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Parabolic category \mathcal{O} , perverse sheaves on Grassmannians, Springer fibres and Khovanov homology

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ABSTRACT

For a fixed parabolic subalgebra \mathfrak{p} of $\mathfrak{gl}(n, \mathbb{C})$ we prove that the centre of the principal block $\mathcal{O}_0^{\mathfrak{p}}$ of the parabolic category \mathcal{O} is naturally isomorphic to the cohomology ring $H^*(\mathcal{B}_{\mathfrak{p}})$ of the corresponding Springer fibre. We give a diagrammatic description of $\mathcal{O}_0^{\mathfrak{p}}$ for maximal parabolic \mathfrak{p} and give an explicit isomorphism to Braden’s description of the category $\text{Perv}_B(G(k, n))$ of Schubert-constructible perverse sheaves on Grassmannians. As a consequence Khovanov’s algebra \mathcal{H}^n is realised as the endomorphism ring of some object from $\text{Perv}_B(G(n, n))$ which corresponds under localisation and the Riemann–Hilbert correspondence to a full projective–injective module in the corresponding category $\mathcal{O}_0^{\mathfrak{p}}$. From there one can deduce that Khovanov’s tangle invariants are obtained from the more general functorial invariants in [C. Stroppel, *Categorification of the Temperley Lieb category, tangles, and cobordisms via projective functors*, Duke Math. J. **126**(3) (2005), 547–596] by restriction.

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Introduction

Let $G = \text{GL}(n, \mathbb{C})$ be the general linear group with subgroup B given by all invertible upper triangular matrices. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and \mathfrak{b} be their Lie algebras and let W be the Weyl group, so $W = S_n$. Let $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ be positive integers summing up to n . Then we have the parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$ of \mathfrak{g} with Levi subalgebra $\mathfrak{gl}_{\mu_1} \oplus \mathfrak{gl}_{\mu_2} \oplus \dots \oplus \mathfrak{gl}_{\mu_r}$, Weyl group $W_{\mathfrak{p}} = S_{\mu_1} \times S_{\mu_2} \times \dots \times S_{\mu_r}$, and P the corresponding parabolic subgroup of G . Let $x_{\mu} \in \mathfrak{g}$ be a nilpotent element whose Jordan normal form has blocks of size μ_i , $1 \leq i \leq r$. Let $u_{\mu} = \text{Id} + x_{\mu}$ be

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the corresponding unipotent element. Let $\mathcal{B} = G/B$, the variety of full flags in \mathbb{C}^n . Associated with μ we have the partial flag variety G/P . On the other hand, we also have the variety \mathcal{F}^{u_μ} of u_μ -fixed points: \mathcal{F}^{u_μ} is the Springer fibre associated with \mathfrak{p} ; we denote it by $\mathcal{B}_\mathfrak{p}$.

Let $\mathcal{O}_0^\mathfrak{p}$ be the principal block in the category of highest weight modules for \mathfrak{g} which are locally finite for \mathfrak{p} . If $\mathfrak{p} = \mathfrak{b}$ then $\mathcal{O}_0^\mathfrak{p}$ is the principal block of the ordinary Bernstein–Gel’fand–Gel’fand (BGG) category \mathcal{O} . The category $\mathcal{O}_0^\mathfrak{p}$ is equivalent to $\text{mod-}A^\mathfrak{p}$, the category of finitely generated modules over the (finite-dimensional) endomorphism algebra $A^\mathfrak{p}$ of a minimal projective generator of $\mathcal{O}_0^\mathfrak{p}$; it is also equivalent to the category of perverse sheaves on G/P , constructible with respect to the Schubert stratification (via localisation and the Riemann–Hilbert correspondence).

Centres and Springer fibres. The first result of this paper generalises Soergel’s results from [Soe90], confirms [Kho04, Conjecture 3] and gives an explicit description of the centre $\mathcal{Z}(A^\mathfrak{p})$ of $A^\mathfrak{p}$.

THEOREM 1. *There is a canonical isomorphism of algebras $H^*(\mathcal{B}_\mathfrak{p}) \cong \mathcal{Z}(A^\mathfrak{p})$.*

In particular, up to isomorphism, $\mathcal{Z}(A^\mathfrak{p})$ only depends on the parts of μ , not on the order in which they appear.

This theorem was independently proved by Brundan [Bru08] using different techniques. His approach also works for singular blocks and provides an explicit description of these centres as quotients of polynomial rings.

The cohomology of Springer fibres $H^*(\mathcal{B}_\mathfrak{p})$ was used by Springer [Spr78] to construct the irreducible representations of the symmetric group S_n . In particular, he defined an S_n -action on $H^*(\mathcal{B}_\mathfrak{p})$. From the isomorphism above we obtain an induced S_n -action on $\mathcal{Z}(A^\mathfrak{p})$. In §3 we give a functorial interpretation of this S_n -action on $\mathcal{Z}(A^\mathfrak{p})$ as follows. Let B_W be the underlying braid group. Jantzen’s translation functors can be used to define a (weak) braid group action on the bounded derived category of $\mathcal{O}_0^\mathfrak{p}$ (see e.g. [BFK99, Rou06, Str05]). The resulting functors are the derived functors of Irving’s shuffling functors (see §3.3). Since these functors are tilting functors, they induce a braid group action on the centre of $\mathcal{O}_0^\mathfrak{p}$, hence on $\mathcal{Z}(A^\mathfrak{p})$. Now the natural map $can : \mathcal{Z}(A^\mathfrak{b}) \rightarrow \mathcal{Z}(A^\mathfrak{p})$ is B_W -equivariant (Theorem 3.4.2) and if $\mathfrak{b} = \mathfrak{p}$, then the braid group action factors through an action of S_n (Lemma 3.4.1). A very recent result of Brundan [Bru06] says that can is surjective, hence there is an S_n -action on $\mathcal{Z}(A^\mathfrak{p})$ as well. This is the S_n -action we are looking for. Together with the theorem above and the main result from either [DP81] or [Tan82] it follows that $\mathcal{Z}(\mathcal{O}_0^\mathfrak{p}) \cong \mathbb{C}[W] \otimes_{\mathbb{C}[W_\mathfrak{p}]} \mathbb{C}_{\text{triv}}$ as W -module. It also shows that the dimension of the centre stays invariant under deformations of $A^\mathfrak{p}$.

The main idea of the proof of Theorem 1 is as follows. From Soergel’s Endomorphismensatz and Struktursatz [Soe90] we obtain an isomorphism $H^*(\mathcal{B}) \cong \mathcal{Z}(A^\mathfrak{b})$ of rings. On the other hand we have the restriction map $\mathcal{Z}(A^\mathfrak{b}) \rightarrow \mathcal{Z}(A^\mathfrak{p})$. We first show that the kernel of the canonical map $H^*(\mathcal{B}) \rightarrow H^*(\mathcal{B}_\mathfrak{p})$ is contained in the kernel of $\mathcal{Z}(A^\mathfrak{b}) \rightarrow \mathcal{Z}(A^\mathfrak{p})$ using deformation theory (following [Soe90]). This is based on the results of [Tan82] and a handy description of $H^*(\mathcal{B}_\mathfrak{p})$ as a quotient of $S(\mathfrak{h})$ along the lines of [GP92]. To show that the induced map $\Phi_\mathfrak{p} : H^*(\mathcal{B}_\mathfrak{p}) \rightarrow \mathcal{Z}(A^\mathfrak{p})$ is *injective* it is enough to show that it is injective on its socle (considered as an $H^*(\mathcal{B})$ -module). The main idea here is that the top degrees of $H^*(\mathcal{B}_\mathfrak{p})$ and $\mathcal{Z}(A^\mathfrak{p})$ coincide (Lemma 4.3.2). This will be used to show that $\Phi_\mathfrak{p}$ is non-zero when restricted to the socle (Propositions 4.2.1 and 4.3.1), and even S_n -equivariant onto its image. Then we use the fact that the socle of $H^*(\mathcal{B}_\mathfrak{p})$ is an irreducible S_n -module and obtain the injectivity. In Theorem 4.3.6 we show that the induced

injective map defines an isomorphism of S_n -modules on the top degree parts

$$H^*(\mathcal{B}_{\mathfrak{p}})_{\text{top}} \xrightarrow{\sim} \mathcal{Z}(A^{\mathfrak{p}})_{\text{top}}.$$

This follows on the one hand from Springer’s construction of irreducible S_n -modules and, on the other hand, from the categorification of irreducible S_n -modules using projective–injective modules in $\mathcal{O}_0^{\mathfrak{p}}$ obtained in [KMS09]. In the maximal parabolic case we give an alternative proof for the injectivity by a deformation argument, since the algebra $A^{\mathfrak{p}}$ can be replaced by a symmetric subalgebra with the same centre (§ 4.5) and the deformation ring is a principal ideal domain. For the applications we have in mind (see below) the maximal parabolic case is enough. As far as we see deformation methods are not sufficient to prove the surjectivity in Theorem 1. Instead, some ‘external’ information is needed which is obtained in [Bru06] from the representation theory of cyclotomic Hecke algebras.

Connection to Khovanov homology. Theorem 1 together with [DP81] and [Tan82] provide an explicit description of $\mathcal{Z}(A^{\mathfrak{p}})$, so we would like to have an explicit description of $A^{\mathfrak{p}}$ as well. In general, this seems to be ambitious, but in the case where \mathfrak{p} is a maximal parabolic subalgebra it has been achieved by Braden [Bra02] using perverse sheaves on Grassmannians. However, Braden’s description is difficult to use for explicit calculations. Moreover, the Koszul grading of $A^{\mathfrak{p}}$ (defined in [BGS96]) is not visible. Therefore, we consider the situation of [Bra02] again and first remark that any indecomposable projective $A^{\mathfrak{p}}$ -module has a commutative endomorphism ring (Proposition 2.8.1). Later on we deduce that each of these endomorphism rings is of the form $\mathbb{C}[X]/(X^2)^{\otimes k}$ for some $k \in \mathbb{Z}_{\geq 0}$. In Corollary 5.7.2 we explain how $A^{\mathfrak{p}}$ becomes a graded algebra using the description of [Bra02]. The intriguing result is, however, Theorem 5.8.1 which gives a purely graphical description of Braden’s algebra $A_{m,m}$ very similar to Khovanov’s approach (see e.g. [Kho04]) which we will describe below.

The crucial fact behind Theorem 1 and its proof is the existence of a bijection between the isomorphism classes $\text{PrInj}(\mathfrak{p})$ of indecomposable projective–injective modules in $\mathcal{O}_0^{\mathfrak{p}}$ and the irreducible components in $\mathcal{B}_{\mathfrak{p}}$. Let us consider the case where $n = 2m$ for some $m \in \mathbb{Z}_{>0}$ and $\mu = (m, m)$. In this case the irreducible components of $\mathcal{B}_{\mathfrak{p}}$ and, hence, the isomorphism classes $\text{PrInj}(\mathfrak{p})$, are in bijection to I , the set of crossingless matchings of $2m$ points. Let $\{T(i)_{2m}\}_{i \in I}$ be a complete minimal set of representatives of $\text{PrInj}(\mathfrak{p})$ and $T_{2m} := \bigoplus_{i \in I} T(i)_{2m}$. In [Kho02], a finite-dimensional \mathbb{C} -algebra \mathcal{H}^m was introduced whose primitive idempotents are naturally indexed by crossingless matchings of $2m$ points. These algebras were used to define the famous Khovanov homology which gives rise to an invariant of tangles and links. It is known [Kho04] that the centre $\mathcal{Z}(\mathcal{H}^m)$ of \mathcal{H}^m is isomorphic to $H^*(\mathcal{B}_{\mathfrak{p}})$.

In Theorem 5.6.2 and Proposition 5.6.4 we verify [Str06, Conjecture 2.9(a)] which is a stronger version of the conjectures formulated in [Bra02] and [Kho04].

THEOREM 2. *For any natural number m there is an isomorphism of algebras*

$$\text{End}_{\mathfrak{g}}(T_{2m}) \xrightarrow{\sim} \mathcal{H}^m.$$

Hence, Khovanov’s algebra \mathcal{H}^m is a subalgebra of $A^{\mathfrak{p}}$, where \mathfrak{p} is the parabolic subalgebra of \mathfrak{gl}_{2m} corresponding to the decomposition $2m = m + m$.

COROLLARY 1. *There is an isomorphism of rings $\mathcal{Z}(\text{End}_{\mathfrak{g}}(T_{2m})) \xrightarrow{\sim} \mathcal{Z}(\mathcal{H}^m)$.*

With [Kho04, Theorem 3] we therefore have an alternative proof of Theorem 1 in this special situation (purely based on [Bra02]) which implies [Kho04, Conjecture 2].

As an application of Theorem 2 one can deduce that Khovanov’s tangle invariants are nothing else than restrictions of the functorial invariants from [Str05] (see [Str06, Conjecture 2.9(b)] for a precise statement). Since the proof is lengthy, this part will be presented in a subsequent paper.

Diagrammatic description of Braden’s algebra. Theorem 2 will be a direct consequence of our diagrammatic description of Braden’s algebra $A_{m,m}$ in the case $\mu = (m, m)$. The primitive idempotents of $A_{m,m}$ or, equivalently, the isomorphism classes of indecomposable projective modules of $\mathcal{O}_0^{\mathfrak{p}}$, are in bijection to the shortest coset representatives of $S_m \times S_m \backslash S_{2m}$: by permuting the entries, the symmetric group S_{2m} acts transitively on the set of $\{+, -\}$ -sequences of length $2m$ with exactly m pluses and m minuses. Since the sequence $\sigma_{\text{dom}} = (+, \dots, +, -, \dots, -)$ has stabiliser $S_m \times S_m$ we obtain a bijection between the primitive idempotents of $A_{m,m}$ and the set $\mathcal{S}(m)$ of $\{+, -\}$ -sequences of length $2m$ with exactly m pluses and m minuses (Proposition 5.2.2). The isomorphism class of the projective generalised Verma module in $\mathcal{O}_0^{\mathfrak{p}}$ is mapped to σ_{dom} under this bijection. For $m = 1$ we have the sequence $(+, -)$ corresponding to the projective Verma module and the sequence $(-, +)$ corresponding to the ‘antidominant projective module’.

We want to associate a cup-diagram to each isomorphism class of indecomposable projective modules. To do so we have to make the $\{+, -\}$ -sequences longer. Putting m minuses in front of a sequence from $\mathcal{S}(m)$ and m pluses afterwards we obtain a distinguished set of $\{+, -\}$ -sequences of length $4m$ with exactly $2m$ pluses. Connecting successively each minus with an orphaned neighbored plus to the right we obtain a collection of crossingless matchings of $4m$ -points. In this way we associate to each primitive idempotent a of $A_{m,m}$ a cup diagram/crossingless matching of $4m$ points. For the sake of argument in this introduction we number the points from 1 to $4m$. In the case $m = 1$ for example, the two sequences $(-, +)$ and $(+, -)$ of length 2 from above become the sequences $(-, -, +, +)$ and $(-, +, -, +)$ of length 4 and we associate the crossingless matchings depicted in § 5, Figure 1.

To a pair (a, b) of two primitive idempotents we obtain a collection of circles as in [Kho02], namely by putting one crossingless matching upside down on top of the other (see § 5, Figure 2 for $m = 1$).

The fundamental difference to [Kho02] is that we additionally introduce a colouring of these circles indicating the position of a circle (§ 5, Figure 5). If a circle connects only points in the interval $[m + 1, 3m]$, then the circle is black. If a circle passes either through at least two points in $[1, m]$ or at least two points in $[3m + 1, 4m]$ then it is red. In all other cases it is green.

The principle idea is that we fix for each allowed colour (black, red, green) a two-dimensional topological quantum field theory (TQFT): red circles correspond to the trivial Frobenius algebra, green circles correspond to the one-dimensional Frobenius algebra, and black circles correspond to the Frobenius algebra $\mathbb{C}[X]/(X^2)$. In § 5.4 we combine these three TQFTs to define an algebra \mathcal{K}^m . If we restrict to idempotents such that only black circles occur then we are in exactly the situation of [Kho02] and we obtain \mathcal{H}^m naturally as a subalgebra of \mathcal{K}^m . However, the colouring carries all of the additional information to give a graphical description of Braden’s algebra $A_{m,m}$ (see Theorem 5.8.1).

THEOREM 3. *For any $m \in \mathbb{Z}_{>0}$ there is an isomorphism of algebras*

$$\mathcal{E} : A_{m,m} \cong \mathcal{K}^m.$$

To determine the dimension of the homomorphism space between two indecomposable projective modules it is enough to take the corresponding two cup diagrams, one upside down on top of the other, and count the numbers of circles for each colour. The dimension of the morphism space is then zero if there is a red circle, otherwise two to the power of the number of black circles. Taking the dimension of these homomorphism spaces is a natural extension of the categorical version of the S_n -invariant bilinear form defined on irreducible S_n -modules described in [KMS09].

The homomorphism space between two indecomposable projective modules P and Q carries a natural \mathbb{Z} -grading induced from the Koszul grading introduced in [BGS96]. It turns out that, up to a shift, the Poincaré polynomial agrees with the intersection theory Poincaré polynomial associated with the intersection of the corresponding two irreducible components of the associated Springer fibre (Theorem 5.9.1).

On the other hand \mathcal{K}^m carries a natural \mathbb{Z} -grading induced from the \mathbb{Z} -grading on $\mathbb{C}[X]/(X^2)$, where X has degree two. The isomorphism from Theorem 3 induces a grading of $A_{m,m}$ which we show is the Koszul grading (Corollary 5.7.2). It follows, in particular, that the arrows in the Ext-quiver of $A_{m,m}$ are given by Braden’s relation \leftrightarrow (Corollary 5.7.3).

Plan. The paper starts by recalling basics from Category \mathcal{O} and its deformation theory in § 2. Section 3 contains general facts about braid group actions on the centres of the categories we are interested in. Starting from § 4 we only consider Type A , the Lie algebra \mathfrak{gl}_n . In § 4 we explain the connection between the centres of blocks of category \mathcal{O} and the cohomology of the Springer fibres. Section 5 contains the connection with Braden’s and Khovanov’s work. We tried to make this part accessible without the Lie theoretic background from the previous sections. We abbreviate $\otimes_{\mathbb{C}}$ as \otimes and $\dim = \dim_{\mathbb{C}}$ denotes the dimension of a complex vector space.

1. Preliminaries

Let G be a complex reductive simply connected algebraic group with a chosen Borel subgroup B and maximal torus T . Let \mathfrak{g} be the corresponding reductive complex Lie algebra, with $\mathfrak{b} \supset \mathfrak{h}$ the Lie algebras of B and T , respectively. For any Lie algebra \mathfrak{l} let $\mathcal{U}(\mathfrak{l})$ be its universal enveloping algebra. We abbreviate $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ and denote by \mathfrak{Z} the centre of \mathcal{U} . For any finite-dimensional complex vector space V let $S(V)$ be the algebra of polynomial functions on V^* , especially $S := S(\mathfrak{h}) = \mathcal{U}(\mathfrak{h})$. We denote by $\Delta \subseteq R^+ \subset R$ the set of simple roots, positive roots and all roots. Let W be the Weyl group and $X = X(R)$ the integral weight lattice. The Weyl group acts naturally on \mathfrak{h}^* ; this action is denoted by $(w, \lambda) \mapsto w(\lambda)$, for $w \in W$, $\lambda \in \mathfrak{h}^*$. We also have the so-called ‘dot-action’ given by $(w, \lambda) \mapsto w \cdot \lambda := w(\lambda + \rho) - \rho$, where ρ is the half-sum of positive roots. For a root $\alpha \in R$ we denote by $\check{\alpha}$ the corresponding coroot with the evaluation pairing $\langle \cdot, \cdot \rangle$. In this paper, a weight $\lambda \in \mathfrak{h}^*$ is called *dominant*, if $\langle \lambda + \rho, \check{\alpha} \rangle \notin \{-1, -2, \dots\}$ for any $\alpha \in R^+$. Let $\mathfrak{h}_{\text{dom}}^*$ be the set of dominant weights.

If $\pi \subseteq \Delta$, then there is a corresponding parabolic subalgebra $\mathfrak{p}_{\pi} = \mathfrak{g}_{\pi} \oplus \mathfrak{h}^{\pi} \oplus \mathfrak{n}^{\pi}$ of \mathfrak{g} , where \mathfrak{g}_{π} is semisimple with simple roots π and Cartan \mathfrak{h}_{π} , and $\mathfrak{h}^{\pi} = \bigcap_{\alpha \in \pi} \ker \alpha$. Denote by $p_{\pi} : \mathfrak{h} \rightarrow \mathfrak{h}^{\pi}$ the projection along \mathfrak{h}_{π} as well as the induced restriction morphism $p_{\pi} : S(\mathfrak{h}) \rightarrow S(\mathfrak{h}^{\pi})$.

Let $W_{\mathfrak{p}_{\pi}}$ be the parabolic subgroup of W associated with π . If $\pi = \emptyset$, then $W_{\mathfrak{p}_{\pi}}$ is trivial. We denote by $W^{\mathfrak{p}_{\pi}}$ the set of shortest coset representatives in $W_{\mathfrak{p}_{\pi}} \backslash W$ with respect to the Coxeter length function l . Let $w_0 \in W$ be the longest element, $w_0^{\mathfrak{p}_{\pi}}$ the longest element in $W_{\mathfrak{p}_{\pi}}$, and finally $[w^{\mathfrak{p}_{\pi}}]$ the representative of the longest element of $W_{\mathfrak{p}_{\pi}} \backslash W$.

2. Deformation

Fix now some $\pi \subseteq \Delta$. To simplify notation we leave out the index π most of the time. In particular, $\mathfrak{p} = \mathfrak{p}_\pi$. Consider the following commutative algebra

$$T = T^\pi := S(\mathfrak{h}^\pi)_{(0)} = \left\{ \frac{f}{g} \mid f, g \in S(\mathfrak{h}^\pi), g(0) \neq 0 \right\},$$

the localisation of $S(\mathfrak{h}^\pi)$ at the augmentation ideal, with maximal ideal \mathfrak{m} .

Let T' be a π -deformation algebra, that is, an associative unitary noetherian commutative T -algebra. The structure morphism $\varphi: T \rightarrow T'$, induces a $\mathcal{U}(\mathfrak{p})$ -module structure on T' via the composition $\mathfrak{p}_\pi \xrightarrow{p} \mathfrak{h}^\pi \xrightarrow{i} S(\mathfrak{h}^\pi) \xrightarrow{i'} T \xrightarrow{\varphi} T'$, where p is the canonical projection, and i, i' the canonical inclusions. We have the set X_π of π -integral weights and the set X_π^+ of (π) -admissible weights, defined as follows:

$$\begin{aligned} X_\pi &:= \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}, \alpha \in \pi\}, \\ X_\pi^+ &:= \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_{\geq 0}, \alpha \in \pi\}. \end{aligned}$$

Recall that there is a natural bijection

$$\{\text{finite-dimensional irreducible } \mathcal{U}(\mathfrak{p})\text{-modules}\} \xleftrightarrow{1:1} X_\pi^+, \quad (2.0.1)$$

by mapping a module to its highest weight. Let $\tilde{E}(\lambda)$ be the irreducible module corresponding to $\lambda \in X_\pi^+$. For any π -deformation algebra T' define the T' -deformed (generalised) Verma module with highest weight $\lambda \in X_\pi^+$ as

$$M_{T'}^{\mathfrak{p}}(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\tilde{E}(\lambda) \otimes T').$$

This is a $\mathcal{U}(\mathfrak{g}) \otimes T'$ -module, where T' is just acting on T' by multiplication. Given a $\mathcal{U}(\mathfrak{g}) \otimes T'$ -module M and $\lambda \in \mathfrak{h}^*$, we denote by

$$M_{T'}^\lambda = \{m \in M \mid h.m = \varphi(\lambda(h) + h)m, \forall h \in \mathfrak{h}\}$$

the λ -weight space of M . (Here $\lambda(h) + h$ is considered as an element of T via the map $i' \circ i \circ p$ and $\varphi(\lambda(h) + h)$ is an element of T' .)

