

Palm measures and rigidity phenomena in point processes*

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Abstract

We study the mutual regularity properties of Palm measures of point processes, and establish that a key determining factor for these properties is the rigidity-tolerance behaviour of the point process in question (for those processes that exhibit such behaviour). Thereby, we extend the results of [23], [2], [20] to new ensembles, particularly those that are devoid of any determinantal structure. These include the zeroes of the standard planar Gaussian analytic function and several others.

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1 Introduction

Our aim in this article is to study the mutual singularity (and continuity) properties of Palm measures of point processes. Roughly speaking, the Palm measure of a point process Π (that lives on a space Ξ) with respect to a vector $\underline{\zeta} \in \Xi^r$ is the law of Π conditioned to contain the points in Ξ which are the co-ordinates of $\underline{\zeta}$. In subsequent discussions in Section 3.1, we will provide a rigorous description of Palm measures.

Let $\mathbb{P}_{\underline{\zeta}}$ denote the Palm measure of Π with respect to $\underline{\zeta}$. We are interested in the mutual singularity (and continuity) of $\mathbb{P}_{\underline{\zeta}_1}$ and $\mathbb{P}_{\underline{\zeta}_2}$ for two different vectors $\underline{\zeta}_1$ and $\underline{\zeta}_2$. According to the heuristic description of the Palm measure (as also the rigorous definition to follow in Section 3.1), a random point configuration ξ sampled from the Palm measure necessarily includes the points corresponding to $\underline{\zeta}$, which makes the above question somewhat trivial - roughly speaking, we can decide the identity of the measure by examining whether it contains the points from $\underline{\zeta}_1$ or $\underline{\zeta}_2$. However, it is often customary to think of the Palm measure to be the law of $(\xi \setminus \text{the points of } \underline{\zeta})$. Under this identification, the question of mutual regularity becomes an interesting one, and different answers can be obtained in different natural models.

For the Poisson process, which is the most commonly studied model of point processes, the answer to the question of mutual regularity is trivial: any two Palm measures are always mutually absolutely continuous. This follows from the spatial independence property of the Poisson process, and is valid under mild assumptions on the intensity measure of the Poisson process (principally entailing that the intensity measure does not contain atoms).

In [23], Osada and Shirai studied this question with respect to the Ginibre ensemble, which is a determinantal point process arising out of the eigenvalues of non-Hermitian random matrices. They found a very interesting behaviour: any two Palm measures of the

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Ginibre ensemble are mutually absolutely continuous if the lengths of the conditioning vectors are equal, and they are mutually singular otherwise. This indicates that the study of point processes with strong spatial correlation can throw up surprising answers to the question of comparing Palm measures.

Before moving ahead, we formally state their result (Theorem 1.1 in [23]) below, where we denote by \mathcal{G} the law of the Ginibre ensemble:

Theorem 1.1. *Assume that $\underline{x} \in \mathbb{C}^l$ and $\underline{y} \in \mathbb{C}^n$. If $l = n$, then $\mathcal{G}_{\underline{x}}$ and $\mathcal{G}_{\underline{y}}$ are mutually absolutely continuous. In addition, if $l \neq n$, then $\mathcal{G}_{\underline{x}}$ and $\mathcal{G}_{\underline{y}}$ are singular to each other.*

Results on the similar comparative behaviour of reduced Palm measures have been established by Bufetov in [2] for a large class of determinantal point processes on \mathbb{R} with integrable projection kernels (including the sine, Airy and Bessel processes), by Bufetov and Qiu for a large class of determinantal processes on \mathbb{C} ([4]), and for the Gamma process in 1D, a similar result goes back to the work of Olshanski ([20]).

A key feature of these results is that they depend crucially on the determinantal structure of these models. In this work, we obtain results comparing Palm measures, similar in spirit to (and extending) Theorem 1.1, for a wider class of point processes, particularly those not having any determinantal structure. Key examples of this include the zeroes of the standard planar Gaussian analytic function and its generalizations. Our results also exhibit more delicate dependence on the vectors \underline{x} and \underline{y} for absolute continuity (as contrasted with the simple dependence on dimension for the determinantal processes discussed above).

The study of spatial conditioning in point processes with strong correlation structures has attracted a fair amount of interest in recent years. Principal examples of models studied in this regard include the Ginibre ensemble, the zeroes of the standard planar Gaussian analytic function (henceforth abbreviated as GAF), the sine kernel process on the real line, the Airy and Bessel processes, and so on. In [10], the authors showed that in the Ginibre ensemble, the points outside a domain D determine exactly the number of points in D . For the standard planar GAF zero process, they showed that the points outside D determine the number as well as the sum of the points in D . Furthermore, they showed that the point configuration outside D does not determine anything more about the points inside D . To give an idea of the precise technical sense in which these assertions hold, we will quote the relevant theorems for the Ginibre and the planar GAF zero ensembles (denoted by \mathcal{G} and \mathcal{Z} respectively).

In order to do so, we need to introduce some notation, which will come in handy for understanding our main results as well. A (simple) point process Π is a random locally finite point configuration on some metric space Ξ that is equipped with a regular Borel measure μ . A point process can equivalently be looked upon as a random counting measure, with atoms corresponding to the points. For more details on point processes, we refer the reader to [6] and [15]. In particular, we will be making use of the notion of the r -point intensity measures of a point process, for a concrete definition of which we direct the reader to [12] (Chapter 1, Section 1.2).

The space \mathcal{S} of locally finite point configurations on Ξ is a Polish space, and a point process Π on Ξ can be thought of as a probability measure on \mathcal{S} . Let $\mathcal{D} \subset \Xi$ be a bounded open set. The decomposition $\Xi = \mathcal{D} \cup \mathcal{D}^c$ induces a factorization $\mathcal{S} = \mathcal{S}_{\text{in}} \times \mathcal{S}_{\text{out}}$, where \mathcal{S}_{in} and \mathcal{S}_{out} are respectively the spaces of finite point configurations on \mathcal{D} and locally finite point configurations on \mathcal{D}^c . This immediately leads to the natural decomposition $\Upsilon = (\Upsilon_{\text{in}}, \Upsilon_{\text{out}})$ for any $\Upsilon \in \mathcal{S}$, and consequently a decomposition of the point process Π as $\Pi = (\Pi_{\text{in}}, \Pi_{\text{out}})$.

We are now ready to state from [10] the results for the Ginibre and the planar GAF zero ensembles; in what follows, \mathcal{D} is a bounded open set in \mathbb{C} .

Theorem 1.2. *For the Ginibre ensemble, there is a measurable function $N : \mathcal{S}_{\text{out}} \rightarrow \mathbb{N} \cup \{0\}$ such that a.s.*

$$\text{Number of points in } \mathcal{G}_{\text{in}} = N(\mathcal{G}_{\text{out}}) .$$

Since a.s. the length of $\underline{\zeta}$ equals $N(\mathcal{G}_{\text{out}})$, we can assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ is supported on $\mathcal{D}^{N(\mathcal{G}_{\text{out}})}$.

Theorem 1.3. *For the Ginibre ensemble, \mathbb{P} -a.s. the measure $\rho(\mathcal{G}_{\text{out}}, \cdot)$ and the Lebesgue measure \mathcal{L} on $\mathcal{D}^{N(\mathcal{G}_{\text{out}})}$ are mutually absolutely continuous.*

In the case of the GAF zero process, we prove that the points outside D determine the number as well as the centre of mass (or equivalently, the sum) of the points inside D , and “nothing more”.

Theorem 1.4. *For the GAF zero ensemble,*

(i) *There is a measurable function $N : \mathcal{S}_{\text{out}} \rightarrow \mathbb{N} \cup \{0\}$ such that a.s.*

$$\text{Number of points in } \mathcal{Z}_{\text{in}} = N(\mathcal{Z}_{\text{out}}) .$$

(ii) *There is a measurable function $S : \mathcal{S}_{\text{out}} \rightarrow \mathbb{C}$ such that a.s.*

$$\text{Sum of the points in } \mathcal{Z}_{\text{in}} = S(\mathcal{Z}_{\text{out}}) .$$

For a possible value Υ_{out} of \mathcal{Z}_{out} , define the set of admissible vectors of inside points (obtained by considering all possible orderings of such inside point configurations)

$$\Sigma_{S(\Upsilon_{\text{out}})} := \{ \underline{\zeta} \in \mathcal{D}^{N(\Upsilon_{\text{out}})} : \sum_{j=1}^{N(\Upsilon_{\text{out}})} \zeta_j = S(\Upsilon_{\text{out}}) \}$$

where $\underline{\zeta} = (\zeta_1, \dots, \zeta_{N(\Upsilon_{\text{out}})})$.

