) + \beta(3 - 5w)]\rho \equiv 8\pi\beta_0\rho. \tag{139}$$

When $\beta \to 0$, $\beta_0 \to 1 - 3w$, and $R\left(g, \tilde{\Gamma}\right) \to -8\pi T$, respectively. Substituting Eq. (139) back into the expressions of $f(R, T)$ and F, we obtain

$$f = 8\pi\left[\beta_0 + \frac{\beta_0^2}{2}\alpha\rho + \beta(3w - 1)\right]\rho \tag{140}$$

and

$$F = 1 + \beta_0\alpha\rho. \tag{141}$$

Substituting Eqs. (139)–(141) into the Friedmann Eq. (102), we find

$$\left[H + \frac{\beta_0\alpha\dot{\rho}}{2(1 + \beta_0\alpha\rho)}\right]^2 = \frac{1 + \frac{\beta}{2}(3 - w) + \frac{\beta_0^2}{4}\alpha\rho}{(1 + \beta_0\alpha\rho)}\frac{8\pi\rho}{3}. \tag{142}$$

Substitution of the expression of f_T into the balance equation (114) gives

$$\dot{\rho} = -\beta_1 H\rho, \tag{143}$$

where we have denoted

$$\beta_1 \equiv \frac{3(1 + \beta)(1 + w)}{1 + \frac{\beta}{2}(3 - w)}. \tag{144}$$

When $\beta \to 0$, then $\beta_1 = 3(1 + w)$. Substituting Eq. (143) into Eq. (142), we find

$$\frac{\left[1 + \beta_0 \left(1 - \frac{\beta_1}{2}\right) \alpha \rho\right]^2}{1 + \beta_0 \alpha \rho} H^2$$
$$= \frac{8\pi\rho}{3} \left[1 + \frac{\beta}{2}(3 - w) + \frac{\beta_0^2}{4} \alpha \rho\right]. \tag{145}$$

Taking into account the limit $\alpha\rho \to 0$, we have the series expansion

$$\frac{(1 + \beta_0 \alpha \rho) \left[1 + \frac{\beta}{2}(3 - w) + \frac{\beta_0^2}{4}\alpha\rho\right]}{\left[1 + \beta_0 \left(1 - \frac{\beta_1}{2}\right)\alpha\rho\right]^2} \approx \beta_2 + \beta_3\alpha\rho, \tag{146}$$

where we have denoted $\beta_2 = 1 + \beta(3 - w)/2$ and $\beta_3 = \beta_0 \{\beta_0/4 - [1 + \beta(3 - w)/2](1 - \beta_1)\}$. Then Eq. (145) takes the form

$$H^2 = \frac{8\pi\rho}{3} (\beta_2 + \beta_3\alpha\rho). \tag{147}$$

Equation (143) can be immediately integrated to give

$$\rho = \rho_0 a^{-\beta_1}. \tag{148}$$

Hence Eq. (147) becomes a first-order differential equation,

$$\dot{a} = \sqrt{\frac{8\pi}{3}\rho_0 a^{1-\beta_1}}\sqrt{\beta_2 a^{\beta_1} + \beta_3\alpha\rho_0}, \tag{149}$$

with the general solution given by

$$a(t) = \left(\frac{2\pi}{3}\beta_1^2\beta_2\rho_0 t^2 - \frac{\beta_3}{\beta_2}\alpha\rho_0\right)^{\frac{1}{\beta_1}}. \tag{150}$$

In the limit of large times we obtain $a(t) \sim t^{2/\beta_1}$, and $H(t) = (2/\beta_1)(1/t)$, respectively. The deceleration parameter in this model is given by $q = \beta_1/2 - 1$, and once the coefficient β_1 satisfies the condition $\beta_1/2 < 1$, the Universe will experience an accelerated evolution. For an arbitrary w, the condition for a power law type accelerated expansion is $\beta < -(1 + 3w)/4w$, a condition which shows that, for $w > 0$, β must take negative values.

4.6.2 $f(R, T) = R\left(g, \tilde{\Gamma}\right) - \frac{\alpha^2}{3R(g,\tilde{\Gamma})} + 8\pi\beta g(T)$

Now let us consider the following $f(R, T)$ gravity Palatini type model:

$$f\left(R\left(g, \tilde{\Gamma}\right), T\right) = R\left(g, \tilde{\Gamma}\right) - \frac{\alpha^2}{3R\left(g, \tilde{\Gamma}\right)} + 8\pi\beta g(T), \tag{151}$$

which immediately gives $F = 1 + \alpha^2 R^{-2}\left(g, \tilde{\Gamma}\right)/3$.