2.1 The deformed (parabolic) category \mathcal{O}

Let $\mathcal{O}_T^{\mathfrak{p}}$ be the full subcategory of the category of $\mathcal{U}(\mathfrak{g}) \otimes T$ -modules defined by the set of objects M satisfying:

- M is finitely generated;
- $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_T^\lambda$ as T -module; and
- $(\mathcal{U}(\mathfrak{p}) \otimes T)m$ is a finitely generated T -module for all $m \in M$.

In particular, $M_T^{\mathfrak{p}}(\lambda) \in \mathcal{O}_T^{\mathfrak{p}}$ for any $\lambda \in X_\pi^+$. Note that the third condition is equivalent to saying that M is locally $\mathcal{U}(\mathfrak{g}_\pi \oplus \mathfrak{n}^\pi) \otimes T$ -finite.

If we replace T by \mathbb{C} in all of the definitions, then $\mathcal{O}_T^{\mathfrak{p}}$ is the ordinary parabolic category $\mathcal{O}^{\mathfrak{p}}$ as defined in [Roc80, Section 3]. In particular, $\mathcal{O}_{\mathbb{C}}^{\mathfrak{b}}$ is the ordinary BGG-category $\mathcal{O}^{\mathfrak{b}}$ from [BGG76]. The generalised Verma modules $M_{\mathbb{C}}^{\mathfrak{p}}(\lambda)$ are abbreviated as $M^{\mathfrak{p}}(\lambda)$.

2.2 Weight and root decompositions of $\mathcal{O}_T^{\mathfrak{p}}$

If $\lambda \in \mathfrak{h}^*$ we denote by $\bar{\lambda}$ its class in \mathfrak{h}^*/X and by $\tilde{\lambda}$ its class in $\mathfrak{h}^*/\mathbb{Z}R$. We have the following weight decomposition: $\mathcal{O}_T^{\mathfrak{p}} = \bigoplus_{\Lambda \in \mathfrak{h}^*/X} \mathcal{O}_{T,\Lambda}^{\mathfrak{p}}$, where $M \in \mathcal{O}_{T,\Lambda}^{\mathfrak{p}}$ if and only if $M_T^\lambda \neq \{0\} \Rightarrow \bar{\lambda} = \Lambda$, and the

finer root decomposition $\mathcal{O}_T^{\mathfrak{p}} = \bigoplus_{\Lambda \in \mathfrak{h}^*/\mathbb{Z}R} \mathcal{O}_{T,\Lambda}^{\mathfrak{p}}$, where $M \in \mathcal{O}_{T,\Lambda}^{\mathfrak{p}}$ if and only if $M_T^\lambda \neq \{0\} \Rightarrow \tilde{\lambda} = \Lambda$. Both decompositions are far away from giving block decompositions, hence we have to refine them once more.

2.3 Central character and block decomposition

Let $\xi^\sharp : \mathfrak{Z} \rightarrow S$ be the Harish-Chandra homomorphism normalised by $\xi^\sharp(z) - z \in \mathcal{U}(\mathfrak{g})\mathfrak{n}$, where $\mathfrak{n} = \mathfrak{n}^\emptyset$. Hence, $\xi^\sharp : \mathfrak{Z} \rightarrow S^{(W;\cdot)}$, the W -invariants under the dot-action. This is the comorphism for $\xi : \mathfrak{h}^* \rightarrow \text{Max}(\mathcal{Z})$ which induces a bijection between (W, \cdot) -orbits of \mathfrak{h}^* and maximal ideals $\text{Max}(\mathcal{Z})$ of \mathfrak{Z} . For any weight λ , we have $+\lambda : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$, $\mu \mapsto \lambda + \mu$. Let $\lambda^\sharp : S \rightarrow S$ be the corresponding comorphism.

Since deformed Verma modules are generated by their highest weight space we have a canonical isomorphism of rings $\text{End}_{\mathfrak{g} \otimes T}(M_T^{\mathfrak{p}}(\lambda)) = T$. Let $\chi_\lambda : \mathfrak{Z} \otimes T \rightarrow T$ be such that $z.m = \chi_\lambda(z)m$ for any $z \in \mathfrak{Z} \otimes T$, $m \in M_T^{\mathfrak{p}}(\lambda)$. Explicitly, the morphism χ_λ is given by

$$\mathfrak{Z} \otimes T \xrightarrow{(\lambda^\sharp \circ \xi^\sharp) \otimes \text{id}} S \otimes T \xrightarrow{i^\sharp \otimes \text{id}} S(\mathfrak{h}^\pi) \otimes T \xrightarrow{j \otimes \text{id}} T \otimes T \xrightarrow{m} T, \tag{2.3.1}$$

where $i : (\mathfrak{h}^\pi)^* \rightarrow \mathfrak{h}^*$ is the canonical embedding, j is the canonical embedding into its localisation, and m is the multiplication map (for details see [Soe90, § 2]).

Let $\mu \in \mathfrak{h}^*$ and consider the support $\text{supp } M_T^{\mathfrak{p}}(\lambda) \subseteq \text{Spec}(\mathfrak{Z} \otimes T)$ of $M_T^{\mathfrak{p}}(\lambda)$ viewed as a $\mathfrak{Z} \otimes T$ -module. By definition, $\text{supp } M_T^{\mathfrak{p}}(\lambda)$ is the closed subset of the spectrum of $\mathfrak{Z} \otimes T$ given by all prime ideals containing $\text{Ann}_{\mathfrak{Z} \otimes T}(M_T^{\mathfrak{p}}(\lambda)) = \text{Ann}_{\mathfrak{Z} \otimes T}(M_T^{\mathfrak{p}}(\lambda)^\lambda) = \ker \chi_\lambda$. Unlike in the non-deformed situation, $\ker \chi_\lambda$ does not need to be a maximal ideal. However, the homomorphism theorem implies that there is a homeomorphism between the spectrum of T and $\text{supp } M_T^{\mathfrak{p}}(\lambda)$. Hence, $\text{supp } M_T^{\mathfrak{p}}(\lambda)$ contains exactly one closed point, namely $\xi(\lambda) \otimes T + \mathfrak{Z} \otimes \mathfrak{m}$, since T is local with maximal ideal \mathfrak{m} . If $\lambda, \mu \in X_\pi^+$, then

$$\text{supp } M_T^{\mathfrak{p}}(\lambda) \cap \text{supp } M_T^{\mathfrak{p}}(\mu) \neq \emptyset \Leftrightarrow \xi(\lambda) = \xi(\mu) \Leftrightarrow \lambda \in W \cdot \mu.$$

For χ , a maximal ideal of \mathfrak{Z} , let $\mathcal{O}_{T,\chi}^{\mathfrak{p}}$ be the full subcategory of $\mathcal{O}_T^{\mathfrak{p}}$ given by all objects having support contained in $\bigcap_{\xi(\mu)=\chi} \text{supp } M_T^{\mathfrak{p}}(\mu)$. For $\lambda \in \mathfrak{h}_{\text{dom}}^*$ let $\mathcal{O}_{T,\lambda}^{\mathfrak{p}} = \mathcal{O}_{T,\xi(\lambda)}^{\mathfrak{p}} \cap \mathcal{O}_{T,\Lambda}^{\mathfrak{p}}$, such that $\Lambda = \tilde{\lambda}$. We have the following ‘block’ decomposition:

$$\mathcal{O}_T^{\mathfrak{p}} = \bigoplus_{\lambda \in \mathfrak{h}_{\text{dom}}^*} \mathcal{O}_{T,\lambda}^{\mathfrak{p}} = \bigoplus_{\lambda \in \mathfrak{h}_{\text{dom}}^* \cap X_\pi^+} \mathcal{O}_{T,\lambda}^{\mathfrak{p}}.$$

Strictly speaking, this is not a block decomposition, since the summands might decompose further. This is, however, not the case if $\mathfrak{p} = \mathfrak{b}$ or $\lambda = 0$, where $\mathcal{O}_{T,\lambda}^{\mathfrak{p}}$ is in fact a block. Since we are mainly interested in this case we call it ‘block’ decomposition.

2.4 The ordinary parabolic category $\mathcal{O}_0^{\mathfrak{p}}$

Let us stop for a moment and recall the structure of the principal block $\mathcal{O}_0^{\mathfrak{p}}$ of $\mathcal{O}_{\mathbb{C}}^{\mathfrak{p}} = \mathcal{O}^{\mathfrak{p}}$ from [Roc80]. The generalised Verma modules in $\mathcal{O}_0^{\mathfrak{p}}$ are exactly the $M_{\mathbb{C}}^{\mathfrak{p}}(\lambda)$, where $\lambda \in X_\pi^+ \cap W \cdot 0$, or in other words λ is of the form $\lambda = w \cdot 0$, where $w \in W^{\mathfrak{p}}$. The simple objects in $\mathcal{O}_0^{\mathfrak{p}}$ are exactly the simple quotients $L(w \cdot 0)$, $w \in W^{\mathfrak{p}}$ of these generalised Verma modules. (There is only one finite-dimensional simple module, namely the trivial module.) The category $\mathcal{O}_0^{\mathfrak{p}}$ has enough projectives; for $w \in W^{\mathfrak{p}}$ let $P^{\mathfrak{p}}(w \cdot 0)$ be the projective cover of $L(w \cdot 0)$ in $\mathcal{O}_0^{\mathfrak{p}}$.

2.5 Specialisations: \mathbb{C} and $\mathcal{Q} = \text{Quot } T$

For any morphism $f : T \rightarrow T'$ of deformation algebras let $\mathcal{O}_{T',\lambda}^{\mathfrak{p}}$ denote the image of $\mathcal{O}_{T,\lambda}^{\mathfrak{p}}$ under the specialisation functor $- \otimes_T T'$.

From [Fie03, Proposition 2.6 and §2.4] it follows that if $f : T \rightarrow T/\mathfrak{m} = \mathbb{C}$ is the canonical projection then the image of $\mathcal{O}_T^{\mathfrak{p}}$ under the \mathbb{C} -specialisation is the ordinary parabolic category $\mathcal{O}_{\mathbb{C}}^{\mathfrak{p}}$ with the usual decomposition $\mathcal{O}_{\mathbb{C}}^{\mathfrak{p}} = \bigoplus_{\lambda \in \mathfrak{h}_{\text{dom}}^* \cap X_{\pi}^+} \mathcal{O}_{\mathbb{C},\lambda}^{\mathfrak{p}}$ (see e.g. [Jan83, 4.4]). It follows directly from the definitions that $M_T^{\mathfrak{p}}(\lambda) \otimes_T \mathbb{C} \cong M_{\mathbb{C}}^{\mathfrak{p}}(\lambda)$ as \mathfrak{g} -modules.

On the other hand we could consider the specialisation functor $- \otimes_T \mathcal{Q}$, where $\mathcal{Q} = \text{Quot } T$ is the quotient field of T . We identify $(\mathfrak{h} \otimes_{\mathbb{C}} \mathcal{Q})^* := \text{Hom}_{\mathcal{Q}}(\mathfrak{h} \otimes \mathcal{Q}, \mathcal{Q}) = \mathfrak{h}^* \otimes_{\mathbb{C}} \mathcal{Q}$. Let $\tau \in (\mathfrak{h} \otimes \mathcal{Q})^*$ be the tautological weight, i.e. restricted to \mathfrak{h} it is just the projection onto $\mathfrak{h}^{\pi} \subseteq S(\mathfrak{h}^{\pi}) \subset T \subset \mathcal{Q}$. From the definitions we have $M_T^{\mathfrak{p}}(\lambda) \otimes_T \mathcal{Q} \cong M_{\mathcal{Q}}^{\mathfrak{p}}(\lambda + \tau)$ as $\mathfrak{g} \otimes \mathcal{Q}$ -modules. If $\lambda \in X_{\pi}^+$ for all $\alpha \in \pi$, then $\langle \lambda + \tau, \check{\alpha} \rangle = \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$ by definition of \mathfrak{h}^{π} and τ , hence $\lambda + \tau$ is an admissible weight for the Lie algebra $\mathfrak{g} \otimes \mathcal{Q}$. If $\alpha \in \Delta - \pi$, then $\tau(\alpha) \neq 0$ (since the elements from Δ are linearly independent) and therefore $\langle \lambda + \tau, \check{\alpha} \rangle \notin \mathbb{Z}$ for any $\alpha \in \Delta - \pi$. It follows that $M_{\mathcal{Q}}^{\mathfrak{p}}(\lambda + \tau)$ is simple [Jan83, (1.17)]. In particular, the image of $\mathcal{O}_T^{\mathfrak{p}}$ under the \mathcal{Q} -specialisation functor is semisimple with simple objects $M_{\mathcal{Q}}^{\mathfrak{p}}(\lambda + \tau)$, $\lambda \in X_{\pi}^+$.

2.6 Translation functors

Let $\lambda, \mu \in \mathfrak{h}_{\text{dom}}^*$ such that $\mu - \lambda \in X(R)$. For any deformation algebra T' let

$$\theta_{\lambda,T'}^{\mu} : \mathcal{O}_{T',\lambda}^{\mathfrak{p}} \rightarrow \mathcal{O}_{T',\mu}^{\mathfrak{p}}$$

be the translation functor defined as $M \mapsto \text{pr}_{\mu}(M \otimes E)$, where E is the finite-dimensional \mathfrak{g} -module with extremal weight $\mu - \lambda$ and pr_{μ} is the projection to the summand $\mathcal{O}_{T',\mu}^{\mathfrak{p}}$. Since $\mathcal{O}_{T'}^{\mathfrak{p}}$, the direct sum of all blocks, is closed under tensoring with finite-dimensional \mathfrak{g} -modules, the definition makes sense. Obviously, $\theta_{\lambda,T'}^{\mu}$ commutes with base change, i.e. there is a natural isomorphism

$$\theta_{\lambda,T'}^{\mu}(M \otimes_T T') \cong (\theta_{\lambda,T}^{\mu} M) \otimes_T T'.$$

2.7 Deformed projectives

For the reader's convenience we recall some fundamental properties of the deformed parabolic categories, but omit the proofs. The arguments can be found in [Fie03, Soe90, Soe92].

PROPOSITION 2.7.1. *Let T' be any π -deformation algebra.*

- (i) *The category $\mathcal{O}_{T'}^{\mathfrak{p}}$ has enough projectives.*
- (ii) *The category $\mathcal{O}_{T'}^{\mathfrak{p}}$ is closed under taking direct summands and finite direct sums.*
- (iii) *If $\lambda \in X_{\pi}^+ \cap \mathfrak{h}_{\text{dom}}^*$, then $M_{T'}^{\mathfrak{p}}(\lambda)$ is projective in $\mathcal{O}_{T',\lambda}^{\mathfrak{p}}$ and in $\mathcal{O}_{T'}^{\mathfrak{p}}$.*
- (iv) *Any projective module in $\mathcal{O}_{T'}^{\mathfrak{p}}$ is obtained by applying translation functors to some $M_{T'}^{\mathfrak{p}}(\lambda)$, $\lambda \in X_{\pi}^+ \cap \mathfrak{h}_{\text{dom}}^*$, taking finite direct sums and direct summands.*
- (v) *Any projective object in $\mathcal{O}_{T'}^{\mathfrak{p}}$ has a Verma flag, i.e. a filtration with subquotients isomorphic to various deformed generalised Verma modules.*
- (vi) *The weight spaces of projective objects in $\mathcal{O}_{T'}^{\mathfrak{p}}$ are free T' -modules of finite rank.*
- (vii) *The specialisation functor $- \otimes_T \mathbb{C}$ defines a bijection between the (indecomposable) projective objects in $\mathcal{O}_T^{\mathfrak{p}}$ and the (indecomposable) projective objects in $\mathcal{O}_{\mathbb{C}}^{\mathfrak{p}}$.*
- (viii) *If $M, N \in \mathcal{O}_{T'}^{\mathfrak{p}}$ are projective, then $\text{Hom}_{\mathfrak{g} \otimes T'}(M, N)$ is a free T' -module of finite rank.*

(ix) If $M, N \in \mathcal{O}_T^{\mathfrak{p}}$ are projective, then the canonical map

$$\begin{aligned} \Psi : \mathrm{Hom}_{\mathfrak{g} \otimes T}(M, N) \otimes_T T' &\cong \mathrm{Hom}_{\mathfrak{g} \otimes T'}(M \otimes_T T', N \otimes_T T') \\ f \otimes t &\mapsto f \otimes t \cdot \mathrm{id}. \end{aligned}$$

is an isomorphism.

If $P \in \mathcal{O}_{T'}^{\mathfrak{p}}$ has a Verma flag, then we denote by $[P : M_T^{\mathfrak{p}}(\lambda)]$ the multiplicity of a $M_T^{\mathfrak{p}}(\lambda)$ as a subquotient of an (arbitrarily chosen) Verma flag of P . Note that this number is stable under changes of the deformation ring (by Proposition 2.7.1(vi)). For simplicity we restrict our attention to the principal blocks $\mathcal{O}_{T',0}^{\mathfrak{p}}$ in the following.

2.8 Commutativity of the endomorphism rings

In this section we use the deformation theory to obtain (as an easy application) the commutativity of certain endomorphism rings.

PROPOSITION 2.8.1. *Let $P \in \mathcal{O}_0^{\mathfrak{p}}$ be an indecomposable projective module.*

- (i) *If $[P : M^{\mathfrak{p}}(\lambda)] \leq 1$ for any $\lambda \in \mathfrak{h}^*$, then $\mathrm{End}_{\mathfrak{g}}(P)$ is commutative.*
- (ii) *If $\mathfrak{p} = \mathfrak{b}$, then the following are equivalent:*
 - (a) $[P : M^{\mathfrak{b}}(\lambda)] \leq 1$ for any λ ;
 - (b) $\mathrm{End}_{\mathfrak{g}}(P)$ is commutative;
 - (c) \mathfrak{z} surjects onto $\mathrm{End}_{\mathfrak{g}}(P)$ canonically.

Proof. Let P_T be the T -deformation of P given by Proposition 2.7.1(vii). From Proposition 2.7.1(ix) we obtain an isomorphism of rings

$$\mathrm{End}_{\mathfrak{g} \otimes T}(P_T) \otimes_T T' \cong \mathrm{End}_{\mathfrak{g} \otimes T'}(P_T \otimes_T T')$$

for any T -algebra T' . If we choose $T' = \mathbb{C}$, then the commutativity of $\mathrm{End}_{\mathfrak{g} \otimes T}(P_T)$ implies the commutativity of $\mathrm{End}_{\mathfrak{g}}(P)$. On the other hand, we could choose $T' = \mathcal{Q}$, the ring of fractions of T . The category $\mathcal{O}_{0,\mathcal{Q}}$ is semisimple, with simple objects being the \mathcal{Q} -specialised deformed Verma modules (§ 2.5). They all have commutative endomorphism rings isomorphic to \mathcal{Q} . Set $J = \{\lambda \in \mathfrak{h}^* \mid [P_T : M_T^{\mathfrak{p}}(\lambda)] \neq 0\}$. By our assumption on the multiplicities we obtain $P_T \otimes_T \mathcal{Q} \cong \bigoplus_{\lambda \in J} M_T^{\mathfrak{p}}(\lambda) \otimes_T \mathcal{Q} \cong \bigoplus_{\lambda \in J} M_{\mathcal{Q}}^{\mathfrak{p}}(\lambda + \tau)$, and $\mathrm{End}_{\mathfrak{g} \otimes \mathcal{Q}}(P \otimes_T \mathcal{Q}) \cong \bigoplus_{\lambda \in J} \mathcal{Q}$ is commutative. Proposition 2.7.1(viii) and (ix) provide an inclusion

$$\begin{aligned} \mathrm{End}(P_T) &\longrightarrow \mathrm{End}(P_T) \otimes_T \mathcal{Q} \cong \mathrm{End}(P \otimes_T \mathcal{Q}) \\ f &\longmapsto f \otimes 1. \end{aligned}$$

So, $\mathrm{End}_{\mathfrak{g} \otimes T}(P_T)$ is a subring of a commutative ring, hence itself commutative. The first part of the proposition follows. The second part is [Str03b, Theorem 7.1]. \square

We do not know whether Proposition 2.8.1(ii) is true for general \mathfrak{p} . A famous example for a module P satisfying the conditions of the proposition in case $\mathfrak{p} = \mathfrak{b}$ is the ‘antidominant projective module’ $P(w_0 \cdot 0) \in \mathcal{O}_0^{\mathfrak{b}}$ (see § 3 below). The following are further examples (see also Proposition 4.2.1).

PROPOSITION 2.8.2. *Let $\mathfrak{g} = \mathfrak{gl}_n$ and \mathfrak{p}_{π} a maximal parabolic, i.e. $\pi = \Delta - \{\alpha_s\}$ for some simple reflection s . Then $\mathrm{End}_{\mathfrak{g}}(P)$ is commutative for any indecomposable projective object $P \in \mathcal{O}_0^{\mathfrak{p}}$.*

Proof. With this choice of a parabolic subalgebra, the assumptions of Proposition 2.8.1(i) are satisfied [Bre02, Theorem 5.1]. \square

In §5 we show that the endomorphism rings appearing in Proposition 2.8.2 are of the form $(\mathbb{C}[X]/(X^2))^{\otimes k}$ for some $k \in \mathbb{Z}_{\geq 0}$.

3. The centre as a module for the Weyl group

Recall that the centre $\mathcal{Z}(\mathcal{A})$ of an abelian category \mathcal{A} is the endomorphism ring of the identity functor $\text{id} = \text{id}_{\mathcal{A}}$. If, for instance, $\mathcal{A} \cong \text{mod-}A$, the category of finitely generated right modules over some unitary \mathbb{C} -algebra A , then the centre of \mathcal{A} is naturally isomorphic to the (ordinary) centre $\mathcal{Z}(A)$ of the algebra A . The isomorphism associates to a natural transformation f the value $f_A(1)$.

3.1 The cohomology ring of the flag variety

Let us consider for a moment the case $\mathfrak{p} = \mathfrak{b}$. It is well known that the centre of $\mathcal{O}_{\mathbb{C},0}^{\mathfrak{b}}$ is naturally isomorphic to $\mathfrak{C} = S/(S_+^W)$, the ring of coinvariants ([Soe90, Endomorphismensatz and Struktursatz] together with [MS08, Theorem 5.2(2)]). The natural action of the Weyl group W on $S = S(\mathfrak{h})$ gives rise to an action of W on \mathfrak{C} . In the deformed situation, the picture is similar: the centre of the deformed category $\mathcal{O}_{0,T}^{\mathfrak{b}}$ is naturally isomorphic to $S \otimes_{S^W} T$ [Soe92, Theorem 9, Corollary 1], and hence carries obviously the structure of a W -module. To obtain an explicit description of the isomorphism we first note that each element of the centre of $\mathcal{O}_{0,T}^{\mathfrak{b}}$ defines an element of the endomorphism ring E of the ‘antidominant projective’ in $\mathcal{O}_{0,T}^{\mathfrak{b}}$ by restriction, defining an isomorphism between the centre of the category and E (see e.g. [Str06, Theorem 1.8]). On the other hand, Soergel showed in [Soe92, Theorem 9] that E is canonically isomorphic to $T \otimes_{T^W} T = S \otimes_{S^W} T$. Moreover, since the specialisation functor $- \otimes_T \mathbb{C}$ maps E surjectively onto the endomorphism ring of the antidominant projective module in $\mathcal{O}_{\mathbb{C},0}^{\mathfrak{b}}$, the principal block of the ordinary category \mathcal{O} , the centre of $\mathcal{O}_{T,0}^{\mathfrak{b}}$ maps surjectively onto the centre of $\mathcal{O}_{\mathbb{C},0}^{\mathfrak{b}}$.