Since a.s. the length of $\underline{\zeta}$ equals $N(\Upsilon_{\text{out}})$, we can assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ gives us the distribution of a random vector in $\mathcal{D}^{N(\Upsilon_{\text{out}})}$ supported on $\Sigma_{S(\Upsilon_{\text{out}})}$.

Theorem 1.5. *For the GAF zero ensemble, \mathbb{P} -a.s. the measure $\rho(\mathcal{Z}_{\text{out}}, \cdot)$ and the Lebesgue measure \mathcal{L}_{Σ} on $\Sigma_{S(\mathcal{Z}_{\text{out}})}$ are mutually absolutely continuous.*

We refer to as “rigidity” the phenomenon in which the point configuration outside D exactly determines certain statistics of the points inside D . By “tolerance”, we refer to the phenomenon in which, subject to the constraints imposed by the rigidity properties, points inside D can be found in “almost any” possible configuration.

Formally, we define rigidity and tolerance as follows.

Definition 1.6. *A measurable function $f_{\text{in}} : \mathcal{S}_{\text{in}} \rightarrow \mathbb{C}$ is said to be **rigid** with respect to the point process X on S if there is a measurable function $f_{\text{out}} : \mathcal{S}_{\text{out}} \rightarrow \mathbb{C}$ such that a.s. we have $f_{\text{in}}(X_{\text{in}}) = f_{\text{out}}(X_{\text{out}})$.*

In order to give a rigorous definition of tolerance, we restrict our scope a little bit, though the present definition still captures all the known examples, and is easily amenable to generalization.

Definition 1.7. *Let Π be a point process on Ξ and $\varphi : \Xi \rightarrow \mathbb{C}$ be a measurable function. Then the linear statistics $\Lambda(\varphi)[\Pi]$ is defined to be the random variable*

$$\Lambda(\varphi)[\Pi] := \int_{\Xi} \varphi(z) d[\Pi](z),$$

where $d[\Pi]$ is the (random) counting measure naturally associated with Π .

Definition 1.8. Let Π be a point process on a Riemannian manifold Ξ with volume measure μ . Let $\mathcal{D} \subset \Xi$ be a bounded open set, and let $\{\Lambda(\Phi_j)[\Pi_{\text{in}}]\}_{j=0}^t$ be rigid linear statistics, with $\Phi_0 \equiv 1$ and $\Phi_1, \dots, \Phi_t : \mathcal{D} \rightarrow \mathbb{C}$ smooth functions. For an integer $m \geq 0$ and $\underline{s} := (s_1, \dots, s_t) \subset \mathbb{C}^t$, consider the submanifold of \mathcal{D}^m

$$\Sigma_{m,\underline{s}} := \{\underline{\zeta} = (\zeta_1, \dots, \zeta_m) \in \mathcal{D}^m : \Lambda(\Phi_j)[\delta_{\underline{\zeta}}] = s_j; 1 \leq j \leq t\},$$

where $\delta_{\underline{\zeta}} \in \mathcal{S}$ is the point configuration corresponding to the point set $\{\zeta_i\}_{i=1}^m$.

Then Π is said to be tolerant subject to $\{\Lambda(\Phi_j)[\Pi_{\text{in}}]\}_{j=0}^t$ if the conditional distribution $(\Pi_{\text{in}} | \Pi_{\text{out}} = \omega)$ is mutually absolutely continuous with the point process of $\Lambda(\Phi_0)[\Pi_{\text{in}}] = N(\omega)$ points sampled independently from the submanifold $\Sigma_{N(\omega),\underline{s}}$ (where $s_i = \Lambda(\Phi_i)[\Pi_{\text{in}}] = S_i(\omega), 1 \leq i \leq t$) equipped with the restriction of the volume measure $\mu^{\otimes N(\omega)}$.

Remark 1.9. One can generalise the above notion of tolerance by demanding constraints on more general functionals than linear statistics. For example, for a fixed positive integer k , one can consider a smooth function $\Psi : \mathcal{D}^k \rightarrow \mathbb{C}$ and demand that the functional $\Lambda^k(\Psi)[\Pi_{\text{in}}] := \int \dots \int \Psi d[\Pi_{\text{in}}]^{\otimes k}$ is rigid.

Finally, we define a regular collection of smooth functions:

Definition 1.10. Consider a collection of smooth functions $\{\Phi_1, \dots, \Phi_k\}$, each mapping $\Xi \rightarrow \mathbb{C}$. We also consider the associated function

$$\Psi_r : \Xi^r \rightarrow \mathbb{C}^k$$

given by

$$\Psi_r(\underline{\zeta}) := (\Lambda(\Phi_1)[\delta_{\underline{\zeta}}], \dots, \Lambda(\Phi_k)[\delta_{\underline{\zeta}}]).$$

We call such a collection to be regular if, for each $r \geq k$, the Jacobian of the function Ψ_r is of full rank a.e.

The phenomena of rigidity and tolerance have been used to understand various questions regarding point processes, particularly those with a stochastic geometric flavour. In [7], the rigidity of the sine kernel process was used in order to settle a natural completeness question regarding random exponential functions arising out of the sine process. More generally (Theorem 1.3 therein), a positive resolution was obtained with regard to a natural completeness question for determinantal point processes, under the assumption that the point process in question exhibits rigidity with regard to the number of points. In [9], the authors used rigidity and tolerance phenomena from [10] to study continuum percolation on the Ginibre and the standard planar GAF zero ensembles, in particular to establish the uniqueness of the infinite cluster in the supercritical regime. In [21], Osada used a related quasi-Gibbs property in order to study dynamics on the Ginibre ensemble. In [1] and [3], Bufetov et al. examine further interesting models of point processes from the perspective of rigidity, and obtained proofs of the rigidity of the number of points for the Airy and the Bessel processes, among others.

2 Statement of main results

In this article, we explore the connections between rigidity phenomena and results of the nature of Theorem 1.1. More specifically, we extend the results of [23] to point processes exhibiting rigidity and tolerance phenomena of a given nature. In heuristic terms, we show that for a point process exhibiting rigidity behaviour with respect to the statistics $\{m_i\}_{i=0}^k$ (and “nothing more”), the Palm measures \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually absolutely continuous if $m_i(\zeta_1) = m_i(\zeta_2), 0 \leq i \leq k$, and they are mutually singular otherwise. This fits in nicely with Theorem 1.1, given the fact that Theorems 1.2 and 1.3

show that the Ginibre ensemble exhibits rigidity in the number of points (and “nothing more”).

However, it also enables us to obtain similar theorems about the mutual regularity of different Palm measures with respect to many other models, which often exhibit a much more complicated correlation behaviour than the Ginibre (already discernible in the joint density structure of the finite particle approximations). E.g., the zeroes of the standard planar GAF exhibit interactions of all orders (as opposed to pairwise interactions like in the Ginibre ensemble).