From the trace of (15) of the Palatini field equations we obtain

$$R^2\left(g, \tilde{\Gamma}\right) + \Phi R\left(g, \tilde{\Gamma}\right) - \alpha^2 = 0, \tag{152}$$

where we have denoted

$$\Phi = 8\pi \left[T - (T + \Theta)\beta g_T + 2\beta g(T)\right]. \tag{153}$$

The algebraic equation (152) has two distinct solutions. However, only one of them can be adopted as the physical solution, more exactly the one which in the limit $f \to R$, would give $R = -8\pi T$, which is the trace of the Einstein field equation. Hence when $\Phi \leq 0$, the physical solution of Eq. (152) is [99]

$$R\left(g, \tilde{\Gamma}\right) = -\frac{1}{2}\left(\Phi - \sqrt{\Phi^2 + 4\alpha^2}\right). \tag{154}$$

In the following we will study this cosmological model under the approximation $\alpha \gg |\Phi|$ [99], which allows us to find the evolution of the Universe at later times when $R\left(g, \tilde{\Gamma}\right)$ is relatively small. Under the adopted approximation we can expand $R\left(g, \tilde{\Gamma}\right)$ to the order of $O(|\Phi|)$, and we obtain

$$R\left(g, \tilde{\Gamma}\right) \approx -\frac{1}{2}\Phi + \alpha, \tag{155}$$

$$f\left(R\left(g, \tilde{\Gamma}\right), T\right) \approx -\frac{2}{3}\Phi + \frac{2}{3}\alpha + 8\pi\beta g(T), \tag{156}$$

$$F \approx \frac{1}{3\alpha}\Phi + \frac{4}{3}. \tag{157}$$

For simplicity, we set again $g(T) = T$. Using the approximations (155), (156) and (157), respectively, we can easily obtain the Palatini $f(R, T)$ field equations for this model:

$$\tilde{R}_{\mu\nu} = \frac{g_{\mu\nu}}{4}\alpha + 8\pi\left(\frac{3}{4}T_{\mu\nu} - \frac{5g_{\mu\nu}}{16}T\right) - 8\pi\beta$$
$$\times \left[\left(\frac{3}{4}\Theta_{\mu\nu} - \frac{5g_{\mu\nu}}{16}\Theta\right) + \left(\frac{3}{4}T_{\mu\nu} - \frac{g_{\mu\nu}}{16}T\right)\right]. \tag{158}$$

When $f(R, T) \to f(R)$, $\beta \to 0$, and the above field equations reduce to the field equations considered within the Palatini formulation of $f(R)$ gravity, considered in [99],[2]:

$$\tilde{R}_{\mu\nu} = \frac{g_{\mu\nu}}{4}\alpha + 8\pi\left(\frac{3}{4}T_{\mu\nu} - \frac{5g_{\mu\nu}}{16}T\right). \tag{159}$$

With the equation of state $w = P/\rho$, and since $T = -\rho + 3P$ for a perfect fluid, we have $\Phi = -8\pi\beta_0\rho$; besides, we already know from Eq. (143) that $\dot{\rho} = -\beta_1 H\rho$. Thus, similar to Eq. (145), we have

$$\frac{\left[1 - \pi\beta_0(2-\beta_1)\frac{\rho}{\alpha}\right]^2}{1 - 2\pi\beta_0\frac{\rho}{\alpha}}H^2$$
$$= \left[1 + (3+4\beta)w + \frac{2}{3}\beta_0\right]\pi\rho + \frac{\alpha}{12}. \tag{160}$$

In the first-order approximation in ρ/α we obtain

$$H^2 = \frac{\alpha}{12} + \left[\frac{1}{6}\beta_0(5-\beta_1) + (3+4\beta)w + 1\right]\pi\rho. \tag{161}$$

