Universidade de Aveiro Departamento de Matemática
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Paraconsistência em Lógica Híbrida

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o júri

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palavras-chave Paraconsistência, lógica híbrida, lógica quase-híbrida, bistrutura, frame de Kripke, satisfação dissociada, satisfação forte, modelo, medida de inconsistência.

resumo O uso de lógicas híbridas permite a descrição de estruturas relacionais, ao mesmo tempo que permite estabelecer relações de acessibilidade entre estados, e, para além disso, nomear e fazer referência ao que acontece em estados específicos.

No entanto, a informação que recolhemos está sujeita a inconsistências, isto é, a procura de diferentes fontes de informação pode levar à recolha de contradições. O que nos dias de hoje, com tantos meios de divulgação disponíveis, acontece frequentemente.

O objetivo deste trabalho é desenvolver ferramentas capazes de lidar com informação contraditória que possa ser descrita através de fórmulas de lógicas híbridas. Construir modelos e comparar a inconsistência de diferentes bases de dados e ver a aplicabilidade deste método no dia-a-dia são a base para o desenvolvimento desta dissertação.

keywords Paraconsistency, hybrid logic, quasi-hybrid logic, bistructure, Kripke frame, decoupled satisfaction, strong satisfaction, model, measure of inconsistency.

abstract The use of hybrid logics allows the description of relational structures, at the same time that allows establishing accessibility relations between states and, furthermore, nominating and making mention to what happens at specific states.

However, the information we collect is subject to inconsistencies, namely, the search for different information sources can lead us to pick up contradictions. Nowadays, by having so many means of dissemination available, that happens frequently.

The aim of this work is to develop tools capable of dealing with contradictory information that can be described as hybrid logics' formulas. To build models, to compare inconsistency in different databases, and to see the applicability of this method in day-to-day life are the basis for the development of this dissertation.

Contents

Chapter 1

Introduction

Information is nowadays, and progressively more, accessible to everyone, thanks to all transmission means existent. Actually, sharing information is crucial in a huge variety of areas, from the health area, where any development concerning the discovery of a new cure or the appearence of a new illness should be reported globally, until the economics area, because presently the successes and insuccesses of different markets should be thoroughly followed since all of them have an impact in our lives. In summary, each and every information is serviceable in some sense. However, one does not need to go so far, it takes only a few minutes of conversation with a friend so that information is exchanged. Problems arise when from different sources one obtains contradictory information, in fact, gathering inconsistent information is the rule rather than the exception. Personal opinions change from one person to another, and not even scientific areas escape the occurrence of inconsistencies. This is the main reason why the study and development of flexible logical systems able to handle heterogeneous and complex data has become more and more relevant for the last two decades, resorting to interdisciplinary research in linguistics, computer sciences, mathematics and even philosophy.

Paraconsistent logics were created with the purpose of allowing inconsistencies without producing the collapse of the whole system. Discussed for almost a century, paraconsistent logics have been a growing topic of interest, as the increase in the number of publications, the number of dedicated specialized journals and of international conferences (for example the World Congress of Paraconsistency) confirms. For the last sixty years, many philosophers, logicians and mathematicians have become involved in the area. The Brazilian logic school (Newton da Costa, Walter Carnielli, Jean-Yves Béziau, João Marcos, etc.) is prominent among them. There have been also developed many systems of paraconsistent logic, either to meet different aims or to target specific applications.

Paraconsistent logics exclude the *Principle of Non-Contradiction* ([43]) which states that from contradictory premises any formula can be derived, thus turning the logic trivial. The first person to discuss the possibility of violating this principle was Łukasiewicz, however he did not develop any logical system to formalize his studies. It was his disciple, Stanisław Jaśkowski, the one who constructed the first system of propositional paraconsistent logic.

Whilst paraconsistent logics allow storaging and reasoning with inconsistent information, such would never be possible within classical logics where a formula can only be true or false, cannot be both nor anything else. With the intention of removing inconsistencies there were created techniques, but all of them showed limited applicability, as well as not to be very reliable because often *hidden* inconsistencies were not eliminated. So, in conclusion, the safest and more adequate way to deal with inconsistency, for example in sets of information (knowledgebases), is in fact using paraconsistent logics, where inconsistencies are seen as facts rather than anomalies.

Among the areas where paraconsistent logics have been successfully applied are the ones of Computer Science, Medicine and Robotics. For example in the medical practice, consulting two or more physicians may lead to contradictory diagnoses, none of them to be dismissed ([57]). In Physics, through the years there have been made discoveries that collide with each others, namely some aspects of quantum mechanics are being discussed ([45]). In Computer Science, subdomains like requirements engineering ([51]), artificial intelligence ([52]) and automated reasoning within information processing knowledgebases ([47]), are among the most relevant areas in which paraconsistent logic can address theoretical difficulties raised by inconsistent data. Other applications are discussed in [91]. All of this has contributed to transform paraconsistent logics in a mature, independent, and well established field of mathematical logic as the entry "03B53 - Paraconsistent logics" in the 2010 Mathematics Subject Classification evidences.

One of the systems of paraconsistent logic was projected by Grant and Hunter in [58], they call it *Quasi-Classical Logic*, and it consists on *adapting* a first-order logic (with functions of arity 0) so that it is possible to enable both a proposition and its negation to be true. This is accomplished by resorting to two valuations, one for the positive literals, and another for the negative ones. From a set of informations, a model which will be a set of literals satisfying all formulas in the initial set is defined, later the ones with smaller cardinality are considered and then the ones with smallest measure of inconsistency are handled. This *measure of inconsistency* is a ratio between 0 and 1, where de numerator is the number of inconsistencies in the model, and the denominator is the possible number of inconsistencies in it. This is not a new idea, it has been already used in [11] and [12], however, being able to measure inconsistency is a great feature which allows comparing between models and deciding which one is less inconsistent.

Different researches led to the appearance of *hybrid logics* almost seven decades ago, motivated by the necessity of – in addition to describing relational structures and taking in consideration the context in which the evaluation of a formula takes place, which modal logics already could do – being able to make assertions about each state and to define accessibility relations and equalities between states. This is possible due to the introduction of a new set of atomic formulas, called *nominals*, such that each of them is true exactly at the unique state it names, and a new operator, @, called the *satisfaction operator*. The basic version of hybrid logic, although being more expressive than the basic version of modal logic, does not increase complexity. There are some other versions, where binders are introduced, but where the complexity augments.

The challenge was to unite both paraconsistency and hybrid logics, thus creating a paraconsitent version for hybrid logics, and the aim of this work is to present it. Following the work of Hunter and Grant in [58], this dissertation introduces the *Quasi-Hybrid (Basic) Logic*. Without loss of generality, formulas are in negation normal form. Once again, the idea is to allow both a propositional symbol and its negation to be true by resorting to two valuations. The positive valuation of a propositional symbol is the set of states where the propositional symbol is true, while the negative one is the set of states where the negation of the propositional symbol is true. The *"valuation"* of nominals is not subject to inconsistencies, so it is considered disjoint from the valuations of propositional symbols, and it is called *"nomination"*, since for every nominal there is only one state where the nominal is true, which is the state it names. Then, most concepts defined for Quasi-Classical Logic are adapted, such as the concepts of structure, bistructure, literal, model, ... The measure of inconsistency is also described in Quasi-Hybrid Logic, again as a ratio between 0 and 1, and the results for the intrinsic and extrinsic inconsistencies in Quasi-Classical Logic are also suited and proved in the context of basic hybrid logic with paraconsistency as introduced.

Outline

The **First Chapter** introduces the concept of paraconsistency, accompanied with its emergence and then a *road map* of schools of paraconsistent logic is presented. An example of paraconsistent logic, the Quasi-Classical Logic, which is the basis for the developments in Chapter 3, is broadly introduced. To conclude, applications of paraconsistent logics are announced.

In the **Second Chapter** the subject to be approached is hybrid logic. There will be introduced both the *basic hybrid logic* and the *strong Priorean logic*. The concept of bisimulation is given, and some results are presented. Then there are defined the *standard and hybrid translations*, which will help understanding the relation between hybrid and first-order logics.

In the **Third Chapter**, which was the target of investigation for the last months, arises the idea of considering *paraconsistency in hybrid logic*. This idea brings a new definition for Quasi-Hybrid Logic, as well as new notions of satisfaction, namely the presence of two valuations that allow local inconsistency without destroying the validity of the entire theory. The concept of model is introduced, and subsequently the concepts of minimal model and preferred model appear. Meanwhile, it is presented the measure of inconsistency of a model, fundamental to compare between theories. Along the Chapter, various examples are given in order to illustrate the theory.

In order to evidence the relevance of this theme, and to conclude, the **Fourth Chapter** introduces *applications*, namely in the fields of robotics and medical care.

In the **Fifth Chapter**, conclusions are stated, and some ideas for future research are presented.

Chapter 2

Paraconsistency

Paraconsistent (or inconsistency-tolerant) logics distinguish themselves from the classical ones through their ability to reason about inconsistent information without slipping into absurdity. Their underlying idea is not new; in fact, history tells us that Aristotle already used it up to some extent. However, they only became an active scientific topic in the beginning of the 20th century, and received their name in 1976, in the Third Latin America Conference on Mathematical Logic when the Peruvian philosopher Francisco Miró Quesada introduced it as a possible solution to a request by da Costa, [43].

The prefix *"para"*, of Greek origin, has three synonyms: (1) "against", as in *"paradox"*, Greek for "against the common sense"; (2) "beyond", as in *"paranormal"*; and finally (3) "very similar", "connected" or "nearby" as in *"parallel"* and *"parabola"*.

In *classical logic* – the one developed by Boole, Frege, Russell *et al.* in the late 1800s, and typically taught in the university courses – as well as in *intuitionistic* and most other logics, [66], contradictions are linked to everything. Paraconsistent logics accommodate inconsistency in a sensible manner that treats inconsistent information as informative. So, when in need of a controlled and discriminative environment where we can keep sensible and reasonable even when contradictions arise, the answer is precisely the huge universe of paraconsistent logics.

In [81], Priest declares that paraconsistent logics' major motivation is the thought that in certain circumstances we may be in a situation where our information or theory is inconsistent, and yet we are required to draw inferences in a controlled way. He gives numerous examples of inconsistent situations: information in a computer data base (this one is fairly obvious), various scientific theories (for example, Bohr's theory of the atom which required bound electrons both to irradiate energy (by Maxwell's equations) and not to (since they do not spiral inwards towards the nucleus)), constitutions and other legal documents (constitutions that give different kinds of people different privileges, for example, when a person of type A can do something and a person of type B cannot do that same thing, then a person who is both types appears), descriptions of fictional (and other non-existent) objects (for example characterizations by means of a novel or a myth), and descriptions of counterfactual situations (when form an impossible situations we make a proposal, for example, *"if you draw a square circle, I would give you my money"*) are some of them.

Formally, a *logic* is composed of (but not only) a set of well-formed formulas, together with an inference relation, $\vdash$, following the so called *Tarskian conditions*:

1. Reflexivity: If $A \in \Gamma$, then $\Gamma \vdash A$;

2. Monotonicity: If $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash A$;

3. Transitivity: If $\Gamma \vdash A$ and $\Delta \cup \{A\} \vdash B$, then $\Gamma \cup \Delta \vdash B$.

It is possible to characterize an enormous amount of distinct logics using these conditions, but they can also be made more permissive. One example is the weakening of the monotonicity requirement, which paves the way for a whole new family of logics, the non-monotonic ones.

This *inference relation*, also called relation of logical consequence, details which formulas (*conclusions*) follow from which others (*premises*). If the formula B follows from the set of formulas $\{A_0, A_1, A_2, \ldots, A_n\}$, we write $\{A_0, A_1, A_2, \ldots, A_n\} \vdash B$ and say that B is a *consequence* of $\{A_0, A_1, A_2, \ldots, A_n\}$. When B is a consequence of $\emptyset$, we say that B is a *theorem.* A set of formulas closed under $\vdash$, *i.e.*, such that $\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma$, is called a *theory.*

A relation of logical consequence is called *explosive* if the entailment $\{A, \neg A\} \vdash B$ holds, for every A and B. This phenomenon is known as the *principle of explosion* or *ex contradictione quodlibet* (ECQ) (latin for "from a contradiction anything follows"), [38]. Thereby, if a theory contains a single inconsistency then it has every formula as a theorem. Actually, this is the only inconsistent theory in non-paraconsistent logics, called the trivial one. The opposite happens in paraconsistent logics: they are capable of distinguishing between inconsistent theories; such feature is interesting because when reasoning with a paraconsistent logic one can begin with inconsistent premises and still reach sensible conclusions.

As pointed out, all approaches to paraconsistency seek inference relations that do not explode. This can be accomplished by idealizing new meanings for the relation of logical consequence, which, obviously, will not be explosive, or, through any other process which removes causes of explosion.

One may say that paraconsistency carries out a *tradeoff.* Removing the potential for explosion requires dropping at least one of the following principles: *disjunction introduction* ($A \vdash A \vee B$), *disjunctive syllogism* ($\{A \vee B, \neg A\} \vdash B$), or *transitivity* ($\Gamma \vdash A,\ A \vdash B \Rightarrow \Gamma \vdash B$). Frequently logicians choose to drop the disjunctive syllogism. One, however, may choose to maintain it along with transitivity and exclude disjunction introduction instead.

Moreover, the next three principles cannot coexist: *reductio ad absurdum* ($A \to (B \wedge \neg B) \vdash \neg A$), *rule of weakening* ($A \vdash B \to A$) and *double negation elimination* ($\neg\neg A \vdash A$). So, at least one should be abandoned too.

2.1 History of Paraconsistent Logic

In the words of Aristotle, *"it is impossible that the same thing should be necessitated by the being and by the not-being of the same thing"*. This principle, known as the *connexive principle*, became a topic of debates in the Middle Ages, [73]. Two of its biggest debaters, Boethius (480 — 524 or 525) and Abelard (1079 — 1142) took the principle and considered two sorts of consequences. The first one was a similar view of the notion of truth-preservation that fails to meet the connexive principle. The second one is the *containment account* which does not permit an inference whose conclusion is arbitrary, thus capturing Aristotle's idea.

To reject the *connexive principle* is an approach that has become most influential. The followers of Adam Parvipontanus (or Adam of Balsham, 12th century) embraced the truth-preservation account which states that it is impossible for the premises to be true but the conclusion false, and the 'paradoxes' that are associated with it, [78]. On the other hand, the *containment account* did not disappear. John Duns Scotus (1266 — 1308), and his followers accepted the philosophy that the sense of the premises contains the sense of the conclusions, [92]. All of this is, of course, related to the paraconsistent logics' history, which is detailed next.

The idea of abandoning the principle of explosion occurred to different people at different times and places independently of each other during the last century, triggered by several distinct motivations. The Russian Vasil'év proposed, in 1910, a modified Aristotelian syllogistic including statements of the form: *"S is both P and not P"*, which seems to be the earliest paraconsistent logic in the contemporary era. However the proposal did not make any significant impact at the time. Later on, the Polish logician Stanisław Jaśkowski, disciple of Łukasiewicz, developed the first (formal) system of propositional paraconsistent logic ([64]).

Paraconsistent logics were independently developed in South America by Asenjo (1954) and especially Newton da Costa (1963) in their doctoral dissertations, with an emphasis on mathematical applications. By the mid-1970s, the development of paraconsistent logic became more intensive. Afterwards, for the last sixty years, many philosophers, logicians and mathematicians have become involved in the area with the Brazilian logic school (Newton da Costa, Walter Carnielli, Jean-Yves Béziau, João Marcos, etc.) taking a prominent role. The theory of paraconsistent logic developed considerably as witnessed by both the increase in the number of publications, the number of dedicated journals, and international conferences: in 1997, the First World Congress on Paraconsistency was held at the University of Ghent in Belgium, [15]; the second was held in São Sebastião (São Paulo, Brazil) in 2000, [36]; the third in Toulouse (France) in 2003, [21]; and the fourth in Melbourne (Australia) in 2008, [87]. The fifth took place in Kolkate (India), this year (2014), [20].

Our attention is now turned into the large variety of ways paraconsistent logic can be embedded into. The most relevant ones are introduced.

2.2 Schools of Paraconsistent Logic

Through the years, many ideas of how and where can paraconsistency be used have been developed all over the world as the interest in paraconsistent logic continues to grow. Here are presented some ways of viewing paraconsistency.

2.2.1 Dialetheism

The formal definition of a paraconsistent logic is the following:

Definition I: A logic is paraconsistent iff it is not the case that for all formulas A, B, $\{A, \neg A\} \vdash B$.

This is simply the denial of *ex contradictione quodlibet*, *i.e.*, that a logic is paraconsistent iff it does not validate explosion. However, things are a little different in this second definition:

Definition II: A logic is paraconsistent iff there are some formulas A, B such that $\vdash A$ and $\vdash \neg A$, but not $\vdash B$.

A logic that satisfies the second definition automatically satisfies the first one, but the converse is not true. Actually the subtle difference between them is what will help distinguishing between two main *degrees* of paraconsistency: weak and strong.

Roughly speaking, *weak* paraconsistency is the concept that insists that any apparent contradictions are always due to human error. In a better world where humans did not err, we would use classical logic because no true theory would ever contain an inconsistency. But in the real world sometimes information systems contain regrettable but inevitable errors, and paraconsistent logics are tools for damage control. Weak paraconsistentists look for ways to restore consistency to the system or to make the system work as consistently as possible.

On the other hand, *strong* paraconsistency comprehends ideas like: *"some contradictions may not be errors"*; *"some true theories may actually be inconsistent"*. A strong paraconsistentist considers relaxing the law of non-contradiction in some way, either by dropping it entirely, so that $\neg(A \wedge \neg A)$ is not a theorem, or by holding that the law can itself figure into contradictions of the form "always, not (A and not A), and sometimes, both A and not A". The assumption that true contradictions exist in reality led to the establishment of the philosophical school of *dialetheism*, most notably advocated by Graham Priest and properly introduced now.

Pioneered by Richard Routley and Graham Priest in Australia in the 1970s, the *dialetheism* is not a logic, it is rather a *movement* where the belief is that contradictions truly exist in real life, [17]. Do not confuse dialetheism with paraconsistency, the latter can be seen as a property of an inference relation, whereas the former can be assumed as a view about truth. Indeed, in a paraconsistent logic assuming that a formula is both true and false does not mean that the contradiction is true *per se*, as supported by the school of dialetheism. This is why paraconsistency must be distinguished from dialetheism. A paraconsistent logician may feel some pull towards dialetheism, however we need to keep in mind that most paraconsistent logics are not 'dialetheic' logics.

Despite the need for dialetheism and paraconsistency to be distinguished, dialetheism can be a motivation for paraconsistent logic. One example of a *dialetheia* is the *Liar Paradox*, expressed in the sentence *"This sentence is not true"* (A). The explanation is simple: if we consider that (A) is true, then *"This sentence is not true"* is true; therefore (A) is false, however by hypothesis, (A) is true, which is a contradiction. On the other hand, by considering that (A) is false, then *"This sentence is not true"* is false, thus (A) is true, which contradicts the hypothesis that (A) is false. So, (A) is both true and false, that is, a *dialetheia*.

2.2.2 Discussive logic

As said before, the first formal paraconsistent logic was projected by Jaśkowski, in 1948, (cf. [44]). The logician's idea was to imagine a group of people having a *discussion* where each participant contributes with some information. Each assertion is true according to the participant. However, even though each participant's opinion is self-consistent, the group as a whole (*i.e.*, the sum of all assertions) may bring inconsistency. In fact, one ought to expect that participants disagree on some issue, as it happens in our daily lives: by reading new articles, blogs and opinion pieces, we take in contradictions.

In order to see the relevance of paraconsistent logics in discussions, consider that a participant says that A holds, while another one says that $\neg A$ holds. Then both A and $\neg A$ hold in the discourse, yet, a certain sentence B may not be supported by any of the participants, so the *discussive logic* invalidades the principle of explosion.

2.2.3 Preservationism

Preservationism appeared *circa* 1978, pioneered by Ray Jennings and Peter Soch, two logicians from Canada. It is an approach to modal logic and paraconsistency with a certain connection with discussive logic, [65]. The idea of preservationism is to, given a set of inconsistent premises Γ, define maximally consistent subsets Γ' such that if $A \in \Gamma \backslash \Gamma'$, then $\{A\} \cup \Gamma'$ is inconsistent. They do not reason over an inconsistent set of premises, they reason over consistent subsets of premises.

The level of an inconsistent set of premises is defined as the least number of subsets into which the initial inconsistent set must be divided in order to get internal consistency in all subsets. The consequence relation, called *forcing*, is defined over some maximally consistent subset, in terms of a logic X, as follows:

A set of formulass Γ forces A iff there is at least one subset $\Delta \subseteq \Gamma$ such that A is an X-valid inference from Δ.

This consequence relation preserves the level of the maximally consistent subset, which is why this is known as preservationism.

More recently, Payette showed that two logics are identical if and only if they assign any set of formulas the same level, [65].

2.2.4 Adaptive logics

Developed by Diderik Batens in Belgium, the primary motivation for *adaptive logics* is that we should treat a formula or a theory as consistently as possible, [14]. An adaptive logic adapts itself to the situation when the inference rules are about to be applied. They are typically used to explain the many interesting dynamic consequence relations that occur in *human* reasoning. As humans, our knowledge is not closed under logical consequence because we can always add something to the things that we already know and also because all of our actions have consequences and we are not fully aware of all of them.

Our way of reasoning is dynamic in two aspects:

1. *Externally dynamic:* if conclusions may be discarded in view of new information. This means that the consequence relation is nonmonotonic ($\Gamma \vdash A$ and $\Gamma \cup \Delta \nvdash A$ for some Γ, Δ, A).