We formally state our main theorem as follows:

Theorem 2.1. *Let Π be a point process on a Riemannian manifold Ξ (without boundary) with volume measure μ , and having r -point intensity measures ρ_r mutually absolutely continuous w.r.t. $\mu^{\otimes r}$ for all r . Let $\Phi_0 \equiv 1$, and let Φ_1, \dots, Φ_k be a regular collection of smooth functions mapping $\Xi \rightarrow \mathbb{C}$ such that, for any bounded open set $\mathcal{D} \subset \Xi$, the linear statistics $\{\Lambda(\Phi_i)[\Pi_{\text{in}}]\}_{i=0}^k$ are rigid, and Π is tolerant subject to $\{\Lambda(\Phi_i)[\Pi_{\text{in}}]\}_{i=0}^k$. Then, for a.e. pair of vectors $(\zeta_1, \zeta_2) \in \Xi^m \times \Xi^n$, the reduced Palm measures \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} of Π (at ζ_1, ζ_2 respectively) are mutually singular if*

$$\left(\Lambda(\Phi_i)[\delta_{\zeta_1}]\right)_{i=0}^k \neq \left(\Lambda(\Phi_i)[\delta_{\zeta_2}]\right)_{i=0}^k.$$

Conversely, for every $r \geq k$ and a.e. $\underline{a} \in \mathbb{C}^k$, \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually absolutely continuous for a.e. pair $(\zeta_1, \zeta_2) \in \Xi^r$ such that $\Lambda(\Phi_0)[\delta_{\zeta_1}] = \Lambda(\Phi_i)[\delta_{\zeta_1}] = r$ and

$$\left(\Lambda(\Phi_i)[\delta_{\zeta_1}]\right)_{i=1}^k = \left(\Lambda(\Phi_i)[\delta_{\zeta_2}]\right)_{i=1}^k.$$

Remark 2.2. For the singularity statement in Theorem 2.1, the pair (ζ_1, ζ_2) are a.e. with respect to the measure $\rho_m \times \rho_n$ (equivalently, $\mu^{\otimes m} \times \mu^{\otimes n}$) on $\Xi^m \times \Xi^n$. Recall the map Ψ from Definition 1.10. For the absolute continuity statement, $\underline{a} \in \mathbb{C}^k$ is a.e. with respect to the push-forward of ρ_r (equivalently, $\mu^{\otimes r}$) under Ψ and (ζ_1, ζ_2) are a.e. with respect to the induced measure (from Ξ^r) on the submanifold

$$\mathcal{M}_{\underline{a}} = \{\zeta : \left(\Lambda(\Phi_i)[\delta_{\zeta}]\right)_{i=1}^k = \underline{a}\} \subset \Xi^r.$$

Remark 2.3. Theorem 2.1 goes through verbatim (with the same proof) if each Φ_i maps into \mathbb{R} instead of \mathbb{C} . We use \mathbb{C} in the present article because many of our interesting examples, including the zeroes of Gaussian analytic functions, are naturally covered in that setting.

One of the foremost instances where Theorem 2.1 extends the state of the art is the case of the standard planar GAF zero process. In [8], the authors introduce a family of point processes, which are zeroes of Gaussian analytic functions indexed by a parameter α . These ensembles are called α -GAFs, and they establish that for $\alpha \in (\frac{1}{k}, \frac{1}{k-1}]$, the α -GAF zero process exhibits rigidity at level k . That is, the configuration outside a domain determines the number and the first $k - 1$ moments of the inside points, and “nothing more”. Consequently, our result implies that for a.e. ζ_1 and ζ_2 , the measures \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually absolutely continuous if the first k moments of (the co-ordinates of) ζ_1 and ζ_2 are the same, and they are mutually singular otherwise. This shows, in particular, that the mutual regularity properties of the different Palm measures of a point process can depend on the conditioning vector in an arbitrarily complicated manner (the complexity of the dependence being measured by the number of statistics that need to be matched in order to ensure absolute continuity).

In [13] and [22], a *quasi Gibbs* property is established for sine, Airy ($\beta = 1, 2, 4$) and Bessel ($\beta = 2$) point processes. Rigidity of the number of points for these processes

was established in [1], and tolerance of these point processes (subject to the number of points) can be deduced from this quasi Gibbs property. Consequently, we can invoke our Theorem 2.1 to obtain a new proof of the analogue of Theorem 1.1 for these processes.

We formally state these results as follows (we denote by $\zeta(i)$ the i -th co-ordinate of the vector $\underline{\zeta}$):

Theorem 2.4. *Let Π be a point process on a Riemannian manifold Ξ with volume measure μ , and let $\zeta_1 \in \Xi^m, \zeta_2 \in \Xi^n$. Then the following statements are true about the reduced Palm measures \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} :*

- *When Π the standard planar GAF zero process on \mathbb{C} ,*
 - *For Lebesgue-a.e. $s \in \mathbb{C}$ and a.e. ζ_1, ζ_2 such that $|\zeta_1| = |\zeta_2|$ and $\sum_{i=1}^{|\zeta_1|} \zeta_1(i) = \sum_{i=1}^{|\zeta_2|} \zeta_2(i) = s$, \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually absolutely continuous.*
 - *For a.e. ζ_1, ζ_2 such that $|\zeta_1| \neq |\zeta_2|$ or $\sum_{i=1}^{|\zeta_1|} \zeta_1(i) \neq \sum_{i=1}^{|\zeta_2|} \zeta_2(i)$, \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually singular.*
- *When Π the α -GAF zero process on \mathbb{C} ,*
 - *For Lebesgue-a.e. $\underline{m} \in \mathbb{C}^{\lfloor \frac{1}{\alpha} \rfloor}$ and a.e. ζ_1, ζ_2 such that $|\zeta_1| = |\zeta_2|$ and $\sum_{i=1}^{|\zeta_1|} \zeta_1(i)^j = \sum_{i=1}^{|\zeta_2|} \zeta_2(i)^j = \underline{m}(j)$ for all $1 \leq j \leq \lfloor \frac{1}{\alpha} \rfloor$, \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually absolutely continuous.*
 - *For a.e. ζ_1, ζ_2 such that $|\zeta_1| \neq |\zeta_2|$ or $\sum_{i=1}^{|\zeta_1|} \zeta_1(i)^j \neq \sum_{i=1}^{|\zeta_2|} \zeta_2(i)^j$ for some $1 \leq j \leq \lfloor \frac{1}{\alpha} \rfloor$, \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually singular.*
- *For Π the i.i.d. perturbation of \mathbb{Z}^2 (resp., \mathbb{Z}) with random variables having a non-vanishing density on \mathbb{R}^2 (resp. \mathbb{R}) with a finite second (resp., first) moment, we have, for Lebesgue-a.e. $(\zeta_1, \zeta_2) \in \mathbb{C}^m \times \mathbb{C}^n$ (resp., $\mathbb{R}^m \times \mathbb{R}^n$),*
 - \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually singular if $m \neq n$
 - \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually absolutely continuous if $m = n$

Remark 2.5. For i.i.d. perturbations of \mathbb{Z}^d by d -dimensional Gaussians having small enough variance, a similar conclusion as the 1 or 2D lattice perturbations above holds.

This theorem follows from our main Theorem 2.1, coupled with the results on the rigidity and tolerance properties of these ensembles, as in [10] (Theorem 1.1 - Theorem 1.4) and [8] (Theorem 2.1). Rigidity of the number of points for i.i.d. lattice perturbations satisfying the conditions in the statement of Theorem 2.4 has been established in [24]; the tolerance (in our terminology) is a simple consequence of the fact that the perturbations are independent and have a positive density a.e. with respect to the Lebesgue measure. The remark about Gaussian perturbations also follows from rigidity established in [24] and a tolerance statement that follows from independence considerations.

Here we illustrate the details in the case of the standard planar GAF zero process; the details in the other cases are on similar lines. Our goal is to verify that the standard planar GAF zero process satisfies the conditions of Theorem 2.1. For this, we will make use of the rigidity-tolerance behaviour of this point process, which was established in [10]; for convenience the relevant results have been quoted here as Theorems 1.4 and 1.5. In terms of the conditions laid out in Theorem 2.1, clearly $\Xi = \mathbb{C}$ in this case, with μ the Lebesgue measure on \mathbb{C} . It is well known (also easy to see from the definition of the GAF) that the r -point intensity measures of the GAF zeros have densities with respect to the Lebesgue measure on \mathbb{C}^r (for a specific reference, see [12]). We put $k = 1$ and

$\Phi_1(z) = z$. Clearly, the function Φ_1 is a *regular* as per Definition 1.10, as can be seen from the fact that the Jacobian of the map

$$\Psi_r : (z_1, \dots, z_r) \mapsto \sum_{i=1}^r z_i$$

is $[1, 1, \dots, 1]$, which is always of full rank. Theorem 1.4 and 1.5 are equivalent to the statement that for any bounded open set $D \subset \mathbb{C}$, the statistics $(\Lambda(\Phi_i)[\Pi_{\text{in}}])_{i=0}^1 = (N(\Pi_{\text{in}}), S(\Pi_{\text{in}}))$ are rigid, and the GAF zero process is tolerant subject to $(N(\Pi_{\text{in}}), S(\Pi_{\text{in}}))$ (recall the Definitions 1.6 and 1.8 of rigidity and tolerance respectively). This verifies the conditions of Theorem 2.1 for the GAF zero process.