Hence in the present model a cosmological constant $\alpha/12$ is automatically generated, due to the $1/R$ modification of the gravitational Lagrangian. Hence for $\rho \to 0$, the Universe will end in a de Sitter phase, with $H = H_0 = \sqrt{\alpha/12} =$ constant. For $\rho \neq 0$, we have $\rho = \rho_0 a^{-\beta_1}$, and the evolution of the scale factor is determined by the equation

$$\dot{a} = a\sqrt{\frac{\alpha}{12} + \frac{\beta_4\rho_0}{a^{\beta_1}}}, \tag{162}$$

where we have denoted $\beta_4 = [\beta_0(5-\beta_1)/6 + (3+4\beta)w + 1]\pi$, with the general solution given by

$$a(t) = 12^{\frac{1}{\beta_1}}\left\{\sqrt{\frac{\beta_4\rho_0}{\alpha}}\sinh\left[\sinh^{-1}\left(\frac{a_0^{\frac{\beta_1}{2}}}{2\sqrt{3}\sqrt{\frac{\beta_4\rho_0}{\alpha}}}\right)\right.\right.$$
$$\left.\left. + \frac{\sqrt{\alpha}\beta_1(t-t_0)}{4\sqrt{3}}\right]\right\}^{\frac{2}{\beta_1}}, \tag{163}$$

where we have used the initial condition $a(t_0) = a_0$.

For the Hubble function we obtain

$$H(t) = \frac{\sqrt{\alpha}}{2\sqrt{3}}\coth\left[\frac{\sqrt{\alpha}\beta_1(t-t_0)}{4\sqrt{3}}\right.$$
$$\left. + \sinh^{-1}\left(\frac{a_0^{\beta_1/2}}{\sqrt{12\beta_4\rho_0/\alpha}}\right)\right], \tag{164}$$

while the deceleration parameter of this model is given by

$$q(t) = \frac{1}{2}\beta_1\mathrm{sech}^2\left[\frac{\sqrt{\alpha}\beta_1(t-t_0)}{4\sqrt{3}}\right.$$
$$\left. + \sinh^{-1}\left(\frac{a_0^{\beta_1/2}}{\sqrt{12\beta_4\rho_0/\alpha}}\right)\right] - 1, \tag{165}$$

where $\mathrm{sech}\, t = 1/\cosh t$. In the limit $t \to \infty$, $q \to -1$, and hence the Universe ends in a de Sitter type accelerating phase, independent of the matter equation of state.

a. The matter dominated phase The matter dominated phase corresponds to the choice $w = 0$ in the matter equation of state, that is, to a Universe filled with pressureless baryonic matter. In order to investigate the behavior of the cosmological model during matter domination, we consider the series expansion of the cosmological parameters. Thus we obtain

$$a(t) \approx a_0 + \frac{a_0^{1-\frac{\beta_1}{2}}\beta_4\rho_0\sqrt{\frac{\alpha a_0^{\beta_1}}{\beta_4\rho_0} + 12}}{2\sqrt{3}}(t-t_0)$$
$$+ \frac{1}{24}a_0^{1-\beta_1}\left[\alpha a_0^{\beta_1} - 6(\beta_1-2)\beta_4\rho_0\right](t-t_0)^2 + \cdots, \tag{166}$$

$$H(t) \approx \frac{a_0^{-\frac{\beta_1}{2}}\sqrt{\alpha a_0^{\beta_1} + 12\beta_4\rho_0}}{2\sqrt{3}} - \frac{1}{2}\beta_1\beta_4\rho_0 a_0^{-\beta_1}$$
$$\times (t-t_0) + \frac{\beta_1^2\beta_4\rho_0 a_0^{-\frac{3\beta_1}{2}}\sqrt{\alpha a_0^{\beta_1} + 12\beta_4\rho_0}}{8\sqrt{3}}$$
$$\times (t-t_0)^2 + \cdots, \tag{167}$$

$$q(t) \approx \frac{6(\beta_1-2)\beta_4\rho_0 - \alpha a_0^{\beta_1}}{\alpha a_0^{\beta_1} + 12\beta_4\rho_0}$$
$$- \frac{\sqrt{3}\alpha\beta_1^2 a_0^{\frac{\beta_1}{2}}(\beta_4\rho_0)\sqrt{\alpha a_0^{\beta_1} + 12\beta_4\rho_0}}{\left(\alpha a_0^{\beta_1} + 12\beta_4\rho_0\right)^2}(t-t_0)$$
$$+ \frac{\alpha\beta_1^3\beta_4\rho_0\left(\alpha a_0^{\beta_1} - 6\beta_4\rho_0\right)}{4\left(\alpha a_0^{\beta_1} + 12\beta_4\rho_0\right)^2}(t-t_0)^2 + \cdots. \tag{168}$$