Let $\mathcal{B} = G/B$ be the flag variety corresponding to \mathfrak{g} (i.e. the variety of Borel subalgebras in \mathfrak{g}) and $H^*(\mathcal{B})$ its cohomology algebra with complex coefficients. The Weyl group acts on $H^*(\mathcal{B})$. Note that \mathfrak{C} has an even \mathbb{Z} -grading coming from the grading on S , where \mathfrak{h} is concentrated in degree two. We recall the following well-known fact (see e.g. [DP81, Section 4.1] or [Spr76, Proposition 7.2]).

PROPOSITION 3.1.1. *There is a W -equivariant isomorphism of graded algebras $\psi : \mathfrak{C} \cong H^*(G/B)$, and $\mathfrak{C} \cong \mathbb{C}[W]$ as W -modules.*

Via the natural isomorphism $\mathfrak{C} = \mathcal{Z}(\mathcal{O}_0^{\mathfrak{b}})$, the centre of $\mathcal{O}_0^{\mathfrak{b}}$ inherits an action of W giving rise to the regular representation. In the following we explain how this W -action on the centre can be obtained via braid group actions on the bounded derived category $\mathcal{D}^b(\mathcal{O}_0^{\mathfrak{b}})$, inducing a W -action on the centre of $\mathcal{O}_{\mathbb{C},0}^{\mathfrak{b}}$ and then finally also on the centre of $\mathcal{O}_{\mathbb{C},0}^{\mathfrak{p}}$.

3.2 Braid group actions on the centre of a category

Before we pass to derived categories, we want to give the main idea behind this braid group action on the centre of $\mathcal{O}_0^{\mathfrak{p}}$ by first assuming a simplified situation. Let \mathcal{C} be an abelian \mathbb{C} -linear category. Let $F : \mathcal{C} \rightarrow \mathcal{C}$ be a functor. Assume that F is invertible. Then the centre of \mathcal{C} is isomorphic to $\text{End}(F)$ in two ways: first by mapping an element c in the centre to $F(c)$ and second by mapping c (naively) to the endomorphism given by multiplication with c . Now given c in the centre of \mathcal{C} there is a unique c' in the centre of \mathcal{C} such that $F(c)$ is given by multiplication

with c' . In particular, there is an automorphism $\Psi_F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}(\mathcal{C})$ which maps c to c' . Hence, c and c' are related by the formula $c'_{F(M)} = F(c_M)$ for any object M .

Assume now that G is a group acting on \mathcal{C} , i.e. for any $g \in G$, there is an (invertible) functor $F_g : \mathcal{C} \rightarrow \mathcal{C}$ such that $F_e \cong \text{id}$, $F_{gh} \cong F_g \circ F_h$. We obtain the corresponding automorphisms Ψ_{F_g} which give rise to an action of G on the centre of \mathcal{C} . For further details we refer the reader to [Kho04].

3.3 The Irving shuffling functors

Let $s \in W$ be a simple reflection and choose $\lambda \in \mathfrak{h}^*$ an integral weight with stabiliser $\{e, s\}$. Let $\theta_s = \theta_\lambda^0 \theta_0^\lambda : \mathcal{O}_0^{\mathfrak{b}} \rightarrow \mathcal{O}_0^{\mathfrak{b}}$ be the translation functor through the s -wall. Let $a_s : \text{id} \rightarrow \theta_s$ be the adjunction morphism. Consider the functor $C_s = \text{coker}(a_s)$. This is a right exact functor such that its left derived functor $\mathcal{L}C_s$ induces an equivalence on the bounded derived category $\mathcal{D}^b(\mathcal{O}_0)$ (see [MS05, Theorem 5.7]). It is quite easy to see that they satisfy braid relations in the weak sense (see [KM05, Theorem 2] and [MOS09, Section 6.5]), which means if we have a braid relation $st \dots = ts \dots$ then there is an isomorphism of functors $\mathcal{L}C_s \mathcal{L}C_t \cong \dots \cong \mathcal{L}C_t \mathcal{L}C_s \dots$. (Although this weak version is enough for our purposes we want to point out that Rouquier showed that the isomorphisms of functors can be chosen in a compatible way [Rou06].)

Since the translation functors preserve the parabolic categories, the functors C_s induce functors $C_s : \mathcal{O}_0^{\mathfrak{p}} \rightarrow \mathcal{O}_0^{\mathfrak{p}}$ and the left derived functors $\mathcal{L}C_s$ are auto-equivalences of $\mathcal{D}^b(\mathcal{O}_0^{\mathfrak{p}})$ (cf. [MS05, § 4]).

Each of the categories $\mathcal{O}_0^{\mathfrak{p}}$ is equivalent to $\text{mod-}A^{\mathfrak{p}}$ for some finite-dimensional algebra $A^{\mathfrak{p}}$ (see e.g. [Str03b, § 2.1]). Under this equivalence, the functors C_s become so-called tilting functors, given by tensoring with some tilting complex (see [Ric94] and [MS05, § 5]). Hence, we have a braid group action via tilting auto-equivalences on $\mathcal{D}^b(\mathcal{O}_0^{\mathfrak{p}})$. Since these equivalences are given by tilting complexes we obtain an induced braid group action on the centre of the underlying abelian category $\mathcal{O}_0^{\mathfrak{p}}$ (see [Ric89, Theorem 9.2]).

3.4 The action of the Weyl group on $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{b}})$

Let us now construct this action explicitly. We first consider the case $\mathfrak{p} = \mathfrak{b}$ and recall some results from [Soe90]: let $P(w_0 \cdot 0) \in \mathcal{O}_0^{\mathfrak{b}}$ be the projective cover of the simple Verma module $M(w_0 \cdot 0) = L(w_0 \cdot 0)$. Consider Soergel's *Strukturfunctor*

$$\mathbb{V} = \text{Hom}_{\mathfrak{g}}(P(w_0 \cdot 0), -) : \mathcal{O}_0^{\mathfrak{b}} \longrightarrow \text{mod-End}_{\mathfrak{g}}(P(w_0 \cdot 0)).$$

By Soergel's Endomorphismensatz we have $\text{End}_{\mathfrak{g}}(P(w_0 \cdot 0)) \cong \mathfrak{C} = \mathfrak{C}^{op}$ canonically and under this identification we obtain a functor $\mathbb{V} : \mathcal{O}_0^{\mathfrak{b}} \rightarrow \mathfrak{C}\text{-mod}$. There is an isomorphism $\mathbb{V}\theta_s \cong \Theta_s \mathbb{V}$, where $\Theta_s : \mathfrak{C}\text{-mod} \rightarrow \mathfrak{C}\text{-mod}$, $M \mapsto \mathfrak{C} \otimes_{\mathfrak{C}^s} M$ and \mathfrak{C}^s denotes the s -invariants of \mathfrak{C} . Under this isomorphism, the adjunction morphism $a_s : \text{id} \rightarrow \theta_s$ corresponds to the morphism a_s given by

$$\begin{aligned} a_s(M) : \quad \mathfrak{C} \otimes_{\mathfrak{C}} M &\rightarrow \mathfrak{C} \otimes_{\mathfrak{C}^s} M, \\ 1 \otimes m &\mapsto X \otimes m + 1 \otimes Xm, \quad m \in M, \quad X = \check{\alpha}_s \end{aligned} \tag{3.4.1}$$

see [Str05, Lemma 8.2]. Let coker_s be the functor of taking the cokernel of $a_s : \text{id} \rightarrow \theta_s$. By construction, $\mathbb{V}C_s \cong \text{coker}_s \mathbb{V}$ when restricted to the additive subcategory of $\mathcal{O}_0^{\mathfrak{b}}$ generated by $P(w_0 \cdot 0)$.

LEMMA 3.4.1. *Let $c \in \mathfrak{C}$ with the corresponding element m_c of the centre of $\mathfrak{C}\text{-mod}$ given by multiplication with c . Then $\text{coker}_s(m_c) = m_{s(c)}$.*

Proof. It is enough to check this on the regular module \mathfrak{C} . Let $\mathfrak{C} \xrightarrow{a_s} \mathfrak{C} \otimes_{\mathfrak{C}^s} \mathfrak{C} \xrightarrow{p} \text{coker}_s(\mathfrak{C})$ be the defining sequence of $\text{coker}_s(\mathfrak{C})$. Note that $\text{coker}_s(\mathfrak{C}) \cong \mathfrak{C}$ as a \mathfrak{C} -module, and is generated by the image of $1 \otimes 1$ under p . Let $c \in \mathfrak{C}$, homogeneous of degree one. Then we have

$$\Theta_s(m_c)(1 \otimes 1) = 1 \otimes m_c(1) = 1 \otimes c. \quad (3.4.2)$$

On the other hand $s(c) - c = rX$ for some $r \in \mathbb{C}$ and $X = \check{\alpha}_s$. Since \mathfrak{C} is a free \mathfrak{C}^s module on basis $1, X$, we therefore obtain

$$1 \otimes c = \frac{1}{2}(1 \otimes (s(c) + c) - 1 \otimes (s(c) - c)) = \frac{1}{2}((s(c) + c) \otimes 1 - 1 \otimes rX),$$

because $c + s(c)$ is s -invariant. However,

$$\begin{aligned} \frac{1}{2}((s(c) + c) \otimes 1 - 1 \otimes rX) &\equiv \frac{1}{2}((s(c) + c) \otimes 1 + rX \otimes 1) \\ &= \frac{1}{2}((s(c) + c + s(c) - c) \otimes 1) \\ &= s(c) \otimes 1, \end{aligned}$$

where \equiv means equality modulo the image of a_s . The lemma follows. \square

Let us summarise: on $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$, the centre of $\mathcal{O}_0^{\mathfrak{p}}$, there is an action of the braid group B_W which underlies W . This action is induced from the braid group action of the left derived functors of Irving's shuffling functors (§ 3.3).

THEOREM 3.4.2. *Let $\mathfrak{p} \subseteq \mathfrak{g}$ be any parabolic subalgebra containing \mathfrak{b} .*

- (i) *The action of the braid group on $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{b}})$ factors through W .*
- (ii) *The canonical isomorphism $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{b}}) = \mathfrak{C}$ is W -equivariant.*
- (iii) *The canonical restriction morphism $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{b}}) \rightarrow \mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$ is B_W -equivariant. In particular, the image becomes a W -module.*

Proof. The first two statements hold because of Lemma 3.4.1 and the natural identification of the centre with the endomorphism ring of the antidominant projective module by restriction [MS08, Theorem 5.2(2)]. The last statement follows directly from the definition of the braid group actions. \square

Remark 3.4.3. Theorem 3.4.2(i) and (ii) hold analogously for the deformed categories $\mathcal{O}_{T,0}^{\mathfrak{p}}$. If we consider the corresponding semisimple category $\mathcal{O}_{\mathcal{Q},0}^{\mathfrak{p}}$, then we have isomorphisms

$$\begin{aligned} \mathcal{Z}(\mathcal{O}_{\mathcal{Q},0}^{\mathfrak{p}}) &\cong \bigoplus_{x \in W^{\mathfrak{p}}} \text{End}_{\mathfrak{g} \otimes \mathcal{Q}} M_{\mathcal{Q}}^{\mathfrak{p}}(x \cdot 0 + \tau) = \bigoplus_{w \in W^{\mathfrak{p}}} \mathcal{Q} \\ z &\mapsto (z_x)_{x \in W^{\mathfrak{p}}}, \end{aligned}$$

where $z_x \in \mathcal{Q}$ is the image of the natural transformation z applied to $M_{\mathcal{Q}}(x \cdot 0 + \tau)$ evaluated at $1 \otimes 1 \otimes 1 \otimes 1 \in M_{\mathcal{Q}}(x \cdot 0 + \tau)$. The \mathcal{Q} -version of \mathcal{LC}_s maps $M_{\mathcal{Q}}(x \cdot 0)$ to $M_{\mathcal{Q}}(xs \cdot 0)$ if $x, xs \in W^{\mathfrak{p}}$ and to $M_{\mathcal{Q}}(x \cdot 0)[1]$ otherwise. Hence, $s(z_x) = z_{xs}$ if $xs \in W^{\mathfrak{p}}$ and $s(z_x) = z_x$ otherwise. Therefore, $\mathcal{Z}(\mathcal{O}_{\mathcal{Q},0}^{\mathfrak{p}}) \cong \mathbb{C}[W] \otimes_{\mathbb{C}[W^{\mathfrak{p}}]} \mathbb{C}_{\text{triv}}$ as a W -module.

4. Type A: the centres and the Springer fibres

From now on we stick to the special case where $\mathfrak{g} = \mathfrak{gl}_n$ with standard Borel subalgebra \mathfrak{b} given by the upper triangular matrices. We would like to generalise Proposition 3.1.1 to the parabolic case.

Let $G = \mathrm{GL}(n, \mathbb{C})$ with Borel subgroup B given by the upper triangular matrices and Lie algebra \mathfrak{b} . Then $\mathcal{B} = G/B$ is the variety of complete flags in \mathbb{C}^n . Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra containing \mathfrak{b} , hence \mathfrak{p} is given by upper diagonal $(\mu_1, \mu_2, \dots, \mu_r)$ -block matrices, where $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ is a composition of n . For example, $\mathfrak{p} = \mathfrak{b}$ corresponds to the composition (1^n) of n . Any maximal parabolic subalgebra corresponds to a two part composition. Let $\lambda(\mu) = \lambda(\mathfrak{p})$ be the partition obtained from μ by reordering the parts. Let $x = x_{\mathfrak{p}} \in G$ be the nilpotent element of G in Jordan normal form such that the Jordan blocks are of size μ_1, μ_2 , etc. Let $\mathcal{B}_{\mathfrak{p}}$ be the Springer fibre corresponding to x , that means the subvariety of \mathcal{B} of all flags fixed by the unipotent element $u = \mathrm{Id} + x$. Note that $\mathcal{B}_{\mathfrak{b}} = \mathcal{B}$.

4.1 The main result: centres via cohomology

Let $H^*(\mathcal{B}_{\mathfrak{p}})$ denote the cohomology of $\mathcal{B}_{\mathfrak{p}}$. Springer defined an S_n -action on $H^*(\mathcal{B}_{\mathfrak{p}})$ and proved that the top part $H^*(\mathcal{B}_{\mathfrak{p}})_{\mathrm{top}}$ is the irreducible representation of S_n corresponding to the partition $\lambda(\mathfrak{p})$ (see [Spr76] or [HS77]). The embedding of $\mathcal{B}_{\mathfrak{p}}$ into \mathcal{B} induces a morphism $h : H^*(\mathcal{B}_{\mathfrak{b}}) \rightarrow H^*(\mathcal{B}_{\mathfrak{p}})$, which is surjective [HS77] and W -equivariant [DP81, Tan82].

THEOREM 4.1.1. *Let $\mathfrak{g} = \mathfrak{gl}_n$ with parabolic subalgebra \mathfrak{p} containing \mathfrak{b} and Weyl group $W = S_n$.*

- (i) *The canonical map $H^*(\mathcal{B}) = \mathfrak{C} = \mathcal{Z}(\mathcal{O}_0) \rightarrow \mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$ factors through $H^*(\mathcal{B}_{\mathfrak{p}})$ and induces an isomorphism of rings*

$$\Phi_{\mathfrak{p}} : H^*(\mathcal{B}_{\mathfrak{p}}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}}).$$

- (ii) $\Phi_{\mathfrak{p}}$ is W -equivariant and $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}}) \cong \mathbb{C}[W] \otimes_{\mathbb{C}[W_{\mathfrak{p}}]} \mathbb{C}_{\mathrm{triv}}$ as a W -module.
- (iii) Up to isomorphism, $H^*(\mathcal{B}_{\mathfrak{p}})$ and $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$ only depend on $\lambda(\mathfrak{p})$.

Proof. Let R denote the regular functions on $\mathfrak{h}^* \oplus U$, where $U := \{\lambda \in \mathfrak{h}^* \mid \lambda(\mathfrak{h}_{\pi}) = 0\} = (\mathfrak{h}^{\pi})^*$. Hence, $R = S \otimes S(\mathfrak{h}^{\pi})$. We fix the standard basis $\epsilon_i, 1 \leq i \leq n$ for \mathfrak{h}^* with its set of fundamental weights $\omega_i = \sum_{k=1}^i \epsilon_k$. The $\omega_{i_1}, \dots, \omega_{i_r}$ contained in U form a basis of U . Let

$$I = \{f \in R \mid f(w(\lambda), \lambda) = 0, \text{ for all } \lambda \in U, w \in W\} \subseteq R \subseteq S \otimes T$$

and put $K = \{f(-, 0) \mid f \in I\} \subset S = S(\mathfrak{h})$.

Claim 1: $K = \ker(S \rightarrow \mathfrak{C} \rightarrow H^*(\mathcal{B}_{\mathfrak{p}})) =: \ker$.

We start by showing that K contains \ker . Thanks to [Tan82] we have an explicit set of generators for \ker : Let $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_{r'})$ be the dual partition of $\lambda(\mu)$. If we identify S with $\mathbb{C}[x_1, \dots, x_n]$ in the usual way by taking the dual standard basis vectors ϵ_i^* as generators, then \ker is generated by all l th elementary symmetric functions $e_l(\mathcal{X}), k > 0, l > 0$ where

$$\mathcal{X} \subseteq \{x_1, \dots, x_n\}, \quad |\mathcal{X}| = k, \quad k \geq l > k - (\mu'_{n-k+1} + \mu'_{n-k+2} + \dots + \mu'_{r'}).$$

Therefore, it is enough to show that these $e_l(\mathcal{X})$ are contained in K . Thanks to the W -invariance, we only have to consider the cases where \mathcal{X} consists of the first k variables x_1, x_2, \dots, x_k . Taking the dual basis of the $\omega_{i_j} \in U, 1 \leq j \leq r$, we identify $R = \mathbb{C}[x_1, x_2, \dots, x_n] \otimes \mathbb{C}[y_1, y_2, \dots, y_r]$, and for any choice of k and l from the allowed range construct a polynomial $f = f_{k,l} \in R$ with the following properties:

- (i) $f(x_1, x_2, \dots, x_n, 0, 0, \dots, 0) = e_l(x_1, x_2, \dots, x_k)$;
- (ii) $f(b_1, b_2, \dots, b_n, a_1, a_2, \dots, a_r) = 0$ for any point (b_1, b_2, \dots, b_n) where μ_i of the coordinates are equal to a_i for $1 \leq i \leq r$.

Then $f \in I$ and so $e_l(x_1, x_2, \dots, x_k) = f(x_1, x_2, \dots, x_n, 0, 0, \dots, 0) \in K$.

Let now k and l be fixed from the allowed range. The construction of this polynomial goes along the lines of [GP92]. For $1 \leq i \leq r$ let m_i be the maximum of $\mu_i - n + k$ and zero, and define $d := \sum_{i=1}^r m_i$. Note that $d = (\mu'_{n-k+1} + \mu'_{n-k+2} + \cdots + \mu'_r)$ as above. To construct the polynomial f we first define $P(t) = \prod_{i=1}^k (t + x_i)$, a polynomial in t with coefficients from $\mathbb{C}[x_1, x_2, \dots, x_k]$, and consider the polynomial $Q(t) = \prod_{i=1}^r (t + y_i)^{m_i}$ of degree d in t with coefficients from $\mathbb{C}[y_1, y_2, \dots, y_r]$. We perform the long division of $P(t)$ by $Q(t)$ and obtain $P(t) = q(t)Q(t) + r(t)$, where $r(t) = \sum_{s=0}^{d-1} r_s t^s$ is a polynomial in t with coefficients being homogeneous polynomials r_s in the x and y . We claim that $f = r_{k-l}$ does the job.

If we set all $y_i = 0$, then $Q(t) = t^d$, and $r_s = e_{k-s}(x_1, \dots, x_k)$, hence $r_{k-l} = e_l(x_1, \dots, x_k)$ and property (i) follows. Now let $(a_1, \dots, a_r) \in \mathbb{C}^r = U$ and $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{C}^n = \mathfrak{h}^*$ such that μ_i of the coordinates are equal to a_i for $1 \leq i \leq r$, hence at least $m_i = \mu_i - (n - k)$ of the first k coordinates are equal to a_i . In particular, $P(t)$ evaluated at the first k coordinates of \mathbf{b} , is divisible by $Q(t)$ if $y_i = a_i$, i.e. if $Q(t)$ is evaluated at \mathbf{b} . Hence, $r(t)$ becomes zero when evaluated at $x_i = b_i$, $y_i = a_i$, and so property (ii) holds. This implies $\ker \subseteq K$.

It is left to show that the inclusion is in fact an equality. Let $u \in U$ be a generic point and define $K_u = \{f(-, u) = 0 \mid f \in I\} \subseteq S$. Since u is generic, S/K_u is the coordinate algebra of $|W/W_{\mathfrak{p}}|$ distinct points in \mathfrak{h}^* , and hence $\dim(S/K_u) = |W/W_{\mathfrak{p}}|$, and also $\dim(S/\text{gr}K_u) = |W/W_{\mathfrak{p}}|$, where the associated graded gr is taken with respect to the canonical grading of S . Moreover, one can easily see that $K \subseteq \text{gr}(K_u)$ by mapping $f(-, 0)$ to $f(-, u)$. Altogether, the natural surjection from S to S/K factors through S/\ker and the natural surjection from S to $S/\text{gr}K_u$ factors through S/K . Therefore, $\dim(S/\ker) \geq \dim(S/K) \geq \dim(S/\text{gr}K_u) = |W/W_{\mathfrak{p}}|$. By the main result of [Tan82], we have $\dim(S/\ker) = |W/W_{\mathfrak{p}}|$, hence all of the dimensions agree and Claim 2 follows.

Claim 2: $K \subseteq \ker(S \rightarrow \mathcal{Z}(\mathcal{O}_0) \rightarrow \mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}}))$.

Let P_T be a projective generator of $\mathcal{O}_{T,0}^{\mathfrak{p}}$. We consider the following commutative diagram.

$$\begin{array}{ccccc} \mathfrak{z} \otimes T & \xrightarrow{\xi^{\sharp}} & S^W \otimes T & \xrightarrow{h} & \text{End}_{\mathfrak{g} \otimes T}(P_T) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ \mathfrak{z} \otimes \mathcal{Q} & \xrightarrow{\xi^{\sharp}} & S^W \otimes \mathcal{Q} & \xrightarrow{h} & \text{End}_{\mathfrak{g} \otimes \mathcal{Q}}(P_{\mathcal{Q}}) \end{array}$$

The two maps labelled h are given by applying the product of the two factors to the module (so that $h \circ \xi^{\sharp}$ is the canonical map), whereas the vertical maps are the obvious inclusions (using Proposition 2.7.1 (ix)). By § 2.5 we have an isomorphism

$$\epsilon: \text{End}_{\mathfrak{g} \otimes \mathcal{Q}}(P_T \otimes_T \mathcal{Q}) \cong \text{End}_{\mathfrak{g} \otimes \mathcal{Q}}\left(\bigoplus a_x M_{\mathcal{Q}}^{\mathfrak{p}}(x \cdot 0 + \tau)\right),$$

where a_x is the multiplicity of $M_{\mathcal{Q}}^{\mathfrak{p}}(x \cdot 0)$ as a subquotient of a Verma flag of P_T (see § 2.5). The map γ is injective (by Proposition 2.7.1 (viii)). Hence the kernel of the upper can is the kernel of $\text{can} \circ \beta$.