We now interpret the conclusions of Theorem 2.1 for the GAF zero process. For a vector $\underline{\zeta} \in \mathbb{C}^r$, denote by $|\underline{\zeta}|$ the dimension and by $S(\underline{\zeta})$ the sum of the co-ordinates of $\underline{\zeta}$. Then Theorem 2.1 implies that for a.e.-pair $(\underline{\zeta}_1, \underline{\zeta}_2) \in \mathbb{C}^m \times \mathbb{C}^n$ such that $(|\underline{\zeta}_1|, S(\underline{\zeta}_1)) \neq (|\underline{\zeta}_2|, S(\underline{\zeta}_2))$, the reduced Palm measures $\mathbb{P}_{\underline{\zeta}_1}$ and $\mathbb{P}_{\underline{\zeta}_2}$ are mutually singular. In particular, this implies that if $m \neq n$, then $\mathbb{P}_{\underline{\zeta}_1}$ and $\mathbb{P}_{\underline{\zeta}_2}$ are mutually singular for Lebesgue a.e. $(\underline{\zeta}_1, \underline{\zeta}_2) \in \mathbb{C}^m \times \mathbb{C}^n$. This brings us to the situation $m = n$. In this scenario, there are two possibilities: $S(\underline{\zeta}_1) \neq S(\underline{\zeta}_2)$ and $S(\underline{\zeta}_1) = S(\underline{\zeta}_2)$. Regarding the former possibility, for Lebesgue a.e. $(\underline{\zeta}_1, \underline{\zeta}_2) \in \mathbb{C}^m \times \mathbb{C}^m$, Theorem 2.1 states that $\mathbb{P}_{\underline{\zeta}_1}$ and $\mathbb{P}_{\underline{\zeta}_2}$ are mutually singular. A particularly interesting case of this is when $m = n = 1$, which we state as a separate corollary:

Corollary 2.6. *For the standard planar GAF zero process, the reduced Palm measures \mathbb{P}_z and \mathbb{P}_w are mutually singular for Lebesgue a.e. pair $(z, w) \in \mathbb{C} \times \mathbb{C}$.*

This contrasts markedly with the analogous comparison of Palm measures in most determinantal processes, including the Ginibre process (Theorem 1.1).

This leaves us with the final case: $m = n$ and $S(\underline{\zeta}_1) = S(\underline{\zeta}_2)$. Denoting $S(\underline{\zeta}_1) = S(\underline{\zeta}_2) = s \in \mathbb{C}$, we consider the manifold

$$\mathcal{M}_s := \{\underline{\zeta} \in \mathbb{C}^m : S(\underline{\zeta}) = s\}.$$

\mathcal{M}_s carries a natural Lebesgue measure, induced from the Lebesgue measure on \mathbb{C}^m , denote this measure by l_s . Then Theorem 2.1 says that for Lebesgue a.e. s , we have that for l_s -a.e. pair $(\underline{\zeta}_1, \underline{\zeta}_2) \in \mathcal{M}_s$, the reduced Palm measures $\mathbb{P}_{\underline{\zeta}_1}$ and $\mathbb{P}_{\underline{\zeta}_2}$ are mutually absolutely continuous.

3 Proof of Theorem 2.1

In this section, we prove our main Theorem 2.1. We will first analyze the support properties of Palm measures, and then connect it with rigidity properties of point processes, in two subsections.

3.1 Palm measures and their support

Let, as before, \mathcal{S} denote the Polish space of all locally finite point configurations on Ξ , and $\mathcal{B}(\Xi)$ denote the Borel sigma field on \mathcal{S} . We begin with the 1-point Campbell measure and a rigorous definition of the 1-point Palm measure of point processes, we refer the reader to [15] Chapter 10 for a more detailed treatment than we present here. The 1-point Campbell measure $\mu^{(1)}$ of a point process Π (whose law we denote by \mathbb{P}) is the measure defined on $\Xi \times \mathcal{S}$ given by

$$\int f(s, \xi) d\mu^{(1)}(s, \xi) := \int \left(\int f(s, \xi) d[\xi](s) \right) d\mathbb{P}(\xi).$$

In the above equation, the measure $[\xi]$ is the counting measure that naturally corresponds to $\xi \in \mathcal{S}$. The 1-point Palm measures $\{\mathbb{P}_s : s \in \mathcal{S}\}$ (that include the points in the

conditioning vector) are defined by a decomposition of the measure $\mu^{(1)}$ into a regular conditional measure with respect to the first co-ordinate:

$$d\mu^{(1)}(s, \xi) = d\rho_1(s) \times d\bar{\mathbb{P}}_s(\xi).$$

In other words, we define the 1-point Palm measure by the integral formulation

$$\int f(s, \xi) d\mu^{(1)}(s, \xi) = \int \left(\int f(s, \xi) d\bar{\mathbb{P}}_s(\xi) \right) d\rho_1(s).$$

Here ρ_1 is the 1-point intensity measure of Π .

For any simple locally finite point configuration $\xi \in \mathcal{S}$ and an integer $r \geq 1$, let us denote by $[\xi]^{\wedge r}$ the counting measure on all possible ordered r -tuples of distinct points of ξ . Then, for any integer $r \geq 1$, the r -point Campbell measure $\mu^{(r)}$ can be defined as a measure on $\Xi^r \times \mathcal{S}$ given by

$$\int f(\underline{s}, \xi) d\mu^{(r)}(s, \xi) := \int \left(\int f(\underline{s}, \xi) d[\xi]^{\wedge r}(\underline{s}) \right) d\mathbb{P}(\xi).$$

Consequently, one can define $\bar{\mathbb{P}}_{\underline{s}}$, the r -point Palm measure at \underline{s} (that includes the points in \underline{s}) by

$$d\mu^{(r)}(\underline{s}, \xi) = d\rho_r(\underline{s}) \times d\bar{\mathbb{P}}_{\underline{s}}(\xi),$$

or equivalently,

$$\int f(\underline{s}, \xi) d\mu^{(r)}(\underline{s}, \xi) = \int \left(\int f(\underline{s}, \xi) d\bar{\mathbb{P}}_{\underline{s}}(\xi) \right) d\rho_r(\underline{s}),$$

where ρ_r is the r -point intensity measure of Π .

Since $\pi(\zeta) \subset \xi$ for each $\xi \in \text{Supp}(\bar{\mathbb{P}}_{\zeta})$, therefore, we can equivalently consider the law of $\xi \setminus \zeta$. We call this measure the *reduced* Palm measure of Π at ζ . We will denote this measure by \mathbb{P}_{ζ} .

Let Q be a countable dense subset of Ξ . We will call a subset G of Ξ to be *good* if G is the union of finitely many disjoint open balls with distinct centres in Q and identical rational radius. We will say that a nested sequence $\{G_n\}$ of good subsets of Ξ (having a fixed number m of constituent balls) *converge* to $\underline{p} \in \Xi^m$ if $G_{n+1} \subset G_n$ and the centres of the constituent balls of G_n converge to \underline{p} (in some ordering of the co-ordinates). In such a situation, we will say that $\underline{p} \in \Xi^m$ is a *limit* of $\{G_n\}$. Finally, we will say that $\zeta \in \Xi^r$, $r \leq m$ (with distinct co-ordinates) *belongs* to the limit \underline{p} of such a sequence G_n of good sets (equivalently, we say that \underline{p} *contains* ζ) if the co-ordinates of ζ are a subset of those of \underline{p} .

Recall, from Section 2, the notation that $\underline{m}(\zeta)$ denotes the vector $(m_1(\zeta), \dots, m_k(\zeta))$, where $m_i(\zeta) = \Lambda(\Phi_i)[\delta_{\zeta}]$. For any bounded open set $D \subset \Xi$, the number of points of our process Π that lie in D will be denoted by $N(D)$. This quantity is a measurable function of the point configuration θ in $\mathcal{D}^{\mathbb{G}}$ because of the rigidity of the number of points (which corresponds to the functional $\Psi_0 \equiv 1$), and we will denote this function by $m_0(\theta; D)$. Similarly, $\underline{m}(\Pi_{\text{in}})$ (where Π_{in} denotes the points of Π inside D in uniform random order) is a measurable function of θ , and we denote this by $\mathfrak{M}(\theta; D)$.

Let $r \leq p$ be positive integers. For a good set $G \subset \Xi$ having p constituent balls, we define the event $\mathcal{A}(G, r) \in \mathcal{B}(\Xi)$, which entails that $\xi \in \mathcal{A}(G, r)$ if

- ξ is supported on $G^{\mathbb{G}}$
- $m_0(\xi; G) = r$.