These equations describe the main cosmological parameters during the matter dominated era. The expansion is decelerating, and, depending on the model parameters, the deceleration parameter can have a large range of positive values. The transition to the accelerating phase occurs at a time interval t_{tr}, which in the first-order approximation is found as

$$t_{tr} \approx t_0 + \frac{a_0^{-\frac{\beta_1}{2}}\sqrt{\frac{\alpha a_0^{\beta_1}}{3\beta_4\rho_0} + 4}\left[6(\beta_1-2)\beta_4\rho_0 - \alpha a_0^{\beta_1}\right]}{\alpha\beta_1^2\sqrt{\beta_4\rho_0}}. \tag{169}$$

Since t_{tr} must be greater than t_0, it follows that in order for the model to admit a matter dominated era followed by a transition to an accelerated phase, the model parameters must satisfy the condition $6(\beta_1-2)\beta_4\rho_0 - \alpha a_0^{\beta_1} > 0$, or, equivalently,

$$\left(1 + \frac{3\beta}{2}\right)\frac{\alpha a_0^{\beta_1}}{6\beta_4\rho_0} < 1, \tag{170}$$

a condition that can easily be satisfied by appropriately choosing the free parameters in the gravitational action.

5 Discussions and final remarks

In the present paper we have considered in the framework of the Palatini formalism the gravitational field equations for the modified gravity $f(R, T)$ theory, implying a geometry–matter coupling, with the trace of the energy-momentum tensor included as a field variable in the gravitational action. We have derived the field equations by independently varying the metric and the connection in the $f(R, T)$ type gravitational action, and we have formulated them in both the initial metric frame and the conformal one, in which the independent connection can be expressed as the Levi-Civita connection of an auxiliary, energy-momentum trace dependent metric, which is related to the physical metric by a conformal transformation. Similar to the metric case [46] the energy-momentum tensor of the matter is not conserved, and the energy and momentum balance equations take the same form as in the metric theory. Generally, Palatini type theories have a number of special properties that make them especially attractive for analyzing strong gravity phenomena, like, for example, the dynamics of the early Universe or stellar collapse processes [100–107]. The coupling of the trace of the energy-momentum tensor with the curvature scalar generates some extra terms in the gravitational field equations, which strongly depend on the possible functional forms for the geometry– energy momentum trace coupling. If, for example, it would be possible to generate through the geometry–energy momentum trace coupling some repulsive forces, then one could obtain cosmological models that are non-singular at extremely high densities and high geometric curvatures, or one could even construct models for non-singular collapsing stars as viable alternatives for the black hole paradigm.

To obtain such repulsive gravitational forces, in modified gravity theories with geometry–matter coupling no new degrees of freedom in the matter side (exotic sources) or in the gravitational side are required in the total action. In these models the extra force is simply induced by the coupling between matter and geometry. Our present results show that Palatini type theories might play an important role in the phenomenology of gravity at both high densities (energies), as well as in the very low density limit. On the other hand, in the variational process the assumption of the independence between metric and connection is essential to obtain second-order differential equations for the metric tensor. It is thus possible to assume that at large/small scales the effective descriptions of the gravitational forces, going beyond standard general relativity, could come from the Palatini formulation of gravity theories.

In the Palatini type formulation of $f(R, T)$ gravity, the equation of motion of massive particles is non-geodesic, and in three dimensions and in the Newtonian limit, Eq. (66) can be formally represented as an ordinary vector equation in three dimensions of the form $\vec{a} = \vec{a}_N + \vec{a}_f$ where $\vec{a}$ represents the total acceleration of the particle, $\vec{a}_N$ denotes the Newtonian gravitational acceleration, while $\vec{a}_f$ is the acceleration due to the presence of the extra force induced by the coupling between geometry and matter. This shows that one observational or experimental possibility of testing the effects of the coupling between geometry and the trace of the energy-momentum tensor could be in the physical domain of extremely small particle accelerations, with values of the order of 10^{-10} m/s^2. Such an acceleration could explain the observed behavior of test particles rotating around galaxies, which is usually explained by postulating the existence of dark matter. However, as a possible astrophysical application of the gravitational field equations derived with the Palatini formalism one may consider an alternative view to the dark matter problem, in which the mass discrepancy in galaxies and clusters of galaxies as well as the galactic rotation curves are explained by the existence of a non-minimal coupling between matter and geometry.