An element $z \otimes t \in \mathfrak{z} \otimes T$ acts on $M_{\mathcal{Q}}^{\mathfrak{p}}(x^{-1} \cdot 0 + \tau)$ by multiplication with $(x^{-1} \cdot 0 + \tau)(\xi^{\sharp}(z))\tau(t)$. On the other hand $(x^{-1} \cdot 0 + \tau) \circ \xi^{\sharp}(z) = \xi^{\sharp}(z)(x^{-1} \cdot 0 + \tau) = (x \cdot x^{-1} \cdot 0 + x(\tau)) \circ \xi^{\sharp}(z) = x(\tau) \circ \xi^{\sharp}(z)$, so $z \otimes t$ acts by multiplication with $x(\tau) \circ \xi^{\sharp}(z)$. On the other hand

$$j_x: S \otimes T \xrightarrow{x^{\sharp} \otimes \text{id}} S \otimes T \xrightarrow{p_{\pi} \otimes \text{id}} S(\mathfrak{h}^{\pi}) \otimes T \xrightarrow{\text{mult}} T \longrightarrow \mathcal{Q}, \quad (4.1.1)$$

where $x^\sharp : S \rightarrow S$ is the comorphism given by the action of x . Hence, $z \otimes t$ acts by multiplication with $\tau \circ j_x(\xi^\sharp(z) \otimes t)$.

To pass from $\mathfrak{Z} \otimes T$ to $S \otimes T$ note that the map ξ^\sharp induces an isomorphism $\psi : \mathfrak{Z} \otimes T / \xi^{\sharp^{-1}}(J) \cong S \otimes T / J$, where $J = \bigcap_{x \in W} \ker j_x \subseteq S \otimes T$ (see [Soe90, p. 429]). (Note that we use here the assumption that our block $\mathcal{O}_0^{\mathfrak{p}}$ is regular, and so ξ is étale at 0.)

Let now $f \in K$ and find $\tilde{f} \in I$ such that $f = \tilde{f}(-, 0)$. Using the explicit formula before (4.1.1) and the map ψ it follows that \tilde{f} acts on $M_{\mathcal{O}}(x^{-1} \cdot 0 + \tau)$ by multiplication with the function $\lambda \mapsto \tilde{f}(x(\lambda), \lambda)$, hence by zero. So, \tilde{f} is in the annihilator of P_T . By the definition of K , we can write $f = f \otimes 1 + g \in S \otimes T$, where $g \in S \otimes \mathfrak{m}$. So, g induces an endomorphism on P_T which specialises to the zero endomorphism of $P_{\mathbb{C}}$. Claim 2 follows.

Therefore, the canonical map $H^*(\mathcal{B}) = \mathfrak{C} = \mathcal{Z}(\mathcal{O}_0) \rightarrow \mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$ factors through $H^*(\mathcal{B}_{\mathfrak{p}})$ and induces the map $\Phi_{\mathfrak{p}}$. The latter is surjective by [Bru06, Theorem 5.11]. The injectivity will be proved at the end of the section. Let us assume for the moment we have proved this already (so statement (i) holds). Thanks to Theorem 3.4.2, $\Phi_{\mathfrak{p}}$ is W -equivariant, and statement (ii) is true if $\mathfrak{p} = \mathfrak{b}$ by Proposition 3.1.1. Hence, the image of $\Phi_{\mathfrak{p}}$, i.e. $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$, is isomorphic to $\mathbb{C}[W] \otimes_{\mathbb{C}[W]} \mathbb{C}_{\text{triv}}$ as W -module by [HS79, Corollary 8.5] or [Tan82, Theorem 1]. Statement (ii) of the theorem follows. Part (iii) is clear from [Tan82], but also has a direct proof from the categorical side by [MS08, Theorems 5.4 and 5.2]. \square

In the following two sections we prove the injectivity of $\Phi_{\mathfrak{p}}$. This result follows in fact directly by dimension arguments, since Brundan [Bru06] showed that the dimension of the centre of $\mathcal{O}_0^{\mathfrak{p}}$ is equal to the dimension of $H^*(\mathcal{B}_{\mathfrak{p}})$. However, our approach gives a distinguished basis of the top degree part of $H^*(\mathcal{B}_{\mathfrak{p}})$ and shows in a nice way how the categorification of the symmetric group action comes into play.

4.2 A generalised antidominant projective module

The missing part in the proof of Theorem 4.1.1 will be deduced from several non-trivial results which we recall first. We start with the following fact.

PROPOSITION 4.2.1. *Let $\mathfrak{g} = \mathfrak{g}_{\mathfrak{n}}$ and \mathfrak{p} be any parabolic subalgebra. Then there is always an indecomposable projective module $P \in \mathcal{O}_0^{\mathfrak{p}}$ such that:*

- P is injective; and
- the natural action of \mathfrak{C} defines a surjection $\mathfrak{C} \twoheadrightarrow \text{End}_{\mathfrak{g}}(P)$, in particular $\text{End}_{\mathfrak{g}}(P)$ is commutative.

Proof. By [IS88, Corollary p. 327] there is an integral dominant weight λ such that $\mathcal{O}_{\lambda}^{\mathfrak{p}}$ contains a simple projective module $N = L(\mu)$, $\mu = x \cdot \lambda$ for some $x \in W$. Hence, N is also injective and a Verma module. Then $\theta_{\lambda}^0(N)$ is projective and injective, and has a (generalised) Verma flag satisfying the assumptions of Proposition 2.8.1 (see also [Jan83, 4.13(1)]). Hence, there is some projective and injective module $P = \theta_{\lambda}^0 N$ with commutative endomorphism ring. If $\Gamma : \mathcal{O}_{\lambda} \rightarrow \mathcal{O}_{\lambda}^{\mathfrak{p}}$ is the functor which picks out the largest quotient contained in $\mathcal{O}_{\lambda}^{\mathfrak{p}}$, then $\Gamma M(\mu) = N$. Since Γ commutes with translation functors, we have $\Gamma \theta_{\lambda}^0 M(\mu) \cong \theta_{\lambda}^0 \Gamma M(\mu) \cong \theta_{\lambda}^0 N = P$. On the other hand, Γ commutes (by definition) with the action of the centre and thanks to the existence of the canonical projection $\text{End}_{\mathfrak{g}}(\theta_{\lambda}^0 M(\mu)) \rightarrow \text{End}(\theta_{\lambda}^0 N)$, it is enough to show that \mathfrak{C} surjects onto $\text{End}_{\mathfrak{g}}(\theta_{\lambda}^0 M(\mu))$ naturally. However, $M(\mu) \cong T_x M(\lambda)$, where T_x is the twisting functor as studied in [AL03, AS03]. Now, T_x commutes with the action of the centre (see the definition of the functors in [AS03]), it is therefore enough to show that \mathfrak{C} surjects onto $\text{End}_{\mathfrak{g}}(\theta_{\lambda}^0 M(\lambda))$ naturally.

However, $\theta_\lambda^0 M(\lambda)$ satisfies the assumptions of Proposition 2.8.1 for $\mathfrak{p} = \mathfrak{b}$. Then the statement follows from Proposition 2.8.1(ii). \square

DEFINITION 4.2.2. An object $M \in \mathcal{O}_0^{\mathfrak{p}}$ is called *projective–injective* if it is both projective and injective in $\mathcal{O}_0^{\mathfrak{p}}$.

Proposition 4.2.1 ensures the existence of projective–injective modules.

4.3 Grading and Loewy length

Let $P \in \mathcal{O}_0^{\mathfrak{p}}$ be projective, then $\text{End}_{\mathfrak{g}}(P)$ has a natural non-negative \mathbb{Z} -grading induced from the Koszul grading [BGS96] of $A^{\mathfrak{p}} = \text{End}_{\mathfrak{g}}(P_{\text{gen}})$, where $P_{\text{gen}} \in \mathcal{O}_0^{\mathfrak{p}}$ is a minimal projective generator. This Koszul grading induces a \mathbb{Z} -grading on $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$, the centre of $\mathcal{O}_0^{\mathfrak{p}}$.

PROPOSITION 4.3.1. *Assume that \mathfrak{g} is any reductive complex Lie algebra and \mathfrak{p} some parabolic subalgebra. Let $P_i, i \in I$, be a complete system of representatives for the isomorphism classes of indecomposable projective–injectives in $\mathcal{O}_0^{\mathfrak{p}}$.*

- (i) *The centre of $\mathcal{O}_0^{\mathfrak{p}}$ is the centre of $\text{End}_{\mathfrak{g}}(\bigoplus_{i \in I} P_i)$.*
- (ii) *The Loewy lengths of all P_i agree and equal the maximal possible Loewy length ll . The maximal degree of $\text{End}_{\mathfrak{g}}(P_i)$, considered as a graded ring, is equal to $\text{ll} - 1$.*
- (iii) *Consider $\mathcal{Z}^{\mathfrak{p}} := \mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$ as a graded ring with its top degree part $\mathcal{Z}_{\text{top}}^{\mathfrak{p}}$. Then $\dim_{\mathbb{C}} \mathcal{Z}_{\text{top}}^{\mathfrak{p}} = |I|$ and $\text{top} = \text{ll} - 1$.*

Proof. The first two statements of the proposition were proved in [MS08, Theorem 5.2]. Now consider $D := \text{End}_{\mathfrak{g}}(\bigoplus_{i \in I} P_i)$ as a graded ring. Let \tilde{P}_i be a graded lift of P_i (in the sense of [BGS96] or [Str03a, §3.1]). Then the radical, socle and the grading filtrations of \tilde{P}_i agree up to a shift of the grading [BGS96, Proposition 2.4.1], since P_i has simple head and simple socle (the latter by [Irv85, Appendix]). If now $f \in D$ is of maximal degree, then f is contained in the span of the maps g_i where g_i maps the head of P_i to its socle and is zero on all other summands by the second statement. On the other hand, the g_i are all contained in the centre of B . Therefore, we have an isomorphism of vector spaces $D_{\text{top}} \cong \mathcal{Z}_{\text{top}}^{\mathfrak{p}}$ and the proposition follows. \square

LEMMA 4.3.2. *Let $\mathfrak{g} = \mathfrak{gl}_n$ and \mathfrak{p} be any parabolic subalgebra. Then the top degree of $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$ agrees with the top degree of $\mathcal{H}^*(\mathcal{B}_{\mathfrak{p}})$.*

Proof. From [IS88, Proposition and Corollary 3.1] (see also Remark 4.4.3) we have an explicit formula for the Loewy length of a projective–injective module in $\mathcal{O}_0^{\mathfrak{p}}$, hence for the top degree of $\mathcal{Z}(\mathcal{O}_0^{\mathfrak{p}})$ by Proposition 4.3.1. The formula agrees with [HS79, Lemma 1.3] and implies the assertion. \square

LEMMA 4.3.3. *Let $\mathfrak{g} = \mathfrak{gl}_n$ and \mathfrak{p} be any parabolic subalgebra. Then the following numbers coincide:*

- *the number of isomorphism classes of indecomposable projective–injective modules in $\mathcal{O}_0^{\mathfrak{p}}$;*
- *the dimension of $\mathcal{Z}_{\text{top}}^{\mathfrak{p}}$, where $\mathcal{Z}^{\mathfrak{p}}$ is the centre of $\mathcal{O}_0^{\mathfrak{p}}$, considered as a graded algebra;*
- *the number of irreducible components of $\mathcal{B}_{\mathfrak{p}}$, and hence the dimension of $H^*(\mathcal{B}_{\mathfrak{p}})_{\text{top}}$;*
- *the dimension of the irreducible representation $S^{\lambda(\mathfrak{p})}$ of the symmetric group S_n .*

Proof. The first two agree by Proposition 4.3.1, the last two by Springer’s construction of irreducible S_n -modules [HS79, Proposition 7.1]. Thanks to the main result of [Irv85], the indecomposable projective–injective modules in $\mathcal{O}_0^{\mathfrak{p}}$ are indexed by elements of some right cell of S_n . The statement follows, since the cell modules are exactly the irreducible S_n -modules [Nar89]. \square

The importance of the bijections in the previous lemma becomes apparent in the fact that the category of indecomposable projective–injective modules in $\mathcal{O}_0^{\mathfrak{p}}$ together with the translation functors θ_s , $s \in W$ a simple reflection, categorifies the irreducible representation $S^{\lambda(\mathfrak{p})'}$ of S_n , where $\lambda(\mathfrak{p})'$ denotes the dual partition of $\lambda(\mathfrak{p})$.

More precisely: consider the additive category $\mathcal{C}^{\mathfrak{p}}$ of projective–injective modules in $\mathcal{O}_0^{\mathfrak{p}}$. By Lemma 4.3.3, the complexified (split) Grothendieck group $K_0(\mathcal{C}^{\mathfrak{p}})$ (that is, the complexification of the free abelian group generated by the isomorphism classes $[M]$ of objects M in $\mathcal{C}^{\mathfrak{p}}$ modulo the relation $[M] + [N] = [M \oplus N]$) is isomorphic to the corresponding Specht module as a complex vector space. Moreover, $\mathcal{C}^{\mathfrak{p}}$ is stable under translations θ_s through walls. Since the functors θ_s are exact, they induce endomorphisms $[\theta_s]$ of $K_0(\mathcal{C}^{\mathfrak{p}})$. Let $T_s = [\theta_s] - \text{id} : K_0(\mathcal{C}^{\mathfrak{p}}) \rightarrow K_0(\mathcal{C}^{\mathfrak{p}})$, then the statement is as follows.

PROPOSITION 4.3.4 [KMS09]. *Let $\mathfrak{g} = \mathfrak{gl}_n$ with \mathfrak{b} the standard Borel and \mathfrak{p} a parabolic subalgebra. Let $\lambda(\mathfrak{p})'$ be the dual partition of $\lambda(\mathfrak{p})$ and $S^{\lambda(\mathfrak{p})'}$ the corresponding Specht module. Then there is an isomorphism of right S_n -modules*

$$\epsilon : S^{\lambda(\mathfrak{p})'} \cong K_0(\mathcal{C}^{\mathfrak{p}}).$$

The S_n -module structure on the right-hand side is induced by the T_s .

Let us finish the proof of Theorem 4.1.1 as follows.

THEOREM 4.3.5. *Let $\mathfrak{g} = \mathfrak{gl}_n$ and let \mathfrak{p} be any parabolic subalgebra. Then the map $\Phi_{\mathfrak{p}}$ from Theorem 4.1.1 is an inclusion which induces an isomorphism of S_n -modules $H^*(\mathcal{B}_{\mathfrak{p}})_{\text{top}} \cong \mathcal{Z}_{\text{top}}^{\mathfrak{p}}$.*

Proof. We know that $\Phi_{\mathfrak{p}}$ is a homomorphism of \mathfrak{C} -modules. Hence, it is enough to show that $\Phi_{\mathfrak{p}}$ is injective when restricted to the socle of $H := H^*(\mathcal{B}_{\mathfrak{p}})$. By [Gor03, Theorem 6.6(vi)] the socle of H agrees with the part of highest degree H_{top} . We have $\Phi_{\mathfrak{p}}(H_{\text{top}}) \subseteq \mathcal{Z}_{\text{top}}^{\mathfrak{p}}$ by Lemma 4.3.2 and obtain $\Phi_{\mathfrak{p}}(H_{\text{top}}) \neq 0$ by Propositions 4.2.1 and 4.3.1. On the other hand $\Phi_{\mathfrak{p}}$ is B_W -equivariant and even W -equivariant onto its image (Theorem 3.4.2). Since H_{top} is an irreducible W -module, $\Phi_{\mathfrak{p}}$ defines an inclusion $H^*(\mathcal{B}_{\mathfrak{p}})_{\text{top}} \hookrightarrow \mathcal{Z}_{\text{top}}^{\mathfrak{p}}$, and $\Phi_{\mathfrak{p}}$ is injective. Moreover, $H^*(\mathcal{B}_{\mathfrak{p}})_{\text{top}} \hookrightarrow \mathcal{Z}_{\text{top}}^{\mathfrak{p}}$ must be an isomorphism by Lemma 4.3.3. \square

As a consequence (independent of [Bru06, Theorem 5.11]) we obtain the following categorical construction of the Springer representations.

COROLLARY 4.3.6. *There is an isomorphism $\mathcal{Z}_{\text{top}}^{\mathfrak{p}} \cong S^{\lambda(\mathfrak{p})}$ of S_n -modules.*

Proof. This follows directly from Theorem 4.3.5 and Springer’s construction of the irreducible S_n -modules [Spr78], since $\Phi_{\mathfrak{p}}$ is W -equivariant. \square

4.4 A few remarks on the singular case

Let still $\mathfrak{g} = \mathfrak{gl}_n$. Let $\nu \in \mathfrak{h}^*$ be an integral dominant weight. Let $W_{\nu} = \{w \in W \mid w \cdot \nu = \nu\}$. Then $\mathcal{Z}(\mathcal{O}_{\nu}^{\mathfrak{b}}) = \mathfrak{C}^{W_{\nu}}$, the W_{ν} invariants of \mathfrak{C} (see [Soe90]). Let \mathfrak{p}^{ν} be the parabolic subalgebra of \mathfrak{g} such that $W_{\mathfrak{p}^{\nu}} = W_{\nu}$ and denote by P^{ν} the subgroup of $\text{GL}(n, \mathbb{C})$ with Lie algebra \mathfrak{p}^{ν} . Let $\mathcal{P}^{\nu} = G/P^{\nu}$

be the corresponding partial flag variety. The canonical projection $\mathcal{B} \rightarrow \mathcal{P}^\nu$ gives rise to an inclusion $i_{\mathfrak{p}}^\nu : H^*(\mathcal{P}^\nu) \hookrightarrow H^*(\mathcal{B})$ with image \mathfrak{C}^{W_ν} (see [HS79, Lemma 8.1]). For any parabolic \mathfrak{p} of \mathfrak{g} let $\mathcal{P}_\mathfrak{p}^\nu$ be the associated fixed point variety. The image of the corresponding inclusion

$$i_{\mathfrak{p}}^\nu : H^*(\mathcal{P}_\mathfrak{p}^\nu) \hookrightarrow H^*(\mathcal{B}_\mathfrak{p})$$

are the W_ν -invariants $H^*(\mathcal{B}_\mathfrak{p})^{W_\nu}$ of $H^*(\mathcal{B}_\mathfrak{p})$ (see [HS79, Lemma 8.1]).

Theorem 4.1.1 and Lemma 4.3.3 are generalised as follows.

THEOREM 4.4.1. *Let $\mathfrak{g} = \mathfrak{gl}_n$ with the standard Borel \mathfrak{b} contained in a fixed parabolic $\mathfrak{p} = \mathfrak{p}_\pi$. Let $\nu \in \mathfrak{h}^*$ be a dominant integral weight.*

- (i) *The canonical map $H^*(\mathcal{P}^\nu) = \mathfrak{C}^{W_\nu} = \mathcal{Z}(\mathcal{O}_\nu^{\mathfrak{b}}) \rightarrow \mathcal{Z}(\mathcal{O}_\nu^{\mathfrak{p}})$ factors through $H^*(\mathcal{P}_\mathfrak{p}^\nu)$ and induces a ring homomorphism*

$$\Phi_{\mathfrak{p}}^\nu : H^*(\mathcal{P}_\mathfrak{p}^\nu) \rightarrow \mathcal{Z}(\mathcal{O}_\nu^{\mathfrak{p}}).$$

- (ii) *$\Phi_{\mathfrak{p}}^\nu$ is a surjection and an isomorphism if $\mathfrak{p} = \mathfrak{b}$.*

Proof. The canonical projection of $\mathcal{B}_\mathfrak{p}$ onto $\mathcal{P}_\mathfrak{p}^\nu$ induces an inclusion $H^*(\mathcal{P}_\mathfrak{p}^\nu) \rightarrow H^*(\mathcal{B}_\mathfrak{p})$. By [HS79, Lemma 8.1] we know that the image are exactly the W_ν -invariants of $H^*(\mathcal{B}_\mathfrak{p})$. Assume that z is in the kernel of the canonical map $H^*(\mathcal{P}^\nu) \rightarrow H^*(\mathcal{P}_\mathfrak{p}^\nu)$. Assume that z acts non-trivially on $\mathcal{O}_\nu^{\mathfrak{p}}$. Then there is some module $M \in \mathcal{O}_\nu^{\mathfrak{p}}$, such that z acts non-trivially on M . Hence, $\theta_0^\nu(z) \in \text{End}_{\mathfrak{g}}(\theta_0^\nu M)$ is non-trivial, since θ_0^ν is exact and does not annihilate modules. By [Soe90, Lemma 8], $\theta_0^\nu(z)$ is just given by multiplication with $z \in \mathfrak{C}$. Hence, z acts non-trivially on $\mathcal{O}_\nu^{\mathfrak{p}}$. This is a contradiction to Theorem 4.1.1 and the first statement of the theorem follows.

The map $\Phi_{\mathfrak{p}}^\nu$ is an isomorphism by [Soe90, Endomorphismensatz]. On the other hand $\Phi_{\mathfrak{p}}^\nu$ is surjective by [Bru06, Theorem 5.11]. \square

LEMMA 4.4.2. *Let \mathfrak{g} be any reductive complex Lie algebra with Borel \mathfrak{b} and some parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}$. Let $\nu \in \mathfrak{h}^*$ be a dominant integral weight. Then $P \in \mathcal{O}_\nu^{\mathfrak{p}}$ is indecomposable projective–injective if and only if $\theta_\nu^0 P \in \mathcal{O}_\nu^{\mathfrak{p}}$ is.*

Proof. Let $L \in \mathcal{O}_\nu^{\mathfrak{p}}$ be a simple module. Then $\theta_0^\nu L = L'$ is simple or zero [Jan83, 4.12(3)], and $\text{Hom}_{\mathfrak{g}}(\theta_\nu^0 P, L) = \text{Hom}_{\mathfrak{g}}(P, \theta_0^\nu L) = \text{Hom}_{\mathfrak{g}}(P, L') \neq 0$ only if P is the projective cover of L' . Using again [Jan83, 4.12(3)] we deduce that $\theta_\nu^0 P$ has simple top, and is therefore indecomposable. Since θ_ν^0 does not annihilate any module, P is indecomposable if and only if $\theta_\nu^0 P$ is indecomposable. Since $\theta_0^\nu \theta_\nu^0$ is isomorphic to a direct sum of copies of the identity functor (see [Jan83, 4.13(2)] and [BG80, Theorems 3.3 and 3.5]) and translation functors preserve projectivity and injectivity, P is projective–injective if and only if $\theta_\nu^0 P$ is. \square

Remark 4.4.3. Proposition 4.3.4 together with Proposition 4.4.2 give the dimension of the top degree part of $\mathcal{Z}(\mathcal{O}_\nu^{\mathfrak{p}})$, namely the number of standard ν -tableaux of shape $\lambda(\mathfrak{p})'$, the dual partition of $\lambda(\mathfrak{p})$. Using a graded version of θ_ν^0 (in the sense of [Str03a, Definition 3.3]) one can deduce from Proposition 4.4.2 a formula for the top degree of $\mathcal{Z}(\mathcal{O}_\nu^{\mathfrak{p}})$: consider $\lambda(\mathfrak{p})'$ and $w_0^{\mathfrak{p}'}$ the longest element in the corresponding symmetric group $S_{\lambda(\mathfrak{p})'_1} \times \cdots \times S_{\lambda(\mathfrak{p})'_r}$. If ν is regular, then $\text{top} = 2(l(w_0^{\mathfrak{p}'})$ by [IS88, Proposition and Corollary 3.1]. If ν is not necessarily regular, let w_ν^0 be the longest element in W_ν . Then $\text{top} = 2(l(w_0^{\mathfrak{p}'}) - l(w_\nu^0))$, because a graded version of θ_ν^0 adds $2l(w_\nu^0)$ degrees if we apply it to a simple module (see [Str03a, Theorem 8.2(4)] for a special case). If either $\mathfrak{p} \neq \mathfrak{b}$ or ν is not regular, then the number of simple objects in $\mathcal{O}_\nu^{\mathfrak{p}}$ is in fact

(see [Bru06, Theorem 2]) strictly smaller than the dimension of $H^*(\mathcal{P}_{\mathfrak{p}}^\nu)$ (see § 2.4 and [HS79, Remark 8.6]). In particular, the map $\Phi_{\mathfrak{p}}^\nu$ is not injective in these cases [Bru06, Theorem 2]. A detailed description of these resulting proper quotients of $H^*(\mathcal{P}_{\mathfrak{p}}^\nu)$ can be found in [Bru08].