2. *Internally dynamic:* if a conclusion may be removed in view of the better understanding of the premises provided by a sequel of the reasoning, *i.e.*, if we are forced to infer a contradiction at a later stage, our reasoning has to adapt itself so that an application of the previously used inference rule is withdrawn.

A system of an adaptive logic can be characterized by three elements: the lower limit logic, a set of abnormalities, and an adaptive strategy. The lower limit logic determines which consequences hold regardless of any assumptions (or conditions); it is not subject to adaptation. A set of abnormalities is a set of formulas presupposed as absurd at the beginning of reasoning until they are shown to be otherwise. Finally, an adaptive strategy will pick, based on the set of abnormalities, one specific way of applying the inference rules.

2.2.5 Logics of Formal Inconsistency

Logics of Formal Inconsistency, LFIs, are a family of paraconsistent logics where each of its components constitutes a consistent fragment of classical logic. As a family of paraconsistent logics, LFIs reject explosion principle when a contradiction is present, [35]. They appeared recently, as Marcos, Carnielli and others worked on developing a broad generalization of the *C-systems*, (cf. [43], [90]), investigated by Newton C. A. da Costa, one of the first pioneeres of paraconsistent logic in Brazil back in the 1950s.

When working on his C-systems, da Costa established that ([46]):

1. in this kind of logic, the principle of contradiction should not be generally valid;
2. in general, it should not be possible to deduce any conclusion from two contradictory premises;
3. extending these logics to obtain quantified logics should be immediate.

The idea was to study inconsistent theories without assuming that they were necessarily trivial, and that consistent formulas and inconsistent formulas should be separated. They have in mind that classical logic should be respected as much as possible, and that we should divert from this classical approach only when contradictions appear. This means that the ECQ is valid in the absence of contradictions.

LFIs have been useful on modelling some actual mathematics. In particular, da Costa and his students developed new ideas in the areas of arithmetics (cf. [54]), infinitesimal calculus (cf. [48]) and model theory (cf. [37]).

2.2.6 Relevant logics

Relevant logics, ([3], [4]), were introduced by Anderson and Belnap in 1975, in Pittsburgh. Their underlying idea is that, for an argument to be valid, premises and conclusions must have a significative connection.

Formally, a logic is relevant iff it satisfies the condition:

if $A \to B$ is a theorem, then A and B share a non-logical constant.

It follows that $\{p, \neg p\} \nvdash q$, so the principle of explosion does not hold.

Relevant logics provide an implication connective that obeys *modus ponens* ($A \to B$, A, therefore B), which evidently restores a lot of power lost in the invalidity of disjunctive syllogism. But from there on, one has several different axiomatizations for a relevant logic [50]. The relevant approach has been used in the study of set theory and arithmetics [74].

2.3 Quasi-Classical Logic

So far, a large number of formal techniques to invalidate ECQ have been developed. The need to isolate an inconsistency is the main purpose to reject ECQ. [77] is an interesting reading in the way it takes the ideas discussed in Section 2.2 and introduces formal methods for most of them. However in this Section, only the Quasi-Classical Logic, a version of a paraconsistent system developed by Grant and Hunter ([58]), is addressed.

The *Quasi-Classical Logic*, a gateway to deal with inconsistencies in classical logic, was developed by Grant and Hunter in the 21th century, [58]. This work will be the basis to the developments in Chapter 4, where it will be introduced a paraconsistent version of basic hybrid logic, capable of processing inconsistencies. It will be named *Quasi-Hybrid Logic* since it follows the approach of Grant and Hunter. But before presenting it, let us focus on the Quasi-Classical Logic.

The idea behind Quasi-classical Logic is to decouple the link between a proposition and its negation. For that, two valuations are required: one for dealing with the positive propositions and another one for dealing with the negative propositions. The satisfaction is defined with resource to this two valuations, plus a feature called focus, which asserts the validity of disjunctive syllogism in Quasi-Classical Logic. Since it is possible to use the method of Robinson diagrams, models are defined as sets of literals. The measure of inconsistency is introduced as the ratio between the number of inconsistencies in the model, and the total number of inconsistencies that the model could have. Models with least cardinality are called minimal and models with least inconsistency measure are called preferred. Using the last ones, it is defined the extrinsic and intrinsic inconsistency of a theory, which are sequences of inconsistency measures. For those sequences, several results are announced and proved.

It will be considered a first-order language that contains, in terms of vocabulary, the following logical symbols: a countable set of variables V, denoted $x, y, \dots$; ordinary connectives $\{\neg, \vee, \wedge, \rightarrow\}$; two quantifiers $\{\exists, \forall\}$; punctuation symbols for readability; and in terms of extra-logical symbols: a set of constants $\mathcal{C}$, denoted as $a, b, c, \dots$; along with a set of predicate symbols, $\mathcal{P}$, denoted $P(\cdot), R(\cdot), \dots$, with their arities in parentheses;

Note that there are only allowed function symbols of arity 0, *i.e.* constants.

A term is either a variable or a constant symbol. An atom is a formula of the form $P(t_1, \dots, t_n)$, where P is a predicate symbol of arity n and $t_1, \dots, t_n$ are terms. And a literal is an atom or the negation of an atom. It is also assumed that all formulas are in *prenex conjunctive normal form*, this is possible because every first-order formula can be transformed into this form.

Although it is not part of the language, the $\sim$ operator is introduced in order to make some definitions clearer. Let α be a literal and let $\sim$ be a complementation operation such that $\sim \alpha = \neg\alpha$ and $\sim (\neg\alpha) = \alpha$.

Consider also the definition for the focus, which will take an important role in the definition of satisfaction of a disjunction of literals.

Definition 2.3.1. *Let $\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n$ be a disjunction of literals, with $n > 1$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ all distinct. The* focus *of $\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n$ by α_i, denoted $\otimes(\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n, \alpha_i)$ is the clause obtained by removing α_i from $\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n$.*

It is now defined a classical structure for a language, a pair with a domain and an interpretation function.

Definition 2.3.2. *A* classical structure *for a language L is a pair (D, I), where*

1. *$D \neq \emptyset$ is the domain;*
2. *I is the interpretation function, assigning elements and predicates over D to symbols in L as follows:*
   - *For $c \in \mathcal{C}$, $I(c) \in D$;*
   - *For $P(n)$ an n-ary predicate symbol, $I(P) : D^n \to \{0, 1\}$ is an n-ary predicate over D;*

To make notation easier to follow, it will be assumed that I maps constants to themselves. This leads to assume that $\mathcal{C} \subseteq D$. Going even further, by introducing a new constant for each element in $D \backslash \mathcal{C}$, it is assumed that we deal with languages $L(D) = \langle \mathcal{P}, D \rangle$, where D is the domain of interpretation.

In order to accommodate inconsistencies, it is required a *bistructure* which is a tuple (D, I^+, I^-), where (D, I^+) and (D, I^-) are classical structures.

The definition of an assignment is the usual one: an *assignment* is a function $g : V \to D$.

Given an assignment g and $x \in V$, an *x-variant assignment* g' of g is an assignment such that:

$$g'(y) = g(y), \text{ } for \text{ } all \text{ } y \in V, y \neq x$$

The extension g (abusing notation) of an assignment to terms is defined by

$$g(t) = d, \text{ } if \text{ } t = x \text{ } and \text{ } g(x) = d$$

$$g(t) = d, \text{ } if \text{ } t = d$$

Next, it is introduced the satisfaction in this quasi-classical language. First, the decoupled version for literals:

Definition 2.3.3. *Given a bistructure $E = (D, I^+, I^-)$ and an assignment g, the* decoupled satisfaction, *$\models_d$, for literals in $L(D)$ is defined as follows:*

$$\begin{aligned} E, g \models_d P(t_1, t_2, \ldots, t_n) \quad &\text{iff} \quad I^+(P)(g(t_1), g(t_2), \ldots, g(t_n)) = 1 \\ E, g \models_d \neg P(t_1, t_2, \ldots, t_n) \quad &\text{iff} \quad I^-(P)(g(t_1), g(t_2), \ldots, g(t_n)) = 1 \end{aligned}$$

Note that both an atom and its negation may be true in a bistructure: this constitutes the justification for the term decoupled satisfaction and, also, provides the basis for paraconsistent reasoning. The strong satisfaction can now be defined.

Definition 2.3.4. *A satisfiability relation* $\models_s$*, called* strong satisfaction*, is defined as follows:*

1. $E, g \models_s \alpha$ *iff* $E, g \models_d \alpha$*, for* α *a literal;*

2. $E, g \models_s \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n$ *iff* $[E, g \models_s \alpha_1$ *or* $E, g \models_s \alpha_2$ *or* $\cdots$ *or* $E, g \models_s \alpha_n]$ *and for all* $1 \leq i \leq n, [E, g \models_s \sim \alpha_i$ *implies* $E, g \models_s \otimes(\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n, \alpha_i)]$*, for* $\alpha_1, \ldots, \alpha_n$ *literals;*

3. $E, g \models_s \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n$ *iff* $E, g \models_s \phi_1$ *and* $E, g \models_s \phi_2$ *and* $\cdots$ *and* $E, g \models_s \phi_n$ *for* $\phi_1, \ldots, \phi_n$ *disjunctions of literals;*

4. $E, g \models_s \exists x.\theta$ *iff for some* $x-$*variant assignment* g' *of* g*,* $E, g' \models_s \theta$ *for* θ *any formula;*

5. $E, g \models_s \forall x.\theta$ *iff for all* $x-$*variant assignment* g' *of* g*,* $E, g' \models_s \theta$ *for* θ *any formula;*

Strong validity in a bistructure is defined by:

$$E \models_s \theta \text{ iff for all assignment } g,\ E, g \models_s \theta$$

For Δ a finite set of formulas and E a bistructure; E is a QC *model of* Δ iff for all $\theta \in \Delta, E \models_s \theta$.

For the rest of the discussion on this theme, it is necessary to present the concepts of *ground atom* and *ground literal*:

For a language $L = \langle \mathcal{P}, \mathcal{C} \rangle$ and a domain D,

- $GrdAt(L, D) = \{P(t_1, \ldots, t_n) \mid P(n) \in \mathcal{P},\ t_1, \ldots, t_n \in D\}$;
- $GrdLt(L, D) = GrdAt(L, D) \cup \{\neg\alpha \mid \alpha \in GrdAt(L, D)\}$;

A QC model $E = (D, I^+, I^-)$ may be equivalently represented by a set of ground literals as follows:

$$\{\alpha \mid E \models_s \alpha \text{ and } \alpha \in GrdLt(L, D)\}.$$

It is possible to adopt this representation of models because earlier it was assumed that for any language $L = \langle \mathcal{P}, \mathcal{C} \rangle$ and domain D, $\mathcal{C} \subseteq D$. Furthermore, it is essential to recall the presence of *Robinson diagrams* in classical logic.

It is assumed that all languages $L = \langle \mathcal{P}, \mathcal{C} \rangle$ are finite (*i.e.* $\mathcal{P}$ is finite and $\mathcal{C}$ is finite) and that all domains D are finite. This means that all QC models considered here are finite.

The following notation will be used:

- QC(L, Δ, D) is the class of QC models of Δ, where the formulas are in L and the domain of the QC models is D;

- $\mathcal{M}$ denotes a QC model in the form of ground literals.

A QC model for a set Δ may have more information than it is needed to satisfy all the elements in Δ. This is the reason why it is introduced the following definition for minimal QC model.

Definition 2.3.5. *Let L be a language and Δ a set of formulas in L. The set of* minimal QC models *with domain D is* $\mathrm{MQC}(L, \Delta, D) := \{\mathcal{M} \in \mathrm{QC}(L, \Delta, D) \mid \mathcal{M}' \subset \mathcal{M}$ *implies* $\mathcal{M}' \notin \mathrm{QC}(L, \Delta, D)\}$*;*

In some sense, these are models without irrelevant, useless information and restricting attention to them does not affect reasoning.

It follows an example illustrating the concepts introduced. Namely, it exemplifies the role of the focus of a disjunction of literals when determining models.

Example 1. *Let $L = \langle\{P(1), Q(1)\}, \{a\}\rangle$ and $\Delta = \{P(a) \vee Q(a), \neg Q(a)\}$.*

In this case there is only one minimal QC *model for any domain, which is: $\mathcal{M} = \{\neg Q(a), P(a)\}$. That is because in a* QC *model where $\neg Q(a)$ holds, $\otimes(P(a) \vee Q(a), \neg Q(a))$ which is $P(a)$ must also be true.*

A QC *model (not minimal) would be, for example, $\mathcal{M}' = \{\neg Q(a), P(a), Q(a)\}$.*

The *measure for the inconsistency* of a model is crucial in a diverse range of applications in artificial intelligence to compare between knowledgebases. It may be a useful tool in analysing various information types, such as news reports, software specifications, integrity constraints and e-commerce protocols, [63]. Depending on the language L used and the domain D, a knowledgebase may have many different minimal QC models. The measure will be a ratio between 0 and 1, with denominator the total possible number of inconsistencies in the bistructure;

Given a QC model $\mathcal{M}$, its *conflictbase* is defined by

$$Cnfl(\mathcal{M}) := \{\alpha \mid \alpha \in \mathcal{M} \text{ and } \neg\alpha \in \mathcal{M}\};$$

The *measure of inconsistency* for a model $\mathcal{M}$ in the context of L and D (*i.e.*, $Cnfl(\mathcal{M}) \subseteq GrdAt(L, D)$) is defined by

$$ModInc(\mathcal{M}, L, D) := \frac{|Cnfl(\mathcal{M})|}{|GrdAt(L, D)|}$$

To exemplify, here are given some examples:

Example 2. *Resuming to Example 1, $Cnfl(\mathcal{M}) = \emptyset$. Considering $D = \{a\}$, the measure of inconsistency is:*

$$ModInc(\mathcal{M}, L, D) = \frac{0}{2} = 0$$

For the model $\mathcal{M}'$, taking the same domain, the measure of inconsistency is:

$$ModInc(\mathcal{M}, L, D) = \frac{1}{2}$$

In the next example, it will be notorious that two minimal models do not always have the same measure of inconsistency:

Example 3. *Let $L = \langle\{P(1), Q(1)\}, \{a\}\rangle$, $\Delta = \{\exists x.P(x), \neg P(a) \vee Q(a), \neg Q(a)\}$ and $D = \{a, b\}$.*

So, $\mathrm{MQC}(L, \Delta, D) = \{\mathcal{M}_1, \mathcal{M}_2\}$*, where:*

$$\mathcal{M}_1 = \{P(a), \neg P(a), Q(a), \neg Q(a)\}$$

$$\mathcal{M}_2 = \{P(b), \neg P(a), \neg Q(a)\}$$

Hence, $ModInc(\mathcal{M}_1, L, D) = \frac{2}{4} > \frac{0}{4} = ModInc(\mathcal{M}_2, L, D)$

Since not all minimal QC models have the same conflictbase, thus the same measure of inconsistency, it will be considered a new class of models, the preferred QC models, constituted by minimal models with minimal conflictbase. In essence, there will be considered the least inconsistent models. Formally,

Definition 2.3.6. *Let L be a language and Δ a set of formulas in L. The set of* preferred QC models *with domain D is* $\mathrm{PQC}(L, \Delta, D) := \{\mathcal{M} \in \mathrm{MQC}(L, \Delta, D) \mid$ *for all* $\mathcal{M}' \in \mathrm{MQC}(L, \Delta, D),\ |Cnfl(\mathcal{M})| \leq |Cnfl(\mathcal{M}')|\}$;

One example of a preferred QC model, based on the previous one:

Example 4. *From Example 3,* $\mathrm{PQC}(L, \Delta, D) = \{\mathcal{M}_2\}$.

It seems quite straightforward to say that two preferred QC models for the same set Δ, with the same language and domain have the same measure of inconsistency. Furthermore, this is valid also when there are considered two different domains, but with the same size.

Now it is possible to define, for a theory in a language, a sequence of inconsistency ratios:

Definition 2.3.7. *We define the* extrinsic inconsistency *of a theory Δ in a language L, $TheoryInc(\Delta, L)$ as a sequence $\langle r_1, \ldots, r_n\rangle$ where for all $n \geq 1$, let W_n be a domain of size n. If there is a model $\mathcal{M} \in \mathrm{PQC}(L, \Delta, W_n)$, then let $r_n = ModInc(\mathcal{M}, L, W_n)$, otherwise, let $r_n = *$. We use $*$ as a kind of a null value.*

This sequence captures how the inconsistency of a theory Δ in a language L evolves with increasing domain size. At one extreme, there are cases where we do not have inconsistencies for any domain size; for example for the trivial case when $\Delta = \emptyset$, $TheoryInc(\Delta, L) = \langle 0, 0, \dots \rangle$. At the other extreme, there are theories Δ which are completely inconsistent, *i.e.*, $TheoryInc(\Delta, L) = \langle 1, 1, \dots \rangle$, such as $\Delta = \{\forall x(P(x) \wedge \neg P(x))\}$, in a language where $\mathcal{P} = \{P(1)\}$.

Some examples on the computation of the extrinsic inconsistency of some theories:

Example 5. *Let* $L = \langle \{P(1)\}, \{a, b\} \rangle$ *and* $\Delta = \{\neg P(a) \vee P(b), P(a)\}$.

$$TheoryInc(\Delta, L) = \langle *, 0, 0, 0, \dots \rangle$$

Example 6. *Let* $L = \langle \{P(1), Q(1)\}, \{a, b\} \rangle$ *and* $\Delta = \{\neg P(a), \neg Q(b), \forall x.P(x) \wedge \neg P(x) \wedge Q(x)\}$.

$$TheoryInc(\Delta, L) = \langle *, \frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \dots \rangle$$

Some properties of $TheoryInc$ are now announced and proved:

Proposition 2.3.8. *Let* $TheoryInc(\Delta, L) = \langle x_1, x_2, \dots \rangle$. *If* $|\mathcal{C}| = k$ *then for all* i *such that* $1 \leq i < k, x_i = *$ *and for all* $i \geq k, x_i \neq *$.

Proof. The reason for the asterisks is that the domain, according to a previous definition, must have at least as many elements as the number of constants in L, that is, $|\mathcal{C}|$. And it is always possible to increase the size of the domain.

□

Another interesting proposition is the following:

Proposition 2.3.9. *Let* $TheoryInc(\Delta, L) = \langle r_1, r_2, \dots \rangle$. *If there is an* $r_i \in \{r_1, r_2, \dots\}$, *such that* $r_i = 0$, *then* $\langle r_1, r_2, \dots \rangle$ *is of the form* $\langle *, \dots, *, 0, \dots, 0, \dots \rangle$.

Adopting a *lexicographic ordering*, denoted by the relation $\preceq$ over the tuples generated by the $TheoryInc$ function, comes as:

Definition 2.3.10. *Let* $TheoryInc(\Delta_1, L_1) = \langle r_1, r_2, \dots \rangle$ *and* $TheoryInc(\Delta_2, L_2) = \langle s_1, s_2, \dots \rangle$.
$TheoryInc(\Delta_1, L_1) \preceq TheoryInc(\Delta_2, L_2)$ *iff for all* $i \geq 1$, $r_i \leq s_i$ *or* $r_i = *$ *or* $s_i = *$.

Writting $TheoryInc(\Delta_1, L_1) \prec TheoryInc(\Delta_2, L_2)$ abbreviates $TheoryInc(\Delta_1, L_1) \preceq TheoryInc(\Delta_2, L_2)$ and $TheoryInc(\Delta_1, L_1) \neq TheoryInc(\Delta_2, L_2)$.

In case $L_1 = L_2 (= L)$, it is said that Δ_1 has smaller than or equal inconsistency as Δ_2 iff $TheoryInc(\Delta_1, L) \preceq TheoryInc(\Delta_2, L)$ and this is denoted as $\Delta_1 \leq^L_{inc} \Delta_2$.

To accompany this idea, an example:

Example 7. *Let* $L = \langle\{P(1)\}, \{a\}\rangle, \Delta_1 = \{P(a)\}$ *and* $\Delta_2 = \{P(a), \neg P(a)\}$*.*

$$TheoryInc(\Delta_1, L) = \langle 0, 0, 0, \dots\rangle$$

$$TheoryInc(\Delta_2, L) = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\rangle$$

Then $\Delta_1 \leq^L_{inc} \Delta_2$*.*

The $TheoryInc$ function is monotonic with respect to Δ and antimonotonic with respect to L, *i.e.*, if $\Delta_1 \subseteq \Delta_2$, then $TheoryInc(\Delta_1, L) \preceq TheoryInc(\Delta_2, L)$, and if $L_1 \subseteq L_2$ ($\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{C}_1 \subseteq \mathcal{C}_2$) then $TheoryInc(\Delta, L_2) \preceq TheoryInc(\Delta, L_1)$.

The definition of QC-equivalent sets is presented.

Definition 2.3.11. *For* $\Delta_1, \Delta_2 \subseteq \mathrm{Form}(L), \Delta_1$ *is* QC-equivalent *to* Δ_2 *if, for all* $\mathcal{M}$, $\mathcal{M}$ *is a* QC *model of* Δ_1 *iff* $\mathcal{M}$ *is a* QC *model of* Δ_2*.*

Thus, the following statement is true:

Proposition 2.3.12. *Let* Δ_1, Δ_2 *be sets of formulas in the language* L*. If* Δ_1 *is* QC*-equivalent to* Δ_2*, then*

$$TheoryInc(\Delta_1, L) = TheoryInc(\Delta_2, L).$$

Resembling the definition of extrinsic inconsistency, it is introduced the definition of *intrinsic inconsistency* of a set of formulas Δ, for a specific language L defined with recourse to Δ, as follows:

Definition 2.3.13. *For a given theory* Δ*, let* L^Δ *be the language that contains exactly the predicate symbols and constants that occur in* Δ*. The* intrinsic inconsistency of Δ*, is given by* $TheoryInc(\Delta)$*, as* $TheoryInc(\Delta) = TheoryInc(\Delta, L^\Delta)$*.*

So the measure of intrinsic inconsistency of a theory $TheoryInc(\Delta)$, delineates the degree of the theory in its own terms, whereas $TheoryInc(\Delta, L)$, the extrinsic inconsistency of a theory, delineates the degree of the theory with respect to the language L.