We have also briefly investigated the intriguing feature of the non-conservation of the energy-momentum tensor of the matter in $f(R, T)$ gravity theory by interpreting it in the framework of the thermodynamics of open systems. We have interpreted this effect as describing phenomenologically the particle production in the cosmological fluid filling the Universe, with the extra terms induced by the non-minimal coupling between R and T assumed to describe particle creation processes, with the gravitational field acting as a source for particles. We have explicitly obtained the particle creation rates, the entropy flux, the creation pressure and entropy generation rate in a covariant form, as functions of the Lagrangian density $f(R, T)$ of the theory, and of its derivatives, respectively. On the other hand it is natural to assume that such particle production processes are of the same nature as the similar processes that appear in the framework of quantum field theory in curved space-times. A static gravitational field does not produce particles. But a time dependent gravitational field can generate new particles. This interesting analogy between gravitational theories with geometry–matter coupling and quantum field theory in curved space-times may open the possibility of an effective classical description of quantum gravity on small geometric scales.

As a cosmological application of the Palatini formalism of $f(R, T)$ theory we have briefly considered two classes of cosmological models, corresponding to two choices of the gravitational Lagrange density $f(R, T) = k(R) + g(T)$, with $k(R) \sim R^2$ and $k(R) \sim 1/R$, respectively. In both cases we have assumed for the function $g(T)$ the simple

form $g(T) \sim T$. We have explicitly shown that both models can generate an accelerating expansion of the Universe, with a power law and an exponential form of the scale factor, respectively.

In both metric and Palatini formulation of $f(R, T)$ gravity, dark energy is interpreted as a material–geometrical fluid, with a negative parameter of the equation of state, for which the function $f(R, T)$ is not known *a priori*. Hence, similar to the case of $f(R)$ gravity, there is a need of a model independent reconstruction of the Lagrangian of theory, which can be done by using some relevant cosmographic techniques to determine which $f(R, T)$ model is favored with respect to others. In the case of $f(R)$ gravity, a cosmographic approach was introduced in [16], by assuming that the cosmological principle is valid, and that dark energy can be described as a geometric fluid. Then, after expanding the cosmological observables (the Hubble parameter, the luminosity distance, the apparent magnitude modulus, the effective pressure etc.) into a Taylor series, and matching the derivatives of the expansions with cosmological data, one can obtain some model independent constraints on the gravitational theory. The coefficients of the power series of the expansion of the scale factor, calculated at present time (at redshift $z = 0$) are known as the cosmographic series. The importance of the cosmographic approach is that it does not need the assumption of a specific cosmological model. If the scalar curvature is negligible, the Taylor series of the scale factor around $\Delta t = t - t_0 = 0$ can be represented as [108]

$$\frac{1 - a(t)}{H_0} \approx \Delta t + \frac{q_0}{2} H_0 \Delta t + \frac{j_0}{6} H_0^2 \Delta t^3 - \frac{s_0}{24} H_0^3 \Delta t^4 + \cdots ,$$

(171)

where the jerk parameter j is defined as $j(t) = (1/aH^3)(d^3 a/dt^3)$, while the snap parameter s is given by $s(t) = (1/aH^4)(d^4a/dt^4)$ [110]. A strategy to infer the transition redshift z_{da}, which indicates the passage of the Universe from a decelerating to an accelerating phase, was proposed, in the framework $f(R)$ gravity, in [109]. This goal can be achieved by numerically reconstructing $f(z)$, that is, the corresponding gravitational Lagrangian $f(R)$ re-expressed as a function of the redshift z, and by matching $f(z)$ with cosmography. The high-redshift $f(R)$ cosmography was considered in [110], by adopting the technique of polynomial reconstruction. Instead of considering the Taylor expansions that proved to be non-predictive for redshifts $z > 1$, the Padé rational approximations were considered, by performing series expansions that converge in the domains of high redshifts. A first step in this strategy is the reconstruction of the function $f(z)$, by assuming that the Ricci scalar can be inverted with respect to the redshift z.

The cosmographic approach developed in [108–110] for the case of $f(R)$ gravity can also be extended to both metric and Palatini $f(R, T)$ gravities. To be more specific, such an approach requires one to rewrite $f(R, T)$ or $f\left(R\left(g, \tilde{\Gamma}\right), T\right)$ into a function $f(z)$. Similar to the approach introduced in [16] for $f(R)$ gravity in the metric formulation, in order to handle high-z data one can rewrite the $f(R, T)$ function into an $f(z)$ function generally through the use of Padé polynomials. As a next step data fitting based on some general $f(z)$ models is required. Hence one can generalize the approaches of [108–110] for the case of $f(R, T)$ theory, in both metric and Palatini formulations, and numerically determine the coefficients of the series expansions for R and T in the $f(R, T)$ models through cosmological data fitting. The cosmographic approach could help distinguish between the roles and weights of the functions R and T in the gravitational action, and it could lead to a full comparison of the theory with the cosmological observations.