4.5 The maximal parabolic case

The proof of the injectivity of $\Phi_{\mathfrak{p}}$ in Theorem 4.1.1 was quite involved. In the case $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathfrak{p} \neq \mathfrak{g}$ is a maximal parabolic subalgebra, there is an alternative proof which we now present.

LEMMA 4.5.1. *Let B be a finite-dimensional complex algebra. Assume that B is symmetric, i.e. there is a non-degenerate associative symmetric \mathbb{C} -bilinear form $\mathbf{b} : B \times B \rightarrow \mathbb{C}$. Then there is an isomorphism of vector spaces*

$$\begin{aligned} \mathcal{Z}(B) &\cong (B/[B, B])^* \\ z &\mapsto \mathbf{b}(z, -). \end{aligned}$$

Proof. Let $z \in \mathcal{Z}(B)$, the centre of B and $a, b \in B$. Then $\mathbf{b}(z, ab - ba) = \mathbf{b}(z, ab) - \mathbf{b}(z, ba) = \mathbf{b}(z, ab) - \mathbf{b}(zb, a) = \mathbf{b}(z, ab) - \mathbf{b}(a, zb) = \mathbf{b}(z, ab) - \mathbf{b}(a, bz) = \mathbf{b}(z, ab) - \mathbf{b}(ab, z) = 0$. Hence, $\mathbf{b}(z, -) \in (B/[B, B])^*$. On the other hand if $\mathbf{b}(z, -) \in (B/[B, B])^*$, then $\mathbf{b}(z, ab - ba) = 0$ for any $a, b \in B$. Hence, $\mathbf{b}(za, b) = \mathbf{b}(z, ab) = \mathbf{b}(z, ba) = \mathbf{b}(ba, z) = \mathbf{b}(b, az) = \mathbf{b}(az, b)$ for all $b \in B$ and so $az = za$, since \mathbf{b} is non-degenerate, and therefore $z \in \mathcal{Z}(B)$. The claim of the lemma follows. \square

THEOREM 4.5.2. *Let $\mathfrak{g} = \mathfrak{gl}_n$ and \mathfrak{p}_π a maximal parabolic, i.e. $\pi = \Delta - \{\alpha_s\}$. Then $\Phi_{\mathfrak{p}}$ is an isomorphism.*

Proof. The surjectivity is given by [Bru06]. Let $P_i, i \in I$, be a complete system of representatives for the isomorphism classes of indecomposable projective-injectives in $\mathcal{O}_0^{\mathfrak{p}}$. Set $P = \bigoplus_{i \in I} P_i$ and let P_T be the T -deformation of P given by Proposition 2.7.1(vii). Then $A_T = \text{End}_{\mathfrak{g} \otimes T}(P_T)$ is a free T -module of finite rank (Proposition 2.7.1(viii)). Let $D_T := [A_T, A_T]$. Since $\mathfrak{p} = \mathfrak{p}_\pi$ is assumed to be maximal parabolic, \mathfrak{h}^π is one-dimensional and $S(\mathfrak{h}^\pi)$ is a principal ideal domain. Therefore, D_T is a free T -module as well. We have canonical isomorphisms $A_T \otimes_T T' \cong \text{End}_{\mathfrak{g} \otimes T}(P_T) \otimes_T T' \cong \text{End}_{\mathfrak{g} \otimes T'}(P_T \otimes_T T')$ for $T' = \mathbb{C}$ or $T' = \mathcal{Q}$ (Proposition 2.7.1(ix)). Set $A_T \otimes_T T' = A_{T'}$ for $T' = \mathbb{C}$ or $T' = \mathcal{Q}$ and note that $D_{T'} := D_T \otimes_T T' = [A_T, A_T] \otimes_T T'$ surjects onto $[A_{T'}, A_{T'}]$ canonically. For $T = \mathcal{Q}$ we even have an isomorphism, since D_T is free as a T -module. We deduce that

$$\begin{aligned} \dim_{\mathbb{C}} \text{End}_{\mathfrak{g}}(P) &= \dim_{\mathcal{Q}} \text{End}(P_T \otimes_T \mathcal{Q}) = \text{rank}_T \text{End}(P_T) = \text{rank}_T A_T \\ \dim_{\mathbb{C}} D_{\mathbb{C}} &\leq \dim_{\mathcal{Q}} D_{\mathcal{Q}} = \text{rank}_T D_T. \end{aligned} \tag{4.5.1}$$

Since $\text{End}_{\mathfrak{g} \otimes \mathcal{Q}}(P_{\mathcal{Q}})$ is a product of $|W^{\mathfrak{p}}|$ matrix rings (see § 2.5), we have $\dim_{\mathcal{Q}}(A_{\mathcal{Q}}) - \dim_{\mathcal{Q}} D_{\mathcal{Q}} = \dim_{\mathcal{Q}}(A_{\mathcal{Q}}/D_{\mathcal{Q}}) = |W^{\mathfrak{p}}|$. Hence, (4.5.1) implies $|W^{\mathfrak{p}}| \leq \dim_{\mathbb{C}}(A_{\mathbb{C}}) - \dim_{\mathbb{C}} D_{\mathbb{C}} = \dim_{\mathbb{C}}(A_{\mathbb{C}}/D_{\mathbb{C}})$. Since the algebra $A_{\mathbb{C}}$ is symmetric [MS08, Theorem 5.2], we obtain

$$\dim_{\mathbb{C}} \mathcal{Z}(A_{\mathbb{C}}) \geq |W^{\mathfrak{p}}|.$$

Since the map $\Phi_{\mathfrak{p}}$ is surjective, $\mathcal{Z}(A_{\mathbb{C}})$ is a quotient of $\mathbb{C}[W] \otimes_{\mathbb{C}[W_{\mathfrak{p}}]} \mathbb{C}_{\text{triv}}$ as W -module by Theorem 3.4.2 and [HS79, Corollary 8.5]. Comparing the dimensions we are done. \square

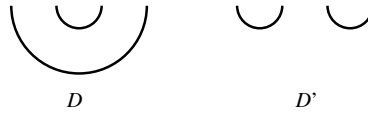


FIGURE 1. The elements of $\text{Cup}(2)$.

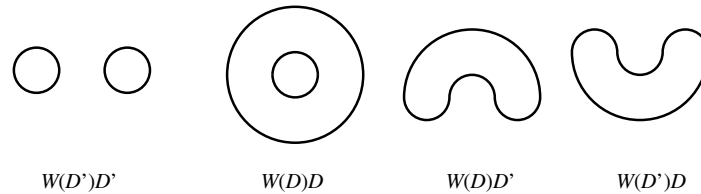


FIGURE 2. Gluing two cup diagrams gives a collection of circles.

5. Diagrammatic approach and Khovanov homology

In this section we prove Theorems 2 and 3 from the introduction. We proceed as follows: first we recall the definition of the algebras \mathcal{H}^n . Then we state some combinatorial results which will be used to give a purely diagrammatic description of $\mathcal{O}^{\mathfrak{p}}(\mathfrak{gl}_n)$ for maximal parabolic subalgebras $\mathfrak{p} \neq \mathfrak{g}$. We explicitly describe how Braden’s presentations can be transformed into this diagrammatic framework, where almost everything is computable. This will finally improve the presentation of [Bra02] drastically in several ways: we are able to see the Koszul grading and obtain a usual Ext-quiver with *homogeneous* relations, we give a very easy recipe to compute dimensions of homomorphism spaces between projective modules, and deduce that the endomorphism rings of projective–injective modules are all isomorphic.

For the rest of the paper we fix $n \in \mathbb{Z}_{>0}$ and $\mathfrak{g} = \mathfrak{gl}_{2n}$ with standard Borel \mathfrak{b} and $\mathfrak{p} = \mathfrak{p}_n$ the parabolic subalgebra where $W_{\mathfrak{p}} = S_n \times S_n$ and denote $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{gl}_{2n})$ by $\mathcal{O}_0^{n,n}$. This is the category which plays an important role.

5.1 Khovanov’s algebras \mathcal{H}^n

We recall the basic definitions from [Kho00], but refer to that paper for details. From now on let $R := \mathbb{C}[X]/(X^2)$ be the ring of dual numbers. This is a commutative Frobenius algebra, hence defines a two-dimensional TQFT \mathcal{F} . In other words, \mathcal{F} is a monoidal functor from the category of oriented cobordisms between 1-manifolds to the category of finite-dimensional complex vector spaces. The Frobenius algebra structure of R is given by:

- the associative multiplication $\mathbf{m} : R \otimes R \rightarrow R$, $r \otimes s \mapsto rs$;
- the comultiplication map $\Delta : R \rightarrow R \otimes R$, $1 \mapsto X \otimes 1 + 1 \otimes X$, $X \mapsto X \otimes X$ (note that this is just a special case of (3.4.1) for $\mathfrak{g} = \mathfrak{gl}_2$);
- the unit map $\epsilon : \mathbb{C} \rightarrow R$, $1 \mapsto 1$;
- the counit or trace map $\delta : R \rightarrow \mathbb{C}$, $1 \mapsto 0$, $X \mapsto 1$.

The functor \mathcal{F} associates to k disjoint circles the vector space $R^{\otimes k}$, to the cobordisms of ‘pair of pants shape’ the multiplication map \mathbf{m} and the comultiplication Δ , respectively. To the cobordisms connecting one circle with the empty manifold, \mathcal{F} associates the trace map δ and the unit map ϵ .

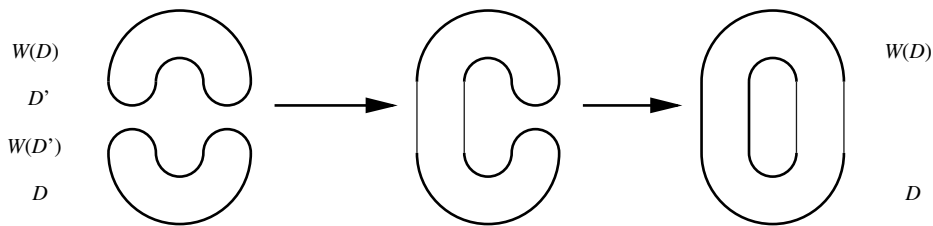


FIGURE 3. Composition of maps: $R \otimes R \xrightarrow{m} R \xrightarrow{\Delta} R \otimes R$.

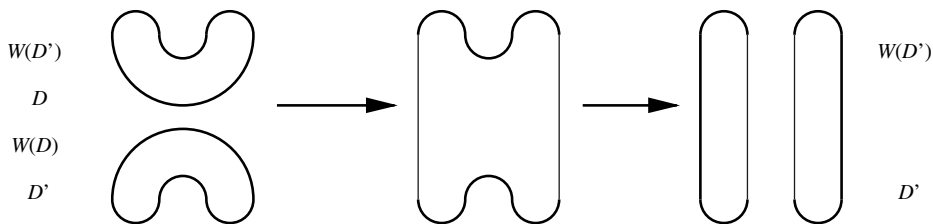


FIGURE 4. Composition of maps: $R \otimes R \xrightarrow{m} R \xrightarrow{\Delta} R \otimes R$.

Let $\text{Cup}(n)$ be the set of crossingless matchings of $2n$ points (see Figure 1). For $a, b \in \text{Cup}(n)$ let $W(b)$ be the reflection of b in the horizontal axis and $W(b)a$ the closed 1-manifold obtained by gluing $W(b)$ and a along their boundaries (see Figure 2). Given $a, b, c \in \text{Cup}(n)$, there is the cobordism from $W(c)bW(b)a$ to $W(c)a$ which contracts $bW(b)$ (see Figures 3 and 4 where the relevant parts are drawn as thin (grey) lines). This cobordism induces a homomorphism of vector spaces

$$\mathcal{F}(W(c)b) \otimes \mathcal{F}(W(b)a) \rightarrow \mathcal{F}(W(c)a). \tag{5.1.1}$$

The algebra \mathcal{H}^n introduced in [Kho00] is defined as follows: as a vector space it is

$$\mathcal{H}^n = \bigoplus_{a,b \in \text{Cup}(n)} {}_b\mathcal{H}_a^n = \bigoplus_{a,b \in \text{Cup}(n)} \mathcal{F}(W(b)a). \tag{5.1.2}$$

The elements from $\text{Cup}(n)$ should be thought of as being primitive idempotents of \mathcal{H}^n , and the spaces $\mathcal{F}(W(b)a) =: {}_b\mathcal{H}_a^n$ are the morphisms from the indecomposable projective left \mathcal{H}^n -module indexed by a to that indexed by b . Therefore, one defines the product $fg = 0$, if $f \in {}_c\mathcal{H}_d^n, g \in {}_b\mathcal{H}_a^n$, where $a, b, c, d \in \text{Cup}(n), b \neq d$. In the case $b = d$, the product is given by (5.1.1).

If we consider R as a graded vector space with the basis vector $1 \in R$ in degree -1 and the basis vector $X \in R$ in degree 1 , then the vector space \mathcal{H}^n inherits a natural \mathbb{Z} -grading. To make it compatible with the algebra structure we have to apply an overall shift $\langle n \rangle$ which increases the grading by n . The graded vector space $\mathcal{H}^n = \bigoplus_{a,b \in \text{Cup}(n)} {}_b\mathcal{H}_a^n \langle n \rangle$ with the above multiplication becomes a positively graded algebra [Kho02]. In the following we consider the algebra \mathcal{H}^n with this grading. In particular, there are the subalgebras ${}_a\mathcal{H}_a^n \cong R^{\otimes n} \langle n \rangle$ of \mathcal{H}^n for any $a \in \text{Cup}(n)$ (cf. Lemma 5.4.2 below).

5.2 Combinatorics: tableaux and generalised cup diagrams

In this section we recollect a few combinatorial facts that are needed later. For any positive integer n let $\mathcal{S}(n)$ be the set of all sequences $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{2n})$ where $\sigma_i \in \{+, -\}$ for $1 \leq i \leq 2n$ with exactly n pluses (minuses respectively). Of course, S_{2n} acts transitively on $\mathcal{S}(n)$ from the right-hand side. Let $\sigma_{\text{dom}} := (+, +, \dots, +, -, -, \dots, -) \in \mathcal{S}(n)$.

Let $Y(n)$ be the set of Young diagrams which fit into an $n \times n$ -square of n^2 boxes, i.e. the Young diagram corresponding to the partition (n^n) . Let $Y(n, \text{upper})$ be the set of Young diagrams which fit into the Young diagram corresponding to the partition $(n-1, n-2, n-3, \dots)$.

To $D \in Y(n)$ we associate a sequence $\sigma_D \in \mathcal{S}(n)$ indicating its shape as follows: first embed D into a $n \times n$ -square D' of n^2 boxes, such that their upper left corners coincide. Now there is a unique path p_D from the bottom left corner to the top right corner of D' , along the sides of the boxes, such that the interior of D is completely to the left of the path and all other boxes are to the right. The number of sides involved in the path is always $2n$. Starting from the lower left corner, the path is uniquely determined by giving the direction for each side. We use the rule ‘minus=go up’, ‘plus=go right’. In this way we associate to D first a path p_D , and then a sequence $\sigma_D \in \mathcal{S}(n)$ encoding the path p_D .

Example 5.2.1. It is $Y(2) = \{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \emptyset\}$ and $Y(2, \text{upper}) = \{\square, \emptyset\}$. $\sigma_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = (+, +, -, -)$, $\sigma_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = (+, -, +, -)$, $\sigma_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = (+, -, -, +)$, $\sigma_{\square} = (-, +, +, -)$, $\sigma_{\square} = (-, +, -, +)$, $\sigma_{\emptyset} = (-, -, +, +)$.

Let $\text{PMS}(n)$ (and $\text{PrInj}(n)$ respectively) be the set of iso-classes of indecomposable projective(-injective) modules in $\mathcal{O}_0^{n,n}$.

PROPOSITION 5.2.2. *There are canonical bijections*

$$\begin{array}{ccccccc} Y(n) & \leftrightarrow & \mathcal{S}(n) & \leftrightarrow & S_n \times S_n \backslash S_{2n} & \leftrightarrow & \text{PMS}(n) \\ D & \mapsto & \sigma_D, \sigma_{\text{dom}w} & \mapsto & w & \mapsto & [P^{\mathbb{P}}(w \cdot 0)] \end{array}$$

Proof. The first bijection is clear. The second is obvious, since S_{2n} acts transitively on $\mathcal{S}(n)$ and σ_{dom} has stabiliser $S_n \times S_n$. The third bijection is by definition (see § 2.4). \square

To make the assignment $D \mapsto \sigma_D$ more precise we now follow the setup of Braden, see [Bra02] for details. Put $\mathbb{H} := \mathbb{Z} + \frac{1}{2}$ and for $k, l \in \mathbb{R}$ set $\mathbb{H}[k, l] := \{\alpha \in \mathbb{H} \mid k \leq \alpha \leq l\}$. We generalise the construction above: let λ be a *partition*, by which from now on we mean an infinite decreasing sequence $\lambda = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ of non-negative integers such that $\lambda_i = 0$ for large i . We associate the corresponding Young diagram $D(\lambda)$ with λ_i boxes in the i -row, and also an infinite $\{+, -\}$ -sequence φ_λ indexed by \mathbb{H} (that is, a function $\varphi_\lambda : \mathbb{H} \rightarrow \{-, +\}$) as follows:

$$\varphi_\lambda(\alpha) = \begin{cases} - & \text{if } \alpha = \lambda_i - i + \frac{1}{2} \text{ for some } i \in \mathbb{Z}_{>0}, \\ + & \text{otherwise.} \end{cases}$$

The uncommon indexing set is chosen to make it compatible with [Bra02]. In particular, $D(\lambda)$ fits into a square of $n \times n$ boxes if and only if:

- $\varphi_\lambda(j) = -$ if $j < -n + \frac{1}{2}$;
- $\varphi_\lambda(j) = +$ if $j > n - \frac{1}{2}$;
- the set $\{\varphi_\lambda(j) \mid n - \frac{1}{2} \geq j \geq -n + \frac{1}{2}\}$ contains exactly n pluses and n minuses.

Let $\widetilde{\mathcal{S}}(n)$ be the set of such $\{+, -\}$ -sequences. We have isomorphisms of finite sets $Y(n) \cong \widetilde{\mathcal{S}}(n)$, $D(\lambda) \mapsto \varphi_\lambda$ and $\widetilde{\mathcal{S}}(n) \cong \mathcal{S}(n)$ $\varphi \mapsto (\varphi(-n + \frac{1}{2}), \varphi(-n + \frac{3}{2}), \dots, \varphi(n - \frac{1}{2}))$, the *restriction* of φ to $\mathbb{H}[-n, n]$.

DEFINITION 5.2.3. Let λ be a partition, i.e. $\lambda = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ of non-negative integers such that $\lambda_i = 0$ for large i . Following [Bra02] we call $(\alpha, \beta) \in \varphi_\lambda^{-1}(\{-\}) \times \varphi_\lambda^{-1}(\{+\})$ a λ -pair if $\alpha < \beta$, $\sum_{\alpha \leq \gamma \leq \beta} \varphi_\lambda(\alpha)1 = 0$ and β is minimal with this property. If (α, β) and (α', β') are λ -pairs, then $(\alpha', \beta') > (\alpha, \beta)$ if and only if $\alpha' < \alpha$ and $\beta' > \beta$. If (α', β') is minimal with this

property, then (α', β') is called a *parent* of (α, β) . In the following (α', β') will always denote the parent of (α, β) .

For an example see the first diagram in Figure 7. Each λ -pair has a unique parent [Bra02, § 1.2], so the notation (α', β') makes sense. The set $\mathbb{H}[-n, n]$ labels in a natural way the endpoints of the arcs in any cup diagram from $\text{Cup}(n)$ from the left to the right: if $n = 2$ for instance, then $\text{Cup}(2) = \{D, D'\}$ (see Figure 1), the vertices labelled by $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ from left to right. Then $(-\frac{3}{2}, \frac{3}{2}), (-\frac{1}{2}, \frac{1}{2})$ are λ -pairs for D , whereas $(-\frac{3}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{3}{2})$ are λ -pairs for D' .

If $D \in Y(m, \text{upper})$, then let c_D be the $2m$ -cup diagram $c_D \in \text{Cup}(m)$ where $\alpha, \beta \in \mathbb{H}[-m, m]$ are connected if and only if (α, β) form a λ -pair in φ_λ . (This procedure is well defined since the pairs are nested [Bra02, Lemma 1.2.1].)

The bijections from the Proposition 5.2.2 provide three different labelling sets for the isomorphism classes of indecomposable projective modules in $\text{PMS}(n)$. The subset $\text{PrInj}(n)$ given by projective–injective modules plays an important role, so we would like to have the corresponding labelling sets singled out. The indecomposable projective module $P([w^{\mathfrak{p}}] \cdot 0)$ corresponding to the longest element $[w^{\mathfrak{p}}]$ is always projective–injective in $\mathcal{O}_0^{\mathfrak{p}}$, and all other projective–injective modules are exactly those which occur as direct summands when translation functors are applied to $P([w^{\mathfrak{p}}] \cdot 0)$, and their number is equal to the dimension of the Specht module corresponding to $\lambda(\mathfrak{p})'$ (see [KMS09, Proposition 4.3.4]). Now Proposition 5.2.2 restricts to the following result.

PROPOSITION 5.2.4. *There are canonical bijections*

$$\begin{array}{ccccccc} \text{Cup}(n) & \longleftarrow & Y(n, \text{upper}) & \longleftrightarrow & \mathcal{S}(n)' & \longleftrightarrow & \text{PrInj}(n) \\ c_D & \longleftarrow & D & \longmapsto & \sigma_D, \sigma_{\text{dom} w} & \longmapsto & [P^{\mathfrak{p}}(w \cdot 0)] \end{array}$$

where $\mathcal{S}(n)'$ denotes the subset of sequences $\sigma \in \mathcal{S}(n)$ such that for any plus there are more minuses than pluses appearing prior to the given plus.