This idea is illustrated with another example:

Example 8. *Let* $L = \langle\{P(1), Q(1), R(1)\}, \{a, b\}\rangle$*,* $\Delta = \{P(a), \neg P(b), R(a), \neg R(a)\}$*.*

$$TheoryInc(\Delta, L) = \langle *, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \dots\rangle$$

$$TheoryInc(\Delta) = \langle *, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\rangle$$

When $TheoryInc(\Delta_1) \preceq TheoryInc(\Delta_2)$ it is said that Δ_1 has smaller than or equal inconsistency as Δ_2, and this is denoted as $\Delta_1 \leq_{inc} \Delta_2$.

Remark. Posterior to this work, the duo Grant and Hunter wrote [59]. This new approach considers full first-order logic, it no longer uses the focus, and it introduces classes of models and it also introduces different measures of inconsistency: one for finite sets of finite models, another for sets that include a model of every domain size and finally another for sets that include infinite models.

2.4 Applications

When in the middle of a situation where inconsistencies are not merely due to human errors or confusion, and are not easily removed even upon careful reflection, the answer is paraconsistent logic, because, obviously, we want inconsistency without triviality. In this section there are introduced some fields where paraconsistency is desirable and has already been successfully used.

2.4.1 Linguistics

Studies have been made in this field in connection with paraconsistency. The idea is that lexical features are preserved even in inconsistent contexts. For example the word 'near' has spacial connotations that remain undisturbed even when dealing with impossible spaces, as in *"I am near a blue door that is red"*. Clearly, this is an inconsistency as the door can be only one of two things: blue or red.

Thus, if natural languages would have associated a logic, some sort of paraconsistency ought to be taken into consideration.

2.4.2 Law, Science and Revision

Consider a country with the following laws (in [82]):

1. No non-Caucasian people shall have the right to vote;

2. All landowners shall have the right to vote.

Which are the rights of an individual that is not Caucasian and owns a small farm? The laws, as they stand, are inconsistent. So, either new laws are made, ending inconsistencies, or the current ones must be revised. In either case, though, the law as it stands needs to be dealt with in a discriminating way.

Even in science we hold some laws as true. As new discoveries are made everyday, the scientific process accepts that laws can be, in the light of new evidence, revised, updated, or even rejected completely. At present, it seems extremely likely that different branches of science are inconsistent with one another or even within the same discipline, as is the case in theoretical physics with relativity and quantum mechanics [86]. Even set theory and foundations of mathematics can be considered paraconsistent, (cf. [84], [67]).

In [61] it is explained that Frege was able to explain most mathematical notions with the help of his *comprehension scheme*, which asserts that, for every φ (formula or statement), there should exist a set X such that, for all x, $x \in X$ if and only if $\varphi(x)$ is true (x does not occur free in φ); formally, $\exists X \forall x(x \in X \leftrightarrow \varphi(x))$. Moreover, by the *axiom of extensionality*, this set X is uniquely determined by $\varphi(x)$, formally, $\forall x(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$. A flaw in Frege's system was uncovered by Russell: by taking $\varphi(x)$ to be $\neg(x \in X)$, Frege's system is logically contradictory. This became known as the *Russel's Paradox*: *"Consider M the set of all sets that do not contain themselves as members. Does M belong to itself?"* Formally, A is an element of M if and only if A does not belong to itself: $M = \{A \mid A \notin A\}$. If the answer is yes, than M is not an element of itself according to the definition, which is a contradiction. On the other hand, if the answer is no, then M does not contain itself, so, by the definition of M, M must be an element of M, again another contradiction. Therefore, stating that "M is an element of M" and stating that "M is not an element of M" lead both to inconsistencies. Thus, set theory is considered paraconsistent.

Moreover, it must be said that people have inconsistent beliefs as well.

2.4.3 Automated Reasoning

Another application is in the field that intersects automated reasoning and information processing. As a practical example, take a computer which stores large amounts of information and is capable of performing operations and inferences over such data. Stored information may contain inconsistencies, certainly a problem for database operations with theorem-provers, and so has drawn much attention from computer scientists. For this reason, techniques for removing inconsistent information have been investigated. Yet all have limited applicability, and, in any case, are not guaranteed to produce full consistency. Hence, even if steps are taken to get rid of contradictions when they are found, an underlying paraconsistent logic is desirable so that hidden contradictions cannot invalidate conclusions taken from the theory.

Another example involves a team of researchers in São Paulo, Brazil, which has been using paraconsistent logics in artificial intelligence. One of their results, the autonomous robot EMMY III is designed to be able to navigate through dilemmatic situations, for example, when one sensor detects an obstacle in front of the robot, while the other detects the presence of no objects [89].

2.4.4 Paraconsistent Artificial Neural Networks – PANNets

Paraconsistent Artificial Neural Networks – PANNets – have been highly explored in the last years, (cf. [52]). Among their applications, the use of PANNets for image recognition in medical, biological and odontological diagnoses and for intelligent systems for sound and image recognition and modeling of the brain should be highlighted.

In medical science, a paraconsistent artificial neural network has showed some potential for detecting Alzheimer's disease, as reported in [69].

It was also made a study of brain electroencephalography (EEG) waves through a new paraconsistent artificial neural network which is capable of manipulating concepts like impreciseness, inconsistency, and paracompleteness in a nontrivial manner. In what concerns to applications, the paraconsistent artificial neural network showed the capacity of recognizing children with Dyslexia [1].

PANNets were even used to analyse cephalometric measurements in order to support orthodontics diagnoses. They use the information provided by cephalometric analysis to perform pattern recognition then, from determined parameters, it is established the best treatment for the patient, narrated in [2], and [75].

Chapter 3

Hybrid Logics

Modal Logics, [24], provide a simple formalism for working with relational structures (or multigraphs). However, they lack in mechanisms for naming worlds (or states), asserting equalities and describing accessibility relations between them. *Hybrid logics*, [23], appear as an extension of propositional modal logic with the ability to refer to worlds by considering a new class of atomic formulas, called *nominals*, and using a new operator, @, called *satisfaction operator*. A nominal's interpretation is required to be a singleton, *i.e.*, a nominal is true at exactly one state: the one it names.

Hybrid logics turn out to be the key to improve some already existing logics in modal form, for example feature logic, description logic and temporal logic, [23]. Furthermore, hybrid logics have been an opulent source of inspiration for many researchers in many areas (see examples in [68], [93]).

With hybrid logics we may express what happens at a specific world: for instance $@_i p$ means that the proposition p is true at the world named by i, whereas $\neg @_i p$ (logically equivalent to $@_i \neg p$) denies it. We may also express equality between worlds: $@_i j$ means that the worlds named by i and j coincide, and $\neg @_i j$ (logically equivalent to $@_i \neg j$) says the contrary. It is even possible to express accessibility between worlds: $@_i \Diamond j$ means that j can be reached from i, and $\neg @_i \Diamond j$ (logically equivalent to $@_i \Box \neg j$) says that there is no such connection.

Moreover, hybrid logics are strictly more expressive than its modal fragment. For example, irreflexivity ($i \to \neg \Diamond i$), asymmetry ($i \to \neg \Diamond \Diamond i$) or antisymmetry ($i \to \Box(\Diamond i \to i)$) are properties of the underlying transition structure which are simply no definable in standard modal logic, but straightforward to state in the hybrid family.

We shall examine two different kinds of hybrid languages: we start by introducing, as in [27], the *basic hybrid logic*, and finish with a version with quantification over worlds, the so called *strong Priorean logic*.

Although being stronger than proposition modal logic, the basic hybrid logic does not increase the complexity of the problem of determining whether a formula is valid or not.

Actually, it remains a decidable system. However, in the strong Priorean logic the complexity increases.

But first, some historical details on hybrid logics' introduction and development.

3.1 History of Hybrid Logics

Hybrid logics were introduced by Arthur Prior in the 50's [31]. Prior focused in the framework of temporal logic and, in 1954, at the New Zealand Congress of Philosophy, introduced the I-calculus (which later he calls U-calculus).

In the I-calculus, propositions of the tense calculus are treated as predicates expressing properties of dates, represented by variables. He established that the formula px should be read as "p at x" and considered a binary relation I over dates, where xIy should be read as "y is later than x".

By representing the time of utterance by means of an arbitrary date x, Fp (intuitively interpreted as "the proposition p happens in the future", but that, later Prior explains, means "it is now the case that it will be the case that p happens") is equated with $\exists y(xIy \wedge py)$. And similarly for the past, Pp, equated with $\exists y(yIx \wedge py)$.

The initial idea of Prior was to reconstruct the tense calculus using the I-calculus, but soon he found out that the second was more expressive than the first, and started investigating ways to extend the expressive power of the tense calculus, which led to what we call today very expressive hybrid languages, *i.e.*, hybrid languages including binders ($\forall, \downarrow$).

After Prior's death in 1969, his student Robert Bull continued his work on hybridization. Bull considered a logic containing variables for paths on a model, which he calls "history-propositional" variables and provided it with an axiomatization and proved its completeness [34].

Some years later, in the 80's, the Bulgarian school of logic, also known as Sofia school, (namely Passy, Tinchev, Gargov and Goranko) revived the interest in hybrid logic. They explored the fact that the union of two accessibility relations is definable in the basic modal language in the sense that the formula $\langle T\rangle p \leftrightarrow \langle R\rangle p \vee \langle S\rangle p$ is valid on a frame precisely if R_T, the relation that interprets $\langle T\rangle$, is the union of R_R and R_S, respectively the relation that interprets $\langle R\rangle$ and the relation that interprets $\langle S\rangle$. Yet, and it came as a surprise, the intersection of two accessibility relations does not work in the same way [55]. Later, Gargov, Passy and Tinchev showed in [53] that the intersection can be defined using nominals by stating that $\langle T\rangle i \leftrightarrow \langle R\rangle i \wedge \langle S\rangle i$, where R_T is the relation that interprets $\langle T\rangle$, and is the intersection of R_R and R_S, respectively the relation that interprets $\langle R\rangle$ and the relation that interprets $\langle S\rangle$. The same occurs for complementation: although there is no formula of the basic modal logic that is valid on a frame where the accessibility relation that interprets $\langle R\rangle$

is the complement of the accessibility relation that interprets $\langle S \rangle$, there is a such formula when nominals are added to the language and it is defined that $\langle R \rangle i \leftrightarrow \neg \langle S \rangle i$.

Among the large number of interesting results that Passy and Tinchev proved, there is one which was crucial to the development of Chapter 4. They observed in [80] that named models, *i.e.*, models in which each world is named by a nominal, can be completely described by a set of formulas of the form $(\neg)@_i p$, $(\neg)@_i j$, $(\neg)@_i \Diamond j$. Clearly such property holds in the basic hybrid logic ($\mathcal{H}(@)$) as it depends solely on the nominals and satisfaction operator machinery.

The Sofia tradition in hybrid logic continues with the work of Goranko. Goranko, together with Gargov, investigated the basic modal language extended with nominals and the universal and existential modalities, $\mathcal{H}(E)$. Goranko also investigated the binder ($\downarrow$) in the context of hybrid logic, see [56].

Meanwhile, Patrick Blackburn, Maarten de Rijke and Yde Venema wrote [24] a great manual about modal logics, and Blackburn also wrote [23], a very good introduction to hybrid logics. More recently, Blackburn, Marx and Areces have dedicated their research to prove results on interpolation and complexity of hybrid logics [8, 6, 7]. Another contributor to the development of hybrid logics is Balder ten Cate, who wrote his thesis on model theory for extended modal languages [88], and contributed a lot in the domain of bisimulation. In [27], Blackburn and ten Cate approach several extensions of basic hybrid logic, and provide them with Hilbert axiomatizations. Also, Ian Hodkinson's work on axiomatizing hybrid logics using modal logics is quite interesting [62]. And finally, Torben Braüner worked on defining first-order hybrid logic [32], intuitionistic hybrid logic [33] and many-valued hybrid logic [60]. Works on hierarchical hybrid logic [71] and hybridization [76, 70, 79] have been especially addressed lately.

3.2 Basic Hybrid Logic

To start, it is presented the simplest form of hybrid logic: the *basic hybrid language*, $\mathcal{H}(@)$. The basic hybrid language introduces nominals and the satisfaction operator into the propositional modal logic. However, it is interesting to see how such simple extension (with only nominals and the satisfaction operator) carries such great power in terms of expressivity.

Next it is defined the syntactic structure of hybrid propositional logic:

Definition 3.2.1. *Let* $L = \langle \text{Prop}, \text{Nom} \rangle$ *be a* hybrid similarity type*, where* Prop *is a set of* propositional symbols *and* Nom *is a set disjoint from* Prop*. We use* $p, q, r, etc.$ *to refer to the elements in* Prop*. The elements in* Nom *are called* nominals *and we typically write them as* $i, j, k, etc.$*. The well-formed formulas over* L*,* $\text{Form}_@(L)$*, are defined by the following grammar:*

$$WFF := \; i \mid p \mid \bot \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi \mid @_i\varphi$$

For any nominal i, any formula φ, $@_i\varphi$ is called a *satisfaction statement.*

A hybrid structure is defined as a Kripke frame with some slight differences.

Definition 3.2.2. *Let* $L = \langle \text{Prop}, \text{Nom} \rangle$ *be a hybrid similarity type. A* hybrid structure $\mathcal{H}$ *over* L *is a tuple* (W, R, N, V)*. Here,* W *is a non-empty set called* domain *whose elements are called* states *or* worlds*, and* R *is a binary relation such that* $R \subseteq W \times W$ *and is called the* accessibility relation*.* $N : \text{Nom} \to W$ *is a function called* hybrid nomination *that assigns nominals to elements in* W *such that for any nominal* i, $N(i)$ *is the element of* W *named by* i*. We call this element the denotation of* i *under* N*.* V *is a* hybrid valuation*, which means that* V *is a function with domain* Prop *and range* $Pow(W)$ *such that* $V(p)$ *tells us at which states (if any) each propositional symbol is true.*

The pair (W, R) *is called the* frame *underlying* $\mathcal{H}$ *and* $\mathcal{H}$ *is said to be a* structure *based on this frame.*

The satisfaction relation comes as a generalization of Kripke's satisfaction for modal logic, in the following sense:

Definition 3.2.3 (Satisfaction)**.** *The* local satisfaction *relation* $\models$ *between a hybrid structure* $\mathcal{H} = (W, R, N, V)$*, a state* $w \in W$ *and a hybrid formula is recursively defined by:*

1. $\mathcal{H}, w \models i$ *iff* $w = N(i)$*;*
2. $\mathcal{H}, w \models p$ *iff* $w \in V(p)$*;*
3. $\mathcal{H}, w \models \bot$ *never;*
4. $\mathcal{H}, w \models \top$ *always;*
5. $\mathcal{H}, w \models \neg\varphi$ *iff not* $\mathcal{H}, w \models \varphi$*;*
6. $\mathcal{H}, w \models \varphi \wedge \psi$ *iff* $\mathcal{H}, w \models \varphi$ *and* $\mathcal{H}, w \models \psi$*;*
7. $\mathcal{H}, w \models \varphi \vee \psi$ *iff* $\mathcal{H}, w \models \varphi$ *or* $\mathcal{H}, w \models \psi$*;*
8. $\mathcal{H}, w \models \Diamond\varphi$ *iff* $\exists w' \in W(wRw'$ *and* $\mathcal{H}, w' \models \varphi)$*;*
9. $\mathcal{H}, w \models \Box\varphi$ *iff* $\forall w' \in W(wRw' \Rightarrow \mathcal{H}, w' \models \varphi)$*;*
10. $\mathcal{H}, w \models @_i\varphi$ *iff* $\mathcal{H}, w' \models \varphi$*, where* $w' = N(i)$*;*

If $\mathcal{H}, w \models \varphi$ *it is said that* φ is satisfied in $\mathcal{H}$ at w*. If* φ *is satisfied at all states in a structure* $\mathcal{H}$*, it is written* $\mathcal{H} \models \varphi$*. If* φ *is satisfied at all states in all structures based on a frame* $\mathcal{F}$*, then it is said that* φ is valid on $\mathcal{F}$ *and it is written* $\mathcal{F} \models \varphi$*. If* φ *is valid on all frames, then it is simply said that* φ is valid *and it is written* $\models \varphi$*.*

For $\Delta \subseteq \text{Form}_@(L)$*, it is said that* $\mathcal{H}$ *is a* model of Δ *iff for all* $\theta \in \Delta$, $\mathcal{H} \models \theta$*.*

The idea of two hybrid formulas being logically equivalent is expressed in the next definition.

Definition 3.2.4. *A formula* $\varphi^* \in \text{Form}_{@}(L)$ *is said to be* (logically) equivalent to $\varphi \in \text{Form}_{@}(L)$ *iff for all hybrid structure* $\mathcal{H} = (W, R, N, V)$ *and all* $w \in W$*,*

$$\mathcal{H}, w \models \varphi \textit{ iff } \mathcal{H}, w \models \varphi^*.$$

Given this definition, note that boolean connectives have the usual properties, and that $\Box\varphi$ is (logically) equivalent to $\neg\Diamond\neg\varphi$.

Next lemma states some properties about the satisfaction operator that will be important in the sequel.

Lemma 3.2.5. *Let* φ, ψ *be hybrid formulas. Then,*

1. $@_i(\varphi \vee \psi)$ *is equivalent to* $@_i\varphi \vee @_i\psi$*;*

2. $@_i(\varphi \wedge \psi)$ *is equivalent to* $@_i\varphi \wedge @_i\psi$*;*

3. $@_i@_j\varphi$ *is equivalent to* $@_j\varphi$*;*

4. $\neg @_i\varphi$ *is equivalent to* $@_i\neg\varphi$*;*

5. $@_i(\varphi_1 \wedge \psi_1) \vee @_i(\varphi_2 \wedge \psi_2)$ *is equivalent to* $(@_i\varphi_1 \vee @_i\varphi_2) \wedge (@_i\varphi_1 \vee @_i\psi_2) \wedge (@_i\psi_1 \vee @_i\varphi_2) \wedge (@_i\psi_1 \vee @_i\psi_2)$*.*

The problem of determining the satisfiability of a formula is decidable, in fact, basic hybrid logic is no more complex than basic modal logic.

Theorem 3.2.6. *The satisfiability problem for the basic hybrid logic is* PSPACE*-complete.*

Proof. See [6]. □

A homomorphism between hybrid structures is defined by considering them as first-order structures. Concretely,

Definition 3.2.7. *Let* $L = \langle \text{Prop}, \text{Nom} \rangle$ *be a hybrid similarity type,* $\mathcal{H} = (W, R, N, V)$ *and* $\mathcal{H}' = (W', R', N', V')$ *be two hybrid structures over* L*. A* homomorphism h from $\mathcal{H}$ to $\mathcal{H}'$ *is a map* $h : W \to W'$ *such that*

1. *for any* $p \in \text{Prop}$ *and any* $w \in W$*,* $w \in V(p)$ *iff* $h(w) \in V'(p)$*;*

2. *for any* $i \in \text{Nom}$*,* $h(N(i)) = N'(i)$*;*

3. *for any* $w, s \in W$*,* wRs *implies that* $h(w)R'h(s)$*.*

It is said that h is an embedding *if it is injective and the condition (3) holds in the strong version:*

$$\text{for any } w, s \in W,\ wRs \text{ iff } h(w)R'h(s).$$

3.2.1 Frame Definability

The definition of frame definability is given as follows:

Definition 3.2.8. *A formula φ* defines *a class K of frames of some type if for all of those frames $\mathcal{F}$, one has that $\mathcal{F} \in K \Leftrightarrow \mathcal{F} \models \varphi$.*

Hybrid languages have gained popularity because, as said earlier, many properties of frames that are not modally definable can be defined using nominals, for example irreflexivity ($i \to \neg\Diamond i$) and antisymmetry ($i \to \Box(\Diamond i \to i)$). These formulas do not contain any propositional symbols, so they are called *pure*. Thus, hybrid logics are stronger than modal ones, in the sense that they allow the definition of more properties.

Definition 3.2.9. *A frame $\mathcal{F}$ has the property stated if and only if it validates the hybrid formula listed alongside. We start with the ones also definable in basic modal logic:*

$$\begin{array}{rll}
\forall w\ wRw & \textit{reflexivity} & @_i\Diamond i \\
\forall w,u\ (wRu \to uRw) & \textit{symmetry} & @_i\Box\Diamond i \\
\forall w,u,v\ (wRu \wedge uRv \to wRv) & \textit{transitivity} & @_i\Diamond j \wedge @_j\Diamond k \to @_i\Diamond k \\
\forall w,u\ (wRu \to \exists v(wRv \wedge vRu)) & \textit{density} & \Diamond i \to \Diamond\Diamond i
\end{array}$$

Now there are given some examples of properties that are not definable in basic modal logic but that are easily stated in basic hybrid language:

$$\begin{array}{rll}
\forall w\ \neg wRw & \textit{irreflexivity} & @_i\neg\Diamond i \\
\forall w,u\ (wRu \to \neg uRw) & \textit{asymmetry} & @_i\neg\Diamond\Diamond i \\
\forall w,u\ (wRu \wedge uRw \to u = w) & \textit{antisymmetry} & i \to \Box(\Diamond i \to i) \\
\forall w,u\ (wRu \vee w = u \vee uRw) & \textit{trichotomy} & @_j\Diamond i \vee @_j i \vee @_i\Diamond j
\end{array}$$

However, there are many classes of frames which are not definable using only pure $\mathcal{H}(@)$-axioms. The class of frames in which every world has a predecessor is an example. This class of frames is definable in $\mathcal{H}(@, \forall)$, which will be introduced in Section 3.3, using the formula $\forall s \exists t @_t \Diamond s$.