The cosmology of Palatini $f(R, T)$ gravity represents a promising way for the explanation of the accelerated phases in the dynamics of the Universe, which is characterized by its evolution in both very early and late stages. In the present paper we have introduced some basic theoretical tools necessary for the in-depth investigation of the cosmological and astrophysical aspects of the Palatini formulation of $f(R, T)$ gravity.

Appendix A: $f(R, T)$ field equations in the metric formulation

We first introduce two basic formulas we are going to use,

$$\frac{2\delta\sqrt{-g}}{\sqrt{-g}} = g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu},$$

(A1)

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\lambda} \cdot 2g_{\sigma\lambda}\delta\Gamma^\sigma_{\mu\nu}$$
$$= \frac{1}{2}g^{\rho\lambda}\left(\nabla_\mu\delta g_{\nu\lambda} + \nabla_\nu\delta g_{\lambda\mu} - \nabla_\lambda\delta g_{\mu\nu}\right),$$

(A2)

where in Eq. (A2) we have taken into account the relations

$\delta\left(\nabla_\mu g_{\nu\lambda}\right) = \delta\left(\nabla_\nu g_{\lambda\mu}\right) = \delta\left(\nabla_\lambda g_{\mu\nu}\right) = 0$. A variation $\delta g^{\mu\nu}$ of the metric tensor then leads to

$$\delta T = \delta\left(g^{\alpha\beta}T_{\alpha\beta}\right) = \left(T_{\alpha\beta}\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}} + \Theta_{\mu\nu}\right)\delta g^{\mu\nu}$$

$$= \left(T_{\mu\nu} + \Theta_{\mu\nu}\right)\delta g^{\mu\nu}, \quad \Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} \quad \text{(A3)}$$

and

$$\delta R_{\mu\nu} = \delta\left(\partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\rho}\right)$$

$$= \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho_{\mu\rho}, \quad \text{(A4)}$$

respectively, where the above relation is called the Palatini identity. Then for the variation of the metric we obtain

$$\delta S = \int \frac{\sqrt{-g}}{16\pi}\Big[f_R\left(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right) + f_T\delta T$$

$$-\frac{g_{\mu\nu}}{2}f\delta g^{\mu\nu} - 8\pi T_{\mu\nu}\delta g^{\mu\nu}\Big]\mathrm{d}^4 x$$

$$= \int \frac{\sqrt{-g}}{16\pi}\Big[f_R\left(R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) + f_T\left(T_{\mu\nu} + \Theta_{\mu\nu}\right)$$

$$-\frac{g_{\mu\nu}}{2}f - 8\pi T_{\mu\nu}\Big]\delta g^{\mu\nu}\,\mathrm{d}^4 x, \quad \Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta.$$

$$\text{(A5)}$$

Assuming that the variation of $\delta g^{\mu\nu}$ vanishes at infinity, then

$$\int \frac{\sqrt{-g}}{16\pi}\left(g_{\mu\nu}g^{\alpha\beta}f_R\nabla_\alpha\nabla_\beta - f_R\nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu}\,\mathrm{d}^4 x$$

$$= \int \frac{\sqrt{-g}}{16\pi}\left(-g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha f_R\nabla_\beta\delta g^{\mu\nu} + \nabla_\mu f_R\nabla_\nu\delta g^{\mu\nu}\right)\mathrm{d}^4 x$$

$$= \int \frac{\sqrt{-g}}{16\pi}\left(g_{\mu\nu}g^{\alpha\beta}\delta g^{\mu\nu}\nabla_\beta\nabla_\alpha f_R - \delta g^{\mu\nu}\nabla_\nu\nabla_\mu f_R\right)\mathrm{d}^4 x$$

$$= \int \frac{\sqrt{-g}}{16\pi}\delta g^{\mu\nu}\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f_R\,\mathrm{d}^4 x \quad \text{(A6)}$$

and

$$\delta S = \int \frac{\sqrt{-g}}{16\pi}\Big[\left(R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f_R$$

$$+\left(T_{\mu\nu} + \Theta_{\mu\nu}\right)f_T - \frac{g_{\mu\nu}}{2}f(R,T) - 8\pi T_{\mu\nu}\Big]\delta g^{\mu\nu}\,\mathrm{d}^4 x.$$

$$\text{(A7)}$$

Since $\delta S = 0$, from the above relation we obtain immediately the field equation (8) of $f(R,T)$ gravity theory.