Proof. A sequence $\sigma \in \mathcal{S}(n)$ can give rise to a cup diagram in $\text{Cup}(n)$ if and only if for any plus there are more minuses than pluses appearing prior to the given plus. Hence, the associated path stays above the diagonal, and gives rise to a Young diagram $D \in Y(n, \text{upper})$ such that $\sigma_D = \sigma$. Since the sets $\text{Cup}(n)$, $Y(n, \text{upper})$ and $\mathcal{S}(n)'$ have the same cardinality (the n th Catalan number), the first two bijections follow. The element $[w^{\mathfrak{p}}]$ corresponds to the sequence $(-, -, \dots, -, +, +, \dots, +)$, hence to a $2n$ -cup diagram. The last bijection follows from Proposition 4.3.4, since the S_{2n} -Specht module $S^{(n,n)}$ factors through the specialised Temperley–Lieb algebra TL_{2n} (see [Str05, Theorem 4.1]) and the resulting representation is the regular representation. \square

Example 5.2.5. Let $n = 1$, hence $\mathfrak{g} = \mathfrak{gl}_2$ with Weyl group $S_2 = \langle s \rangle$ and $\mathfrak{p} = \mathfrak{b}$. The only element in $Y(1, \text{upper})$ is the empty diagram and corresponds to the cup diagram with one cup and the sequence $(-, +)$, which then corresponds to the indecomposable projective–injective module $P^{\mathfrak{p}}(s \cdot 0) = P^{\mathfrak{b}}(s \cdot 0) \in \mathcal{O}_0^{1,1}$.

Example 5.2.6. Let $n = 2$, hence $\mathfrak{g} = \mathfrak{gl}_4$ with Weyl group $S_4 = \langle s_1 = (1, 2), s_2 = (2, 3), s_3 = (3, 4) \rangle$ and \mathfrak{p} such that $W_{\mathfrak{p}} = \langle s_1, s_3 \rangle \cong S_2 \times S_2$. The empty Young diagram corresponds to the element D from Figure 1 and the sequence $(-, -, +, +)$, which correspond then to $P^{\mathfrak{p}}(s_2 s_1 s_3 s_2 \cdot 0)$, whereas the one-box diagram corresponds to D' and the sequence $(-, +, -, +)$ which correspond to $P := P^{\mathfrak{p}}(s_2 s_1 s_3 \cdot 0)$. Note that $P \cong \theta_\lambda^0 P^{\mathfrak{p}}(s_2 \cdot \nu)$, where ν is a dominant

integral weight with stabiliser $\langle s_1, s_3 \rangle$ (hence, this illustrates Proposition 4.2.1). It is easy to check that $\text{End}_{\mathfrak{g}}(P) \cong \mathbb{C}[X]/(X^2) \otimes \mathbb{C}[X]/(X^2)$.

5.3 Khovanov's algebra and PrInj

The motivation for the remaining sections is to establish a direct connection between the tangle invariants defined in [Kho02] on the one hand and those defined in [Str05] on the other hand. In this paper we establish the key step from which it can then be deduced that Khovanov's invariants are nothing else than certain restrictions of the functorial invariants from [Str05] (see § 5.10). The key step is to prove the following result (conjectured in [Str06], see also the weaker version in [Bra02]).

THEOREM 5.3.1. *Let $n \in \mathbb{Z}_{>0}$ and $\mathfrak{g} = \mathfrak{gl}_{2n}$. Let $\mathfrak{p} = \mathfrak{p}_n$ and $P^{\mathfrak{p}}(x \cdot 0)$, $x \in I \subseteq W^{\mathfrak{p}}$ be a complete set of representatives for $\text{PrInj}(n)$. Set*

$$D_{n,n} := \text{End}_{\mathfrak{g}}\left(\bigoplus_{x \in I} P^{\mathfrak{p}}(x \cdot 0)\right).$$

Then there is an isomorphism of algebras

$$D_{n,n} \cong \mathcal{H}^n \tag{5.3.1}$$

such that $\text{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(x \cdot 0), P^{\mathfrak{p}}(y \cdot 0))$ is identified with ${}_b\mathcal{H}_a^n$, where $a = \sigma_{\text{dom}x}$, $b = \sigma_{\text{dom}y}$. The isomorphism is even an isomorphism of \mathbb{Z} -graded algebras.

COROLLARY 5.3.2. *In the situation of the theorem we have*

$$\text{End}_{\mathfrak{g}}(P) \cong (\mathbb{C}[X]/(X^2))^{\otimes n}$$

for any indecomposable projective–injective module in $\mathcal{O}_0^{\mathfrak{p}}$.

To prove the theorem we embed the algebra \mathcal{H}^n into a larger algebra \mathcal{K}^n where the primitive idempotents are in bijection to the elements of $Y(n)$ and not just to the elements of $Y(n, \text{upper})$. The actual proof will be given in § 5.7.

5.4 The algebra \mathcal{K}^n , an enlargement of \mathcal{H}^n

Let $a \in \mathcal{S}(n)$. Take the corresponding Young diagram $D \in Y(n)$ (i.e. $a = \sigma_D$) and the corresponding partition λ . We view D as a Young diagram $\tilde{D} \in Y(2n)$ and associate the $\{+, -\}$ -sequence $\tilde{\sigma}_{\lambda} \in \mathcal{S}(2n)$ of length $4n$ by restricting the $\{+, -\}$ -sequence φ_{λ} to $\mathbb{H}[-2n, 2n]$. Alternatively, we could take the sequence a and put n minuses in front and n pluses afterwards to obtain a $\{+, -\}$ -sequence of length $4n$ which is exactly $\tilde{\sigma}_{\lambda}$.

Let $\tilde{a} = \sigma_{\tilde{D}} \in \text{Cup}(2n)$ be the corresponding $4n$ -cup diagram where the arcs correspond to λ -pairs. Hence, given $a, b \in \mathcal{S}(n)$ we have the cup diagrams \tilde{a}, \tilde{b} where the endpoints of the cups are labelled by $\alpha \in \mathbb{H}[-2n, 2n]$ and the arcs correspond to λ -pairs. We call an endpoint *inner* if it is contained in $\mathbb{H}[-n, n]$, *outer left* if it is contained in $\mathbb{H}[-2n, -n]$, *outer right* if it is contained in $\mathbb{H}[n, 2n]$.

Consider $W(\tilde{b})\tilde{a}$. This is a collection of circles which we view as coloured.

- A circle is *black* if it passes through inner points only.
- A circle is *green* if it is not black and passes through at most one outer left point and at most one outer right point.
- A circle is *red* if it is neither black nor green.

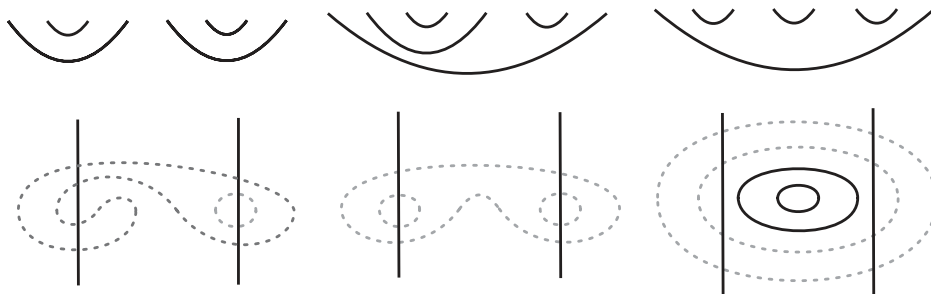


FIGURE 5. The vertical lines separate inner from outer points.

Example 5.4.1. For $n = 2$ and $a = (+, +, -, -)$, $b = (-, +, +, -)$, $c = (+, -, +, -)$, $d = (-, -, +, +)$ we display in Figure 5 the cup diagrams \tilde{a} , \tilde{b} , \tilde{c} as well as the diagrams $W(\tilde{b})\tilde{a}$, $W(\tilde{c})\tilde{a}$ and $W(\tilde{d})\tilde{d}$. (The dotted circles are green; the left most (dashed) circle is red. There are two black circles.)

Let $\mathbf{B}(b, a)$ ($\mathbf{G}(b, a)$, $\mathbf{R}(b, a)$ respectively) be the number of black (green and red, respectively) circles in $W(\tilde{b})\tilde{a}$. To $W(\tilde{b})\tilde{a}$ we associate the complex vector space

$$\mathcal{G}(W(\tilde{b})\tilde{a}) := {}_b\mathcal{K}_a^n := \begin{cases} R^{\otimes \mathbf{B}(b,a)} \otimes \mathbb{C}^{\otimes \mathbf{G}(b,a)} & \text{if } \mathbf{R}(b, a) = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

As a complex vector space, the algebra \mathcal{K}^n is

$$\mathcal{K}^n = \bigoplus_{a,b \in \mathcal{S}(n)} {}_b\mathcal{K}_a^n = \bigoplus_{a,b \in \mathcal{S}(n)} \mathcal{G}(W(\tilde{b})\tilde{a}). \quad (5.4.1)$$

The unit $\epsilon: \mathbb{C} \rightarrow R$, the map $\epsilon': R \rightarrow \mathbb{C}$ $1 \mapsto 1$ $X \mapsto 0$, the inclusion $\{0\} \rightarrow R$ and the zero map $R \rightarrow \{0\}$ give rise to canonical maps $\text{can}: \mathcal{F}(W(\tilde{c})\tilde{a}) \rightarrow \mathcal{G}(W(\tilde{c})\tilde{a})$ and $\text{can}: \mathcal{G}(W(\tilde{c})\tilde{a}) \rightarrow \mathcal{F}(W(\tilde{c})\tilde{a})$ which ‘introduce the colouring’ and ‘forget the colouring’. We turn \mathcal{K}^n into an algebra by putting $fg = 0$ if $f \in {}_c\mathcal{K}_d^n$, $g \in {}_b\mathcal{K}_a^n$, where $a, b, c, d \in \mathcal{S}(n)$, $b \neq d$; and in the case $b = d$, the product is given by the composition

$$\begin{array}{ccc} \mathcal{G}(W(\tilde{c})\tilde{b}) \otimes \mathcal{G}(W(\tilde{b})\tilde{a}) & & \mathcal{G}(W(\tilde{c})\tilde{a}), \\ \downarrow & & \uparrow \\ \mathcal{F}(W(\tilde{c})\tilde{b}) \otimes \mathcal{F}(W(\tilde{b})\tilde{a}) & \rightarrow & \mathcal{F}(W(\tilde{c})\tilde{a}) \end{array} \quad (5.4.2)$$

where the vertical maps are canonical and the horizontal map is the multiplication in \mathcal{H}^{2n} from (5.1.1).

LEMMA 5.4.2. *Let $a \in \mathcal{S}(n)$. Then ${}_a\mathcal{K}_a^n \cong R^{\otimes \mathbf{B}(a,a)} \otimes \mathbb{C}^{\otimes \mathbf{G}(a,a)}$ as algebras and the canonical map ${}_a\mathcal{H}_a^{2n} \rightarrow {}_a\mathcal{K}_a^n$ is surjective.*

Proof. By the definition of \tilde{a} and the colouring rules, $W(\tilde{a})\tilde{a}$ is a union of black and green circles only. Hence, ${}_a\mathcal{K}_a \cong \mathcal{G}(W(\tilde{a})\tilde{a}) = R^{\otimes \mathbf{B}(a,a)} \otimes \mathbb{C}^{\otimes \mathbf{G}(a,a)} \neq \{0\}$. If we number the circles of $W(\tilde{a})\tilde{a}$, first the black and then the green ones, then

$$\begin{aligned} \mathcal{G}(W(\tilde{a})\tilde{a}) \otimes \mathcal{G}(W(\tilde{a})\tilde{a}) &\longrightarrow \mathcal{G}(W(\tilde{a})\tilde{a}), \\ (R^{\otimes \mathbf{B}(a,a)} \otimes \mathbb{C}^{\otimes \mathbf{G}(a,a)}) \otimes (R^{\otimes \mathbf{B}(a,a)} \otimes \mathbb{C}^{\otimes \mathbf{G}(a,a)}) &\longrightarrow R^{\otimes \mathbf{B}(a,a)} \otimes \mathbb{C}^{\otimes \mathbf{G}(a,a)} \end{aligned}$$

by multiplying the i th factor in the first tensor product $\mathcal{G}(W(\tilde{a})\tilde{a})$ with the i th factor in the second $\mathcal{G}(W(\tilde{a})\tilde{a})$. The first statement of the lemma follows. The second follows directly from the definitions. \square

Example 5.4.3. Let $n = 1$ with the two sequences $a = (-, +)$ and $b = (+, -)$. Then \tilde{a} and \tilde{b} are the diagrams as depicted in Figure 1. We obtain ${}_a\mathcal{K}_a^1 = R \otimes \mathbb{C}$ as algebra, ${}_b\mathcal{K}_a^1 = \mathbb{C} = {}_a\mathcal{K}_b^1$ as vector spaces and ${}_b\mathcal{K}_b^1 = \mathbb{C} \otimes \mathbb{C}$ as algebra. Using formula (5.4.2) one easily verifies that the multiplication is given by the following formulas

$$\begin{aligned}
 {}_a\mathcal{K}_a^1 \otimes {}_a\mathcal{K}_b^1 &= R \otimes \mathbb{C} \otimes \mathbb{C} \longrightarrow \mathbb{C} = {}_a\mathcal{K}_b^1 \\
 1 \otimes 1 \otimes 1 &\longmapsto 1 \\
 X \otimes 1 \otimes 1 &\longmapsto 0 \\
 {}_b\mathcal{K}_a^1 \otimes {}_a\mathcal{K}_b^1 &= \mathbb{C} \otimes \mathbb{C} \longrightarrow \mathbb{C} \otimes \mathbb{C} = {}_b\mathcal{K}_b^1 \\
 1 \otimes 1 &\longmapsto 0 \\
 {}_a\mathcal{K}_b^1 \otimes {}_b\mathcal{K}_a^1 &= \mathbb{C} \otimes \mathbb{C} \longrightarrow R \otimes \mathbb{C} = {}_a\mathcal{K}_a^1 \\
 1 \otimes 1 &\longmapsto X \otimes 1 \\
 {}_a\mathcal{K}_b^1 \otimes {}_b\mathcal{K}_b^1 &= \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \longrightarrow \mathbb{C} = {}_a\mathcal{K}_b^1 \\
 1 \otimes 1 \otimes 1 &\longmapsto 1 \\
 {}_b\mathcal{K}_a^1 \otimes {}_a\mathcal{K}_a^1 &= \mathbb{C} \otimes R \otimes \mathbb{C} \longrightarrow \mathbb{C} = {}_b\mathcal{K}_a^1 \\
 1 \otimes 1 \otimes 1 &\longmapsto 1, \\
 1 \otimes X \otimes 1 &\longmapsto 0 \\
 {}_b\mathcal{K}_b^1 \otimes {}_b\mathcal{K}_a^1 &= \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \longrightarrow \mathbb{C} = {}_b\mathcal{K}_a^1 \\
 1 \otimes 1 \otimes 1 &\longmapsto 1.
 \end{aligned}$$

Note that \mathcal{K}^1 is isomorphic to the endomorphism ring $A_{1,1}$ of a minimal projective generator in $\mathcal{O}_0^b(\mathfrak{gl}_2)$ (see [Str03b, § 5.1.1] and also Theorem 5.8.1).

The \mathbb{Z} -grading on H^{2n} (see the paragraph before § 5.2) induces a unique grading on K^n with respect to which both the maps can be graded maps. It is then clear that K^n is itself a graded algebra. We leave it as an exercise to the reader to show that K^1 from Example 5.4.3 is in fact a Koszul algebra.

If $a \in \mathcal{S}(n)$, then we define for $1 \leq i \leq 2n$ the element $\text{can}(X_i(a)) \in {}_a\mathcal{K}_a^n$ as the image of $X_i(a) := 1^{\otimes(i-1)} \otimes X \otimes 1^{\otimes(2n-i)} \in {}_{\tilde{a}}\mathcal{H}_{\tilde{a}}^{2n}$ under the canonical map. The following lemma describes ${}_b\mathcal{K}_a^n$ as a left ${}_b\mathcal{K}_b^n$ - and right ${}_a\mathcal{K}_a^n$ -module.

LEMMA 5.4.4. *Let $a, b \in \mathcal{S}(n)$. Let $f \in {}_b\mathcal{K}_a^n$ and $\text{can}(f)$ its canonical image in ${}_{\tilde{b}}\mathcal{H}_{\tilde{a}}^{2n}$. Then*

$$\begin{aligned}
 \text{can}(X_i(b))f &= \text{can}(X_i(b) \text{can}(f)) \in {}_b\mathcal{K}_a^n \\
 f \text{can}(X_i(a)) &= \text{can}(\text{can}(f)X_i(a)) \in {}_b\mathcal{K}_a^n
 \end{aligned}$$

for $1 \leq i \leq 2n$.

Proof. This follows directly from the definitions (5.1.1) and (5.4.2). \square

5.5 Braden's description of $\mathcal{O}_0^{n,n}$

If P is a minimal projective generator of $\mathcal{O}_0^{n,n}$ and $A_{n,n} = \text{End}_{\mathfrak{g}}(P)$, then $\mathcal{O}_0^{n,n}$ is equivalent to the category of finitely generated right $A_{n,n}$ -modules. (In fact we could also work with left modules,

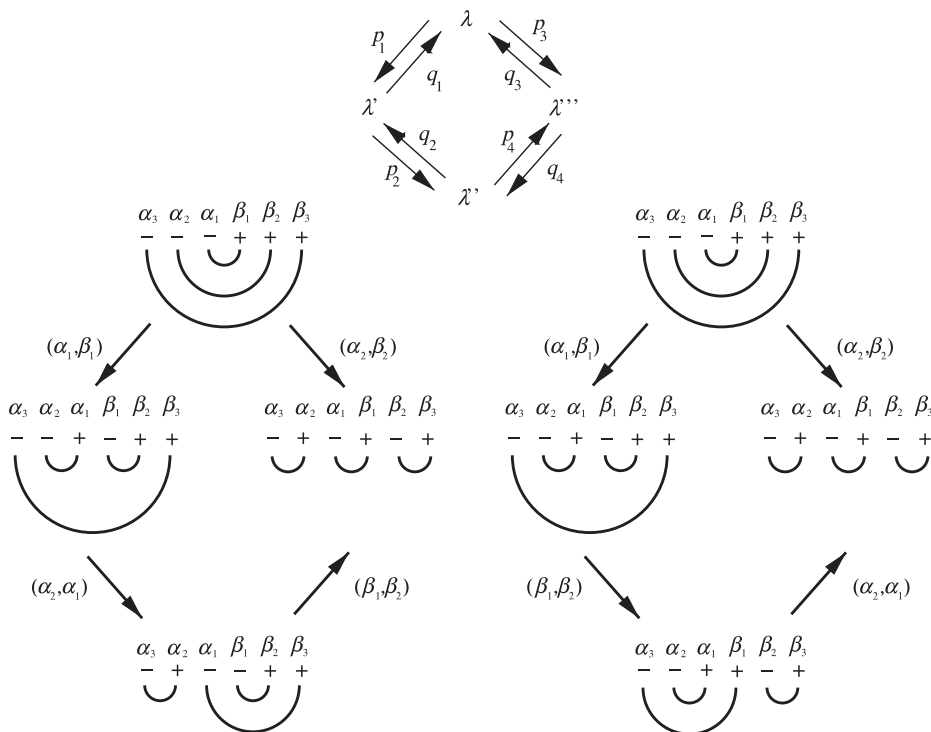


FIGURE 6. Diamonds.

since \mathcal{O} has a contravariant duality which identifies $A_{n,n}$ with its opposite algebra.) In [Bra02], Braden gave an explicit description of $A_{n,n}$ in terms of generators and relations. We briefly recall this description.

DEFINITION 5.5.1. Let λ and ν be partitions. We write $\lambda \rightarrow \nu$ or $\lambda \xrightarrow{(\alpha,\beta)} \nu$ if there is a λ -pair (α, β) such that $\varphi_\lambda(\gamma) = \varphi_\nu(\gamma)$ for any $\gamma \in \mathcal{H}$, $\gamma \neq \alpha, \beta$ and $\varphi_\nu(\alpha) = +$ and $\varphi_\nu(\beta) = -$ (see Figure 7). Note that the λ -pair is uniquely determined by λ and ν . We write $\lambda \leftrightarrow \nu$ if either $\lambda \rightarrow \nu$ or $\nu \rightarrow \lambda$. Similarly, if $\phi_\lambda, \phi_\nu \in \widetilde{\mathcal{S}(n)}$ with corresponding partitions λ and ν , then we write $\phi_\lambda \rightarrow \phi_\nu$ if $\lambda \rightarrow \nu$ and $\phi_\lambda \leftrightarrow \phi_\nu$ if $\lambda \leftrightarrow \nu$. A *diamond* is a tuple $(\lambda, \lambda', \lambda'', \lambda''')$ of four distinct partitions satisfying $\lambda \leftrightarrow \lambda' \leftrightarrow \lambda'' \leftrightarrow \lambda''' \leftrightarrow \lambda$.

Typical diamonds are depicted in Figure 6 where we display the relevant parts of the $\{+, -\}$ -sequences with their cup-diagrams.

PROPOSITION 5.5.2 [Bra02, § 1.3]. *The algebra $A_{n,n}$ is the unitary associative \mathbb{C} -algebra with generators*

$$\left\{ e_\lambda, t_{\alpha,\lambda} \mid \alpha \in \mathbb{H}, \lambda \in \widetilde{\mathcal{S}(n)} \right\},$$

$$\left\{ p(\lambda, \nu), \mu(\lambda, \nu) \mid \lambda, \nu \in \widetilde{\mathcal{S}(n)}, \lambda \leftrightarrow \nu \right\},$$

and relations:

- (i) $\sum_{\lambda \in \widetilde{\mathcal{S}(n)}} e_\lambda = 1$;
- (ii) $e_\lambda e_\nu = 0$ if $\lambda \neq \nu$ and $e_\lambda e_\lambda = e_\lambda$ for any $\lambda \in \widetilde{\mathcal{S}(n)}$;

- (iii) $\mu(\lambda, \nu) = 1 + p(\lambda, \nu)p(\nu, \lambda)$ for any $\lambda \leftrightarrow \nu$;
- (iv) $t_{\alpha, \lambda}p(\lambda, \nu) = p(\lambda, \nu)t_{\alpha, \nu}$ for any $\alpha \in \mathbb{H}$, $\lambda \in \widetilde{\mathcal{S}(n)}$;
- (v) $t_{\alpha, \lambda}t_{\beta, \nu} = 0$ if $\lambda \neq \nu$;
- (vi) the t commute with each other;
- (vii) $t_{\alpha, \lambda}t_{\beta, \lambda} = e_\lambda$ if (α, β) is a λ -pair;
- (viii) $t_{\alpha, \lambda} = e_\lambda$ for any $\lambda \in \widetilde{\mathcal{S}(n)}$ if $\alpha < -n$ or $\alpha > n$;
- (ix) if $\lambda \xrightarrow{(\alpha, \beta)} \nu$ and the λ -pair (α', β') is the parent of (α, β) , then

$$\begin{aligned} \mu(\nu, \lambda)^{\eta(\beta)} &= t_{\alpha, \nu}t_{\beta', \nu} \\ \mu(\lambda, \nu)^{\eta(\beta)} &= t_{\alpha, \lambda}t_{\beta', \lambda} \end{aligned}$$

where $\eta(\beta) = (-1)^{\beta + \frac{1}{2}}$;

- (x) if $(\lambda, \lambda', \lambda'', \lambda''')$ is a diamond with all elements contained in $\widetilde{\mathcal{S}(n)}$, then

$$p(\lambda'', \lambda')p(\lambda', \lambda) = p(\lambda'', \lambda''')p(\lambda''', \lambda).$$

If all of the elements in the diamond except λ''' are in $\widetilde{\mathcal{S}(n)}$, then

$$p(\lambda, \lambda')p(\lambda', \lambda'') = 0 = p(\lambda'', \lambda')p(\lambda', \lambda). \tag{5.5.1}$$

Remark 5.5.3. The labelling of the idempotents in the algebra $A_{n,n}$ is such that $e_\lambda \in A_{n,n} = \text{End}_{\mathfrak{g}}(P)$ is the idempotent projecting P onto its summand $P^p(w \cdot 0)$ where λ corresponds to w according to Proposition 5.2.4. For instance, the empty partition corresponds to the zero-dimensional Schubert cell, whereas the largest possible partition corresponds to the largest Schubert cell. Indeed, the idempotent e_λ is naturally associated with the Schubert cell X_λ in [Bra02, §1.1], which in turn comes along with an intersection homology complex \mathcal{I}_w corresponding to $P^p(w \cdot 0)$ (see [HTT08, Theorem 12.2.5 and Example 12.2.6]). The dictionary between the partition λ and the Weyl group element w is given by the formulas [Ful98, Propositions 8 and 9] together with the duality [Ful98, p. 149].