The union of frame classes can be defined in hybrid logics as follows.

Proposition 3.2.10 ([88])**.** *Let L be a hybrid similarity type. For all formulas $\varphi, \psi \in$* $\mathrm{Form}_{@}(L)$ *that do not share any propositional symbols, and for all distinct nominals i, j*

not occurring in φ and ψ, $@_i\varphi \vee @_j\psi$ defines the union of the frame classes defined by φ and ψ.

An *axiomatization* for basic hybrid language, $\mathbf{K}_{\mathcal{H}(@)}$, is provided in the following table (cf. [27]):

$\mathbf{K}_{\mathcal{H}(@)}$	
Axioms:	
(CT)	$\vdash \phi$, for all classical tautologies ϕ
(Dual)	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$
($K_\Box$)	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
($K_@$)	$\vdash @_i(p \to q) \to (@_ip \to @i_q)$
(Selfdual)	$\vdash \neg @_ip \leftrightarrow @_i\neg p$
(Ref)	$\vdash @_ii$
(Intro)	$\vdash i \wedge p \to @_ip$
(Back)	$\vdash \Diamond @_ip \to @_ip$
(Agree)	$\vdash @_i@_jp \to @_jp$
Rules:	
(MP)	If $\vdash \phi \to \psi$ and $\vdash \phi$ then $\vdash \psi$
(Gen)	If $\vdash \phi$ then $\vdash \Box\phi$
($\text{Gen}_@$)	If $\vdash \phi$ then $\vdash @_i\phi$
(Subst)	If $\vdash \phi$ then $\vdash \phi^\sigma$, where σ is a substitution that uniformly replaces propositional symbols by arbitrary formulas and nominals by nominals.
($\text{Name}_@$)	If $\vdash @_i\phi$ and i does not occur in ϕ then $\vdash \phi$
(BG)	If $\vdash @_i\Diamond j \to @_j\phi$ and $j \neq i$ does not occur in ϕ then $\vdash @_i\Box\phi$

The axiomatization presented is sound and complete with respect to the class of all frames. Recall that an axiomatization is sound for a class of semantic structures if every derivable formula is semantically valid, and the converse of that, *i.e.* saying that every semantically valid formula is derivable, is the definition of a complete axiomatization.

Moreover, completeness is guaranteed for extensions with *pure* axioms. Note that a pure formula in the context of $\mathcal{H}(@)$ is a formula with no propositional symbols, *i.e.*, the only atomic symbols that a pure formula can contain are nominals.

For any set of pure $\mathcal{H}(@)$-formulas Λ, let $\mathbf{K}_{\mathcal{H}(@)} + \Lambda$ denote the above axiomatization extended with the axioms in Λ. Then,

Theorem 3.2.11 (Completeness, [27]). *Let Λ be any set of pure $\mathcal{H}(@)$-axioms. A set of $\mathcal{H}(@)$-formulas Σ is $\mathbf{K}_{\mathcal{H}(@)} + \Lambda$ consistent iff Σ is satisfiable in a model based on a frame satisfying the frame properties defined by Λ.*

For example, by considering $\Lambda = \{@_i\Diamond j \wedge @_j\Diamond k \to @_i\Diamond k\}$, one has that $\mathbf{K}_{\mathcal{H}(@)} + \Lambda$ is complete with respect to the class of transitive frames.

3.2.2 Hybrid Diagrams

In order to define the diagram of a hybrid structure, the concepts of hybrid atom and hybrid literal must be introduced.

Definition 3.2.12. *For a hybrid similarity type $L = \langle \text{Prop}, \text{Nom} \rangle$, we define*

1. *Hybrid atoms over L:*
   $\text{HAt}(L) = \{@_i p,\ @_i j,\ @_i\Diamond j \mid i, j \in \text{Nom}, p \in \text{Prop}\}$;
2. *Hybrid literals over L:*
   $\text{HLit}(L) = \{@_i p,\ @_i\neg p,\ @_i j,\ @_i\neg j, @_i\Diamond j,\ @_i\Box\neg j \mid i, j \in \text{Nom}, p \in \text{Prop}\}$;

An important feature of hybrid logic is the fact that we can specify Robinson diagrams [8]. As in first-order logic, in order to define the diagram of a hybrid structure, the hybrid similarity type L is expanded by adding new nominals for the elements of the domain W. The denotation $L(W)$ is used for this new hybrid similarity type; in other words, $L(W) = \langle \text{Prop}, \text{Nom} \cup W \rangle$.

Given a hybrid structure $\mathcal{H} = (W, R, N, V)$ over L, $\mathcal{H}(W)$ denotes the natural expansion of $\mathcal{H}$ to $L(W)$ by taking N the identity on the new symbols.

The diagram of a hybrid structure $\mathcal{H}$ over L is the set of literals over $L(W)$ that are valid in $\mathcal{H}(W)$. Formally,

Definition 3.2.13. *For a hybrid similarity type, $L = \langle \text{Prop}, \text{Nom} \rangle$, and a hybrid structure over L, $\mathcal{H} = (W, R, N, V)$, the* elementary diagram of $\mathcal{H}$, *diag*$(\mathcal{H})$, *is the set of hybrid literals over $L(W)$ that hold in $\mathcal{H}(W)$, i.e.,*

$$diag(\mathcal{H}) = \{\alpha \in \text{HLit}(L(W)) \mid \mathcal{H}(W) \models \alpha\}$$

Actually, $diag(\mathcal{H})$ behaves like the standard diagram for first-order logic.

Theorem 3.2.14. *Let $L = \langle \text{Prop}, \text{Nom} \rangle$ be a hybrid similarity type and $\mathcal{H} = (W, R, N, V)$, $\mathcal{H}' = (W', R', N', V')$ two hybrid structures over L. Then, there is an embedding from $\mathcal{H}$ to $\mathcal{H}'$ iff $\mathcal{H}'$ can be expanded to a model of $diag(\mathcal{H})$.*

Proof. Let h be an embedding from $\mathcal{H}$ to $\mathcal{H}'$. Define the expansion $\mathcal{H}'^h = (W', R', N'^h, V')$ of $\mathcal{H}'$ to $L(W)$ by extending N to $\mathrm{Nom} \cup W$ by $N'^h(w) = h(w)$ for $w \in W$. It is not hard to show that $\mathcal{H}'^h$ is a model of the diagram of $\mathcal{H}$.

Conversely, let $\overline{\mathcal{H}'} = (W', R', \bar{N}, V')$ be an expansion of $\mathcal{H}'$ to $L(W)$ which is a model of the diagram of $\mathcal{H}$. Define the map $h : W \to W'$ by $h(w) = \bar{N}(w)$. Clearly, h is injective. Moreover, it is not difficult to see that h is an homomorphism. Hence, it is an embedding. □

In Example 9, it is presented the diagram for the hybrid structure represented.

Example 9. *Let* $L = \langle \{p, q\}, \{\} \rangle$*, and* $W = \{u, v, w\}$

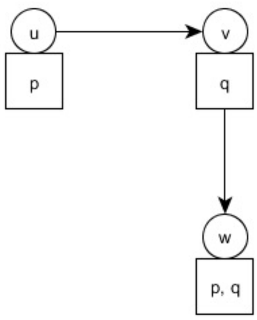

Figure 3.1: A hybrid structure.

The hybrid structure $\mathcal{H}$ *described in Figure 3.1 has the following diagram:*

$$diag(\mathcal{H}) = \{@_u p, @_u \neg q, @_v \neg p, @_v q, @_w p, @_w q$$

$$@_u \neg v, @_u \neg w, @_v \neg u, @_v \neg w, @_w \neg u, @_w \neg v$$

$$@_u \Diamond v, @_u \Box \neg u, @_u \Box \neg w, @_v \Diamond w, @_v \Box \neg u$$

$$@_v \Box \neg w, @_w \Box \neg u, @_w \Box \neg v, @_w \Box \neg w\}$$

Once again, let us recall that given L and W, the diagram of Figure 3.1 is unique.

3.2.3 Bisimulation and Standard Translation

In this section the notion of bisimulation between hybrid structures, sometimes called *bisimulation with constants*, is addressed.

Also, the notion that $\mathcal{H}(@)$ is the bisimulation invariant fragment of a first-order language with constants is explained ([88]).

In [7], bisimulations are seen as binary relations that connect worlds in which the atomic information is the same, and where the accessibility relations match.

In this section it will also be introduced the *standard translation*, which transforms hybrid formulas into first-order formulas. This translation preserves truth, and has a deep connection with bisimulations, as it will be shown.

Definition 3.2.15 (Bisimulation)**.** *Let* $L = \langle \mathrm{Prop}, \mathrm{Nom} \rangle$ *be a hybrid similarity type. Let* $\mathcal{H} = (W, R, N, V)$ *and* $\mathcal{H}' = (W', R', N', V')$ *be two hybrid structures. A* $\mathcal{H}(@)$- bisimulation *between* $\mathcal{H}$ *and* $\mathcal{H}'$ *is a non-empty relation* $Z \subseteq W \times W'$ *such that:*

- *All points named by nominals are related by* Z*, i.e., for each* $i \in \mathrm{Nom}$*,* $N(i) Z N'(i)$*;*
- *for every pair* $(w, w') \in Z$ *we have:*
  - *Atomic conditions:*
    - *for all* $p \in \mathrm{Prop}$*,* $w \in V(p)$ *iff* $w' \in V'(p)$*.*
    - *for all* $i \in \mathrm{Nom}$*,* $N(i) = w$ *iff* $N'(i) = w'$*.*
  - *if* wRu *for some* $u \in W$*, then there is some* $u' \in W'$ *such that* $w'R'u'$ *and* uZu' ***(Zig)****,*
  - *conversely: if* $w'R'u'$ *for some* $u' \in W'$*, then there is some* $u \in W$ *such that* wRu *and* uZu' ***(Zag)****.*

To illustrate this definition, an example of a bisimulation is presented:

Example 10. *Let* $L = \langle \{p, q\}, \{\} \rangle$ *be a hybrid similarity type and* $\mathcal{H} = (W, R, N, V)$ *and* $\mathcal{H}' = (W', R', N', V')$ *be two hybrid structures. A bisimulation* Z *is represented in the following figure:*

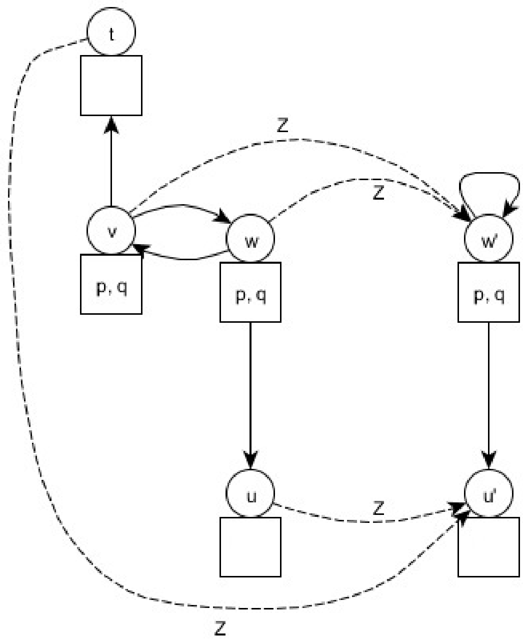

Figure 3.2: A $\mathcal{H}(@)$-bisimulation.

A pointed hybrid structure is a pair $(\mathcal{H}, w)$ where $\mathcal{H}$ is a hybrid structure and w is an element of $\mathcal{H}$.

Two pointed hybrid structures $(\mathcal{H}, w)$ and $(\mathcal{H}', w')$ are *bisimilar*, if there is a $\mathcal{H}(@)$-bisimulation Z between $\mathcal{H}$ and $\mathcal{H}'$ such that wZw'.

It is well known that hybrid satisfaction is invariant under bisimulation (cf. [88]):

Theorem 3.2.16. *Basic hybrid logic is invariant under bisimulation: let L be a hybrid similarity type; if two pointed hybrid structures $(\mathcal{H}, w)$ and $(\mathcal{H}', w')$ are bisimilar, then for any $\varphi \in \mathrm{Form}_@(L)$*

$$\mathcal{H}, w \models \varphi \textit{ iff } \mathcal{H}', w' \models \varphi$$

Proof. The proof follows the idea in [24] and is by induction on φ. Suppose that $(\mathcal{H}, w)$ and $(\mathcal{H}', w')$ are bisimilar, thus wZw'.

For $\varphi = p \in \mathrm{Prop}$,

$$\begin{aligned} \mathcal{H}, w \models p &\Leftrightarrow w \in V(p) \\ &\Leftrightarrow w' \in V'(p) \quad (\textit{atomic cond. in } 3.2.15) \\ &\Leftrightarrow \mathcal{H}', w' \models p \end{aligned}$$

For $\varphi = i \in \mathrm{Nom}$,

$$\begin{array}{rcll} \mathcal{H}, w \models i & \Leftrightarrow & w = N(i) & \\ & \Leftrightarrow & w' = N'(i) & (\textit{atomic cond. in } 3.2.15) \\ & \Leftrightarrow & \mathcal{H}', w' \models i & \end{array}$$

For $\varphi = \Diamond\phi$, with $\phi \in \mathrm{Form}_@(L)$,

$$\begin{array}{rcll} \mathcal{H}, w \models \Diamond\phi & \Leftrightarrow & \text{there exists } v \text{ such that } (wRv \ \&\ \mathcal{H}, v \models \phi) & \\ & \Leftrightarrow & \text{there exists } v' \text{ such that } (w'R'v' \ \&\ \mathcal{H}', v' \models \phi) & (\textbf{Zig/Zag} \textit{ in } 3.2.15 \\ & \Leftrightarrow & \mathcal{H}', w' \models \Diamond\phi & \text{plus the induction hypothesis)} \end{array}$$

For $\varphi = \Box\phi$, with $\phi \in \mathrm{Form}_@(L)$, the proof is analogous.

For φ a boolean case the proof is immediate.

Finally, for the case that $\varphi = @_i\phi$,

$$\begin{array}{rcll} \mathcal{H}, w \models @_i\varphi & \Leftrightarrow & \mathcal{H}, N(i) \models \varphi & \\ & \Leftrightarrow & \mathcal{H}', N'(i) \models \varphi & (\text{for all } i \in \mathrm{Nom},\ N(i)ZN'(i),\ \textit{in } 3.2.15 \\ & \Leftrightarrow & \mathcal{H}', w' \models @_i\varphi & \text{plus induction hypothesis)} \end{array}$$

□

The reciprocal is not true, unless we consider a restriction to image-finite models, [24]. $\mathcal{H}$ is an *image-finite model* iff for each state u in $\mathcal{H}$, the set $\{v_i \in W \mid uRv_i\}$ is finite.

Theorem 3.2.17. *Let L be a hybrid similarity type; $\mathcal{H}$ and $\mathcal{H}'$ are image-finite L-models and $w \in W, w' \in W'$.*

$$(\mathcal{H}, w) \textit{ and } (\mathcal{H}', w') \textit{ are bisimilar iff for all } \varphi \in \mathrm{Form}_@(L),\ \mathcal{H}, w \models \varphi \Leftrightarrow \mathcal{H}', w' \models \varphi$$

Proof. The direction from left to right follows from Theorem 3.2.16. For the other direction, let us prove that the relation of hybrid equivalence on these models is itself a $\mathcal{H}(@)$-bisimulation. The proof for the atomic conditions is obvious. In order to prove **Zig**, assume that for all φ, $\mathcal{H}, w \models \varphi \Leftrightarrow \mathcal{H}', w' \models \varphi$, and that wRv. Suppose there is no v' such that $w'Rv'$ and for all φ, $\mathcal{H}, v \models \varphi \Leftrightarrow \mathcal{H}', v' \models \varphi$. Consider the set $S' = \{u' \mid w'Ru'\}$; S' must be non-empty, because, if it were the case that $S' = \emptyset$, then one would have $\mathcal{H}', w' \models \Box\bot$, which contradicts the fact that $\mathcal{H}, w \models \varphi \Leftrightarrow \mathcal{H}', w' \models \varphi$, since $\mathcal{H}, w \models \Diamond\top$ (because wRv). For $\mathcal{H}'$ image-finite, one must have that S' is a finite set, let us suppose that $S' = \{w'_1, \ldots, w'_n\}$. By assumption, for every $w'_i \in S'$, there is a formula ψ_i such that $\mathcal{H}, v \models \psi_i$, but $\mathcal{H}', w'_i \not\models \psi_i$. It follows that

$$\mathcal{H}, w \models \Diamond(\psi_1 \wedge \cdots \wedge \psi_n) \text{ and } \mathcal{H}', w' \not\models \Diamond(\psi_1 \wedge \cdots \wedge \psi_n),$$

which again contradicts the assumption that $\mathcal{H}, w \models \varphi \Leftrightarrow \mathcal{H}', w' \models \varphi$. Thus we reach absurdity. Therefore, such v' must exist. The idea for proving **Zag** is similar. The first condition of all in the definition of $\mathcal{H}(@)$-bisimulation is achieved because, for all $i \in \text{Nom}$, $\psi \in \text{Form}_@(L)$,

$$\begin{aligned} \mathcal{H}, v \models \psi, v = N(i) &\Leftrightarrow \mathcal{H}, w \models @_i\psi \\ &\Leftrightarrow \mathcal{H}', w' \models @_i\psi \qquad (\mathcal{H}, w \models \varphi \Leftrightarrow \mathcal{H}', w' \models \varphi, \text{ for all } \varphi) \\ &\Leftrightarrow \mathcal{H}, v' \models \psi, v' = N'(i) \end{aligned}$$

Thus, $N(i)ZN'(i)$, for all $i \in \text{Nom}$.

□

Given a structure $\mathcal{H} = (W, R, N, V)$ for the hybrid similarity type $L = \langle \text{Prop}, \text{Nom} \rangle$, its correspondent structure in a first-order language $L^* = \langle \mathcal{P}, \mathcal{C} \rangle$ where $\mathcal{P}$ is a set of predicate symbols and $\mathcal{C}$ is a set of constant symbols, together with a binary relation R, denoted $\mathcal{H}^* = (W, V^*)$, is such that ([29]):

- there is an unary predicate $P \in \mathcal{P}$ for each $p \in \text{Prop}$,
- there is a binary relation $R \in \mathcal{P}$,
- there is a constant $c_i \in \mathcal{C}$ for each $i \in \text{Nom}$,
- $V^*(R) = R$,
- $V^*(P) = \{w \in V(p)\}$.

The link between hybrid and classical languages is that both can talk about relational structures, so it seems likely that there is a connection between them. A *standard translation* is a way of transforming hybrid formulas into first-order formulas capturing in the process the first one's meaning ([5]).

Definition 3.2.18. *The* standard translation*, that maps basic hybrid formulas into first-order formulas, is defined as:*

1. $ST_x(\top) \;=\; \top$
2. $ST_x(\bot) \;=\; \bot$
3. $ST_x(p) \;=\; P(x),\;\; p \in \text{Prop}$

4. $ST_x(i) \;=\; (x \approx c_i),\; i \in \text{Nom}$

5. $ST_x(\neg\varphi) \;= \neg ST_x(\varphi)$

6. $ST_x(\varphi \vee \psi) \;=\; ST_x(\varphi) \vee ST_x(\psi)$

7. $ST_x(\varphi \wedge \psi) \;=\; ST_x(\varphi) \wedge ST_x(\psi)$

8. $ST_x(\Diamond\varphi) \;=\; \exists y(xRy \wedge ST_y(\varphi))$

9. $ST_x(\Box\varphi) \;=\; \forall y(xRy \to ST_y(\varphi))$

10. $ST_x(@_i\varphi) \;=\; (ST_x(\varphi))[c_i/x],\; i \in \text{Nom}$

Here x is a fixed, yet arbitrary, free variable, and y is a variable not used in the translation.

For any modal formula ϕ, $ST_x(\phi)$ is a first-order formula containing exactly one free variable, namely x. Nominals correspond to constants, and satisfaction statements let us switch our perspective from the current state to named states (the substitution $[c_i/x]$, which replaces all free occurrences of x by c_i in the formula, makes it quite clear).

Theorem 3.2.19 ([7])**.** *This translation preserves the truth, in the sense that for all formulas $\varphi \in \text{Form}_@(L)$, given a model $\mathcal{H} = (W, R, N, V)$ for hybrid logic and its correspondent model for first-order logic $\mathcal{H}^* = (W, V^*)$, and for all worlds w of W,*

$$\mathcal{H}, w \models \varphi \text{ iff } \mathcal{H}^* \models ST_x(\varphi)[x \leftarrow w]$$

(where $[x \leftarrow w]$ means assign w to the free variable x).