Appendix B: Divergence of the matter energy-momentum tensor in the metric formalism

By taking the covariant divergence of Eq. (9), with the use of the mathematical identity $\nabla_\mu G^\mu_\nu(g) = 0$ we obtain

$$\nabla_\mu\left[f_R G^\mu_\nu(g)\right] - R_{\mu\nu}(g)\nabla^\mu f_R + \frac{\delta^\mu_\nu}{2}R(g)\nabla_\mu f_R$$

$$= \left[G^\mu_\nu(g) - R^\mu_\nu(g) + \frac{\delta^\mu_\nu}{2}R\right]\nabla_\mu f_R$$

$$= \nabla_\mu\left[8\pi T^\mu_\nu - \left(T^\mu_\nu + \Theta^\mu_\nu\right)f_T\right]$$

$$+\frac{\delta^\mu_\nu}{2}\left(\nabla_\mu f - f_R\nabla_\mu R\right)$$

$$+\left(\Box\nabla_\nu - \nabla_\nu\Box\right)f_R - R_{\mu\nu}(g)\nabla^\mu f_R$$

$$= \nabla_\mu\left[8\pi T^\mu_\nu - \left(T^\mu_\nu + \Theta^\mu_\nu\right)f_T\right] + \frac{f_T}{2}\nabla_\nu T = 0, \quad \text{(B1)}$$

where we have used the relations

$$(\nabla_\nu\Box - \Box\nabla_\nu)\phi = g^{\alpha\beta}\left(\nabla_\nu\nabla_\alpha\nabla_\beta - \nabla_\alpha\nabla_\beta\nabla_\nu\right)\phi$$

$$= g^{\alpha\beta}\left(\nabla_\nu\nabla_\alpha - \nabla_\alpha\nabla_\nu\right)\nabla_\beta\phi$$

$$= g^{\alpha\beta}R^\mu_{\beta\alpha\nu}\nabla_\mu\phi$$

$$= -R_{\mu\nu}\nabla^\mu\phi$$

$$\text{(B2)}$$

and

$$\nabla_\nu f(R,T) = f_R\nabla_\nu R + f_T\nabla_\nu T, \quad \text{(B3)}$$

respectively.

Appendix C: The geometric quantities in the FRW geometry

For the metric (99), we have

$$\partial_\lambda g^{\mu\nu} = \begin{cases} 2a\dot{a}, & \lambda = 0 \text{ and } \mu = \nu \neq 0, \\ 0, & \text{others.} \end{cases} \quad \text{(C1)}$$

Hence the only nonzero components of the connection, $\Gamma^\rho_{\mu\nu}(g) = (g^{\rho\sigma}/2)\left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}\right)$, are now

$$\Gamma^0_{ii} = a\dot{a} = a^2 H \text{ and } \Gamma^i_{0i} = \Gamma^i_{i0} = \frac{\dot{a}}{a} = H, \quad i = 1, 2, 3.$$

$$\text{(C2)}$$

In that case,

$$R_{00}(g) = \partial_\rho\Gamma^\rho_{00} - \partial_0\Gamma^\rho_{0\rho} + \Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{00} - \Gamma^\rho_{0\sigma}\Gamma^\sigma_{0\rho}$$

$$= -3\left(\dot{H} + H^2\right), \quad \text{(C3)}$$

$$R_{ii}(g) = a^2\left(\dot{H} + 3H^2\right), \quad \text{(C4)}$$

and

$$R(g) = g^{\mu\nu}R_{\mu\nu}(g) = g^{00}R_{00}(g) + 3g^{11}R_{11}(g)$$

$$= 6\left(\dot{H} + 2H^2\right). \quad \text{(C5)}$$

Hence

$$G_{00}(g) = R_{00}(g) - \frac{g_{00}}{2}R(g) = 3H^2, \quad \text{(C6)}$$

$$G_{ii}(g) = R_{ii}(g) - \frac{g_{ii}}{2}R(g) = -a^2\left(2\dot{H} + 3H^2\right). \quad \text{(C7)}$$

