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PARACONSISTENCY AND SETTE'S CALCULUS P1

Abstract. In 1973, Sette presented a calculus, called P1, which is recognized as one of the most remarkable paraconsistent systems. The aim of this paper is to propose a new axiomatization of P1. The axiom schemata are chosen to show that P1 behaves in a *paraconsistent way* only at the atomic level, i.e. the rule: $\alpha,\ \sim\alpha\ /\ \beta$ holds in P1 only if α is not a propositional variable.

Keywords: paraconsistent logic; Sette's system; P1

1. Introduction

Apart from being viewed as a paraconsistent system, P1 has a very important property: it is maximal in the sense that if we enrich the system with any classical tautology that is not valid in P1, the resulting system collapses to the classical propositional calculus (cf. [6, 7]).

Although (DSR) $\alpha,\ \sim\alpha\ /\ \beta$ is not an admissible rule in P1, the system is closed under

(DSRn) $\sim\alpha,\ \sim\sim\alpha\ /\ \beta$

and

(DSRi) $(\alpha \to \beta),\ \sim(\alpha \to \beta)\ /\ \gamma$.[1]

Now assume for instance, that the paraconsitent calculus P1 is applied to an inconsistent theory in which both a conditional, say: $\alpha \to \beta$ and its negation $\sim(\alpha \to \beta)$ are accepted as hypotheses. Unfortunately

[1] Cf. Fact 2.1. '[...] in 1997 E. K. Vojshvillo and J-Y. Béziau [...] discovered independently that in P1 from $\sim A$ and $\sim\sim A$ follows B', [4, p. 83].

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the application of (DSRi) entails any formula γ and make the theory trivial (cf. [3, p. 40]). In fact, (DSR) holds in P1 provided α is not a propositional variable (cf. [1, p. 19]).

2. System P1

Let Var denote a non-empty denumerable set of all propositional variables. The set of formulas For is inductively defined (in Backus-Naur Form) as follows:

$$\varphi ::= p_i \mid \sim\alpha \mid \alpha \to \beta,$$

where $p_i \in$ Var and $i \in N$; α, β are formulas; the symbols $\sim$, $\to$ denote negation and implication, respectively.

The calculus P1 is axiomatized by the following axiom schemata:

(A1) $\alpha \to (\beta \to \alpha)$
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
(A3) $(\sim\alpha \to \ \sim\beta) \to ((\sim\alpha \to \ \sim\sim\beta) \to \alpha)$
(A4) $\sim(\alpha \to \ \sim\sim\alpha) \to \alpha$
(A5) $(\alpha \to \beta) \to \ \sim\sim(\alpha \to \beta)$.

The sole rule of inference is (MP): α, $\alpha \to \beta$ / β.[2]

It is noteworthy that $\sim$ and $\to$ are taken here as primitives. The connectives of conjunction, disjunction and equivalence are introduced through the definitions ([7, p. 178]):

$$\alpha \wedge \beta := (((\alpha \to \alpha) \to \alpha) \to \ \sim((\beta \to \beta) \to \beta)) \to \ \sim(\alpha \to \sim\beta)$$
$$\alpha \vee \beta := (\alpha \to \ \sim\sim\alpha) \to (\sim\alpha \to \beta)$$
$$\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$$

In what follows, we will treat each of the connectives as a useful shorthand which formally does not appear in formulas.

DEFINITION 2.1. A formal proof (deduction) within P1 of α from the set formulas of Γ is a finite sequence of formulas, $\beta_1, \beta_2, \ldots, \beta_n$, where $\beta_n = \alpha$ and each of elements in that sequence is either an axiom of P1, or belongs to Γ, or follows from the preceding formulas in the sequence by (MP).

[2] The axiom (A4) is derivable from others and can be eliminated from the system (cf. [8, p. 68]).

DEFINITION 2.2. A formula α is a syntactic consequence within P1 of a set formulas of Γ ($\Gamma \vdash_{\mathrm{P1}} \alpha$, in symbols) iff if there is a formal proof of α from the set Γ within P1. A formula α is a thesis of P1 iff $\emptyset \vdash_{\mathrm{P1}} \alpha$.