The definition of a first-order formula invariant under $\mathcal{H}(@)$-bisimulations is given as follows:

Definition 3.2.20. *[24] A first-order formula $\psi(x) \in L^*$ is invariant under $\mathcal{H}(@)$-bisimulations if for all models $\mathcal{M}$ and $\mathcal{N}$, and all states w in $\mathcal{M}$, v in $\mathcal{N}$, and all $\mathcal{H}(@)$-bisimulations Z between $\mathcal{M}$ and $\mathcal{N}$ such that wZv, we have, for its correspondent models for first-order,*

$$\mathcal{M}^* \models \psi(x)[x \leftarrow w] \text{ iff } \mathcal{N}^* \models \psi(x)[x \leftarrow v].$$

A result that connects bisimulation with standard translation is presented in [88], and is reproduced here:

Theorem 3.2.21. *Let $\psi \in L^*$ be a first-order formula, with at most one free variable. The following are equivalent:*

1. *$\psi(x)$ is equivalent to the standard translation of a basic hybrid formula;*

2. *$\psi(x)$ is invariant under $\mathcal{H}(@)$-bisimulations.*

3.3 Strong Priorean Logic

In this Section it is introduced the *strong Priorean logic*. In order to do that, consider the addition of the set of *world variables*, WVar, (typically written as s, t, u, etc), distinct from both nominals and propositional variables. Such enables nominal binding by making use of world variables and assignments of values to world variables.

The result of this new machinery is a powerful hybrid logic, $\mathcal{H}(@, \forall)$, whose grammar is defined as follows:

Definition 3.3.1. *Let* $L = \langle \text{Prop}, \text{Nom}, \text{WVar} \rangle$ *be a* hybrid similarity type *where* Prop *and* Nom *are as usual the set of propositional variables and the set of nominals, and* WVar *is the set of world variables. The well-formed formulas over* L*,* $\text{Form}_{@,\forall}(L)$*, are defined by the following grammar:*

$$WFF := \ i \mid p \mid s \mid \bot \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi \mid @_i\varphi \mid @_s\varphi \mid \forall s.\varphi \mid \exists s.\varphi$$

Note that @ can make use of world variables.

A hybrid structure is defined in the same pattern as in the previous section:

Definition 3.3.2. *Let* $L = \langle \text{Prop}, \text{Nom}, \text{WVar} \rangle$ *be a hybrid similarity type. A* hybrid structure $\mathcal{H}$ *over* L *is a tuple* (W, R, N, V)*. Here,* W *is the* domain*,* R *is the* accessibility relation*,* N *is the* hybrid nomination *and* V *is the* hybrid valuation*.*

The pair (W, R) *is called the* frame *underlying* $\mathcal{H}$*, and* $\mathcal{H}$ *is said to be a structure based on this frame.*

But now there is a need of a mechanism for coping with free world variables, so it is considered an *assignment* $g : \text{WVar} \to W$.

If g, g' are assignments of values to variables in $\mathcal{H}$, recall that g' is an *s-variant assignment of* g iff $g'(t) = g(t)$, for all $t \in \text{WVar}$, $t \neq s$; in such case we write $g' \overset{s}{\sim} g$.

The satisfaction is defined in the following way:

Definition 3.3.3 (Satisfaction)**.** *The local satisfaction relation* $\models$ *between a hybrid structure* $\mathcal{H} = (W, R, N, V)$*, a state* $w \in W$*, an assignment* g *and a hybrid formula is recursively defined by:*

- $\mathcal{H}, g, w \models s$ *iff* $w = g(s)$*;*
- $\mathcal{H}, g, w \models i$ *iff* $w = N(i)$*;*
- $\mathcal{H}, g, w \models p$ *iff* $w \in V(p)$*;*

- $\mathcal{H}, g, w \models \bot$ *never;*
- $\mathcal{H}, g, w \models \top$ *always;*
- $\mathcal{H}, g, w \models \neg\varphi$ *iff not* $\mathcal{H}, g, w \models \varphi$*;*
- $\mathcal{H}, g, w \models \varphi \wedge \psi$ *iff* $\mathcal{H}, g, w \models \varphi$ *and* $\mathcal{H}, g, w \models \psi$*;*
- $\mathcal{H}, g, w \models \varphi \vee \psi$ *iff* $\mathcal{H}, g, w \models \varphi$ *or* $\mathcal{H}, g, w \models \psi$*;*
- $\mathcal{H}, g, w \models \Diamond\varphi$ *iff* $\exists w' \in W(wRw'$ *and* $\mathcal{H}, g, w' \models \varphi)$*;*
- $\mathcal{H}, g, w \models \Box\varphi$ *iff* $\forall w' \in W(wRw' \Rightarrow \mathcal{H}, g, w' \models \varphi)$*;*
- $\mathcal{H}, g, w \models @_i\varphi$ *iff* $\mathcal{H}, w' \models \varphi$*, where* $w' = N(i)$*;*
- $\mathcal{H}, g, w \models @_s\varphi$ *iff* $\mathcal{H}, g, w' \models \varphi$*, where* $w' = g(s)$*;*
- $\mathcal{H}, g, w \models \forall s.\varphi$ *iff* $\mathcal{H}, g', w \models \varphi$*, for all* $g' \stackrel{s}{\sim} g$*;*
- $\mathcal{H}, g, w \models \exists s.\varphi$ *iff* $\mathcal{H}, g', w \models \varphi$*, for some* $g' \stackrel{s}{\sim} g$*;*

The catch of this extension is that the satisfiability problem is turned EXPTIME-complete ([27]). On the other hand, observe that this kind of quantification covers other extensions of hybrid logic, namely, it is possible to define the $\downarrow$ binder as:

$$\downarrow x.p \equiv \exists x.(x \wedge p)$$

There are some possible complete axiomatizations for $\mathcal{H}(@, \forall)$, one is an extension of the axiomatization for basic hybrid language previously presented, as follows (cf. [27]):

$\mathbf{K}_{\mathcal{H}(@,\forall)}$-I	
Axioms:	
All axioms of $\mathbf{K}_{\mathcal{H}(@)}$	
(including the last two presented)	
(Q1)	$\vdash \forall s.(\phi \to \psi) \to (\phi \to \forall s.\psi)$ where s does not occur free in ϕ
(Q2)	$\vdash \forall s.\phi \to \phi[\tau \leftarrow s]$ where τ (a world variable or nominal) is substitutable for s in ϕ
(Barcan$_@$)	$\vdash \forall s.@_i\phi \leftrightarrow @_i\forall s.\phi$
Rules:	
All rules of $\mathbf{K}_{\mathcal{H}(@)}$	
(Gen$_\forall$)	If $\vdash \phi$ then $\vdash \forall s.\phi$

3.3.1 Standard and Hybrid Translations

It is also possible to extend standard translation referred in Subsection 3.2.3 to the new formulas introduced in this stronger logic, following the work in [26], as:

1. $ST_x(s) \;=\; (x \approx s),\; s \in \text{WVar}$
2. $ST_x(@_s\varphi) \;=\; (ST_x(\varphi))[s/x],\; s \in \text{WVar}$
3. $ST_x(\forall s.\varphi) \;=\; \forall s.(ST_x(\varphi)),\; s \in \text{WVar}$
4. $ST_x(\exists s.\varphi) \;=\; \exists s.(ST_x(\varphi)),\; s \in \text{WVar}$

Where x is a fixed, yet arbitrary, free variable.

World variables are considered to be first-order variables.

This extension of standard translation is also truth-preserving ([88]), as the one in Subsection 3.2.3.

The language $\mathcal{H}(@, \forall)$ is very strong: *any* first-order expression in a language with a binary relation R, so that it is possible to talk about accessibility, and with an unary relation P, so that it is possible to talk about propositional information, can be translated into a formula of $\mathcal{H}(@, \forall)$. This translation became known as *hybrid translation.*

Definition 3.3.4. *The* hybrid translation *between first-order formulas and hybrid formulas is defined in the following way:*

$$\begin{array}{rcl} HT(\top) & = & \top \\ HT(\bot) & = & \bot \\ HT(sRt) & = & @_s\Diamond t \\ HT(P(s)) & = & @_s p \\ HT(s \approx t) & = & @_s t \\ HT(\neg\varphi) & = & \neg HT(\varphi) \\ HT(\varphi \wedge \psi) & = & HT(\varphi) \wedge HT(\psi) \\ HT(\exists v.\varphi) & = & \exists v.HT(\varphi) \\ HT(\forall v.\varphi) & = & \forall v.HT(\varphi) \end{array}$$

Note that the absence of @ (either as a primitive or defined using some sort of modality), does not make it possible to define such translation [25].

Assuming the same denotation for $\mathcal{H}, \mathcal{H}^*$ as in Subsection 3.2.3, one can state that HT is also truth-preserving.

Theorem 3.3.5 ([5])**.** *Given $\mathcal{H}$ a model for hybrid logic, $\mathcal{H}^*$ its first-order correspondent, for any first-order formula $\varphi \in L^*$, any assignment g,*

$$\mathcal{H}^*, g \models \varphi \textit{ iff } \mathcal{H}, g \models HT(\varphi).$$

3.3.2 Strong Bisimulation

For simplicity, a less expressive language will be considered in this Subsection, namely $\mathcal{H}(E)$. Before introducing the definition of strong bisimulation, the syntax of $\mathcal{H}(E)$ is defined, as well as the satisfaction of $\mathcal{H}(E)$ formulas. Posteriorly, the standard translation is presented and a result connecting it with bisimulation is announced.

For more details on bisimulation and standard translation for others less expressive languages than $\mathcal{H}(@, \forall)$, consult [88].

The syntax of the language $\mathcal{H}(E)$ is now introduced.

Definition 3.3.6. *Let* $L = \langle \mathrm{Prop}, \mathrm{Nom} \rangle$ *be a* hybrid similarity type *where* Prop *is the set of propositional variables and* Nom *is the set of nominals. The well-formed formulas over* L, $\mathrm{Form}_E(L)$, *are defined by the following grammar:*

$$WFF := \; i \mid p \mid \bot \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi \mid E\varphi \mid A\varphi$$

A *structure* is defined as usual, $\mathcal{H} = (W, R, N, V)$, and the *satisfaction* of $E\varphi$ and $A\varphi$ which are new formulas, is defined by:

1. $\mathcal{H}, w \models E\varphi$ iff for some $v \in W$, $\mathcal{H}, v \models \varphi$;
2. $\mathcal{H}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{H}, v \models \varphi$.

Observe that $A\varphi$ is equivalent to $\neg E \neg \varphi$.

This is an interesting approach to hybrid logics, since it is possible to define the @ operator by means of the modalities E and A, namely, $E(i \wedge \varphi)$ means that "somewhere in the model there is a world where i and φ are both true", which has the same meaning as $@_i\varphi$. Thus,

$$@_i\varphi \overset{def}{=} E(i \wedge \varphi)$$

and alternatively,

$$@_i\varphi \overset{def}{=} A(i \rightarrow \varphi).$$

$A(i \rightarrow \varphi)$ says that "at all worlds in the model where i is true, φ is also true".

An axiomatization for $\mathcal{H}(E)$ can be found at [88].

The definition of *strong bisimulation* between $\mathcal{H}(E)$ models is the same as the one for $\mathcal{H}(@)$ plus the fact that it is a left-total and surjective relation.

Definition 3.3.7 (Strong Bisimulation)**.** *Let* $L = \langle \text{Prop}, \text{Nom} \rangle$ *be a hybrid similarity type. Let* $\mathcal{H} = (W, R, N, V)$ *and* $\mathcal{H}' = (W', R', N', V')$ *be two hybrid structures. A* $\mathcal{H}(E)$- bisimulation *between* $\mathcal{H}$ *and* $\mathcal{H}'$ *is a non-empty relation* $Z \subseteq W \times W'$ *such that:*

- $\forall w \in W, \exists w' \in W'$ *such that* wZw' *and* $\forall w' \in W', \exists w \in W$ *such that* wZw'*;*
- *if* $w = N(i)$ *and* $w' = N'(i)$ *for some* $i \in \text{Nom}$*, then* wZw'*;*
- *for every pair* $(w, w') \in Z$ *we have:*
  - *atomic conditions:*
    - $w \in V(p)$ *iff* $w' \in V'(p)$*, for all* $p \in \text{Prop}$*;*
    - $N(i) = w$ *iff* $N'(i) = w'$*, for all* $i \in \text{Nom}$*;*
  - *if* wRu *for some* $u \in W$*, then there is some* $u' \in W'$ *such that* $w'R'u'$ *and* uZu' ***(Zig)****;*
  - *conversely: if* $w'R'u'$ *for some* $u' \in W'$*, then there is some* $u \in W$ *such that* wRu *and* uZu' ***(Zag)****.*

Note that every $\mathcal{H}(E)$-bisimulation is a $\mathcal{H}(@)$-bisimulation.

Theorem 3.2.16 can be translated into $\mathcal{H}(E)$ as follows (cf. [88]):

Theorem 3.3.8. *$\mathcal{H}(E)$ is invariant under bisimulation: let* $L = \langle \text{Prop}, \text{Nom} \rangle$ *be a hybrid similarity type; if two pointed hybrid structures* $(\mathcal{H}, w)$ *and* $(\mathcal{H}', w')$ *are bisimilar, then for any* $\varphi \in \text{Form}_E(L)$

$$\mathcal{H}, w \models \varphi \textit{ iff } \mathcal{H}', w' \models \varphi$$

Proof. Continuing the proof for Theorem 3.2.16, for $\varphi = E\phi$, we have that:

$$\begin{array}{rcl} \mathcal{H}, w \models E\phi & \Leftrightarrow & \text{for some } v \in W, \mathcal{H}, v \models \phi \\ & \Leftrightarrow & \text{for some } v' \in W', \mathcal{H}', v' \models \phi \quad (*) \\ & \Leftrightarrow & \mathcal{H}', w' \models E\phi \end{array}$$

$(*)$ the direction from right to left is because Z is a left-total relation, and it is considered that v' is the element in W' such that vZv'. The converse is because Z is a surjective relation. □

The *standard translation* for the new formulas introduced in $\mathcal{H}(E)$ is given by:

$$ST_x(E\varphi) = \exists y.ST_y(\varphi)$$

Proposition 3.2.19 can be easily considered for formulas in $\mathrm{Form}_E(L)$, and so does Definition 3.2.20, replacing $\mathcal{H}(@)$-bisimulations for $\mathcal{H}(E)$-bisimulations.

An analogous result to the one in 3.2.3, about the connection between bisimulation and standard translation follows (as in [88]).

Theorem 3.3.9. *Let $\psi \in L^*$ be a first-order formula, with at most one free variable. The following are equivalent:*

1. *$\psi(x)$ is equivalent to the standard translation of a hybrid formula in $\mathrm{Form}_E(L)$ for some hybrid similarity type $L = \langle \mathrm{Prop}, \mathrm{Nom} \rangle$;*
2. *$\psi(x)$ is invariant under $\mathcal{H}(E)$-bisimulations.*

Chapter 4

Paraconsistency in Hybrid Logic

This Chapter is a new contribution in the fields of paraconsistency and hybrid logics. It is inspired by the work of Grant and Hunter in [58]. Although [59] seemed simpler and more straightforward in the approach of the satisfaction in the presence of inconsistencies, the fact that [58] carries the notion of disjunctive syllogism ended being more appealing. The main motivation for this Chapter is that it ought to be possible to reason about information in relational structures, even when there are inconsistencies present.

First of all, it is defined a *Quasi-Hybrid Basic Logic*. Analogously to the assumption in [58], where it is assumed that all formulas are in *prenex conjunctive normal form*, it will be assumed that all formulas are in *negation normal form*. This assumption does not lead to loss of generality since Proposition 4.1.2 shows that any hybrid formula is equivalent to one in negation normal form.

The concepts of bistructure, decoupled and strong satisfaction and QH model are presented. The paraconsistent diagram of a bistructure is defined and a very important theorem concerning the designing of QH models as sets of quasi-hybrid literals is proved. Afterwards, there are considered minimal QH models and some examples with illustrations are presented. The inconsistency measure, a central goal of this work is introduced and consequently, the notion of preferred QH model appears. Finally, the extrinsic and intrinsic inconsistency for preferred QH models is defined, and there are presented some analogous results to the ones in [58].

4.1 Quasi-Hybrid Basic Logic

In order to generalize the approach in [58] to the hybrid case, one must consider formulas in *negation normal form* (*i.e.*, formulas in which the negation symbol occurs immediately before propositional symbols or nominals). As such, an analogous result to the one in [28] for classical propositional logic is established. Essentially it states that any hybrid formula is logically equivalent to its negation normal form. It is important to point out that the same result was presented without proof in [49] for the modal case. Therefore, it is possible to restrict our attention to the formulas in negation normal form. First, the formal definition of such formulas:

Definition 4.1.1. *Let $L = \langle \mathrm{Prop}, \mathrm{Nom} \rangle$ be a hybrid similarity type. The* negation normal form *of a formula, for short* NNF*, is defined just as in propositional logic: a formula is said to be in* NNF *if negation only appears directly before propositional variables and/or nominals. The set of* NNF *formulas over L, $\mathrm{Form}_{\mathrm{NNF}(@)}(L)$, is recursively defined as follows:*

For $p \in \mathrm{Prop}$, $i \in \mathrm{Nom}$,

1. $\bot, \top$ *are in* NNF*;*
2. p, i, $\neg p$, $\neg i$ *are in* NNF*;*
3. *If φ, ψ are formulas in* NNF*, then $\varphi \vee \psi$, $\varphi \wedge \psi$ are in* NNF*;*
4. *If φ is in* NNF*, then $\Box\varphi$, $\Diamond\varphi$ are in* NNF*;*
5. *If $i \in \mathrm{Nom}$ and φ is in* NNF*, then $@_i\varphi$ is in* NNF*.*

The next proposition states that there can be considered only the formulas in negation normal form.

Proposition 4.1.2. *Every formula $\varphi \in \mathrm{Form}_@(L)$ is logically equivalent to a formula $\varphi^* \in \mathrm{Form}_{\mathrm{NNF}(@)}(L)$.*

Proof. The proof is achieved by induction on complexity. The base step is trivial, since an atomic formula is already in negation normal form. Most cases of the induction steps are trivial as well. For instance, if A and B are equivalent respectively to negation-normal formulas A^* and B^*, then $A \wedge B$ and $A \vee B$ are equivalent respectively to $A^* \wedge B^*$ and $A^* \vee B^*$, which are also negation-normal. The non-trivial case is to prove that if A is equivalent to the negation-normal A^* then $\sim A$ is equivalent to some negation-normal $A\dagger$. This divides into seven subcases according to the form of A^*. The case where A^* is atomic is trivial, since we may simply let $A\dagger$ be $\sim A^*$. In case A^* is of form $\sim B$, so that $\sim A^*$ is $\sim\sim B$, we may let $A\dagger$ be B. In case A^* is of form $B \vee C$, so that $\sim A^*$ is $\sim (B \vee C)$, which is logically equivalent to $(\sim B \wedge \sim C)$, by the induction hypothesis the simpler formulas $\sim B$ and $\sim C$ are equivalent to formulas $B\dagger$ and $C\dagger$ of the required form, so we may let $A\dagger$ be $(B \dagger \wedge C\dagger)$. The case of conjunction is similar. In case A^* is of form $\Diamond B$, so that $\sim A^*$ is $\sim \Diamond B$, which is logically equivalent to $\Box \sim B$, by the induction hypothesis the simpler formula $\sim B$ is equivalent to a formula $B\dagger$ of the required form, so we may let $A\dagger$ be $\Box B\dagger$. The case of box is similar. If A^* is of the form $@_i B$, so that $\sim A^*$ is $\sim @_i B$, which is logically equivalent to $@_i \sim B$, and since by hypotheses $\sim B$ is equivalent to a formula $B\dagger$ of the required form, then we can conclude that $A\dagger$ is $@_i B\dagger$. □

Based on this proof, a recursive procedure that transforms formulas into negation normal form can be formulated. Formally, $nnf : \text{Form}_{@}(L) \to \text{Form}_{\text{NNF}(@)}(L)$ is defined as follows:

1. $nnf(l) \overset{def}{=} l$, if l is a literal;
2. $nnf(\psi_1 \vee \psi_2) \overset{def}{=} nnf(\psi_1) \vee nnf(\psi_2)$;
3. $nnf(\psi_1 \wedge \psi_2) \overset{def}{=} nnf(\psi_1) \wedge nnf(\psi_2)$;
4. $nnf(\neg(\psi_1 \vee \psi_2)) \overset{def}{=} nnf(\neg\psi_1) \wedge nnf(\neg\psi_2)$;
5. $nnf(\neg(\psi_1 \wedge \psi_2)) \overset{def}{=} nnf(\neg\psi_1) \vee nnf(\neg\psi_2)$;
6. $nnf(\Box\psi) \overset{def}{=} \Box nnf(\psi)$;
7. $nnf(\neg\Box\psi) \overset{def}{=} \Diamond nnf(\neg\psi)$;
8. $nnf(\Diamond\psi) \overset{def}{=} \Diamond nnf(\psi)$;
9. $nnf(\neg\Diamond\psi) \overset{def}{=} \Box nnf(\neg\psi)$;
10. $nnf(\neg\neg\psi) \overset{def}{=} nnf(\psi)$;
11. $nnf(@_i\psi) \overset{def}{=} @_i nnf(\psi)$;
12. $nnf(\neg @_i\psi) \overset{def}{=} @_i nnf(\neg\psi)$;

Without loss of generality (see Proposition 4.1.2), it will be assumed that all formulas are in negation normal form, i.e, given a hybrid similarity type $L = \langle \text{Prop}, \text{Nom} \rangle$, the set of formulas is $\text{Form}_{\text{NNF}(@)}(L)$.

Let us continue with the definition of the $\sim$ operator, which will make some definitions clearer.

Definition 4.1.3. *Let θ be a formula in* NNF *and let $\sim$ be a complementation operation such that $\sim\theta = nnf(\neg\theta)$.*

Recall that a *hybrid structure* for the hybrid similarity type L is a tuple (W, R, N, V). However, in order to accommodate inconsistencies in a model, one has to consider two valuations for propositions: V^+ and V^-.