Substituting Eqs. (C3) and (C4) into Eq. (30) we obtain

$$
\begin{aligned}
\tilde{R}_{00}\left(\tilde{\Gamma}\right) &= R_{00}(g) + \frac{1}{F}\left[\frac{3\left(\nabla_0 F\right)^2}{2F} - \left(\nabla_0\nabla_0 + \frac{g_{00}}{2}\Box\right)F\right] \\
&= -3\left(\dot{H}+H^2\right) + \frac{1}{F}\left[\frac{3\dot{F}^2}{2F} - \left(\ddot{F}+\frac{1}{2}\ddot{F}+\frac{3H}{2}\dot{F}\right)\right] \\
&= -3\left(\dot{H}+H^2\right) + \frac{3}{2F}\left(\frac{\dot{F}^2}{F} - \ddot{F} - H\dot{F}\right)
\end{aligned}
\tag{C8}
$$

and

$$
\begin{aligned}
\tilde{R}_{ii}\left(\tilde{\Gamma}\right) &= R_{ii}(g) + \frac{1}{F}\left[\frac{3\left(\nabla_i F\right)^2}{2F} - \left(\nabla_i\nabla_i + \frac{g_{ii}}{2}\Box\right)F\right] \\
&= a^2\left(\dot{H}+3H^2\right) + \frac{a^2}{F}\left(H\dot{F}+H\dot{F}\frac{\ddot{F}}{2}+\frac{3H\dot{F}}{2}\right) \\
&= a^2\left[\left(\dot{H}+3H^2\right) + \frac{1}{2F}\left(\ddot{F}+5H\dot{F}\right)\right].
\end{aligned}
\tag{C9}
$$

Similarly, by substituting Eq. (C5) into Eq. (31) we find

$$
\begin{aligned}
R\left(g,\tilde{\Gamma}\right) &= R(g) + \frac{3}{F}\left[\frac{\left(\nabla F\right)^2}{2F} - \Box F\right] \\
&= 6\left(\dot{H}+2H^2\right) - \frac{3}{F}\left(\frac{\dot{F}^2}{2F} - \ddot{F} - 3H\dot{F}\right).
\end{aligned}
\tag{C10}
$$

Some combinations of the above equations lead to

$$
\begin{aligned}
G_{00}\left(g,\tilde{\Gamma}\right) &= \tilde{R}_{00}\left(\tilde{\Gamma}\right) - \frac{g_{00}}{2}R\left(g,\tilde{\Gamma}\right) \\
&= G_{00}(g) + \frac{1}{F}\left\{\left(g_{00}\Box - \nabla_0\nabla_0\right)F\right. \\
&\quad \left. +\frac{3}{2F}\left[\nabla_0 F\,\nabla_0 F - \frac{g_{00}}{2}\left(\nabla F\right)^2\right]\right\} \\
&= 3H^2 + \frac{1}{2F}\left(\frac{3\dot{F}^2}{2F}+6H\dot{F}\right)
\end{aligned}
\tag{C11}
$$

and

$$
\begin{aligned}
G_{ii}\left(g,\tilde{\Gamma}\right) &= \tilde{R}_{ii}\left(\tilde{\Gamma}\right) - \frac{g_{ii}}{2}R\left(g,\tilde{\Gamma}\right) \\
&= G_{ii}(g) + \frac{1}{F}\left\{\left(g_{ii}\Box - \nabla_i\nabla_j\right)F\right. \\
&\quad \left. +\frac{3}{2F}\left[\nabla_i F\,\nabla_i F - \frac{g_{ii}}{2}\left(\nabla F\right)^2\right]\right\} \\
&= a^2\left[-\left(2\dot{H}+3H^2\right) + \frac{1}{2F}\left(\frac{3\dot{F}^2}{2F} - 2\ddot{F} - 4H\dot{F}\right)\right].
\end{aligned}
\tag{C12}
$$

Note that for the above calculations we have used the relations

$$
\nabla_\mu F = \partial_\mu F = \left(\dot{F},\,0,\,0,\,0,\right),
\tag{C13}
$$

$$
\nabla_0\nabla_0 F = \left(\partial_{00} - \Gamma^\lambda_{00}\partial_\lambda\right)F = \partial_{00}F = \ddot{F},
\tag{C14}
$$

$$
\nabla_i\nabla_i F = \left(\partial_{ii} - \Gamma^\lambda_{ii}\partial_\lambda\right)F = -\Gamma^0_{ii}\partial_0 F = -a^2 H\dot{F},
\tag{C15}
$$

and

$$
\begin{aligned}
\Box F &= g^{\mu\nu}\nabla_{\mu\nu}F = g^{\mu\nu}\left(\partial_\mu\partial_\nu - \Gamma^\lambda_{\mu\nu}\partial_\lambda\right)F \\
&= \left(-\partial_{00} - 3g^{11}\Gamma^0_{11}\partial_0\right)F = -\ddot{F} - \frac{3\dot{a}}{a}\dot{F} \\
&= -\ddot{F} - 3H\dot{F},
\end{aligned}
\tag{C16}
$$

respectively.

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