Observe that P1 includes (A1) and (A2). (MP) is the sole rule of inference of the calculus. Hence we have a proof of the deduction theorem, DT, in its usual formulation.

THEOREM 2.1. *$\Gamma \vdash_{\mathrm{P1}} \alpha \to \beta$ iff $\Gamma \cup \{\alpha\} \vdash_{\mathrm{P1}} \beta$, where $\alpha, \beta \in$ For and $\Gamma \subseteq$ For.*

FACT 2.1. *The formulas*

(DSv)	$\alpha \to (\sim\alpha \to \beta)$, *if* $\alpha \notin \mathrm{Var}$
(DSn)	$\alpha \to (\sim\alpha \to (\sim\sim\alpha \to \beta))$
(NN1)	$\sim\sim\alpha \to \alpha$
(NN2)	$\alpha \to \sim\sim\alpha$, *if* $\alpha \notin \mathrm{Var}$

are provable in P1.

PROOF. (DSv). By cases. Let $\alpha \notin \mathrm{Var}$ then

Case 1: α is of the form $\sim\phi$.

(a)	$\sim\phi$	by DT
(b)	$\sim\sim\phi$	by DT
(c)	$\sim\phi \to (\sim\beta \to \sim\phi)$	(A1)
(d)	$\sim\beta \to \sim\phi$	(MP), (c), (a)
(e)	$\sim\sim\phi \to (\sim\beta \to \sim\sim\phi)$	(A1)
(f)	$\sim\beta \to \sim\sim\phi$	(MP), (e), (b)
(g)	$(\sim\beta \to \sim\phi) \to ((\sim\beta \to \sim\sim\phi) \to \beta)$	(A3)
(h)	$(\sim\beta \to \sim\sim\phi) \to \beta$	(MP), (g), (d)
(i)	β	(MP), (h), (f)

Case 2: α is of the form $\phi \to \psi$.

(a)	$\phi \to \psi$	by DT
(b)	$\sim(\phi \to \psi)$	by DT
(c)	$(\phi \to \psi) \to \sim\sim(\phi \to \psi)$	(A5)
(d)	$\sim\sim(\phi \to \psi)$	(MP), (c), (a)
(e)	$\sim(\phi \to \psi) \to (\sim\gamma \to \sim(\phi \to \psi))$	(A1)
(f)	$\sim\gamma \to \sim(\phi \to \psi)$	(MP), (e), (b)
(g)	$\sim\sim(\phi \to \psi) \to (\sim\gamma \to \sim\sim(\phi \to \psi))$	(A1)
(h)	$\sim\gamma \to \sim\sim(\phi \to \psi)$	(MP), (g), (d)
(i)	$(\sim\gamma \to \sim(\phi \to \psi)) \to ((\sim\gamma \to \sim\sim(\phi \to \psi)) \to \gamma)$	(A3)

(j)	$(\sim\gamma \to \sim\sim(\phi \to \psi)) \to \gamma$	(MP), (i), (f)
(k)	γ	(MP), (h), (j)

(DSn). By (A1), (DSv) and (MP).
(NN1) (cf. [7, pp. 174–175]):

(a)	$\sim\sim\alpha$	by DT
(b)	$\sim\sim\alpha \to (\sim\alpha \to \sim\sim\alpha)$	(A1)
(c)	$\sim\alpha \to \sim\sim\alpha$	(MP), (b), (a)
(d)	$\sim\alpha \to \sim\alpha$	Thesis: $\alpha \to \alpha$
(e)	$(\sim\alpha \to \sim\alpha) \to ((\sim\alpha \to \sim\sim\alpha) \to \alpha)$	(A3)
(f)	$(\sim\alpha \to \sim\sim\alpha) \to \alpha$	(MP), (e), (d)
(g)	α	(MP), (f), (c)

(NN2). By cases. Let $\alpha \notin \mathrm{Var}$ then
Case 1: α is of the form $\sim\phi$. See [7, pp. 175–176].
Case 2: α is of the form $\phi \to \psi$. By (A5). □

An important question arises at this point: Can a calculus containing (DSv) as a thesis be an example of paraconsistent logic? There is no doubt that the definition of the connectives should be formulated with the intention to reject possibly many substitutions for the so-called *Duns Scotus'* law, i.e. (DS) $\alpha \to (\sim\alpha \to \beta)$.[3] The phrase 'possibly many substitutions' is referred here to variables only.