5.6 The map from $A_{n,n}$ to \mathcal{K}^n

The following lemma allows us to pass between λ -pairs, arcs and circles and follows directly from the definitions.

LEMMA 5.6.1. *Let $a \in \mathcal{S}(n)$ and let λ be the corresponding partition. Then:*

- (i) *there is a canonical bijection between the λ -pairs (α, β) where $-2n \leq \alpha \leq 2n$ and the cups in \tilde{a} ;*
- (ii) *this bijection induces a canonical bijection between the λ -pairs (α, β) where $-2n \leq \alpha \leq 2n$ and the circles in $W(\tilde{a})\tilde{a}$;*
- (iii) *to any λ -pair (α, β) where $-2n \leq \alpha \leq 2n$ and $b \in \mathcal{S}(n)$, there is a unique circle in $W(\tilde{b})\tilde{a}$ (and in $W(\tilde{a})\tilde{b}$, respectively) which contains the arc in \tilde{a} corresponding to (α, β) .*

Let $a, b \in \mathcal{S}(n)$ and $\lambda, \nu \in \widetilde{\mathcal{S}(n)}$ their extensions. We denote by e_λ, e_ν the corresponding idempotents in \mathcal{K}^n . If Lemma 5.6.1 associates with (α, β) the k th circle in $W(\tilde{a})\tilde{b}$ (or $W(\tilde{b})\tilde{a}$), then denote

$$\begin{aligned} X_\alpha(a, b) &:= \text{can}(1^{\otimes(k-1)} \otimes X \otimes 1^{\otimes(2n-k)}) \in {}_a\mathcal{K}_b^n, \\ X_\alpha(b, a) &:= \text{can}(1^{\otimes(k-1)} \otimes X \otimes 1^{\otimes(2n-k)}) \in {}_b\mathcal{K}_a^n. \end{aligned}$$

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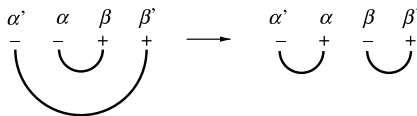


FIGURE 7. The relevant parts involved in $\lambda \xrightarrow{(\alpha, \beta)} \nu$.

If the pair (α, β) is not associated with a circle of $W(\tilde{a})\tilde{b}$ (or $W(\tilde{b})\tilde{a}$, respectively), then set $X_\alpha(a, b) = 0 \in {}_a\mathcal{K}_b^n$ (and $X_\alpha(b, a) = 0 \in {}_b\mathcal{K}_a^n$, respectively).

PROPOSITION 5.6.2. *In the notation of Definition 5.2.3, there is a homomorphism of algebras*

$$\begin{array}{lll} \mathcal{E}: & A_{n,n} & \longrightarrow \mathcal{K}^n \\ & e_\lambda & \longmapsto e_\lambda \qquad \qquad \qquad \in {}_a\mathcal{K}_a^n \\ & t_{\alpha, \lambda} & \longmapsto e_\lambda + \eta(\beta)X_\alpha \qquad \in {}_a\mathcal{K}_a^n \\ & t_{\beta, \lambda} & \longmapsto e_\lambda - \eta(\beta)X_\alpha \qquad \in {}_a\mathcal{K}_a^n \\ & p(\nu, \lambda) & \longmapsto 1 \otimes 1 \otimes \dots \otimes 1 + \frac{1}{2}X_\alpha \qquad \in {}_b\mathcal{K}_a^n \\ & p(\lambda, \nu) & \longmapsto 1 \otimes 1 \otimes \dots \otimes 1 + \frac{1}{2}X_{\alpha'} \qquad \in {}_a\mathcal{K}_b^n \\ & \mu(\lambda, \nu) & \longmapsto e_\lambda + X_\alpha + X_{\alpha'} + X_\alpha \star X_{\alpha'} \qquad \in {}_a\mathcal{K}_a^n \end{array}$$

where $\lambda, \nu \in \widetilde{\mathcal{S}(n)}$ with restrictions $a, b \in \mathcal{S}(n)$ and λ -pairs (α, β) such that $\lambda \xrightarrow{(\alpha, \beta)} \nu$ in the last three cases; and $x \star y$ denotes the component-wise product in ${}_a\mathcal{K}_b^n$ for any $x, y \in {}_a\mathcal{K}_b^n$.

For the proof we need the following lemma which is illustrated in Figure 7.

LEMMA 5.6.3. *Let λ, ν be partitions such that $\lambda \xrightarrow{(\alpha, \beta)} \nu$. Let $j' := (\alpha', \beta')$ be the parent of $j := (\alpha, \beta)$. By definition of \leftrightarrow the λ -pairs j and j' are transformed into the ν -pairs (α', α) , (β, β') . Moreover, $\eta(\beta) = -\eta(\beta')$.*

Proof. The definition of a parent (Definition 5.2.3) implies $\alpha' < \alpha$ and $\beta < \beta'$. The definition of \leftrightarrow gives $\varphi_\nu(\alpha') = -, \varphi_\nu(\alpha) = +, \varphi_\nu(\beta) = -, \varphi_\nu(\beta') = +$. Since (α', β') is the parent of (α, β) , every element $x \in \mathbb{H}[\alpha' + 1, \alpha - 1]$ must be λ -paired with an element $y \in \mathbb{H}[\alpha' + 1, \alpha - 1]$. Therefore, $\sum_{\alpha' \leq \gamma \leq \alpha} \varphi_\nu(\gamma)1 = \sum_{\alpha' < \gamma < \alpha} \varphi_\nu(\gamma)1 = \sum_{\alpha' < \gamma < \alpha} \varphi_\lambda(\gamma)1 = 0$. Similarly, $\sum_{\beta \leq \gamma \leq \beta'} \varphi_\nu(\gamma)1 = 0$. Since going from λ to ν only affects the pairs j and j' we are done. As all $x \in \mathbb{H}[\beta, \beta']$ are ν -paired inside this interval, the cardinality of $\mathbb{H}[\beta, \beta']$ is even. In particular, $\eta(\beta) = -\eta(\beta')$. \square

Proof of Proposition 5.6.2. We have to prove that the map \mathcal{E} is well defined, hence to verify the compatibility with the relations from Proposition 5.5.2. Concerning the relations (i), (ii) and (v), there is nothing to do.

Let now $\lambda \xrightarrow{(\alpha, \beta)} \nu$. Let $j' := (\alpha', \beta')$ be the parent of $j := (\alpha, \beta)$. Consider the relation (iii). Assume first that the circle corresponding to j in $W(\tilde{a})\tilde{b}$ as well as the circle corresponding to j' in $W(\tilde{b})\tilde{a}$ are both black and consider the corresponding subspaces $R \subseteq {}_a\mathcal{K}_b^n$ and $R \subseteq {}_b\mathcal{K}_a^n$. Note that for the description of the maps $\mathcal{E}(p(\lambda, \nu))$ and $\mathcal{E}(p(\nu, \lambda))$ only the circles associated with j and j' are relevant. If we restrict ourselves to the relevant circles, $\mathcal{E}(p(\lambda, \nu))$ and $\mathcal{E}(p(\nu, \lambda))$ become the elements $1 + \frac{1}{2}X \in R$ and their composition is displayed in Figures 3 and 4, hence

explicitly given as

$$\begin{aligned} \Delta \circ m((1 + \frac{1}{2}X) \otimes (1 + \frac{1}{2}X)) &= \Delta((1 + \frac{1}{2}X) \cdot (1 + \frac{1}{2}X)) \\ &= \Delta(1 + X) = X \otimes 1 + 1 \otimes X + X \otimes X. \end{aligned}$$

Hence, relation (iii) is satisfied if all of the relevant circles are black. If there are green circles then the statement follows by applying the canonical maps. Amongst the relevant circles there are no red ones. To see this we consider again the diagrams in Figures 3 and 4. If there is a red circle then at least one of the cup diagrams contains at least two left outer or two right outer points. This, however, is not possible because $a, b \in \mathcal{S}(n)$.

Therefore, \mathcal{E} is compatible with relation (iii). The compatibility with relation (iv) is Lemma 5.4.4. Relation (vi) is clear. If (α, β) is a λ -pair, then

$$\begin{aligned} \mathcal{E}(t_{\alpha,\lambda}t_{\beta,\lambda}) &= (e_\lambda + \eta(\beta)X_\alpha)(e_\lambda - \eta(\beta)X_\alpha) \\ &= e_\lambda - \eta(\beta)X_\alpha + \eta(\beta)X_\alpha = e_\lambda = \mathcal{E}(e_\lambda). \end{aligned}$$

Hence, \mathcal{E} is compatible with relation (vii). Relation (viii) holds by definition. Since $\mu(\lambda, \nu)$ is unipotent [Bra02, Proposition 1.8.2] we could formally take the logarithm \ln of it and replace the relations (ix) by

$$\eta(\beta) \ln(\mu(\nu, \lambda)) = \ln(t_{\alpha,\nu}) + \ln(t_{\beta',\nu}) \tag{5.6.1}$$

$$\eta(\beta) \ln(\mu(\lambda, \nu)) = \ln(t_{\alpha,\lambda}) + \ln(t_{\beta',\lambda}). \tag{5.6.2}$$

Note that for $Y = \eta(\beta)X_\alpha$ we have $\ln(e_\nu + Y) = Y$. Therefore,

$$\begin{aligned} \mathcal{E}(\ln(t_{\alpha,\nu}) + \ln(t_{\beta',\nu})) &= \eta(\beta)X_\alpha - \eta(\beta')X_{\alpha'} \\ &\stackrel{\text{Lemma 5.6.3}}{=} \eta(\beta)X_\alpha + \eta(\beta)X_{\alpha'}. \end{aligned}$$

Now $\mathcal{E}(\ln(\mu(\nu, \lambda))) = \ln(e_\lambda + X_\alpha + X_{\alpha'} + X_\alpha \star X_{\alpha'}) = X_\alpha + X_{\alpha'}$, hence equation (5.6.1), and similarly (5.6.2), hold, and \mathcal{E} is compatible with the relations (ix). It is left to verify the diamond relations (x). If $(\lambda, \lambda', \lambda'', \lambda''')$ is a diamond, then the $\varphi_\lambda, \varphi_{\lambda'}, \varphi_{\lambda''}, \varphi_{\lambda'''}$ agree outside a set $N \subseteq \mathbb{H}$ of cardinality four. On N , each of them takes twice the value, and twice the value $+$. Therefore, given N , there are only six choices for λ . These, together with the possible diamonds, are depicted in Figure 6. Let us first assume that

$$\text{All except the vertex } \lambda''' \text{ of the diagram are contained in } \widetilde{S}(n). \tag{5.6.3}$$

In particular, there are either two left outer or two right outer points in λ''' which are paired, and λ''' is the only vertex in the diamond with this property. Let us consider the first diagram in Figure 6. Let A, B, C, D be the top, left, right, bottom vertex of the diamond, respectively.

If λ''' corresponds to A , then either α_1 and β_1 are both left or both right outside, and then α_1, α_2 are left outside, or β_1, β_2 are right outside in B . This contradicts (5.6.3).

If λ''' corresponds to B and say λ to A , then $\lambda \xrightarrow{(\alpha_1, \beta_1)} \lambda'''$. Hence, either α_1 is outer and β_1 is inner or vice versa. In the first case α_3 and α_2 are both left outer in D , in the second case β_1 and β_2 are both right outer in D . This contradicts (5.6.3).

If λ''' corresponds to D and say λ to B , then $\lambda \xrightarrow{(\alpha_2, \alpha_1)} \lambda'''$. Hence, either α_2 is outer and α_1 is inner or vice versa. In the first case α_3 and α_2 are both left outer in C , in the second case β_2 and β_3 are both right outer in C . This contradicts (5.6.3).

If λ''' corresponds to C and say λ to A , then $\lambda \xrightarrow{(\alpha_2, \beta_2)} \lambda'''$. Hence, either α_2 is outer and β_2 is inner or vice versa. In the first case α_3 and α_2 are both left outer in D . This contradicts (5.6.3).

In the second case $W(A)D$ and $W(D)A$ are both a single circle which is red (since it connects β_2 and β_3). Hence,

$$\mathcal{E}(p(\lambda, \lambda'))\mathcal{E}(p(\lambda', \lambda'')) = 0 = \mathcal{E}(p(\lambda'', \lambda'))\mathcal{E}(p(\lambda', \lambda)).$$

The arguments for the second diamond depicted in Figure 6 are analogous. Hence, \mathcal{E} is compatible with the relations (5.5.1).

Let us now assume that $(\lambda, \lambda', \lambda'', \lambda''')$ is a diamond with all elements contained in $\widetilde{\mathcal{S}(n)}$. Let us consider again the diamonds in Figure 6. For simplicity we first assume that all endpoints of the arcs are inner. Restricting to the relevant circles only we could consider any $\mathcal{E}(p(\nu, \nu'))$, $\nu, \nu' \in \{\lambda, \lambda'\lambda''\lambda'''\}$, as an element of $R \otimes R$. An easy direct calculation shows that the composition of two of them is given by the multiplication

$$(R \otimes R) \otimes (R \otimes R) \rightarrow R \tag{5.6.4}$$

$$r_1 \otimes r_2 \otimes r_3 \otimes r_4 \mapsto r_1 r_2 r_3 r_4 \tag{5.6.5}$$

and the statement follows immediately. If one of the relevant circles is red, then not all elements of the diamond are contained in $\widetilde{\mathcal{S}(n)}$. If there are green circles appearing, then the statement follows by applying the canonical map after and before (5.6.4). (This is enough, because $\epsilon' \circ \epsilon : \mathbb{C} \rightarrow R \rightarrow \mathbb{C}$ is the identity map.) The map \mathcal{E} is therefore compatible with the relations (x).

So, the map \mathcal{E} is well defined and gives rise to a homomorphism of algebras. □

PROPOSITION 5.6.4. *The algebra homomorphism \mathcal{E} is surjective.*

Proof. The algebra \mathcal{K}^n is by construction a graded quotient of the algebra \mathcal{H}^{2n} . Now, the algebra \mathcal{H}^{2n} is generated (over its semisimple degree-zero part) in degrees one and two. To see this, first recall that for any $a \in \text{Cup}(2n)$, the subalgebra $a\mathcal{H}^{2n}a$ is generated in degrees zero and two, whereas the space $a\mathcal{H}^{2n}b$, equipped with its natural $a\mathcal{H}^{2n}a$ -module structure, is generated by its lowest degree element $1 \in a\mathcal{H}^{2n}b$. By [Kho04, Lemma 1], the elements $1 \in a\mathcal{H}^{2n}b$ are contained in the subalgebra of \mathcal{H}^{2n} generated by degree-one elements and so the claim follows. Now the proposition follows from Lemma 5.4.4 and the definition of the map \mathcal{E} , since the image of \mathcal{E} contains the images of the generators of \mathcal{H}^{2n} in the quotient algebra \mathcal{K}^n . □

5.7 The grading of $\mathcal{A}_{n,n}$ and the proof of Theorem 5.3.1

Let $\lambda, \nu \in \widetilde{\mathcal{S}(n)}$ and $\lambda \xrightarrow{(\alpha,\beta)} \nu$ and consider $p := p(\nu, \lambda)$ and $q := p(\lambda, \nu)$. Set $x = qp$ and $y = pq$. One can find finite sums $\tilde{p} := \tilde{p}(\nu, \lambda) := \sum_{k \geq 0} c_k p x^k$ and $\tilde{q} := \tilde{q}(\lambda, \nu) := \sum_{k \geq 0} c_k x^k q$, $c_k \in \mathbb{C}$ such that $\tilde{p}\tilde{q} = \ln(\mu(\nu, \lambda))$ and $\tilde{q}\tilde{p} = \ln(\mu(\lambda, \nu))$. Namely define inductively $c_0 = 1$, and for $k \in \mathbb{Z}_{>0}$

$$c_k = \frac{1}{2} \left((-1)^k \frac{1}{k+1} - \sum_{0 < l, m < k, l+m=k} c_l c_m \right).$$

Then the claimed equalities hold (as formal power series in x and y). Since x and y are nilpotent [Bra02, Proposition 1.8.2] all of the infinite sums are in fact finite. Note also that $1 + x = \exp(\ln(1 + x))$ is contained in the subalgebra generated by 1 and $(\ln(1 + x))$, similarly for $y + 1$. We obtain the following proposition.

PROPOSITION 5.7.1. *The algebra $A_{n,n}$ is generated by*

$$\left\{ e_\lambda, \ln(t_{\alpha,\lambda}) \mid \alpha \in \mathbb{H}, \lambda \in \widetilde{\mathcal{S}(n)} \right\},$$

$$\left\{ \tilde{p}(\lambda, \nu), \ln(\mu(\lambda, \nu)) \mid \lambda, \nu \in \widetilde{\mathcal{S}(n)}, \lambda \leftrightarrow \nu \right\}.$$

Proof. It is enough to show that the original generators are in the subalgebra, call it B , generated by the generators from the proposition. This is clear for the e_λ . We also know it for the μ . For the t it follows then from [Bra02, Proposition 1.8.1]. Finally $\tilde{p} = p(1 + \sum_{k>0} c_k x^k)$ where the second factor is invertible in B , since $x \in B$ is nilpotent. Hence, $p \in B$ and similarly $q \in B$ and the statement follows. \square

COROLLARY 5.7.2. *Putting the generators $e_\lambda, \ln(t_{\alpha,\lambda}), \tilde{p}(\lambda, \nu), \ln(\mu(\lambda, \nu))$ of $A_{n,n}$ from Proposition 5.7.1 in degree zero, two, one, two turns $A_{n,n}$ into a positively graded algebra and \mathcal{E} becomes a homomorphism of \mathbb{Z} -graded algebras. This grading is the Koszul grading.*

Proof. Using the new generators, the relations from Proposition 5.5.2 become homogeneous. This is completely obvious except for relation (x). Let us assume this to be true for the moment, then the relations also show that $A_{n,n}$ becomes a quadratic positively graded algebra, i.e. generated in degree zero and one with relations in degree two. Now we are in the situation of [BGS96, Proposition 2.4.1], i.e. for any graded $A_{n,n}$ -module M with simple head, the radical filtration of M agrees (at least up to a shift in the grading) with the grading filtration. This holds in particular for indecomposable projective modules. The same holds if we equip $A_{n,n}$ with its Koszul grading [BGS96]. By the unicity of gradings [BGS96, § 2.5], the statement follows if we verified relation (x).

Since we only have a case-by-case argument, we sketch the argument for a specific example only and leave it to the reader to figure out all other possibilities. Let $(\lambda, \lambda', \lambda'', \lambda''')$ be the diamond on the left in Figure 6. We use the notation from the small diamond in the middle of Figure 6. We claim that the relation $p_2 p_1 = q_4 p_3$ could be replaced by the relation $\tilde{p}_2 \tilde{p}_1 = \tilde{q}_4 \tilde{p}_3$. Let $x_i = q_i p_i$. Using the relations of Proposition 5.5.2 we obtain

$$\begin{aligned} p_2 p_1 (1 + x_1) &\stackrel{\text{(iii),(ix)}}{=} p_2 p_1 (t_{\alpha_1, \lambda} t_{\beta_2, \lambda})^{\eta(\beta_1)} \stackrel{\text{(iv)}}{=} (t_{\alpha_1, \lambda''} t_{\beta_2, \lambda''})^{\eta(\beta_1)} p_2 p_1 \\ &\stackrel{\text{(vii),(vi)}}{=} (t_{\beta_1, \lambda''} t_{\beta_3, \lambda''})^{-\eta(\beta_1)} p_2 p_1 \stackrel{\text{(iv)}}{=} p_2 p_1 T \\ p_2 (1 + x_2) p_1 &\stackrel{\text{(iii),(ix)}}{=} p_2 (t_{\alpha_2, \lambda'} t_{\beta_2, \lambda'})^{\eta(\alpha_1)} p_1 \stackrel{\text{(iv)}}{=} p_2 p_1 T', \\ q_4 p_3 (1 + x_3) &\stackrel{\text{(iii),(ix)}}{=} q_4 p_3 T' \\ (1 + x_4) q_4 p_3 &\stackrel{\text{(iii),(ix)}}{=} (t_{\beta_1, \lambda'''} t_{\beta_3, \lambda'''})^{\eta(\beta_2)} q_4 p_3 \stackrel{\text{(iv)}}{=} q_4 p_3 T \end{aligned}$$

where $T := (t_{\beta_1, \lambda} t_{\beta_3, \lambda})^{-\eta(\beta_1)}$ and $T' = (t_{\alpha_2, \lambda} t_{\beta_2, \lambda})^{\eta(\beta_2)}$.

Hence, $p_2 p_1 = q_4 p_3$ implies $p_2 (1 + x_2)^k p_1 = q_4 p_3 (1 + x_3)^k$ for any $k \in \mathbb{Z}_{\geq 0}$ and then $p_2 (x_2)^k p_1 = q_4 p_3 (x_3)^k$ by induction. The displayed equations above also imply $p_2 (1 + x_2)^k p_1 (1 + x_1)^l = (1 + x_4)^l q_4 p_3 (1 + x_3)^k$ for any $k, l \in \mathbb{Z}_{\geq 0}$ and then, by induction, $p_2 (x_2)^k p_1 (x_1)^l = (x_4)^l q_4 p_3 (x_3)^k$. It follows $\tilde{p}_2 \tilde{p}_1 = \tilde{q}_4 \tilde{p}_3$.

Assume now that $\tilde{p}_2 \tilde{p}_1 = \tilde{q}_4 \tilde{p}_3$. Then $\tilde{p}_2 (1 + x_2)^k \tilde{p}_1 = \tilde{q}_4 \tilde{p}_3 (1 + x_3)^k$ for any $k \in \mathbb{Z}_{\geq 0}$ with the arguments from above. By induction, we obtain $\tilde{p}_2 (x_2)^k \tilde{p}_1 = \tilde{q}_4 \tilde{p}_3 (x_3)^k$. Using again the arguments from above we obtain $\tilde{p}_2 (x_2)^k \tilde{p}_1 (1 + x_1)^l = (1 + x_4)^l \tilde{q}_4 \tilde{p}_3 (x_3)^k$ for any $k, l \in \mathbb{Z}_{\geq 0}$. By

induction we therefore obtain

$$\tilde{p}_2(x_2)^k \tilde{p}_1(x_1)^l = (x_4)^l \tilde{q}_4 \tilde{p}_3(x_3)^k. \quad (5.7.1)$$

Now let l be maximal such that $p_1 x_1^l \neq 0$ or $x_4^l q_4 \neq 0$. Then choose (if possible) k maximal such that $p_2(x_2)^k p_1(x_1)^l \neq 0$ or $(x_4)^l q_4 p_3(x_3)^k \neq 0$. From (5.7.1) we obtain

$$c_0 c_0 p_2(x_2)^k p_1(x_1)^l = c_0 c_0 (x_4)^l q_4 p_3(x_3)^k. \quad (5.7.2)$$

Then we choose $k' < k$ maximal with the above conditions and deduce that $p_2 x_2^{k'} p_1 x_1^l = x_4^l q_4 p_3 x_3^{k'}$. Inductively, the latter holds for any k' . By double induction on l and k we finally obtain $p_2 p_1 = q_4 p_3$. \square

We obtain a description of the arrows in the Ext-quiver of the algebra $A_{m,m}$.