For every $\zeta \in G^r$, we define the event $\mathcal{A}(G, \zeta) \in \mathcal{B}(\Xi)$, which entails that $\xi \in \mathcal{A}(G, \zeta)$ if

- ξ is supported on $G^{\mathbb{G}}$
- $m_0(\xi; G) = |\underline{\zeta}| = r$
- $\mathfrak{M}(\xi; G) = \underline{m}(\underline{\zeta})$.

Finally, we say that a sequence of sets $\{B_n\}$ exhausts the support of a measure γ if $\gamma(B_n^{\mathbb{G}}) \rightarrow 0$ as $n \rightarrow \infty$.

With these definitions, we are ready to state the following technical result:

Lemma 3.1. *For ρ_r -a.e. η , it holds that, for any nested sequence $\{G_n\}$ of good sets (having $p \geq r$ constituent balls for each n) with a limit that contains η :*

- (i) *The events $\mathcal{A}(G_n, r)$ exhaust the support of \mathbb{P}_η .*
- (ii) *The events $\mathcal{A}(G_n, \eta)$ exhaust the support of \mathbb{P}_η .*

Remark 3.2. Since we assume that the sequence of good sets G_n is nested, therefore η is contained in the limit of $\{G_n\}$ implies that $\eta \subset G_n$ for each n .

Proof. Observe that we trivially have the inclusion of events $\mathcal{A}(G_n, \underline{\zeta}) \subset \mathcal{A}(G_n, r)$. Therefore, it suffices to establish part (ii) of the Lemma, from which part (i) will follow.

We proceed as follows. First of all, for a good set G (with p constituent balls) and $\underline{\zeta} \in G^r$, consider the event $\mathcal{F}(G, \underline{\zeta})$ such that a point configuration $\xi \in \mathcal{F}(G, \underline{\zeta})$ entails that

- ξ is supported on $G^{\mathbb{G}}$
- $(m_0(\xi; G), \mathfrak{M}(\xi; G)) \neq (m_0(\underline{\zeta}), \underline{m}(\underline{\zeta}))$.

We assert that for ρ_r -a.e. $\underline{\zeta} \in G^r$, we have $\mathbb{P}_{\underline{\zeta}}[\mathcal{F}(G, \underline{\zeta})] = 0$. To this end, we observe that

$$\begin{aligned} & \int_{G^r} \mathbb{P}_{\underline{\zeta}}[\mathcal{F}(G, \underline{\zeta})] \rho_r(\underline{\zeta}) dV(\underline{\zeta}) \\ &= \mathbb{P}[\underline{\zeta} \cup \xi \text{ is a realisation of } \Pi \text{ for some } \underline{\zeta} \in G^r \text{ and some } \xi \in \mathcal{F}(G, \underline{\zeta})] \\ &= 0, \end{aligned}$$

where, in the last step we have used the fact that

$$\mathbb{P}[\underline{\zeta} \cup \xi \text{ is a realisation of } \Pi \text{ for some } \underline{\zeta} \in G^r \text{ and some } \xi \in \mathcal{F}(G, \underline{\zeta})] = 0$$

because of the rigidity properties of Π with respect to the set G . More precisely, since $\{\underline{\zeta} \cup \xi\} \cap G = \underline{\zeta}$ and $\{\underline{\zeta} \cup \xi\} \cap G^{\mathbb{G}} = \xi$ (as point sets), therefore by the rigidity of Π we have $(m_0(\xi; G), \mathfrak{M}(\xi; G))$ must equal $(m_0(\underline{\zeta}), \underline{m}(\underline{\zeta}))$ (for \mathbb{P} -a.e. realisation $\omega = \underline{\zeta} \cup \xi$ of the point process such that $\omega \cap G = \underline{\zeta}$ and $\omega \cap G^{\mathbb{G}} = \xi$). This proves the assertion.

Since there are only countably many good sets, we can deduce from the above that for ρ_r -a.e. $\underline{\zeta}$, we have $\mathbb{P}_{\underline{\zeta}}[\mathcal{F}(G, \underline{\zeta})] = 0$ for any good set G such that $\underline{\zeta} \in G^r$.

Now let us consider a $\underline{\zeta}$ satisfying the above assertion, and a nested sequence of good sets $\{G_n\}$ (with p constituent balls each) having a limit that contains $\underline{\zeta}$ (and, consequently, $\underline{\zeta} \subset G_n$ for each n). Consider the event $\mathcal{A}(G_n, \underline{\zeta})^{\mathbb{G}}$, under the reduced Palm measure $\mathbb{P}_{\underline{\zeta}}$. This event can occur only in two ways (respectively corresponding to the defining conditions of the event $\mathcal{A}(G_n, \underline{\zeta})$):

- There is at least one point of the Palm process $\mathbb{P}_{\underline{\zeta}}$ inside G_n .
- $\mathcal{F}(G_n, \underline{\zeta})$ occurs

By choice of $\underline{\zeta}$, we already have $\mathbb{P}_{\underline{\zeta}}[\mathcal{F}(G_n, \underline{\zeta})] = 0$. Thus, recalling that $N(U)$ denotes the number of points of a configuration that lie in the set U , we have

$$\mathbb{P}_{\underline{\zeta}}[\mathcal{A}(G_n, \underline{\zeta})^{\mathbb{G}}] \leq \mathbb{P}_{\underline{\zeta}}[N(G_n) \geq 1] \leq \mathbb{E}_{\underline{\zeta}}[N(G_n)] \downarrow 0$$

as $n \rightarrow \infty$, by the Dominated Convergence Theorem. ■

3.2 Palm measures and rigidity phenomena

3.2.1 Singularity

For $\zeta \in \Xi^p, p \geq 1$, recall the notation $m_i(\zeta) = \Lambda(\Phi_i)[\delta_\zeta], 0 \leq i \leq k$, and $\underline{m}(\zeta) = (m_1(\zeta), \dots, m_k(\zeta))$. Also recall that $m_0(\zeta) = |\zeta|$.

Consider $\zeta_1 \in \mathbb{C}^r$ and $\zeta_2 \in \mathbb{C}^s$ such that $(m_0(\zeta_1), \underline{m}(\zeta_1)) \neq (m_0(\zeta_2), \underline{m}(\zeta_2))$, both satisfying the conclusions of Lemma 3.1 part (ii) (this happens a.e.- $\rho_r \times \rho_s$). We also assume that ζ_1 and ζ_2 have distinct co-ordinates (both within and between themselves), since this also happens a.e.- $\rho_r \times \rho_s$. Let $\{G_n\}$ be a nested sequence of good sets, each having $r + s$ constituent balls (and each ball containing exactly one co-ordinate of either ζ_1 or ζ_2), such that $\{G_n\}$ has the limit (ζ_1, ζ_2) (in the sense of the definitions in the previous subsection).

By Lemma 3.1 part (ii), the support of \mathbb{P}_{ζ_1} is exhausted by $\mathcal{A}(G_n, \zeta_1)$ and the support of \mathbb{P}_{ζ_2} is exhausted by $\mathcal{A}(G_n, \zeta_2)$. But since $(m_0(\zeta_1), \underline{m}(\zeta_1)) \neq (m_0(\zeta_2), \underline{m}(\zeta_2))$, therefore $\mathcal{A}(G_n, \zeta_1) \cap \mathcal{A}(G_n, \zeta_2) = \emptyset$. In other words, $\mathcal{A}(G_n, \zeta_1) \subset \mathcal{A}(G_n, \zeta_2)^c$ and $\mathcal{A}(G_n, \zeta_2) \subset \mathcal{A}(G_n, \zeta_1)^c$.

We make the following claim: for two probability measures μ_1 and μ_2 on the same space, suppose there is a sequence of events B_n such that for any $\varepsilon > 0$, $\mu_1(B_n) > 1 - \varepsilon$ for large enough n , and $\mu_2(B_n) < \varepsilon$ for large enough n . Then μ_1 and μ_2 are mutually singular.