Definition 4.1.4. *A* hybrid bistructure *is a tuple (W, R, N, V^+, V^-) where (W, R, N, V^+) and (W, R, N, V^-) are hybrid structures.*

The map V^+ is the interpretation for positive propositional symbols, and V^- is the interpretation for negative ones. This is formalized in the definition for decoupled satisfaction.

Definition 4.1.5. *For a hybrid bistructure* $E = (W, R, N, V^+, V^-)$*, a satisfiability relation* $\models_d$ *called* decoupled satisfaction *at* $w \in W$ *for propositional symbols and nominals is defined as follows:*

1. $E, w \models_d p$ *iff* $w \in V^+(p)$;
2. $E, w \models_d i$ *iff* $w = N(i)$;
3. $E, w \models_d \neg p$ *iff* $w \in V^-(p)$;
4. $E, w \models_d \neg i$ *iff* $w \neq N(i)$;

Note that the link between a formula and its complement has been decoupled at structural level, in order to allow both a positive and a negative propositional symbol to be satisfiable. In contrast, if a classical hybrid structure satisfies a propositional symbol at some world, it is forced to not satisfy its complement at that world.

This decoupling gives the basis for a semantic for paraconsistent reasoning.

Definition 4.1.6. *A satisfiability relation* $\models_s$*, called* strong satisfaction*, is defined as follows:*

1. $E, w \models_s \top$ *always;*
2. $E, w \models_s \bot$ *never;*
3. $E, w \models_s p$ *iff* $E, w \models_d p$;
4. $E, w \models_s \neg p$ *iff* $E, w \models_d \neg p$;
5. $E, w \models_s i$ *iff* $E, w \models_d i$;
6. $E, w \models_s \neg i$ *iff* $E, w \models_d \neg i$;
7. $E, w \models_s \theta_1 \vee \theta_2$ *iff* $[E, w \models_s \theta_1$ *or* $E, w \models_s \theta_2]$ *and* $[E, w \models_s \sim \theta_1 \Rightarrow E, w \models_s \theta_2]$ *and* $[E, w \models_s \sim \theta_2 \Rightarrow E, w \models_s \theta_1]$;
8. $E, w \models_s \theta_1 \wedge \theta_2$ *iff* $E, w \models_s \theta_1$ *and* $E, w \models_s \theta_2$;
9. $E, w \models_s \Diamond\theta$ *iff* $\exists w'(wRw' \;\&\; E, w' \models_s \theta)$;
10. $E, w \models_s \Box\theta$ *iff* $\forall w'(wRw' \Rightarrow E, w' \models_s \theta)$;
11. $E, w \models_s @_i\theta$ *iff* $E, w' \models_s \theta$ *where* $w' = N(i)$;

Strong validity is set as follows:

$$E \models_s \theta \text{ iff for all } w \in W, E, w \models_s \theta.$$

Analogously to the definition in the basic hybrid case of a model of a set Δ of formulas, it is said that E is a *quasi-hybrid model* of Δ iff for all $\theta \in \Delta, E \models_s \theta$.

To make it easier to follow, it will be assumed that N maps nominals to themselves; hence W will always contain all the nominals in L. This also means that all nominals are mapped to distinct elements, *i.e.*, N is an inclusion map. Hence, for a given hybrid similarity type $L = \langle \text{Prop}, \text{Nom} \rangle$ and a domain W of a bistructure one must have $\text{Nom} \subseteq W$.

Following the assumption that N is injective, in order to define diagrams, hybrid literals regarding equality between nominals, *i.e.*, $@_i j$, $@_i \neg j$ are not needed. Therefore, in this context, the notion of atom and literal is reformulated as follows:

For a hybrid similarity type $L = \langle \text{Prop}, \text{Nom} \rangle$,

1. *Quasi-hybrid atoms over* L:
   $\text{QHAt}(L) = \{@_i p,\ @_i \Diamond j \mid i, j \in \text{Nom},\ p \in \text{Prop}\}$;

2. *Quasi-hybrid literals over* L:
   $\text{QHLit}(L) = \{@_i p,\ @_i \neg p,\ @_i \Diamond j,\ @_i \Box \neg j \mid i, j \in \text{Nom},\ p \in \text{Prop}\}$;

To build the paraconsistent diagram, new nominals are added for the elements of W which are not named yet, and this expanded similarity type is denoted by $L(W)$, *i.e.*, $L(W) = \langle \text{Prop}, W \rangle$ (recall that $\text{Nom} \subseteq W$). As in the standard case, $E(W)$ denotes the natural expansion of the bistructure E to the hybrid similarity type $L(W)$, by taking N the identity for the new nominals. Moreover, it will be once again assumed that Prop, Nom are finite sets for any hybrid similarity type $L = \langle \text{Prop}, \text{Nom} \rangle$, as well as the domain W of any bistructure.

Definition 4.1.7. *Let* $L = \langle \text{Prop}, \text{Nom} \rangle$ *be a hybrid similarity type, and consider a hybrid bistructure over* L, $E = (W, R, N, V^+, V^-)$. *The* elementary paraconsistent diagram *of* E, *denoted by* $Pdiag(E)$, *is the set of quasi-hybrid literals over* $L(W)$ *that hold in* $E(W)$, *i.e.,*

$$Pdiag(E) = \{\alpha \in \text{QHLit}(L(W)) \mid E(W) \models_s \alpha\}$$

The paraconsistent diagram $Pdiag(E)$ defines completely the bistructure E in the sense that, fixing the domain W and N being the identity, there is an unique model of $Pdiag(E)$ (over $L(W)$) with domain W and a hybrid nomination N, which is $E(W)$. Therefore, in the sequel, a bistructure $E = (W, R, N, V^+, V^-)$ will be represented by its (finite) paraconsistent diagram $Pdiag(E)$. This syntactical representation will play an important role throughout this dissertation.

Let $L = \langle \text{Prop}, \text{Nom} \rangle$ be a hybrid similarity type, $\Delta \subseteq \text{Form}_{\text{NNF}(@)}(\text{L})$ and W a finite set. The notation $\text{QH}(L, \Delta, W)$ is used for the set of representations (*i.e.*, paraconsistent diagrams) of hybrid bistructures that are models of Δ with domain W. Recall that the domain and the hybrid similarity type are considered to be finite. This implies that bistructures are finite and consequently representations of QH models are also finite. This fact is relevant in the next section when discussing the measure of inconsistency in a model.

The syntactic representations of models will be denoted by $\mathcal{M}$, $\mathcal{M}_1$, etc. Let $\mathcal{M}$ be the representation of E with domain W. For $w \in W$, one writes $\mathcal{M}, w \models_s \varphi$ if $E, w \models_s \varphi$. Analogously, $\mathcal{M} \models_s \varphi$ defines $E \models_s \varphi$.

In order to make it easier to construct QH models as sets of quasi-hybrid literals, some properties about the satisfaction operator are introduced and proved, and a very important theorem which makes it possible to use only quasi-hybrid literals when transforming a formula in negation normal form into a quasi-equivalent positive boolean combination of QH-literals will be presented.

Definition 4.1.8. *A formula* $\varphi^* \in \text{Form}_{\text{NNF}(@)}(\text{L})$ *is said to be* quasi-equivalent to $\varphi \in \text{Form}_{\text{NNF}(@)}(\text{L})$*, denoted* $\varphi \equiv_q \varphi^*$*, iff for all hybrid bistructure* $E = (W, R, N, V^+, V^-)$ *and any* $w \in W$*,*

$$E, w \models_s \varphi \Leftrightarrow E, w \models_s \varphi^*.$$

Some properties of the satisfaction operator in quasi-hybrid logic are now presented:

Lemma 4.1.9. *Let* $L = \langle \text{Prop}, \text{Nom} \rangle$ *be a hybrid similarity type, and* $\varphi, \psi \in \text{Form}_{\text{NNF}(@)}(\text{L})$ *be hybrid formulas in negation normal form. Then,*

1. $@_i(\varphi \vee \psi) \equiv_q @_i\varphi \vee @_i\psi$;
2. $@_i(\varphi \wedge \psi) \equiv_q @_i\varphi \wedge @_i\psi$;
3. $@_i@_j\varphi \equiv_q @_j\varphi$;
4. $\neg @_i\varphi \equiv_q @_i\neg\varphi$.

Proof. (1) Let E be an arbitrary bistructure, and w be an arbitrary world in E:

$$\begin{aligned}
E, w \models_s @_i(\varphi \vee \psi) \quad &\Leftrightarrow \quad E, w' \models_s \varphi \vee \psi, w' = N(i) \\
&\Leftrightarrow \quad [E, w' \models_s \varphi \text{ or } E, w' \models_s \psi] \\
&\qquad \text{and } [E, w' \models_s nnf(\neg\varphi) \Rightarrow E, w' \models_s \psi] \\
&\qquad \text{and } [E, w' \models_s nnf(\neg\psi) \Rightarrow E, w' \models_s \varphi], w' = N(i) \\
&\Leftrightarrow \quad [E, w \models_s @_i\varphi \text{ or } E, w \models_s @_i\psi] \\
&\qquad \text{and } [E, w \models_s @_i(nnf(\neg\varphi)) \Rightarrow E, w \models_s @_i\psi] \\
&\qquad \text{and } [E, w \models_s @_i(nnf(\neg\psi)) \Rightarrow E, w \models_s @_i\varphi] \\
&\Leftrightarrow \quad [E, w \models_s @_i\varphi \text{ or } E, w \models_s @_i\psi] \\
&\qquad \text{and } [E, w \models_s nnf(\neg(@_i\varphi)) \Rightarrow E, w \models_s @_i\psi] \\
&\qquad \text{and } [E, w \models_s nnf(\neg(@_i\psi)) \Rightarrow E, w \models_s @_i\varphi] \\
&\Leftrightarrow \quad E, w \models_s @_i\varphi \vee @_i\psi
\end{aligned}$$

The proof for (2), (3) and (4) is trivial.

□

The distributive law does not hold as it is shown in the following counter-example:

Example 11. *Let $L = \langle \{p, q, r\}, \{i\} \rangle$ and the bistructure E, with domain $W = \{i\}$, $R = \emptyset$ and the valuation V defined by $V^+(p) = \emptyset, V^-(p) = \{i\}, V^+(q) = V^-(q) = \{i\}, V^+(r) = \{i\}, V^-(r) = \emptyset$.*

Clearly, the formula $@_i p \vee (@_i q \wedge @_i r)$ is valid in E. However, the formula $(@_i p \vee @_i q) \wedge (@_i p \vee @_i r)$ is not valid in E. This shows that in Quasi-Hybrid Logic the distributive law does not hold.

The following theorem is very important to build the representation of hybrid bistructures using quasi-hybrid literals:

Theorem 4.1.10. *Let $L = \langle \mathrm{Prop}, \mathrm{Nom} \rangle$ be a hybrid similarity type, and $\mathcal{M}$ be the representation of a finite hybrid bistructure over L with domain W. Then for all formula $\varphi \in \mathrm{Form}_{\mathrm{NNF}(@)}(L)$, there is a positive boolean combination of quasi-hybrid literals $\bar{\varphi}$ over $L(W)$ such that*

$$\mathcal{M} \models_s \varphi \Leftrightarrow \mathcal{M} \models_s \overline{\varphi}.$$

Proof. One should start by noting that for a QH model with finite domain $W = \{i_1, i_2, \ldots, i_n\}$ one has that,

$$\mathcal{M} \models_s \varphi \Leftrightarrow \mathcal{M} \models_s @_{i_1}\varphi \wedge @_{i_2}\varphi \wedge \cdots \wedge @_{i_n}\varphi.$$

Since the conjunction of positive boolean combinations of quasi-hybrid literals remains a positive boolean combination of quasi-hybrid literals , the proof follows by defining a procedure to transform any formula $@_{i_*}\varphi$ in a quasi-equivalent positive boolean combination of quasi-hybrid literals, PBCL for short.

- if $\varphi = p, @_{i_*}p$ is a PBCL;
- if $\varphi = \neg p, @_{i_*}\neg p$ is a PBCL;
- if $\varphi = i, @_{i_*}i$ is a PBCL;
- if $\varphi = \neg i, @_{i_*}\neg i$ is a PBCL;

For the induction step, suppose that $@_{i_*}\phi, @_{i_*}\psi$ are equivalent to PBCL formulas,

- if $\varphi = \phi \vee \psi$, $@_{i_*}\phi \vee \psi$ is quasi-equivalent to $@_{i_*}\phi \vee @_{i_*}\psi$ by Lemma 4.1.9 which by Ind. Hyp. is quasi-equivalent to a PBCL;

- if $\varphi = \phi \wedge \psi$, $@_{i_*}\phi \wedge \psi$ is quasi-equivalent to $@_{i_*}\phi \wedge @_{i_*}\psi$ by Lemma 4.1.9 which by Ind. Hyp. is quasi-equivalent to a PBCL;

- if $\varphi = \Box\phi$, $@_{i_*}\Box\phi$ is quasi-equivalent to $(@_{i_*}\Box\neg i_1 \vee @_{i_1}\phi) \wedge (@_{i_*}\Box\neg i_2 \vee @_{i_2}\phi) \wedge \cdots \wedge (@_{i_*}\Box\neg i_n \vee @_{i_n}\phi)$ by Lemma 4.1.9 which by Ind. Hyp. is quasi-equivalent to a PBCL;

- if $\varphi = \Diamond\phi$, $@_{i_*}\Diamond\phi$ is quasi-equivalent to $(@_{i_*}\Diamond i_1 \wedge @_{i_1}\phi) \vee (@_{i_*}\Diamond i_2 \wedge @_{i_2}\phi) \vee \cdots \vee (@_{i_*}\Diamond i_n \wedge @_{i_n}\phi)$ by Lemma 4.1.9 which by Ind. Hyp. is quasi-equivalent to a PBCL;

- if $\varphi = @_{i_k}\phi$, $@_{i_*}@_{i_k}\phi$ is quasi-equivalent to $@_{i_k}\phi$ by Lemma 4.1.9 which by Ind. Hyp. is quasi-equivalent to a PBCL.

□

This theorem will take an important role when determining QH models. As it was already pointed out, one can represent bistructures by the quasi-hybrid literals that are true there. Therefore, it will be considered that models are representations of bistructures and consequently, models will be sets of quasi-hybrid literals. So, for a given set Δ, an easier construction of models in the desired form requires the application of this theorem for each formula in Δ.

4.1.1 Minimal QH Models

The next definition is the basis for proving that we can deal only with models with the least number of elements:

Definition 4.1.11. *Let L be a hybrid similarity type and W be a domain. For a set K of* QH *models, the set of* satisfied literals *in K is the set $SLit(K)$ defined as follows:*

$$\mathrm{SLit}(K) = \{\alpha \in \mathrm{QHLit}(L(W)) \mid \forall \mathcal{M} \in K,\ \mathcal{M} \models_s \alpha\}$$

As different hybrid similarity types contain different sets of formulas, it is important to have L as a parameter when discussing concepts about models. Also, it will be assumed that Δ is a set of formulas of L. The domain is also important and therefore will be considered as a parameter.

As referred before, models with the least number of elements are now defined:

Definition 4.1.12. *Let L be a hybrid similarity type, $\Delta \subseteq \mathrm{Form}_{\mathrm{NNF}(@)}(\mathrm{L})$ and W a non-empty set. The set of minimal* QH *models of Δ with domain W is the set* $\mathrm{MQH}(L, \Delta, W)$*, defined as:*

$$\mathrm{MQH}(L, \Delta, W) = \{\mathcal{M} \in \mathrm{QH}(L, \Delta, W) \mid \textit{if}\ \ \mathcal{M}' \subset \mathcal{M}\ \ \textit{then}\ \ \mathcal{M}' \notin \mathrm{QH}(L, \Delta, W)\}$$

Clearly, every QH model contains a minimal QH model, *i.e.*, for all QH model $\mathcal{M}_1$, there is a minimal QH model $\mathcal{M}_2$ such that $\mathcal{M}_2 \subseteq \mathcal{M}_1$.

It is not difficult to see that, if a variable $p \in \text{Prop}$ does not occur in Δ, then p also does not occur in any model $\mathcal{M} \in \text{MQH}(L, \Delta, W)$.

The minimal QH models are just models with no irrelevant and useless information, according to the next theorem:

Theorem 4.1.13. *Let L be a hybrid similarity type, $\Delta \subseteq \text{Form}_{\text{NNF}(@)}(\text{L})$ and W a non-empty set. Then*

$$\text{SLit}(\text{QH}(L, \Delta, W)) = \text{SLit}(\text{MQH}(L, \Delta, W))$$

Proof. Since $\text{MQH}(L, \Delta, W) \subseteq \text{QH}(L, \Delta, W)$, by Galois Connection, the set of satisfied literals in $\text{QH}(L, \Delta, W)$ is a subset of the set of satisfied literals in $\text{MQH}(L, \Delta, W)$,

$$\text{SLit}(\text{QH}(L, \Delta, W)) \subseteq \text{SLit}(\text{MQH}(L, \Delta, W)).$$

To prove the other inclusion, let $\varphi \in \text{SLit}(\text{MQH}(L, \Delta, W))$. So, $\mathcal{M}_i \models_s \varphi$, for all $\mathcal{M}_i \in \text{MQH}(L, \Delta, W)$.

For all $\mathcal{M}'_j \in \text{QH}(L, \Delta, W)$, there is an $\mathcal{N}_j \subseteq \text{QHLit}(L(W))$ and an $\mathcal{M}_i \in \text{MQH}(L, \Delta, W)$ such that $\mathcal{M}_i \cup \mathcal{N}_j = \mathcal{M}'_j$. Since $\mathcal{M}_i \models_s \varphi$, for all $\mathcal{M}_i \in \text{MQH}(L, \Delta, W)$, then $\mathcal{M}_i \cup \mathcal{N}_j \models_s \varphi$, for all $\mathcal{M}_i \in \text{MQH}(L, \Delta, W)$ and any $\mathcal{N}_j \subseteq \text{QHLit}(L(W))$. So, $\mathcal{M}'_j \models_s \varphi$, for all $\mathcal{M}'_j \in \text{QH}(L, \Delta, W)$. Therefore,

$$\text{SLit}(\text{MQH}(L, \Delta, W)) \subseteq SLit(\text{QH}(L, \Delta, W)).$$

□

The previous theorem does not hold if one consider all satisfied formulas, say $SForm(K)$, instead of only satisfied literals. The following is actually a counter-example:

Example 12. *Let $L = (\{p, q, r\}, \{i, j\})$, $W = \{i, j\}$ and $\Delta = \{@_i p \vee @_i q, @_i \neg p \vee @_i \neg q, @_j r\}$.*

The two minimal QH *models of Δ are:*

$\mathcal{M}_1 = \{@_i p, @_i \neg q, @_j r\}$*;*

$\mathcal{M}_2 = \{@_i q, @_i \neg p, @_j r\}$*.*

It is easy to see that:

$@_i r \vee @_j r \in SForm(\text{MQH}(L, \Delta, W))$*.*

However, if one consider the model $\mathcal{M} = \{@_i p, @_i \neg q, @_j r, @_j \neg r\}$, then $\mathcal{M} \not\models_s @_i r \vee @_j r$, because since $\mathcal{M} \models_s @_j r$, $\mathcal{M}$ would have to satisfy $@_i r$, which is false.

So, $@_i r \vee @_j r \notin \text{SForm}(\text{QH}(L, \Delta, W))$*.*

Thus $\text{SForm}(\text{MQH}(L, \Delta, W)) \not\subseteq \text{SForm}(\text{QH}(L, \Delta, W))$*.*

Next, there are presented several examples that illustrate how to build models (as sets of quasi-hybrid literals) for a set of formulas Δ, using Theorem 4.1.10.

Example 13. *Let* $L = \langle\{p, q\}, \{i\}\rangle$*,* $W = \{i\}$*, and* $\Delta = \{@_i(p \wedge q), @_i\neg p\}$*.*

First, notice that the formula $@_i(p \wedge q)$ *is quasi-equivalent to* $@_i p \wedge @_i q$*. Hence, any minimal model of* Δ *must contain the set* $\Omega = \{@_i\neg p, @_i q, @_i p\}$*.*

Seeing that in a minimal model all transitions (or its lack) between states must be specified, one has to explore all possibilities for $W = \{i\}$*. Hence there are exactly two minimal* QH *models of* Δ*:*

$\mathcal{M}_1 = \{@_i\neg p, @_i q, @_i p, @_i\Box\neg i\}$*;*

$\mathcal{M}_2 = \{@_i\neg p, @_i q, @_i p, @_i\Diamond i\}$*.*

The minimal model $\mathcal{M}_1$ *is represented in Figure 4.1:*

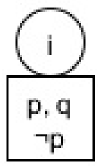

Figure 4.1: The minimal model $\mathcal{M}_1$.

And the minimal model $\mathcal{M}_2$ *is represented in Figure 4.2:*

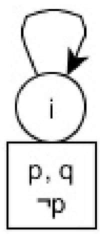

Figure 4.2: The minimal model $\mathcal{M}_2$.

In this example the minimal models have the same number of inconsistencies. However this is not always the case as we will see below.