Sette proved that his calculus was complete with respect to the following matrix

$$\mathfrak{M}_{\mathrm{P1}} := \langle \{0,1,2\}, \{1,2\}, \sim, \to \rangle,$$

where $\{0,1,2\}$ is the set of logical values, $\{1,2\}$ consists of the designated values and the connectives of $\to$ and $\sim$ are defined by the truth-tables:

$\to$	1	2	0
1	1	1	0
2	1	1	0
0	1	1	1

	$\sim$
1	0
2	1
0	1.

Informally speaking, a P1-valuation is any function $v\colon \mathrm{For} \longrightarrow \{1,2,0\}$ compatible with the above truth-tables. A P1-tautology is a formula which under any valuation v takes on the designated values $\{1,2\}$ (cf. [5, p. 18]).

[3] Cf. [3, p. 50]. A similar formulation, for the calculi C_n, can be found in [2, p. 498]: "From two contradictory formulas, α and $\sim\alpha$, it will not in general be possible to deduce an arbitrary formula β [...]".

It can be easily seen that

$$
\begin{array}{ll}
\text{(CI)} & ((\alpha \to \beta) \to \alpha) \to \alpha \\
\text{(CW)} & (\alpha \to \sim\alpha) \to \sim\alpha \\
\text{(CS)} & (\sim\alpha \to \alpha) \to \alpha
\end{array}
$$

are P1-tautologies and, by completeness, they are provable in the calculus.

The truth-value properties of the other connectives are given by the following matrices:

$\wedge$	1	2	0
1	1	1	0
2	1	1	0
0	0	0	0

$\vee$	1	2	0
1	1	1	1
2	1	1	1
0	1	1	0

$\leftrightarrow$	1	2	0
1	1	1	0
2	1	1	0
0	0	0	1.

Notice that all theorems of the positive part of classical propositional logic (CL^+ for short) are P1-tautologies.

3. A New Axiomatization of P1

In this section, we present a new axiomatization of the calculus P1. The propositional connectives of the axiomatization are negation $\sim$ and implication $\to$ which are taken here as primitives.

Let S be a system axiomatized by the following axiom schemas: (A1), (A2), (CI), (CS) and

$$
\begin{array}{ll}
\text{(SAn1)} & \sim\alpha \to (\sim\sim\alpha \to \beta) \\
\text{(SAn2)} & (\alpha \to \beta) \to (\sim(\alpha \to \beta) \to \gamma))
\end{array}
$$

and the rule (MP).

The set of axiom schemata and (MP) define $\vdash_S$ (the consequence relation).

Fact 3.1. (A1), (A2), (CI), (CS), (SAn1) *and* (SAn2) *are* P1-*tautologies and the rule* (MP) *preserves validity.*

Proof. Since every theorem of CL^+ is a P1-tautology and the sole rule of inference of P1 is (MP), all we need is to prove that (CS), (SAn1), (SAn2) are P1-tautologies.

(SAn1): Suppose that $\sim\alpha \to (\sim\sim\alpha \to \beta)$ is not a P1-tautology. Then there is a P1-valuation v such that $v(\sim\alpha \to (\sim\sim\alpha \to \beta)) = 0$. There are two cases to consider, viz.

Case 1. $v(\sim\alpha) = 1$ and $v(\sim\sim\alpha \to \beta) = 0$. If $v(\sim\alpha) = 1$, then $v(\sim\sim\alpha) = 0$ (by the truth-table for negation) and $v(\sim\sim\alpha \to \beta) = 1$ (by the truth-table for implication). A contradiction.

Case 2. $v(\sim\alpha) = 2$ and $v(\sim\sim\alpha \to \beta) = 0$. But it follows from the truth-table for negation that $v(\sim\alpha) = 1$ or $v(\sim\alpha) = 0$, for any $\alpha \in$ For and every P1-valuation v. A contradiction.