Before proving this claim, we note that this suffices to complete the proof of singularity. To see this, set $\mu_1 = \mathbb{P}_{\zeta_1}$, $\mu_2 = \mathbb{P}_{\zeta_2}$ and $B_n = \mathcal{A}(G_n, \zeta_1)$. For any $\varepsilon > 0$ we note that $\mathbb{P}_{\zeta_1}(\mathcal{A}(G_n, \zeta_1)) > 1 - \varepsilon$ for all large enough n because these sets exhaust the support of \mathbb{P}_{ζ_1} . But $\mathcal{A}(G_n, \zeta_1) \subset \mathcal{A}(G_n, \zeta_2)^c$, so $\mathbb{P}_{\zeta_2}(\mathcal{A}(G_n, \zeta_1)) < \varepsilon$ for all large enough n , because $\mathcal{A}(G_n, \zeta_2)$ -s exhaust the support of \mathbb{P}_{ζ_2} . Then, from the above claim, it follows that \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} are mutually singular.

It remains to prove the claim. Let $\mu_1, \mu_2, \{B_n\}$ be as in the claim. Passing to a sub-sequence if necessary, we may assume that $\sum_n \mu_2(B_n) < \infty$. Consider the event

$$B := \overline{\lim} B_n := \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} B_n.$$

Let C_N denote $\bigcup_{n \geq N} B_n$. For $\varepsilon > 0$ and N large enough, $\mu_1(C_N) \geq \mu_1(B_N) > 1 - \varepsilon$. But the C_N -s are decreasing in N , and hence $\mu_1(B) = \lim_{N \rightarrow \infty} \mu_1(C_N) = 1$. On the other hand,

$$\mu_2(C_N) \leq \sum_{n \geq N} \mu_2(B_n).$$

Since $\sum_n \mu_2(B_n) < \infty$, therefore the right hand side can be made arbitrarily small by choosing N large enough. Hence we have $\mu_2(B) = \lim_{N \rightarrow \infty} \mu_2(C_N) = 0$.

The upshot of this is that $\mu_1(B) = 1$, whereas $\mu_2(B) = 0$. Since μ_1, μ_2 are probability measures, this implies that $\mu_1(B^c) = 0$ and $\mu_2(B^c) = 1$. This completes the proof that the measures μ_1 and μ_2 are mutually singular.

This completes the proof that of mutual singularity of \mathbb{P}_{ζ_1} and \mathbb{P}_{ζ_2} .

3.2.2 Absolute continuity

For $\underline{a} \in \mathbb{C}^r$, denote by $\mathcal{M}_{\underline{a}}$ the set $\zeta \in \Xi^r$ such that $\underline{m}(\zeta) = \underline{a}$. Consider the r -point intensity measure $\rho_r dV$ on Ξ^r (where dV is the canonical volume measure on Ξ^r , and ρ_r is the r -point intensity function). Consider the map $\Psi_r : \Xi^r \mapsto \mathbb{C}^k$ given by $\zeta \mapsto \underline{m}(\zeta)$, which is of full rank. This implies that we can decompose $\rho_r dV$ (going to local co-ordinates if necessary) as $\mu(\underline{a}) d\underline{a} \times \nu(\underline{a}, \zeta) d\underline{l}_{\underline{a}}(\zeta)$, where $d\underline{a}$ is Lebesgue measure on \mathbb{C}^k , $\mu(\underline{a}) d\underline{a}$ is the push forward of $\rho_r dV$ to \mathbb{C}^k under Ψ_r , $d\underline{l}_{\underline{a}}$ is the induced measure on $\mathcal{M}_{\underline{a}}$

from dV , and $\nu(\underline{a}, \underline{\zeta})$ is a density (see, e.g., *smooth co-area formula*, [5], Chap. III). Roughly speaking, this corresponds to a foliation of Ξ^r by the level sets of \underline{m} . Since Ξ is covered by a countable union of such charts (i.e., local neighbourhoods), therefore it suffices to work on each such chart.

Since $\rho_r > 0$ a.e. with respect to the canonical volume measure of Ξ^r , therefore for $\mu(\underline{a})d\underline{a}$ -a.e. \underline{a} , we have $\nu(\underline{a}, \underline{\zeta}) > 0$ for $d\underline{a}$ -a.e. $\underline{\zeta}$. Consequently, we deduce the following

Claim 3.3. *For $\mu(\underline{a})d\underline{a}$ -a.e. \underline{a} , it is true that for $d\underline{a} \times d\underline{a}$ -a.e. pair $(\underline{\zeta}_1, \underline{\zeta}_2)$ (so that $\underline{m}(\underline{\zeta}_1) = \underline{m}(\underline{\zeta}_2) = \underline{a}$), we have $\nu(\underline{a}, \underline{\zeta}_1), \nu(\underline{a}, \underline{\zeta}_2) > 0$.*

In light of Lemma 3.1, we deduce that for $\mu(\underline{a})d\underline{a}$ -a.e. \underline{a} , we have that $d\underline{a}$ -a.e. $\underline{\zeta}$ satisfies the support properties as in the conclusion of Lemma 3.1.

Let $\underline{\zeta}_1$ and $\underline{\zeta}_2$ be two configurations such that $\underline{m}(\underline{\zeta}_1) = \underline{m}(\underline{\zeta}_2)$ and they satisfy the properties laid out in Claim 3.3 and Lemma 3.1. Let $\{D(\varepsilon)\}_{\varepsilon \downarrow 0} \subset \Xi$ be a nested sequence of good sets (heuristically speaking, they approximate the ε neighbourhood of the coordinates of $\underline{\zeta}_1$ and $\underline{\zeta}_2$) such that their limit is the $2r$ dimensional configuration $(\underline{\zeta}_1, \underline{\zeta}_2)$.

For brevity, in what follows, we will denote by \mathcal{A}_ε the event $\mathcal{A}(D(\varepsilon), r)$ (as defined in Section 3.1). Recall that, by definition, each point configuration $\xi \in \mathcal{A}_\varepsilon$ satisfies the following conditions:

- ξ is supported on $D(\varepsilon)^\complement$
- $m_0(\xi; D(\varepsilon)) = r$.

We consider the joint law $(\Pi_{\text{in}}, \Pi_{\text{out}})$ of the points in $D(\varepsilon)$ and in $D(\varepsilon)^\complement$ respectively. We denote by E_r the event that there are r points in $D(\varepsilon)$; equivalently $|\Pi_{\text{in}}| = r$. By the rigidity of the number of points in $D(\varepsilon)$, the event E_r is measurable with respect to Π_{out} . We consider the law of $(\Pi_{\text{in}}, \Pi_{\text{out}})$ conditioned on the event E_r . The rigidity properties of Π imply that, on the event E_r , the random variable $\underline{m}(\Pi_{\text{in}})$ is measurable with respect to Π_{out} . This implies that there is a regular conditional probability $Q_\varepsilon(\underline{m}(\Pi_{\text{in}}), d\xi)$ that pertains to the random variable Π_{out} given $\underline{m}(\Pi_{\text{in}})$ and given that E_r occurs.

For a given vector $\underline{a} \in \mathbb{C}^k$, let $\Sigma_{\underline{a}}$ denote the subset of $D(\varepsilon)^r$ such that $\underline{m}(\underline{\zeta}) = \underline{a}$ for all $\underline{\zeta} \in \Sigma_{\underline{a}}$; clearly $\Sigma_{\underline{a}} = D(\varepsilon)^r \cap \mathcal{M}_{\underline{a}}$. From the rigidity properties of our point process, we know that the random variable $\underline{m}(\Pi_{\text{in}})$ is a measurable function of Π_{out} . Moreover, we also know that on the event E_r (measurable with respect to Π_{out}), the conditional law of Π_{in} given Π_{out} , denoted $d\mathbb{P}(\underline{\zeta} | \Pi_{\text{out}} = \xi)$, has a density $f(\underline{\zeta}, \xi)$ with respect to the measure $d\underline{m}(\underline{\zeta})$ on $\Sigma_{\underline{m}(\underline{\zeta})}$ (note here that $\Sigma_{\underline{m}(\underline{\zeta})}$ is determined by ξ because $\underline{m}(\Pi_{\text{in}})$ is measurable with respect to Π_{out}).

We now consider the reduced Palm measure with respect to configurations in $D(\varepsilon)^r$ (on the event \mathcal{A}_ε). In other words, we consider the measures $\mathbb{P}_{\underline{\zeta}}(\cdot \cap \mathcal{A}_\varepsilon)$, where $\underline{\zeta} \in D(\varepsilon)^r$. To introduce our candidate for $\mathbb{P}_{\underline{\zeta}}(\cdot \cap \mathcal{A}_\varepsilon)$, we need to first introduce another quantity.