Example 14. *Let* $L = \langle\{p, q\}, \{i, j\}\rangle$*,* $W = \{i, j\}$*, and* $\Delta = \{p \vee q, @_i\neg p\}$*.*

Not all formulas in Δ *are PBCL. Using the properties of the satisfaction operator and the method described in the proof of Theorem 4.1.10, let us make the necessary adjustments:*

1. $p \vee q \equiv_q (@_i(p \vee q)) \wedge (@_j(p \vee q)) \equiv_q (@_i p \vee @_i q) \wedge (@_j p \vee @_j q)$

Since the formula $@_i\neg p$ *is mandatory in every model, from* $(@_ip \vee @_iq)$ *it follows that* $@_iq$ *is too. The formula* $(@_jp \vee @_jq)$ *is going to be split in two, as there is no restriction to which component consider. Hence, any minimal model of* Δ *must contain one of these sets:* $\Omega_1 = \{@_i\neg p, @_iq, @_jp\}$ *or* $\Omega_2 = \{@_i\neg p, @_iq, @_jq\}$*.*

As pointed out before, all connections (or lack of them) must be specified in minimal QH *models. There will be given some examples of minimal* QH *models for the considered similarity type* L*, set of formulas* Δ*, and domain* W*, in a total of* $2^{2\times 2} = 2^4 = 16$ *possibilities, by combining quasi-hybrid literals of the form* $@_i\Diamond j, @_i\Box\neg j$*.*

For instance, for Ω_1*:*

$\mathcal{M}_1 = \{@_i\neg p, @_iq, @_jp, @_i\Box\neg i, @_i\Box\neg j, @_j\Box\neg i, @_j\Box\neg j\}$;

$\mathcal{M}_2 = \{@_i\neg p, @_iq, @_jp, @_i\Box\neg i, @_i\Diamond j, @_j\Diamond i, @_j\Box\neg j\}$.

And for Ω_2*:*

$\mathcal{M}_3 = \{@_i\neg p, @_iq, @_jq, @_i\Box\neg i, @_i\Box\neg j, @_j\Box\neg i, @_j\Box\neg j\}$;

$\mathcal{M}_4 = \{@_i\neg p, @_iq, @_jq, , @_i\Box\neg i, @_i\Diamond j, @_j\Diamond i, @_j\Box\neg j\}$.

The minimal model $\mathcal{M}_1$ *is represented in Figure 4.3:*

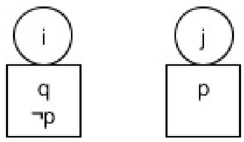

Figure 4.3: The minimal model $\mathcal{M}_1$.

The minimal model $\mathcal{M}_2$ *is represented in Figure 4.4:*

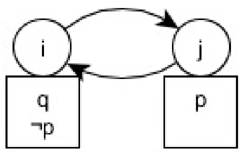

Figure 4.4: The minimal model $\mathcal{M}_2$.

The minimal model $\mathcal{M}_3$ *is represented in Figure 4.5:*

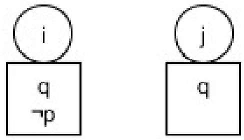

Figure 4.5: The minimal model $\mathcal{M}_3$.

The minimal model $\mathcal{M}_4$ *is represented in Figure 4.6:*

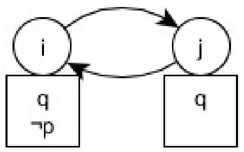

Figure 4.6: The minimal model $\mathcal{M}_4$.

Example 15. *Let* $L = \langle \{p, q\}, \{i, j\} \rangle$*,* $W = \{i, j, k\}$*, and* $\Delta = \{@_i(p \wedge q), @_j(\neg p \wedge p)\}$*.*

None of the formulas in Δ *are PBCL. Let us rearrange it:*

1. $@_i(p \wedge q) \equiv_q @_i p \wedge @_i q$

2. $@_j(\neg p \wedge p) \equiv_q @_j \neg p \wedge @_j p$

The set of formulas that are satisfiable in every minimal QH *model of* Δ *is the set:* $\Omega = \{@_i p, @_i q, @_j p, @_j \neg p\}$*.*

Since there is no information about connections between worlds, there are $2^{3\times 3} = 712$ *combinations of formulas of the form* $@_i \Diamond j, @_i \Box \neg j$*, hence there are* 712 *minimal models.*

One minimal model for Δ *is, for example:*

$$\begin{aligned}\mathcal{M} = \{ & @_i p, @_i q, @_j p, @_j \neg p, @_i \Box \neg i, @_i \Box \neg j, @_i \Box \neg k, \\ & @_j \Box \neg i, @_j \Box \neg j, @_j \Box \neg k, @_k \Box \neg i, @_k \Box \neg j, @_k \Box \neg k\}\end{aligned}$$

The minimal model $\mathcal{M}$ *is represented in Figure 4.7:*

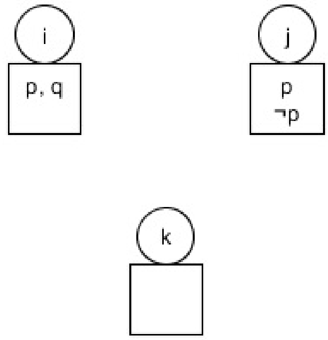

Figure 4.7: The minimal model $\mathcal{M}$.

Example 16. *Let* $L = \langle \{p\}, \{i\} \rangle$, $W = \{i\}$ *and* $\Delta = \{@_i\Box\neg p, @_i\Box p\}$.

There are exactly two minimal QH *models with domain* $W = \{i\}$*, which are:*

$\mathcal{M}_1 = \{@_i\Box\neg i\}$;

$\mathcal{M}_2 = \{@_i\Diamond i, @_i p, @_i\neg p\}$.

The minimal model $\mathcal{M}_1$ *is represented in Figure 4.8:*

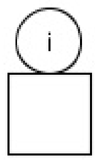

Figure 4.8: The minimal model $\mathcal{M}_1$.

The minimal model $\mathcal{M}_2$ *is represented in Figure 4.9:*

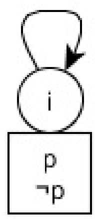

Figure 4.9: The minimal model $\mathcal{M}_2$.

Example 17. *Let* $L = \langle\{p, q, r\}, \{i, j\}\rangle$*,* $W = \{i, j\}$*, and* $\Delta = \{@_i\Diamond j \vee @_j\Diamond i, @_i\Diamond(p \vee q), @_i\Box q, @_i\Box\neg j, @_i\neg q\}$*.*

Some formulas in Δ *are not PBCL. We will make the necessary adjustments:*

1. $@_i\Diamond(p \vee q) \equiv_q @_i\Diamond(p \vee q) \equiv_q (@_i\Diamond i \wedge @_i(p \vee q)) \vee (@_i\Diamond j \wedge @_j(p \vee q))$*;*

2. $@_i\Box q \equiv_q (@_i\Box\neg i \vee @_i q) \wedge (@_i\Box\neg j \vee @_j q)$*.*

There is a set of formulas that are true in every minimal model, which is the set: $\Omega = \{@_i\Box\neg j, @_j\Diamond i, @_i\Diamond i, @_i q, @_i\neg q, @_i p\}$*.*

Again,even though there is already some information about transitions, one needs to complete it in order to have information about all connections (or their lack) between pairs of states. The only pair left is the pair (j, j)*. There are two possibilities: the connection exists, or it does not.*

So, one has the following QH *minimal models:*
$\mathcal{M}_1 = \{@_i\Box\neg j, @_j\Diamond i, @_i\Diamond i, @_i q, @_i\neg q, @_i p, @_j\Diamond j\}$*;*
$\mathcal{M}_2 = \{@_i\Box\neg j, @_j\Diamond i, @_i\Diamond i, @_i q, @_i\neg q, @_i p, @_j\Box\neg j\}$*.*

The minimal model $\mathcal{M}_1$ *is represented in Figure 4.10:*

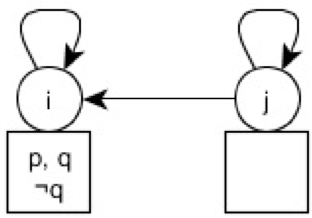

Figure 4.10: The minimal model $\mathcal{M}_1$.

The minimal model $\mathcal{M}_2$ *is represented in Figure 4.11:*

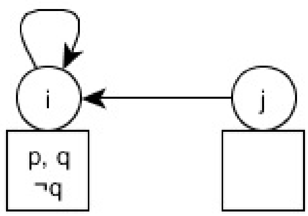

Figure 4.11: The minimal model $\mathcal{M}_2$.

The interest in using $\text{MQH}(L, \Delta, W)$ rather than $\text{QH}(L, \Delta, W)$, for a set of formulas Δ, is that models in $\text{MQH}(L, \Delta, W)$ do not contain irrelevant information for analysing inconsistency, and no useful information is lost.

4.1.2 The Inconsistency Measure

A theory may have different minimal QH models depending on the hybrid similarity type and domain. Now it will be introduced a way to measure the inconsistency of a QH model. This measure is a ratio between 0 and 1 whose numerator is the number of inconsistencies in the model, and whose denominator is the total possible number of inconsistencies in the underlying hybrid similarity type.

To make the notation in the next definition simpler, it is defined the set of *inconsistency literals over L and W* as

$$\text{IL}(L, W) = \{@_i p \mid i \in W, p \in \text{Prop}\}$$

Definition 4.1.14. *For a* QH *model $\mathcal{M}$,*

$$Conflictbase(\mathcal{M}) = \{@_i p \in \text{IL}(L, W) \mid @_i p \in \mathcal{M} \ \& \ @_i \neg p \in \mathcal{M}\}$$

The inconsistency measure comes in the form:

Definition 4.1.15. *The* measure of inconsistency *for a model $\mathcal{M}$ in the context of a hybrid similarity type L and domain W is given by the $ModelInc$ function giving a value between 0 and 1 as follows:*

$$ModelInc(\mathcal{M}, L, W) = \frac{|Conflictbase(\mathcal{M})|}{|\text{IL}(L, W)|}$$

The $ModelInc$ function is anti-monotonic in the following sense:

Theorem 4.1.16. *Let L_1, L_2 be hybrid similarity types and W_1, W_2 non-empty sets. Then,*

- *If $L_1 \subseteq L_2$ then $ModelInc(M, L_1, W) \geq ModelInc(M, L_2, W)$*
- *If $W_1 \subseteq W_2$ then $ModelInc(M, L, W_1) \geq ModelInc(M, L, W_2)$*

Example 18. *In this example there will be considered the minimal models presented in Examples 13–17 and their measures of inconsistency, i.e., their $ModelInc$ function will be computed.*

1. *From Example 13, $Conflictbase(\mathcal{M}_1) = \{@_i p\}$. Then,*

$$ModelInc(\mathcal{M}_1, L, W) = \frac{|Conflictbase(\mathcal{M}_1)|}{|\text{IL}(L, W)|} = \frac{1}{2}$$

The same happens for $\mathcal{M}_2$.

2. *From Example 14,* $Conflictbase(\mathcal{M}_1) = \{\}$. *Then,*

$$ModelInc(\mathcal{M}_1, L, W) = \frac{|Conflictbase(\mathcal{M}_1)|}{|\mathrm{IL}(L, W)|} = \frac{0}{4} = 0$$

The same happens for $\mathcal{M}_2$, $\mathcal{M}_3$ *and* $\mathcal{M}_4$.

3. *From Example 15,* $Conflictbase(\mathcal{M}) = \{@_j p\}$. *Then,*

$$ModelInc(\mathcal{M}, L, W) = \frac{|Conflictbase(\mathcal{M})|}{|\mathrm{IL}(L, W)|} = \frac{1}{4}$$

4. *From Example 16,* $Conflictbase(\mathcal{M}_1) = \{\}$. *Then,*

$$ModelInc(\mathcal{M}_1, L, W) = \frac{|Conflictbase(\mathcal{M}_1)|}{|\mathrm{IL}(L, W)|} = \frac{0}{1} = 0$$

However, for $\mathcal{M}_2$, $Conflictbase(\mathcal{M}_2) = \{@_i p\}$. *Hence,*

$$ModelInc(\mathcal{M}_2, L, W) = \frac{|Conflictbase(\mathcal{M}_1)|}{|\mathrm{IL}(L, W)|} = \frac{1}{1} = 1$$

5. *From Example 17,* $Conflictbase(\mathcal{M}_1) = \{\}$. *Then,*

$$ModelInc(\mathcal{M}_1, L, W) = \frac{|Conflictbase(\mathcal{M}_1)|}{|\mathrm{IL}(L, W)|} = \frac{0}{6} = 0$$

The same happens for $\mathcal{M}_2$.

Example 16 shows that minimal models for a certain Δ, over the same hybrid similarity type and domain, may have different number of inconsistencies and consequently the $ModelInc$ function assigns different values to them.

In order to consider minimal models with the least number of inconsistencies, the class of minimal models will be restricted by considering the so called class of preferred models which are the ones with a minimal conflictbase. This follows the approach of Grant and Hunter in [58]. This idea was already adopted in the context of the minimal four-valued logic [10].

Definition 4.1.17. *Let* L *be a hybrid similarity type,* $\Delta \subseteq \mathrm{Form}_{\mathrm{NNF}(@)}(\mathrm{L})$ *and* W *a non-empty set. The set of* preferred QH models *for* Δ *with domain* W *is given by* $\mathrm{PQH}(L, \Delta, W)$ *as follows:*

$$\begin{aligned} \mathrm{PQH}(L,\Delta,W) &= \{\mathcal{M}\in \mathrm{MQH}(L,\Delta,W)|\ for\ all\ \mathcal{M}'\in \mathrm{MQH}(L,\Delta,W), \\ & \quad |Conflictbase(\mathcal{M})| \leq |Conflictbase(\mathcal{M}')|\} \end{aligned}$$

Theorem 4.1.18. *Let L be a hybrid similarity type, $\Delta \subseteq \mathrm{Form}_{\mathrm{NNF}(@)}(\mathrm{L})$ and W_1, W_2 non-empty sets with the same number of elements.*

If $\mathcal{M}_1 \in \mathrm{PQH}(L,\Delta,W_1)$ and $\mathcal{M}_2 \in \mathrm{PQH}(L,\Delta,W_2)$, then $ModelInc(\mathcal{M}_1, L, W_1) = ModelInc(\mathcal{M}_2, L, W_2)$.

Proof. Suppose that $\mathcal{M}_1 \in \mathrm{PQH}(L,\Delta,W_1)$. Construct a model $\mathcal{M}_2 \in \mathrm{PQH}(L,\Delta,W_2)$ as follows: Define a bijective function $F : W_1 \to W_2$ such that for all $i \in \mathrm{Nom}, F(i) = i$. Write for each QH literal $\alpha \in L(W)$ $F(\alpha)$ for the QH literal where each $w \in W_1$ is replaced by $F(w)$. Let $\mathcal{M}_2 = \{F(\alpha)|\alpha \in \mathcal{M}_1\}$. Clearly, $\mathcal{M}_2 \in \mathrm{QH}(L,\Delta,W_2)$. $\mathcal{M}_2$ must also be minimal because if a proper subset of $\mathcal{M}_2$ was a QH model, by applying F^{-1} one could obtain a QH model with W_1 for the domain, thus making $\mathcal{M}_1$ not minimal. Similarly one can show that $\mathcal{M}_2$ is preferred. The result now follows.

□

Now, for a hybrid similarity type L, it will be defined the extrinsic inconsistency of a set of formulas Δ:

Definition 4.1.19. *We define the* extrinsic inconsistency *of a theory Δ in a hybrid similarity type L, $TheoryInc(\Delta, L)$ as a sequence $\langle r_1, \ldots, r_n\rangle$ where for all $n \geq 1$, let W_n be a domain of size n. If there is a model $\mathcal{M} \in \mathrm{PQH}(L,\Delta,W_n)$, then let $r_n = ModelInc(\mathcal{M}, L, W_n)$, otherwise, let $r_n = *$. We use $*$ as a kind of a null value.*

This sequence captures how the inconsistency of a theory Δ in a hybrid similarity type L evolves with increasing domain size. At one extreme, there are cases where we do not have inconsistencies for any domain size; for example for the trivial case when $\Delta = \emptyset$, $TheoryInc(\Delta, L) = \langle 0, 0, \ldots\rangle$. At the other extreme, there are theories Δ which are completely inconsistent, *i.e.*, $TheoryInc(\Delta, L) = \langle 1, 1, \ldots\rangle$, such as $\Delta = \{p \wedge \neg p : \text{for all } p \in \mathrm{Prop}\}$.

Example 19. *Some examples on the computation of the intrinsic inconsistency of some theories:*

1. *Let $L = \langle\{p\}, \{i, j\}\rangle$ and $\Delta = \{@_i\neg p \vee @_j p, @_i p\}$.*

$$TheoryInc(\Delta, L) = \langle *, 0, 0, 0, \ldots\rangle$$

2. Let $L = \langle\{p, q\}, \{i, j\}\rangle$ *and* $\Delta = \{@_i(p \wedge q), @_j(\neg p \wedge p)\}$.

$$TheoryInc(\Delta, L) = \langle *, \frac{1}{4}, \frac{1}{6}, \dots \rangle$$

3. Let $L = \langle\{p\}, \{i\}\rangle$ *and* $\Delta = \{@_i p, @_i \neg p\}$.

$$TheoryInc(\Delta, L) = \langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle$$

4. Let $L = \langle\{p, q\}, \{i, j\}\rangle$ *and* $\Delta = \{@_i \neg p, @_j \neg q, p \wedge \neg p \wedge q\}$.

$$TheoryInc(\Delta, L) = \langle *, \frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \dots \rangle$$

5. Let $L = \langle\{p\}, \{i, j\}\rangle$ *and* $\Delta = \{i \vee j\}$.

$$TheoryInc(\Delta, L) = \langle *, 0, *, *, \dots \rangle$$

Some properties of $TheoryInc$ are now announced and proved:

Proposition 4.1.20. *Let* $L = \langle \mathrm{Prop}, \mathrm{Nom} \rangle$ *be a hybrid similarity type,* $\Delta \subseteq \mathrm{Form}_{\mathrm{NNF}(@)}(\mathrm{L})$ *and* $TheoryInc(\Delta, L) = \langle x_1, x_2, \dots \rangle$. *If* $|\mathrm{Nom}| = k$ *then for all* i *such that* $1 \leq i < k, x_i = *$*; moreover, if* $x_{k+1} \neq *$ *then for all* $i > k, x_i \neq *$.

Proof. The reason for the asterisks is that the domain, according to our definition, must have at least as many elements as the number of nominals in L, that is, $|\mathrm{Nom}|$.

Suppose now that $x_{k+1} \neq *$. Hence there is a preferred model $\mathcal{M}$ of Δ with domain W which has $k + 1$ elements.

Let u be a new element not in W, *i.e.* the element $k + 1$ and z be the element of W that is not named by a nominal, *i.e.*, the element $k + 2$. We define a new model $\mathcal{M}'$ over L with domain $W \cup \{u\}$, which has cardinality $k + 2$, admitting the following representation:

$$\begin{aligned}\mathcal{M}' = \mathcal{M} &\cup \{@_u \Diamond w \mid w \in W,\ @_z \Diamond w \in \mathcal{M}\} \cup \{@_w \Box \neg u \mid w \in W\} \cup \\ &\{@_u \Box \neg w \mid w \in W,\ @_z \Box \neg w \in \mathcal{M}\} \cup \{@_u p \mid p \in \mathrm{Prop},\ @_z p \in \mathcal{M}\} \cup \\ &\{@_u \neg p \mid p \in \mathrm{Prop},\ @_z \neg p \in \mathcal{M}\}.\end{aligned}$$

The claim is that:

CLAIM: *for any* $\varphi \in \mathrm{Form}_{\mathrm{NNF}(@)}(\mathrm{L})$,
$\mathcal{M}, z \models_s \varphi$ *iff* $\mathcal{M}', u \models_s \varphi$ *and*
$\mathcal{M}, z \models_s \sim \varphi$ *iff* $\mathcal{M}', u \models_s \sim \varphi$.

In fact, this can be proved by induction. The base step is trivial. The steps for the conjunction, the satisfaction operator and the modal operators are also straightforward. Let us see what happens with the disjunction:

Let $\varphi := \varphi_1 \vee \varphi_2$. It is assumed that the claim is true for formulas shorter than φ. (Note that a formula ψ_1 is shorter than a formula ψ_2 iff the formula ψ_1 contains less boolean operators than the formula ψ_2; however, the elements that compose ψ_1 are not necessarily the same elements that compose ψ_2.) Then,

$$
\begin{array}{lll}
\mathcal{M}, z \models \varphi & \text{iff} & \mathcal{M}, z \models_s \varphi_1 \vee \varphi_2 \\
& \text{iff} & [\mathcal{M}, z \models_s \varphi_1 \text{ or } \mathcal{M}, z \models_s \varphi_2] \text{ and} \\
& & [\mathcal{M}, z \models_s \sim \varphi_2 \Rightarrow \mathcal{M}, z \models_s \varphi_1] \text{ and} \\
& & [\mathcal{M}, z \models_s \sim \varphi_1 \Rightarrow \mathcal{M}, z \models_s \varphi_2] \\
& \text{iff}^{(*)} & [\mathcal{M}', u \models_s \varphi_1 \text{ or } \mathcal{M}', u \models_s \varphi_2] \text{ and} \\
& & [\mathcal{M}', u \models_s \sim \varphi_1 \Rightarrow \mathcal{M}', u \models_s \varphi_2] \text{ and} \\
& & [\mathcal{M}', u \models_s \sim \varphi_2 \Rightarrow \mathcal{M}', u \models_s \varphi_1]
\end{array}
$$

The step (*) is by induction hypothesis.

As a consequence, one has that $\mathcal{M}' \models_s \Delta$. And consequently $x_{k+2} \neq *$.

This argument can be recursively applied.

□

Proposition 4.1.21. *Let $L = \langle \text{Nom}, \text{Prop} \rangle$ be a hybrid similarity type with $|\text{Nom}| = k$, $\Delta \subseteq \text{Form}_{\text{NNF}(@)}(\text{L})$ and $TheoryInc(\Delta, L) = \langle x_1, x_2, \dots \rangle$. If $x_{k+1} = 0$ then for all $j > k+1, x_j = 0$.*

Proof. The same construction used in the previous proposition can also be applied here. The model $\mathcal{M}'$ obtained has 0 inconsistencies since $\mathcal{M}$ has 0 inconsistencies.