(SAn2): Assume that $(\alpha \to \beta) \to (\sim(\alpha \to \beta) \to \gamma))$ is not a P1-tautology. Then $v((\alpha \to \beta) \to (\sim(\alpha \to \beta) \to \gamma))) = 0$, for a P1-valuation v.

Case 1. $v(\alpha \to \beta) = 1$ and $v(\sim(\alpha \to \beta) \to \gamma) = 0$. If $v(\alpha \to \beta) = 1$, then $v(\sim(\alpha \to \beta)) = 0$ and $v(\sim(\alpha \to \beta) \to \gamma) = 1$. A contradiction.

Case 2. $v(\alpha \to \beta) = 2$ and $v(\sim(\alpha \to \beta) \to \gamma) = 0$. It follows from the truth-table for implication that $v(\alpha \to \beta) = 1$ or $v(\alpha \to \beta) = 0$, for any α, $\beta \in$ For and every P1-valuation v. A contradiction.

(CS): Suppose that $(\sim\alpha \to \alpha) \to \alpha$ is not a P1-tautology. Then $v((\sim\alpha \to \alpha) \to \alpha) = 0$, for a P1-valuation v.

Case 1. $v(\sim\alpha \to \alpha) = 1$ and $v(\alpha) = 0$. If $v(\alpha) = 0$, then $v(\sim\alpha) = 1$ and $v(\sim\alpha \to \alpha) = 0$. A contradiction.

Case 2. $v(\sim\alpha \to \alpha) = 2$ and $v(\alpha) = 0$. Notice that the following holds: $v(\alpha \to \beta) = 1$ or $v(\alpha \to \beta) = 0$, for any $\alpha, \beta \in$ For and any P1-valuation v. So, in particular: $v(\sim\alpha \to \alpha) = 1$ or $v(\sim\alpha \to \alpha) = 0$, for every P1-valuation v. A contradiction. □

FACT 3.2. (A3), (A4), (A5), (NN1) *and the following formulas*:

(F1)	$(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$
(F2)	$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
(F3)	$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$
(F4)	$((\alpha \to \beta) \to \alpha) \to ((\beta \to \alpha) \to \alpha)$
(F5)	$(\sim\sim\sim(\alpha \to \beta) \to \sim(\alpha \to \beta)) \to ((\alpha \to \beta) \to \sim\sim(\alpha \to \beta))$
(NN2′)	$\sim\alpha \to \sim\sim\sim\alpha$

are provable in S.

PROOF. (F1), (F2) and (F3) by deduction theorem and (MP); (F4) by deduction theorem, (CI) and (MP).

(NN1):

(a)	$\sim\sim\alpha$	by DT
(b)	$\sim\alpha\to(\sim\sim\alpha\to\alpha)$	(SAn1)
(c)	$(\sim\alpha\to(\sim\sim\alpha\to\alpha))\to(\sim\sim\alpha\to(\sim\alpha\to\alpha))$	(F1)
(d)	$\sim\sim\alpha\to(\sim\alpha\to\alpha)$	(MP), (c), (b)
(e)	$\sim\alpha\to\alpha$	(MP), (d), (a)
(f)	$(\sim\alpha\to\alpha)\to\alpha$	(CS)
(g)	α	(MP), (f), (e)

(A3):