We can consider the measure κ on \mathcal{D}^r which is the marginal distribution of the points of Π inside \mathcal{D} on the event E_r (with the points being taken in uniform random order). In other words, this is the measure $\mathbb{P}[E_r]d(\Pi_{\text{in}}|E_r)$, where $(\Pi_{\text{in}}|E_r)$ is the law of Π_{in} given E_r occurs. Clearly, this measure is absolutely continuous with respect to the measure $\rho_r dV$. Consequently, the push forward $(\underline{m})_* \kappa$ is absolutely continuous with respect to the push forward $(\underline{m})_* [\rho_r dV]$. The latter measure, as we may recall is $\mu(\underline{a})d\underline{a}$, which means that there exists a density ϱ such that

$$[(\underline{m})_* d\kappa](\underline{a}) = \varrho(\underline{a})\mu(\underline{a})d\underline{a}. \tag{3.1}$$

Also, it follows from the above discussion that

$$\mathbb{P}[E_r][(\underline{m})_* d(\Pi_{\text{in}}|E_r)](\underline{a}) = [(\underline{m})_* d\kappa](\underline{a}). \tag{3.2}$$

Our candidate for $\mathbb{P}_{\underline{\zeta}}(\cdot \cap \mathcal{A}_\varepsilon)$ (for $\underline{\zeta} \in D(\varepsilon)^r$) is the following:

$$\beta_{\underline{\zeta}}(d\xi) = \varrho(\underline{m}(\underline{\zeta}))f(\underline{\zeta}, \xi)Q(\underline{m}(\underline{\zeta}), d\xi)/\nu(\underline{m}(\underline{\zeta}), \underline{\zeta}), \tag{3.3}$$

and for $\underline{\zeta}$ such that $\nu(\underline{m}(\underline{\zeta}), \underline{\zeta}) = 0$ we simply define $\beta_{\underline{\zeta}}(d\xi)$ to be 0. In other words, we claim that $\mathbb{P}_{\underline{\zeta}}(\cdot \cap \mathcal{A}_\varepsilon) = \beta_{\underline{\zeta}}(\cdot \cap \mathcal{A}_\varepsilon)$ for a.e. $\underline{\zeta} \in D(\varepsilon)^r$.

Let us check that this indeed true for a.e. $\underline{\zeta} \in D^r$. Consider an event $U \times (A \cap \mathcal{A}_\varepsilon)$ such that $U \subset D(\varepsilon)^r$ and A is a measurable set in $\mathcal{B}(\Xi)$. We set $A' = A \cap \mathcal{A}_\varepsilon$, and denote by $M_\varepsilon \subset \mathbb{C}^k$ the image of $D(\varepsilon)^r$ under the map \underline{m} .

We have,

$$\begin{aligned} & \int_U \rho_r(\underline{\zeta}) \left(\int_{A'} \beta_{\underline{\zeta}}(d\xi) \right) dV(\underline{\zeta}) \\ &= \int_{M_\varepsilon} \mu(\underline{a}) \left(\int_{\Sigma_{\underline{a}} \cap U} \nu(\underline{a}, \underline{\zeta}) \left(\int_{A'} \beta_{\underline{\zeta}}(d\xi) \right) d\ell_{\underline{a}}(\underline{\zeta}) \right) d\underline{a} \\ &= \int_{M_\varepsilon} \mu(\underline{a}) \left(\int_{\Sigma_{\underline{a}} \cap U} \left(\int_{A'} \nu(\underline{a}, \underline{\zeta}) \beta_{\underline{\zeta}}(d\xi) \right) d\ell_{\underline{a}}(\underline{\zeta}) \right) d\underline{a} \\ &= \int_{M_\varepsilon} \mu(\underline{a}) \left(\int_{\Sigma_{\underline{a}} \cap U} \left(\int_{A'} \nu(\underline{a}, \underline{\zeta}) \varrho(\underline{a}) f(\underline{\zeta}, \xi) Q_\varepsilon(\underline{a}, d\xi) / \nu(\underline{a}, \underline{\zeta}) \right) d\ell_{\underline{a}}(\underline{\zeta}) \right) d\underline{a} \\ &= \int_{M_\varepsilon} \varrho(\underline{a}) \mu(\underline{a}) \left(\int_{\Sigma_{\underline{a}} \cap U} \left(\int_{A'} f(\underline{\zeta}, \xi) Q_\varepsilon(\underline{a}, d\xi) \right) d\ell_{\underline{a}}(\underline{\zeta}) \right) d\underline{a} \\ &= \int_{M_\varepsilon} \varrho(\underline{a}) \mu(\underline{a}) \left(\int_{A'} \left(\int_{\Sigma_{\underline{a}} \cap U} f(\underline{\zeta}, \xi) d\ell_{\underline{a}}(\underline{\zeta}) \right) Q_\varepsilon(\underline{a}, d\xi) \right) d\underline{a} \text{ \{by Fubini's Theorem\} } \\ &= \int_{M_\varepsilon} \varrho(\underline{a}) \mu(\underline{a}) \left(\int_{A'} \mathbb{P}[\Pi_{\text{in}} \in U | \Pi_{\text{out}} = \xi] Q_\varepsilon(\underline{a}, d\xi) \right) d\underline{a} \text{ \{by definition of } f \text{ \}} \\ &= \int_{M_\varepsilon} \varrho(\underline{a}) \mu(\underline{a}) \mathbb{P}[\Pi_{\text{in}} \in U, \Pi_{\text{out}} \in A' | \underline{m}(\Pi_{\text{in}}) = \underline{a} \cap E_r] d\underline{a} \text{ \{by definition of } Q_\varepsilon \text{ \}} \\ &= \int_{M_\varepsilon} \mathbb{P}[\Pi_{\text{in}} \in U, \Pi_{\text{out}} \in A' | \underline{m}(\Pi_{\text{in}}) = \underline{a} \cap E_r] [(m)_* d\kappa](\underline{a}) \text{ \{by (3.1)\} } \\ &= \int_{M_\varepsilon} \mathbb{P}[\Pi_{\text{in}} \in U, \Pi_{\text{out}} \in A' | \underline{m}(\Pi_{\text{in}}) = \underline{a} \cap E_r] \mathbb{P}[E_r] [(m)_* d(\Pi_{\text{in}} | E_r)](\underline{a}) \text{ \{by (3.2)\} } \\ &= \mathbb{P}[E_r] \mathbb{P}[\Pi_{\text{in}} \in U, \Pi_{\text{out}} \in A' | E_r] \text{ \{by definition of } d(\Pi_{\text{in}} | E_r) \text{ \}} \\ &= \mathbb{P}[\Pi_{\text{in}} \in U, \Pi_{\text{out}} \in A'] \text{ \{ since } \mathbb{P}[\Pi_{\text{in}} \in U, \Pi_{\text{out}} \in A' | E_r^c] = 0 \text{ \}} \\ &= \int_U \rho_r(\underline{\zeta}) \left(\int_{A'} \mathbb{P}_{\underline{\zeta}}(d\xi) \right) dV(\underline{\zeta}) \text{ \{by definition of } \mathbb{P}_{\underline{\zeta}} \text{ \}}. \end{aligned}$$

This shows that, for a.e. $\underline{\zeta} \in D(\varepsilon)^r$, $\mathbb{P}_{\underline{\zeta}}(\cdot \cap \mathcal{A}_\varepsilon) = \beta_{\underline{\zeta}}(\cdot \cap \mathcal{A}_\varepsilon)$. The definition (3.3) of $\beta_{\underline{\zeta}}$ implies that, for $d\ell_{\underline{a}}$ -a.e. $\underline{\zeta}, \underline{\zeta}' \in \Sigma_{\underline{a}}$, we have $\beta_{\underline{\zeta}} \equiv \beta_{\underline{\zeta}'}$. Since $\underline{\zeta}_1$ and $\underline{\zeta}_2$ belong to $\Sigma_{\underline{a}}$ for the same \underline{a} , we deduce that

$$\mathbb{P}_{\underline{\zeta}_1}(\cdot \cap \mathcal{A}_\varepsilon) \equiv \mathbb{P}_{\underline{\zeta}_2}(\cdot \cap \mathcal{A}_\varepsilon). \tag{3.4}$$

For any event $B \in \mathcal{B}(\Xi)$ such that $\mathbb{P}_{\underline{\zeta}_1}(B) = 0$, we have $\mathbb{P}_{\underline{\zeta}_1}(B \cap \mathcal{A}_\varepsilon) = 0$. This implies that $\mathbb{P}_{\underline{\zeta}_2}(B \cap \mathcal{A}_\varepsilon) = 0$, by the mutual absolute continuity of the measures in (3.4). But, as $\varepsilon \rightarrow 0$, we have \mathcal{A}_ε exhausts the support of both $\mathbb{P}_{\underline{\zeta}_1}$ and $\mathbb{P}_{\underline{\zeta}_2}$ (because $\underline{\zeta}_1$ and $\underline{\zeta}_2$ were both chosen to satisfy Lemma 3.1). Letting $\varepsilon \downarrow 0$, we deduce that $\mathbb{P}_{\underline{\zeta}_2}(B) = 0$. This shows that $\mathbb{P}_{\underline{\zeta}_1} \equiv \mathbb{P}_{\underline{\zeta}_2}$.