COROLLARY 5.7.3. *Let $L(v \cdot 0), L(w \cdot 0) \in \mathcal{O}_0^{n,n}$ be simple modules. Consider $\sigma_{\text{dom}} v, \sigma_{\text{dom}} w \in \mathcal{S}(n)$ and let $\nu, \lambda \in \widetilde{\mathcal{S}(n)}$ be their extensions. Then*

$$\text{Ext}_{\mathcal{O}_0^{n,n}}^1(L(v \cdot 0), L(w \cdot 0)) = \begin{cases} \mathbb{C} & \text{if } \nu \leftrightarrow \lambda, \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. Let $P^{(n,n)}(v \cdot 0), P^{(n,n)}(w \cdot 0) \in \mathcal{O}_0^{(n,n)}$ be the projective cover of $L(v \cdot 0)$ and $L(w \cdot 0)$, respectively. Then the dimension of $\text{Ext}_{\mathcal{O}_0^{n,n}}^1(L(v \cdot 0), L(w \cdot 0))$ is equal to the dimension of the subspace M of $\text{Hom}_{\mathcal{O}}(P^{(n,n)}(w \cdot 0), P^{(n,n)}(v \cdot 0))$ spanned by all morphisms f whose image is contained in the radical of $P^{(n,n)}(v \cdot 0)$, but not in the square of the radical. When passing to $A_{n,n}$, this subspace corresponds to the space spanned by all morphisms of degree one, since, for indecomposable projectives, the grading filtration agrees with the radical filtration thanks to [BGS96, Proposition 2.4.1]. By Corollary 5.7.2 there is, up to a non-zero scalar, a unique morphism of degree one if $\nu \leftrightarrow \lambda$ and no morphism otherwise. \square

Proof of Theorem 5.3.1. The homomorphism \mathcal{E} from Proposition 5.6.2 induces a surjective homomorphism of algebras

$$\mathcal{E}' : D_{n,n} \rightarrow \mathcal{H}^n \quad (5.7.3)$$

such that $\text{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(x \cdot 0), P^{\mathfrak{p}}(y \cdot 0))$ is mapped to ${}_a \mathcal{H}_b^n$, where $a = \sigma_{\text{dom}} x, b = \sigma_{\text{dom}} y$ (see Remark 5.5.3). By Corollary 5.7.2 it is only left to show that \mathcal{E}' is an isomorphism. To do so it is enough to compare the dimensions. However, the dimension of ${}_a \mathcal{H}_b^n$ is 2^k , where k is the number of circles in $W(a)b$. We could rephrase this as follows: consider the irreducible right (complex) S_{2n} -module M corresponding to the partition $2n = n + n$. This module has a unique up to a scalar symmetric non-degenerate S_{2n} -invariant bilinear form \mathbf{b} (see [Mur95, § 6]). One can naturally identify the elements of $\text{Cup}(n)$ with the basis of M obtained by specialising the Kazhdan–Lusztig basis in the generic Hecke algebra of S_{2n} such that $\dim_{\mathbb{C}}({}_a \mathcal{H}_b^n) = \mathbf{b}(a, b)$ (see [Fun03, Theorem 7.3] and references therein). On the other hand, we categorified M in Proposition 4.3.4. One can also categorify the bilinear forms as follows: there is a scalar $\gamma \in \mathbb{C}$ such that

$$\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(x \cdot 0), P^{\mathfrak{p}}(y \cdot 0)) = \mathbf{b}(a, b) \gamma \quad (5.7.4)$$

where $a = \sigma_{\text{dom}} x, b = \sigma_{\text{dom}} y$ (see [KMS09, Proposition 4]). The explicit formulas in [IS88, Corollary, p. 327] give the dimension of the endomorphism ring of $P \in \mathcal{O}_0^{n,n}$ as in Proposition 4.2.1, namely as follows: $P \cong \theta_{\nu}^0 M^{\mathfrak{p}}(\nu)$ where $M^{\mathfrak{p}}(\nu)$ is a (simple, projective)

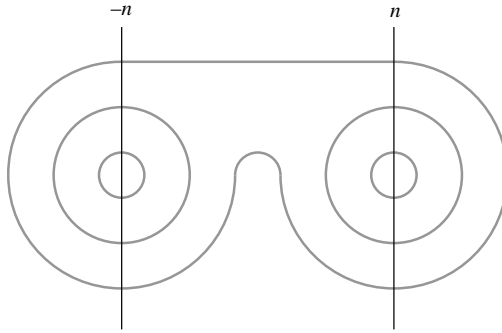


FIGURE 8. If $a_w = +^r -^s +^{s-r}$, then $W(\tilde{a}_w)\tilde{a}_e$ consists of green circles only.

generalised Verma module. By [IS88, Proposition 3.1] the stabiliser of ν is isomorphic to n copies of S_2 . This implies $\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(P, P) = 2^n$ and, hence, $\gamma = 1$. The theorem follows. \square

5.8 A graphical description of $\mathcal{O}_0^{n,n}$

In this section we prove the following.

THEOREM 5.8.1. *For any $n \in \mathbb{Z}_{>0}$ the algebra homomorphism $\mathcal{E} : A_{n,n} \cong \mathcal{K}^n$ is an isomorphism.*

Proof. We use the notation from Proposition 5.2.2 and § 5.4 and set $\mathfrak{p} = \mathfrak{p}_n$. For $w \in W^{\mathfrak{p}}$ put $a_w = \sigma_{\text{dom } w}$. Let us denote by $\varphi_w \in \widetilde{\mathcal{S}(n)}$ the extension of a_w , and let λ_w be the corresponding partition.

Since \mathcal{E} is surjective (Proposition 5.6.4) and the involved algebras are finite dimensional, it is enough to show that for any $v, w \in W^{\mathfrak{p}}$

$$\text{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(v \cdot 0), P^{\mathfrak{p}}(w \cdot 0)) \cong \mathcal{G}(W(\tilde{a}_v)\tilde{a}_w). \tag{5.8.1}$$

Let us first consider the case where $v = e$, the identity of W . By [Bre02, Corollary 5.2] we have $\{0\} \neq \text{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(0), P^{\mathfrak{p}}(w \cdot 0))$ (and then equal to \mathbb{C}) if and only if

$$a_w = (\underbrace{+, \dots, +}_r, \underbrace{-, \dots, -}_s, \underbrace{+, \dots, +}_s, \underbrace{-, \dots, -}_r) =: +^r -^s +^{s-r}$$

for some $r, s \in \mathbb{Z}_{\geq 0}$.

If there are $\gamma, \delta \in \mathbb{H}[-n, 0]$ $\gamma < \delta$ such that $\varphi_w(\gamma) = -$ and $\varphi_w(\delta) = +$, then there is a λ_w -pair (α, β) such that $\alpha, \beta \in \mathbb{H}[-n, 0]$. Moreover, $\varphi_e(\alpha) = + = \varphi_e(\beta)$. The corresponding circle in $W(\tilde{a}_w)\tilde{a}_e$ is then red. Similarly, if there are $\gamma, \delta \in \mathbb{H}[0, n]$ $\gamma < \delta$ such that $\varphi_w(\gamma) = -$ and $\varphi_w(\delta) = +$, then there is a λ_w -pair (α, β) such that $\alpha, \beta \in \mathbb{H}[0, n]$, $\varphi_e(\alpha) = - = \varphi_e(\beta)$. The corresponding circle in $W(\tilde{a}_w)\tilde{a}_e$ is red. Therefore, $\mathcal{G}(W(\tilde{a}_w)\tilde{a}_e) = \{0\}$ if a_w is not of the form $+^r -^s +^{s-r}$ as above. On the other hand, if $a_w = +^r -^s +^{s-r}$, then $W(\tilde{a}_w)\tilde{a}_e$ consists of green circles only (as depicted in Figure 8), and $\mathcal{G}(W(\tilde{a}_w)\tilde{a}_e) = \mathbb{C}$.

Hence, formula (5.8.1) is true for $v = e$ and we can do induction on the length of v . Choose some simple reflection s such that $vs < v$. Then

$$\begin{aligned} & \text{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(v \cdot 0), P^{\mathfrak{p}}(w \cdot 0)) \\ &= \text{Hom}_{\mathfrak{g}}(\theta_s P^{\mathfrak{p}}(vs \cdot 0), P^{\mathfrak{p}}(w \cdot 0)) \\ &= \text{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(vs \cdot 0), \theta_s P^{\mathfrak{p}}(w \cdot 0)) \\ &= \begin{cases} \text{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(vs \cdot 0), P^{\mathfrak{p}}(ws \cdot 0)) & ws \in W^{\mathfrak{p}}, ws > w, \\ \text{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(vs \cdot 0), P^{\mathfrak{p}}(w \cdot 0) \oplus P^{\mathfrak{p}}(w \cdot 0)) & \text{if } ws < w, \\ \{0\} & \text{if } ws \notin W^{\mathfrak{p}} \end{cases} \end{aligned}$$

by [BW01, Theorem 1], the self-adjointness of θ_s and [Soe97, § 3]. On the other hand, if $ws \in W^{\mathfrak{p}}$, $ws > w$, then obviously $\mathcal{G}(W(\tilde{a}_v)\tilde{a}_w) = \mathcal{G}(W(\tilde{a}_{vs})\tilde{a}_{ws})$ which is zero if $ws \notin W^{\mathfrak{p}}$. If $ws \in W^{\mathfrak{p}}$ and $ws < w$, then $\mathcal{G}(W(\tilde{a}_v)\tilde{a}_{ws})$ differs from $\mathcal{G}(W(\tilde{a}_v)\tilde{a}_w)$ by a black circle. Formula (5.8.1) follows therefore from the induction hypothesis. \square

Remark 5.8.2. Theorem 5.8.1 can be generalised to all maximal parabolic subalgebras or perverse sheaves on Grassmannians $\text{Perv}_B(\text{Gr}(k, n))$. In general, one has to take $\{+, -\}$ -sequences of length n with k pluses and $n - k$ minuses. They define the inner points. Then we add $n - k$ minuses to the left and $n - (n - k)$ pluses to the right and proceed as before.

5.9 Intersections of components of the Springer fibre

In the first three sections of the paper we used Soergel’s Endomorphismensatz [Soe90] which, in particular, implies that the endomorphism ring of the only indecomposable projective–injective module in \mathcal{O}_0 has commutative endomorphism ring, isomorphic to $H^*(\mathcal{B})$ (see Proposition 3.1.1 and the preceding paragraph). On the other hand, we know by Proposition 2.8.2 that the endomorphism ring of any indecomposable projective module in a $\mathcal{O}^{\mathfrak{p}}$ for maximal parabolic \mathfrak{p} is commutative, and Theorem 5.3.1 gives an explicit algebraic description of the endomorphism rings of indecomposable projective–injective modules. In fact, it describes the space of homomorphisms between two indecomposable projective–injective modules as a bimodule over their endomorphism rings. A geometrical interpretation of these bimodules in terms of cohomology rings is still missing. We would like to finish this paper by formulating a conjectural interpretation based on the following.

THEOREM 5.9.1. *Let $\mathfrak{g} = \mathfrak{gl}_n$ and let \mathfrak{p} be some maximal parabolic subalgebra. Let $\text{Irr}(\mathcal{B}_{\mathfrak{p}})$ denote the set of irreducible components of $\mathcal{B}_{\mathfrak{p}}$ and $\text{PrInj}(\mathfrak{p})$ the set of isomorphism classes of indecomposable projective–injective modules in $\mathcal{O}_0^{\mathfrak{p}}$. Then there is a bijection*

$$\psi : \text{PrInj}(\mathfrak{p}) \cong \text{Irr}(\mathcal{B}_{\mathfrak{p}})$$

such that there is an isomorphism of vector spaces

$$\text{Hom}_{\mathfrak{g}}(P, Q) \cong H^*(\psi(P) \cap \psi(Q))$$

for any $P, Q \in \text{PrInj}(\mathfrak{p})$.

Proof. By results of Vargas and of Spaltenstein, there is an explicit bijection between the irreducible components of $\mathcal{B}_{\mathfrak{p}}$ and standard tableaux of shape $\lambda(\mathfrak{p})$. In our special situation we have to consider only tableaux with two rows, and we refer to [Fun03, Theorem 5.2], where the bijection is made explicit. We restrict ourselves to the case where $\mu = \lambda(\mathfrak{p})$ is already a partition. This is possible thanks to [MS08, Theorem 5.4].

Let us first consider the case $\lambda(\mathbf{p}) = (n, n)$. Given a standard tableaux T with two rows of length $\lambda_1 = n$ and $\lambda_2 = n$, one can associate a cup diagram in $\text{Cup}(n)$ as follows: T has the entries $1, 2, \dots, 2n$ so that the numbers are decreasing from left to right in each row, and decreasing from top to bottom in each column. The cup diagram C_T has $2n$ vertices, labelled by 1 to $2n$ from the left, so that the left endpoint of each cup is labelled by a number appearing in the bottom row of T , whereas the right endpoints of the cups are labelled by elements from the top row of T . (So the endpoints of each cup are in different rows of T). For example, if $n = 2$ then we have the standard tableaux

$$\begin{array}{|c|c|} \hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 4 & 2 \\ \hline 3 & 1 \\ \hline \end{array}$$

to which we associate D and D' displayed in Figure 1. It is easy to see that this procedure provides a bijection between standard tableaux of shape (n, n) (and, hence, of irreducible components of $\mathcal{B}_{\mathbf{p}}$), and cup diagrams from $\text{Cup}(n)$. Now we apply Proposition 5.2.4 and obtain a bijection between the irreducible components and the isomorphism classes of indecomposable projective–injective modules $\text{PrInj}(n)$. Thanks to Theorem 5.3.1 the dimension of the homomorphism spaces between projective–injective modules can be computed using the diagram calculus, which is also used in [Fun03, Theorems 7.2 and 7.3] to compute the dimension of the cohomology of the intersection of two components. This settles the case of the partition (n, n) and gives an explicit way to compute the dimensions of the vector spaces.

In general, one should argue as follows: first of all we have [KMS09, Proposition 4] the categorification of the Specht modules for the symmetric group (Proposition 4.3.4), but also of its invariant bilinear form (5.7.4) by taking dimensions of the homomorphism spaces between indecomposable projective modules. On the other hand, the dimension of the cohomology of the intersection of components is computed in the same way [Fun03, Theorems 7.2 and 7.3], so that the statement follows up to a multiplication with a common factor. However, one easily checks that this common factor must be equal to one by computing one of the endomorphism rings explicitly. \square

Up to a shift, the isomorphism from the theorem is an isomorphism of \mathbb{Z} -graded vector spaces, where the grading on the hom-space is given as in § 4.3.

We conjecture that this isomorphism is compatible with the multiplicative structure as well.

CONJECTURE 5.9.2. *The isomorphism $\text{End}_{\mathfrak{g}}(P) \cong H^*(\psi(P))$ is a ring homomorphism and*

$$\text{Hom}_{\mathfrak{g}}(P, Q) \cong H^*(\psi(P) \cap \psi(Q))$$

as $(\text{End}_{\mathfrak{g}}(P), \text{End}_{\mathfrak{g}}(Q)) = (H^*(\psi(P)), H^*(\psi(Q)))$ -module.

5.10 Connection with Khovanov homology

Using Theorem 5.3.1 one can deduce that the full conjecture [Str06, Conjecture 2.9] holds. Roughly speaking this says that the functorial tangle invariants defined in [Kho02] are obtained by restricting the functorial tangle invariants from [Str05] to a certain subcategory, invariant under all of these functors. In particular, the resulting homological invariants are the same. The proof is quite technical and lengthy and will appear in a subsequent paper. We expect that Conjecture 5.9.2 provides the basis for a geometric interpretation of Khovanov homology using categories of sheaves related to the Springer fibres.

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REFERENCES

- AL03 H. H. Andersen and N. Lauritzen, *Twisted Verma modules*, in *Studies in Memory of Issai Schur*, Progress in Mathematics, vol. 210 (Birkhäuser, Basel, 2003), 1–26.
- AS03 H. H. Andersen and C. Stroppel, *Twisting functors on \mathcal{O}* , Represent. Theory **7** (2003), 681–699.
- BGS96 A. Beilinson, V. Ginzburg and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–527.
- BFK99 J. Bernstein, I. Frenkel and M. Khovanov, *A categorification of the Temperley–Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors*, Selecta Math. (N.S.) **5** (1999), 199–241.
- BG80 J. N. Bernstein and S. I. Gel’fand, *Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras*, Compositio Math. **41** (1980), 245–285.
- BGG76 I. N. Bernstein, I. M. Gel’fand and S. I. Gel’fand, *A certain category of \mathfrak{g} -modules*, Funkcional. Anal. i Priložen. **10** (1976), 1–8.
- BW01 S. C. Billey and G. S. Warrington, *Kazhdan–Lusztig polynomials for 321-hexagon-avoiding permutations*, J. Algebraic Combin. **13** (2001), 111–136.
- Bra02 T. Braden, *Perverse sheaves on Grassmannians*, Canad. J. Math. **54** (2002), 493–532.
- Bre02 F. Brenti, *Kazhdan–Lusztig and R -polynomials, Young’s lattice, and Dyck partitions*, Pacific J. Math. **207** (2002), 257–286.
- Bru06 J. Brundan, *Centers of degenerate cyclotomic Hecke algebras and parabolic category \mathcal{O}* , Preprint (2006), math.RT/0607717.
- Bru08 J. Brundan, *Symmetric functions, parabolic category \mathcal{O} and the Springer fiber*, Duke Math. J. **143** (2008), 41–79, math.RT0608235.
- DP81 C. De Concini and C. Procesi, *Symmetric functions, conjugacy classes and the flag variety*, Invent. Math. **64** (1981), 203–219.
- Fie03 P. Fiebig, *Centers and translation functors for the category \mathcal{O} over Kac–Moody algebras*, Math. Z. **243** (2003), 689–717.
- Ful98 W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 2 2nd edition (Springer, Berlin, 1998).
- Fun03 F. Fung, *On the topology of components of some Springer fibers and their relation to Kazhdan–Lusztig theory*, Adv. Math. **178** (2003), 244–276.
- GP92 A. M. Garcia and C. Procesi, *On certain graded S_n modules and q -Kostka polynomials*, Adv. Math. **94** (1992), 83–138.
- Gor03 I. Gordon, *Baby Verma modules for rational Cherednik algebras*, Bull. London Math. Soc. **35** (2003), 321–336.
- HS79 R. Hotta and N. Shimomura, *The fixed-point subvarieties of unipotent transformations on generalized flag varieties and the Green functions*, Math. Ann. **241** (1979), 193–208.

- HS77 R. Hotta and T. A. Springer, *A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups*, *Invent. Math.* **41** (1977), 113–127.
- HTT08 R. Hotta, K. Takeuchi and T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, *Progress in Mathematics*, vol. 236 (Birkhäuser, Basel, 2008), Translated from the 1995 Japanese edition by Takeuchi.
- Irv85 R. Irving, *Projective modules in the category \mathcal{O}_S : self-duality*, *Trans. Amer. Math. Soc.* **291** (1985), 701–732.
- IS88 R. S. Irving and B. Shelton, *Loewy series and simple projective modules in the category \mathcal{O}_S* , *Pacific J. Math.* **132** (1988), 319–342.
- Jan83 J. C. Jantzen, *Einhüllende Algebren halbeinfacher Lie-Algebren*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, vol. 3 (Springer, Berlin, 1983).
- KM05 O. Khomenko and V. Mazorchuk, *On Arkhipov’s and Enright’s functors*, *Math. Z.* **249** (2005), 357–386.
- Kho00 M. Khovanov, *A categorification of the Jones polynomial*, *Duke Math. J.* **101** (2000), 359–426.
- Kho02 M. Khovanov, *A functor-valued invariant of tangles*, *Algebr. Geom. Topol.* **2** (2002), 665–741 (electronic).
- Kho04 M. Khovanov, *Crossingless matchings and the cohomology of (n, n) Springer varieties*, *Commun. Contemp. Math.* **6** (2004), 561–577.
- KMS09 M. Khovanov, V. Mazorchuk and C. Stroppel, *A categorification of integral Specht modules*, *Proc. Amer. Math. Soc.* **136** (2008), 1163–1169.
- MOS09 V. Mazorchuk, S. Ovsienko and C. Stroppel, *Quadratic duals, Koszul dual functors, and applications*, *Trans. Amer. Math. Soc.* **361** (2009), 1129–1172, math.RT/0603475.
- MS05 V. Mazorchuk and C. Stroppel, *Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module*, *Trans. Amer. Math. Soc.* **357** (2005), 2939–2973.
- MS08 V. Mazorchuk and C. Stroppel, *Projective–injective modules, Serre functors and symmetric algebras*, *J. Reine Angew. Math.* **616** (2008), 131–165, math.RT0508119.
- Mur95 G. E. Murphy, *The representations of Hecke algebras of type A_n* , *J. Algebra* **173** (1995), 97–121.
- Nar89 H. Naruse, *On an isomorphism between Specht module and left cell of S_n* , *Tokyo J. Math.* **12** (1989), 247–267.
- Ric89 J. Rickard, *Morita theory for derived categories*, *J. London Math. Soc. (2)* **39** (1989), 436–456.
- Ric94 J. Rickard, *Translation functors and equivalences of derived categories for blocks of algebraic groups*, in *Finite-dimensional algebras and related topics (Ottawa, 1992)*, NATO Advanced Science Institute Series C: Mathematical and Physical Sciences, vol. 424 (Kluwer, Dordrecht, 1994), 255–264.
- Roc80 A. Rocha-Caridi, *Splitting criteria for \mathfrak{g} -modules induced from a parabolic and the Bernstein–Gel’fand–Gel’fand resolution of a finite-dimensional, irreducible \mathfrak{g} -module*, *Trans. Amer. Math. Soc.* **262** (1980), 335–366.
- Rou06 R. Rouquier, *Categorification of the braid groups*, in *Trends in Representation Theory of Algebras and Related Topics*, *Contemporary Mathematics*, vol. 406 (American Mathematical Society, Providence, RI, 2006), 137–167.
- Soe90 W. Soergel, *Kategorie \mathcal{O} , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, *J. Amer. Math. Soc.* **3** (1990), 421–445.
- Soe92 W. Soergel, *The combinatorics of Harish-Chandra bimodules*, *J. Reine Angew. Math.* **429** (1992), 49–74.

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- Soe97 W. Soergel, *Kazhdan–Lusztig polynomials and a combinatorics for tilting modules*, Represent. Theory **1** (1997), 83–114 (electronic).
- Spr76 T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207.
- Spr78 T. A. Springer, *A construction of representations of Weyl groups*, Invent. Math. **44** (1978), 279–293.
- Str03a C. Stroppel, *Category \mathcal{O} : gradings and translation functors*, J. Algebra **268** (2003), 301–326.
- Str03b C. Stroppel, *Category \mathcal{O} : quivers and endomorphism rings of projectives*, Represent. Theory **7** (2003), 322–345 (electronic).
- Str05 C. Stroppel, *Categorification of the Temperley–Lieb category, tangles, and cobordisms via projective functors*, Duke Math. J. **126** (2005), 547–596.
- Str06 C. Stroppel, *TQFT with corners and tilting functors in the Kac–Moody case*, Preprint (2006), math.RT/0605103.
- Tan82 T. Tanisaki, *Defining ideals of the closures of the conjugacy classes and representations of the Weyl groups*, Tôhoku Math. J. (2) **34** (1982), 575–585.

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