(a)	$\sim\alpha\to\sim\beta$	by DT
(b)	$\sim\alpha\to\sim\sim\beta$	by DT
(c)	$\sim\beta\to(\sim\sim\beta\to\alpha)$	(SAn1)
(d)	$(\sim\beta\to(\sim\sim\beta\to\alpha))\to(\sim\sim\beta\to(\sim\beta\to\alpha))$	(F1)
(e)	$\sim\sim\beta\to(\sim\beta\to\alpha)$	(MP), (d), (c)
(f)	$(\sim\alpha\to\sim\sim\beta)\to((\sim\sim\beta\to(\sim\beta\to\alpha))\to(\sim\alpha\to(\sim\beta\to\alpha)))$	(F2)
(g)	$(\sim\sim\beta\to(\sim\beta\to\alpha))\to(\sim\alpha\to(\sim\beta\to\alpha))$	(MP), (f), (b)
(h)	$\sim\alpha\to(\sim\beta\to\alpha)$	(MP), (g), (e)
(i)	$(\sim\alpha\to(\sim\beta\to\alpha))\to(\sim\beta\to(\sim\alpha\to\alpha))$	(F1)
(j)	$\sim\beta\to(\sim\alpha\to\alpha)$	(MP), (h), (i)
(k)	$\sim\alpha\to(\sim\alpha\to\alpha)$	(F2), (MP), (a), (j)
(l)	$(\sim\alpha\to(\sim\alpha\to\alpha))\to(\sim\alpha\to\alpha)$	(F3)
(m)	$\sim\alpha\to\alpha$	(MP), (k), (l)
(n)	$(\sim\alpha\to\alpha)\to\alpha$	(CS)
(o)	α	(MP), (m), (n)

(NN2′): By (MP), (A1), (NN1), and a substitution of (A3):

$$(\sim\sim\sim\sim\alpha\to\ \sim\alpha)\to((\sim\sim\sim\sim\alpha\to\ \sim\sim\alpha)\to\sim\sim\sim\alpha).$$

(F5):

(a)	$\sim\sim\sim(\alpha\to\beta)\to\sim(\alpha\to\beta)$	by DT
(b)	$\alpha\to\beta$	by DT
(c)	$(\alpha\to\beta)\to(\sim(\alpha\to\beta)\to\sim\sim(\alpha\to\beta))$	(SAn2)
(d)	$\sim(\alpha\to\beta)\to\sim\sim(\alpha\to\beta)$	(MP), (b), (c)
(e)	$\sim\sim\sim(\alpha\to\beta)\to\sim\sim(\alpha\to\beta)$	(F2), (MP), (a), (d)
(f)	$\sim\sim(\alpha\to\beta)$	(A3), (MP), (a), (e)

(A5): By DT, (F5), (NN1) and (MP).

(A4):

(a)	$\sim(\alpha \to \sim\sim\alpha)$	by DT
(b)	$\sim(\alpha \to \sim\sim\alpha) \to (\sim\sim(\alpha \to \sim\sim\alpha) \to \alpha)$	(SAn1)
(c)	$\sim\sim(\alpha \to \sim\sim\alpha) \to \alpha$	(MP), (b), (a)
(d)	$(\alpha \to \sim\sim\alpha) \to \sim\sim(\alpha \to \sim\sim\alpha)$	(A5)
(e)	$((\alpha \to \sim\sim\alpha) \to \sim\sim(\alpha \to \sim\sim\alpha)) \to$ $\to ((\sim\sim(\alpha \to \sim\sim\alpha) \to \alpha) \to ((\alpha \to \sim\sim\alpha) \to \alpha)))$	(F2)
(f)	$(\alpha \to \sim\sim\alpha) \to \alpha$	(MP), (e), (d), (c)
(g)	$((\alpha \to \sim\sim\alpha) \to \alpha) \to ((\sim\sim\alpha \to \alpha) \to \alpha)$	(F4)
(h)	$(\sim\sim\alpha \to \alpha) \to \alpha$	(MP), (f), (g)
(i)	$\sim\sim\alpha \to \alpha$	(NN1)
(j)	α	(MP), (h), (i) □

Fact 3.3. S = P1.

Proof. (⊆). By Fact 3.1.

(⊇) What is desired is to show that (A3), (A4) and (A5) are provable in S. Obvious by Fact 3.2. □

It is an immediate consequence of Fact 3.3 that P1 is axiomatizable by (A1), (A2), (CI), (CS), (SAn1), (SAn2), and (MP). Moreover, by facts 2.1 and 3.3, we obtain:

Fact 3.4. *The axiom schemata* (SAn1) *and* (SAn2) *can be replaced with* (DSv), *and the resulting system is equivalent to the system* S.

A simple conclusion from Facts 3.3 and 3.4 is that the calculus P1 is axiomatized by (A1), (A2), (DSv), (CI), (CS), and (MP).

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