4 Extensions

Theorem 2.1 can be extended in several directions. One immediate direction is the case when Ξ is a countable discrete set, e.g. \mathbf{Z}, \mathbf{Z}^d or a subset thereof. In many ways, this situation is technically simpler than the continuum setting in which Theorem 2.1 is stated and proved. The volume measure on Ξ (and its subsets) will be naturally replaced by the counting measure on those sets, and there would be no regularity assumptions on the functions Φ . The proof would be the same as the proof of Theorem 2.1.

Another pertinent question to ask is about the situation when $|\underline{\zeta}_1| = |\underline{\zeta}_2|, \underline{m}(\underline{\zeta}_1) \neq \underline{m}(\underline{\zeta}_2)$, but some subsets of coordinates of these two vectors match. This set will be of zero ρ_r measure, and hence is not covered by Theorem 2.1 as is. However, under a mild regularity assumption on (Φ_1, \dots, Φ_k) , we can deal with this scenario as well. Recall that the assumption in Theorem 2.1 is that the functions (Φ_1, \dots, Φ_k) constitute a *regular* collection of functions (in particular, recall Definition 1.10). To address the finer question, we make the additional assumption that $(\Phi_i)_{i \in S}$ is a regular collection of functions for each subset $S \subset \{1, \dots, k\}$. Consider, for $\underline{\zeta} \in \Xi^r$, the vector $s(\underline{\zeta})$ given by the co-ordinates of $\underline{m}(\underline{\zeta})$ whose indices are in S . Also consider, for $\underline{a} \in \mathbb{C}^{|S|}$, the sub-manifold $\mathcal{M}_{\underline{a}}^{[S]}$ formed by $\underline{\zeta} \in \Xi^r$ such that $s(\underline{\zeta}) = \underline{a}$. The fact that $(\Phi_i)_{i \in S}$ is a regular collection of functions allows us to (locally) make a decomposition of the ρ_r in terms of $d\underline{a} \times dl_{\underline{a}}$ (where $l_{\underline{a}}$ is the induced volume measure on $\mathcal{M}_{\underline{a}}^{[S]}$). This would enable us to refine the statement of Lemma 3.1 to the assertion that for $\mu(\underline{a})d\underline{a}$ a.e.- \underline{a} , it is true that for $dl_{\underline{a}}$ -a.e. $\eta \in \mathcal{M}_{\underline{a}}^{[S]}$, the events $\mathcal{A}(G_n, \eta)$ exhaust the support of \mathbb{P}_η . We can then run the same argument as in Section 3.2.1, and conclude that for a.e. sub-manifold $\mathcal{M}_{\underline{a}}^{[S]}$ with identical values of the statistics corresponding to $(\Phi_i)_{i \in S}$, the Palm measures \mathbb{P}_{η_1} and \mathbb{P}_{η_2} are mutually singular for a.e. pair $(\eta_1, \eta_2) \in \mathcal{M}_{\underline{a}}^{[S]} \times \mathcal{M}_{\underline{a}}^{[S]}$.

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References

- [1] A. Bufetov: Rigidity of Determinantal Point Processes with the Airy, the Bessel and the Gamma Kernel, *Bulletin of Mathematical Sciences*, 6.1 (2016), 163–172. MR-3472852
- [2] A. Bufetov: Quasi-symmetries of determinantal point processes, arXiv:1409.2068 (2014).
- [3] A. Bufetov, Y. Dabrowski and Y. Qiu: Linear rigidity of stationary stochastic processes, <http://arxiv.org/abs/1507.00670>.
- [4] A. Bufetov and Y. Qiu: Equivalence of Palm measures for determinantal point processes associated with Hilbert spaces of holomorphic functions, *Comptes Rendus Mathematique*, 353.6 (2015), 551–555. MR-3348991
- [5] I. Chavel: Riemannian geometry: a modern introduction, Vol. 98. Cambridge University Press, (2006). MR-2229062
- [6] D. Daley, D. Vere Jones: An Introduction to the Theory of Point Processes (Vols. I & II), Springer, (1997). MR-0950166
- [7] S. Ghosh: Determinantal processes and completeness of random exponentials: the critical case, *Probability Theory and Related Fields*, 163 (3-4), (2015), 643–665. MR-3418752
- [8] S. Ghosh and M. Krishnapur: Rigidity hierarchy in random point fields: random polynomials and determinantal processes, <https://arxiv.org/abs/1510.08814>.
- [9] S. Ghosh, M. Krishnapur and Y. Peres: Continuum Percolation for Gaussian zeroes and Ginibre eigenvalues, *The Annals of Probability*, 44 (5), (2016), 3357–3384. MR-3551199

- [10] S. Ghosh and Y. Peres: Rigidity and Tolerance in point processes: Gaussian zeroes and Ginibre eigenvalues, <http://arxiv.org/abs/1211.2381>.
- [11] J. Ginibre: Statistical ensembles of complex, quaternion, and real matrices, *Journal of Mathematical Physics*, (1965). MR-0173726
- [14] A. Holroyd and T.Soo: Insertion and deletion tolerance of point processes, *Electron. J. Probab*, 18.74 (2013), 1–24. MR-3091720
- [13] R. Honda and H. Osada: Infinite-dimensional stochastic differential equations related to Bessel random point fields, *Stochastic Processes and their Applications*, 125.10 (2015), 3801–3822. MR-3373304
- [12] J.B. Hough, M.Krishnapur, Y.Peres and B.Virag: Zeros of Gaussian Analytic Functions and Determinantal Point Processes, *A.M.S.*, (2010). MR-2552864
- [15] O. Kallenberg: *Random Measures*, Akademie-Verlag · Berlin and Academic Press, (1983). MR-0854102
- [16] R. Lyons: Determinantal Probability Measures, *Publ. Math. Inst. Hautes Etudes Sci.*, 98, (2003), 167–212. MR-2031202
- [17] F. Nazarov and M.Sodin: Random Complex Zeroes and Random Nodal Lines, *Proceedings of the International Congress of Mathematicians*, Vol. III, (2010), 1450–1484. MR-2827851
- [18] F. Nazarov and M. Sodin: Fluctuations in Random Complex Zeroes: Asymptotic Normality Revisited, *Int. Math. Res. Not.*, Vol. 2011, No. 24, (2011), 5720–5759. MR-2863379
- [19] F. Nazarov, M. Sodin and A. Volberg: Transportation to random zeroes by the gradient flow, *Geometric and Functional Analysis*, Vol 17-3, (2007), 887–935. MR-2346279
- [20] G. Olshanski: The quasi-invariance property for the Gamma kernel determinantal measure, *Advances in Mathematics*, 226.3 (2011), 2305–2350. MR-2739779
- [21] H. Osada: Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials, *Annals of Probability*, 41.1 (2013), 1–49. MR-3059192
- [22] H. Osada and H. Tanemura: Infinite-dimensional stochastic differential equations arising from Airy random point fields, arXiv preprint arXiv:1408.0632 (2014).
- [23] H.Osada and T.Shirai: Absolute continuity and singularity of Palm measures of the Ginibre point process, *Probab. Theory Related Fields* 165, no. 3–4, (2016), 725–770. MR-3520017
- [24] Y. Peres and A.Sly: Rigidity and tolerance for perturbed lattices, <https://arxiv.org/abs/1409.4490>.