□

There can be adopted a *lexicographic ordering*, denoted by the symbol $\preceq$ over the tuples generated by the $TheoryInc$ function:

Definition 4.1.22. *Let L_1 and L_2 be hybrid similarity types and $\Delta_1, \Delta_2 \subseteq \text{Form}_{\text{NNF}(@)}(\text{L})$. Let $TheoryInc(\Delta_1, L_1) = \langle r_1, r_2, \dots \rangle$ and $TheoryInc(\Delta_2, L_2) = \langle s_1, s_2, \dots \rangle$. One says that $TheoryInc(\Delta_1, L_1) \preceq TheoryInc(\Delta_2, L_2)$ iff for all $i \geq 1$, $r_i \leq s_i$ or $r_i = *$ or $s_i = *$.*

Writting $TheoryInc(\Delta_1, L_1) \prec TheoryInc(\Delta_2, L_2)$ abbreviates $TheoryInc(\Delta_1, L_1) \preceq TheoryInc(\Delta_2, L_2)$ and $TheoryInc(\Delta_1, L_1) \neq TheoryInc(\Delta_2, L_2)$.

In case $L_1 = L_2 (= L)$ one says that Δ_1 has smaller than or equal inconsistency as Δ_2 iff $TheoryInc(\Delta_1, L) \preceq TheoryInc(\Delta_2, L)$ and this is denoted by $\Delta_1 \leq^L_{inc} \Delta_2$.

Example 20. *Let $L = \langle \{p\}, \{i\} \rangle, \Delta_1 = \{@_i p\}$ and $\Delta_2 = \{@_i p, @_i \neg p\}$.*

$$TheoryInc(\Delta_1, L) = \langle 0, 0, 0, \dots \rangle$$

$$TheoryInc(\Delta_2, L) = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$$

Then $\Delta_1 \leq^L_{inc} \Delta_2$.

Proposition 4.1.23. *Let* L *be a hybrid similarity type,* $\Delta_1, \Delta_2 \subseteq \mathrm{Form}_{\mathrm{NNF}(@)}(\mathrm{L})$. *If* $\Delta_1 \subseteq \Delta_2$, *then* $TheoryInc(\Delta_1, L) \preceq TheoryInc(\Delta_2, L)$.

Proof. Additional statements may add but cannot subtract inconsistencies.

□

Proposition 4.1.24. *Let* $L_1 = \langle \mathrm{Prop}_1, \mathrm{Nom}_1 \rangle$, *and* $L_2 = \langle \mathrm{Prop}_2, \mathrm{Nom}_2 \rangle$ *be two hybrid similarity types, and* $\Delta \subseteq \mathrm{Form}_{\mathrm{NNF}(@)}(L_1)$. *If* $L_1 \subseteq L_2$, *i.e.,* $\mathrm{Prop}_1 \subseteq \mathrm{Prop}_2, \mathrm{Nom}_1 \subseteq \mathrm{Nom}_2$, *then*

$$TheoryInc(\Delta, L_2) \preceq TheoryInc(\Delta, L_1).$$

Proof. Let n be such that $r_n \neq *$. For a domain W_n, such that $|W_n| = n$, $\mathrm{IL}(L_1(W_n)) \subseteq \mathrm{IL}(L_2(W_n))$.

The fact that $\mathrm{Prop}_1 \subseteq \mathrm{Prop}_2$ does not interfere with the construction of preferred QH models of Δ.

By Theorem 4.1.18, two preferred models for the same set of formulas Δ and domain W_n have the same size of $Conflictbase$. Hence, for each W_n, for $\mathcal{M}_{1,n} \in \mathrm{PQH}(L_1, \Delta, W_n), \mathcal{M}_{2,n} \in \mathrm{PQH}(L_2, \Delta, W_n)$, one has that $|Conflictbase(\mathcal{M}_{1,n})| = |Conflictbase(\mathcal{M}_{2,n})|$. But the denominator of $ModelInc(\mathcal{M}_{1,n}, L_1, W_n), \mathrm{IL}(L_1(W_n))$, is smaller than the denominator of $ModelInc(\mathcal{M}_{2,n}, L_2, W_n), \mathrm{IL}(L_2(W_n))$, for all W_n. Which means that $TheoryInc(\Delta, L_2) \preceq TheoryInc(\Delta, L_1)$.

□

The next example shows that if $\Delta_1 \subseteq \Delta_2$ and $L_1 \subseteq L_2$ then it is not necessarily the case that $TheoryInc(\Delta_1, L_1) \preceq TheoryInc(\Delta_2, L_2)$.

Example 21. *Let*

$$L_1 = \langle \{p\}, \{i\} \rangle \quad and \quad \Delta_1 = \{@_i p\}$$

$$L_2 = \langle \{p\}, \{i\} \rangle \quad and \quad \Delta_2 = \{@_i p, @_i \neg p\}$$

$$L_3 = \langle \{p, q\}, \{i\} \rangle \quad and \quad \Delta_3 = \{@_i p, @_i \neg p, @_i q\}$$

In this case,

$$TheoryInc(\Delta_1, L_1) \prec TheoryInc(\Delta_2, L_2)$$

$$TheoryInc(\Delta_3, L_3) \prec TheoryInc(\Delta_2, L_2)$$

Definition 4.1.25. *For* $\Delta_1, \Delta_2 \subseteq \text{Form}_{\text{NNF}(@)}(L), \Delta_1$ *is* QH-equivalent *to* Δ_2 *if, for all* $\mathcal{M}$*,*

$$\mathcal{M} \textit{ is a } \text{QH} \textit{ model of } \Delta_1 \textit{ iff } \mathcal{M} \textit{ is a } \text{QH} \textit{ model of } \Delta_2.$$

Proposition 4.1.26. *Let L be a hybrid similarity type,* $\Delta_1, \Delta_2 \subseteq \text{Form}_{\text{NNF}(@)}(\text{L})$*. If* Δ_1 *is* QH*-equivalent to* Δ_2*, then*

$$TheoryInc(\Delta_1, L) = TheoryInc(\Delta_2, L).$$

Proof. Since Δ_1 is QH-equivalent to Δ_2, any preferred QH model $\mathcal{M}'$ of Δ_1 is a preferred QH model of Δ_2.

Therefore, for a fixed domain W_n one may consider the same preferred model and consequently the n^{th} element in the sequences $TheoryInc(\Delta_1, L)$ and $TheoryInc(\Delta_2, L)$ are equal. □

Resembling the definition of extrinsic inconsistency, it is introduced the definition of intrinsic inconsistency of a set of formulas Δ, for a specific similarity type L defined with recourse to Δ, as follows:

Definition 4.1.27. *For a given theory Δ, let L^{Δ} be a hybrid similarity type that contains exactly the propositional variables and nominals that occur in Δ. The* intrinsic inconsistency *of Δ, $TheoryInc(\Delta)$, is defined as $TheoryInc(\Delta) = TheoryInc(\Delta, L^{\Delta})$.*

So the measure of intrinsic inconsistency of a theory $TheoryInc(\Delta)$, delineates the degree of the theory in its own terms; whereas $TheoryInc(\Delta, L)$, the extrinsic inconsistency of a theory, delineates the degree of the theory with respect to the hybrid similarity type L.

Example 22. *Let* $L = \langle \{p, q, r\}, \{i, j\} \rangle, \Delta = \{@_i p, @_j \neg p, @_i r, @_i \neg r\}$*.*

$$TheoryInc(\Delta, L) = \langle *, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \dots \rangle$$

$$TheoryInc(\Delta) = \langle *, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots \rangle$$

When $TheoryInc(\Delta_1) \preceq TheoryInc(\Delta_2)$ it is said that Δ_1 has smaller than or equal inconsistency as Δ_2.

4.2 Inconsistency and Bisimulation

In this section the notion of bisimulation for hybrid bistructures is generalized.

Definition 4.2.1. *Let* $L = \langle \text{Prop}, \text{Nom} \rangle$ *be a hybrid similarity type and* $E = (W, R, N, V^+, V^-)$, $E' = (W', R', N', V'^+, V'^-)$ *be two hybrid bistructures. A relation* $Z \subset W \times W'$ *is a* paraconsistent bisimualtion *if* Z *is a bisimulation between the hybrid structures* $E = (W, R, N, V^+)$ *and* $E' = (W', R', N', V'^+)$ *and also a bisimulation between* $E = (W, R, N, V^-)$ *and* $E' = (W', R', N', V'^-)$.

The following proposition reformulates the notion of bisimulation in terms of representation of bistructures.

Proposition 4.2.2. *Let* $L = \langle \text{Prop}, \text{Nom} \rangle$ *be a hybrid similarity type,* E, E' *be two hybrid bistructures and* $\mathcal{M}$, $\mathcal{M}'$ *their syntactic representations. A relation* $Z \subseteq W \times W'$ *is a paraconsistent bisimulation between* E *and* E' *iff*

- *for each* $i \in \text{Nom}$ $N(i) Z N'(i)$*;*
- *for every pair* $(w, w') \in Z$ *we have:*
  - *Atomic conditions:*
    - $@_w p \in \mathcal{M}$ *iff* $@_{w'} p \in \mathcal{M}'$, *for all* $p \in \text{Prop}$.
    - $@_w \neg p \in \mathcal{M}$ *iff* $@_{w'} \neg p \in \mathcal{M}'$, *for all* $p \in \text{Prop}$.
    - *for all* $i \in \text{Nom}$, $N(i) = w$ *iff* $N'(i) = w'$.
  - *if* $@_w \Diamond u \in \mathcal{M}$ *for some* $u \in W$, *then there is some* $u' \in W'$ *such that* $@_{w'} \Diamond u' \in \mathcal{M}'$ *and* $u Z u'$ ***(Zig)****,*
  - *if* $@_{w'} \Diamond u' \in \mathcal{M}'$ *for some* $u' \in W'$, *then there is some* $u \in W$ *such that* $@_w \Diamond u$ *and* $u Z u'$ ***(Zag)****.*

Two bisimilar bistructures may have different conflictbases. Clearly, this is the case when the witness bisimulation is not total or not surjective. Moreover, these two conditions, together, are not sufficient as shown by the following examples:

Example 23. *The bisimulation represented in Figure 4.12, despite being a total and surjective relation, does not imply the same conflictbase in both bistructures.*

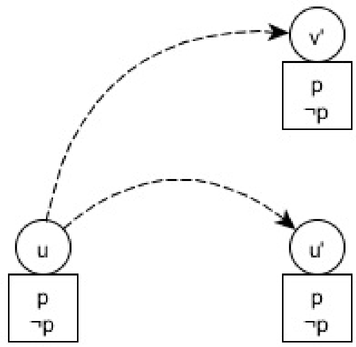

Figure 4.12: Bisimulation 1.

Example 24. *Once again, the bisimulation represented in Figure 4.13, a surjective function, does not guarantee a common conflictbase for bisimilar bistructures.*

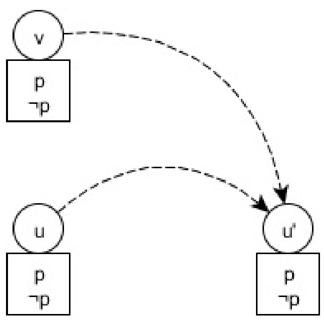

Figure 4.13: Bisimultion 2.

Example 25. *However, a bisimulation which is a bijective function between bistructures allows having always the same conflictbase in bisimilar bistructures. The result is proved below, and an example is given in Figure 4.14.*

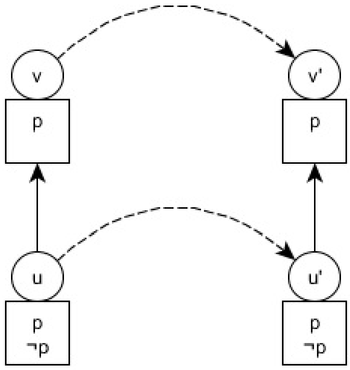

Figure 4.14: Bisimulation 3.

Next theorem states that two bisimilar bistructures such that the bisimulation is a bijective bounded morphism have the same conflictbase.

Theorem 4.2.3. *Let* $L = \langle \mathrm{Prop}, \mathrm{Nom} \rangle$ *be a hybrid similarity type,* $E = (W, R, N, V^+, V^-)$*,* $E' = (W', R', N', V'^+, V'^-)$ *be two hybrid bistructures and* $\mathcal{M}$*,* $\mathcal{M}'$ *their representations.*

If E *and* E' *are bisimilar via a bijective function* Z*, then*

$$ModelInc(\mathcal{M}, L, W) = ModelInc(\mathcal{M}', L, W').$$

Proof. A bijective function between W and W' means that both have the same cardinality.

Moreover, from the atomic conditions in 4.2.2, the conflictbases for each bistructure coincide.

□

Chapter 5

Applications

Several variants of paraconsistent logics have been proposed to answer different problems in specific applications [77]. There are many fields where a paraconsistent version of hybrid logics is welcome. For now, let us concentrate in the Quasi-Hybrid (Basic) Logic.

In Computer Science, subdomains like requirements engineering ([51]), artificial intelligence ([52]) and automated reasoning within information processing knowledgebases ([47]), are among the most relevant areas in which paraconsistent logic can address theoretical difficulties raised by inconsistent data. As an example, in Section 5.1 it will be considered a robot which needs to make decisions based on inconsistent informations. This is an adaptive approach, since it will be considered that at each state the robot has more information, and makes decisions based on them, sometimes even excludes some informations face to new evidences, as reported in Subsection 2.2.4.

In the medical practice, two or more physicians may give two different and even contradictory diagnoses for the same symptoms, none of them to be dismissed, which is the perfect example of a discussive approach, described in Subsection 2.2.2. An application in the field of medicine concerns the health care flow of a patient in a hospital and it is described in Section 5.2.

5.1 Robotics

Recall the robot EMMY III ([89]), which uses paraconsistent logic in order to determine its movements. It is based on it that this small example is presented. The main goal is to observe how can a robot that may receive contradictory information about the presence (or not) of objects in from of it find a way out of a sort of labyrinth.

There will be made a schematic approach rather than a formal one.

Consider a robot with two positioning sensors, covering an amplitude of 90^o: the one on the left covers the first half, and the one on the right covers the second, as in Figure 5.1.

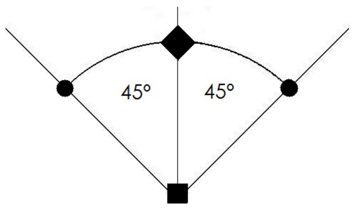

Figure 5.1: The range of each sensor of the robot.

In order to decide if there is something in front of it, the robot will have to consider both evaluations from the sensors: if one of the sensors determines the presence of an object, the robot can not move forward, until that information is denied. The range of the sensors is obviously limited, and will be the measure of the smallest diagonal in the grid presented in Figure 5.2.

The robot is asked to go from the start point to the finish point in the following grid, where the squares represent empty and distinct places, and the octagons represent two distinct places occupied by objects.

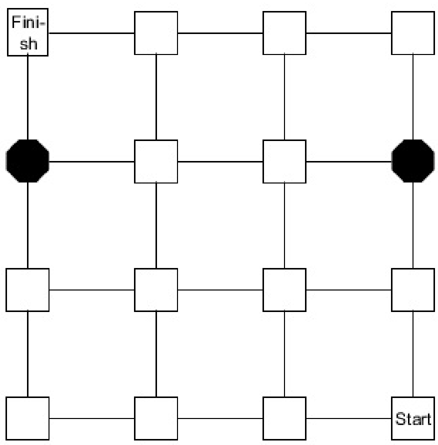

Figure 5.2: The grid.

The robot can rotate precisely 45° at a time, but can only move in four directions: north, south, east and west, and preferentially north and west; the robot can never move in the diagonal.

In order to simplify the schemes obtained, at each position the robot will only collect information regarding the direction it is heading to. In case it finds an object, it has to change direction and furthermore, will collect information at each 45° of the turn. A black small triangle will be used to denote that the sensors did not detect any object, and a black small square will denote otherwise.

Starting by heading west, the first three steps are trivial:

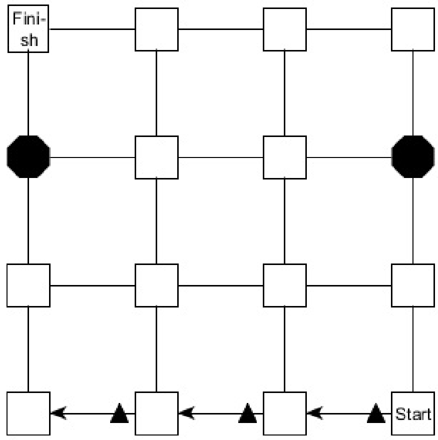

Figure 5.3: Robot's position after 3 steps.

It is now the case that the robot will move north. Then it will face some changes of scenario, as represented in the following scheme:

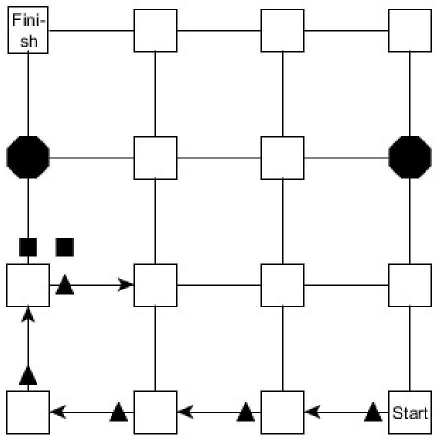

Figure 5.4: Robot's position after 5 steps.

Since the robot detected an object, it is obliged to turn right, and, as explained before, in the middle of the turn, the robot captured the information about the objects detected.

Let us see what happens next:

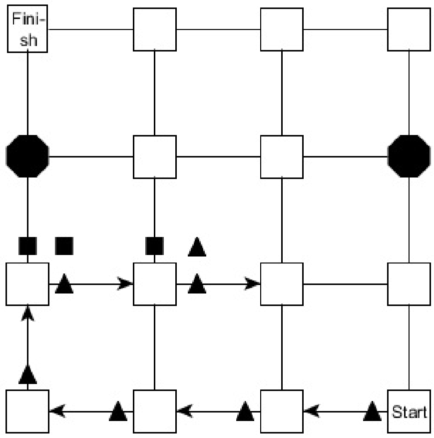

Figure 5.5: Robot's position after 6 steps.

The robot is required to preferentially move north and left, but since going left means going back and that makes no sense since there are other options to explore, and going north is not possible, the robot will continue moving to the right side.

The next step is very interesting:

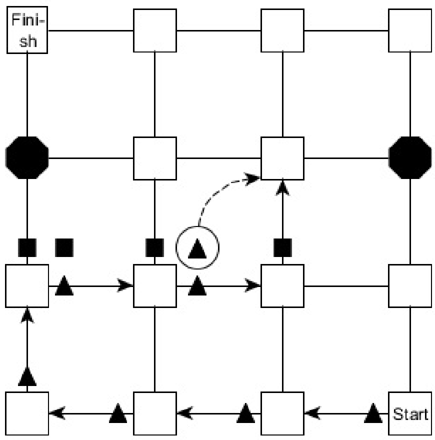

Figure 5.6: Robot's position after 7 steps.

The robot heads north and is confronted with an inconsistency, because an object was detected, but, according to the information gathered in the previous state, that place was not occupied. This inconsistency is due to the fact that only one sensor captured an object, in this case, the sensor on the right side. The presence of an object is thus denied, and the robot can move directed north.

The remaining steps are trivial, however the final step is a little tricky but it is assumed that the finish point can not be occupied by an object, and so the final scheme comes as:

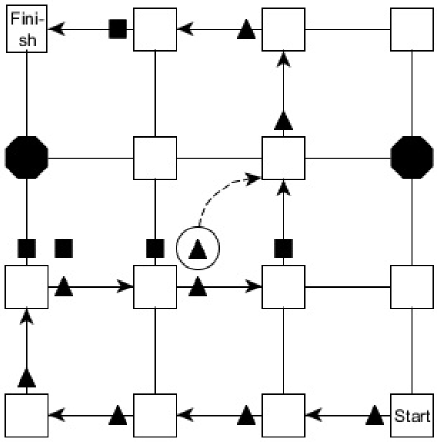

Figure 5.7: The Robot's path from the start point to the finish point.

Based on that, considering that the domain is the number of places represented, namely 16, and that the only propositional symbol corresponds to the presence of an object, thus inconsistencies occur when an object is detected and later not detected, or *vice-versa*, which happens twice, at the finish point, and at the middle point explained after Figure 5.6, one has that the measure of inconsistency for this model is given by:

$$ModelInc(\mathcal{M}, L, W) = \frac{|Conflictbase(\mathcal{M})|}{|IL(L, W)|} = \frac{2}{16} = \frac{1}{8}$$

Programming a robot capable of moving in all directions would be much more challenging, but this intended to be a small example, inspired by the robot EMMY III ([89]).

5.2 Health Care Flow of a Patient

Inconsistencies appear frequently in knowledge representation in the health care area. Medical Informatics deals with health care knowledge that represents the daily behavior of a patient in the health system and an effective procedure to manage such flow of information should be studied. Moreover, Medical Informatics is one field where the ability to reason with inconsistent information is crucial because through the health care process in a hospital, a patient can receive different, even contradictory, diagnoses from different physicians, and the same can happen with medical treatments: they can exhibit contradictory results. Therefore, it is worth to develop easy mechanisms that offer a safe way to 'live' with inconsistency. Paraconsistent reasoning should help in this context, namely in prevention of diseases, diagnosis and therapy of patients.

Being such an important issue, together with Professor Manuel A. Martins, we submitted a paper entitled *"Inconsistencies in Health Care Knowledge"*.

Let us consider, just to exemplify the idea, a small fragment of the clinical flow of a patient in a central hospital, represented in Figure 5.8.

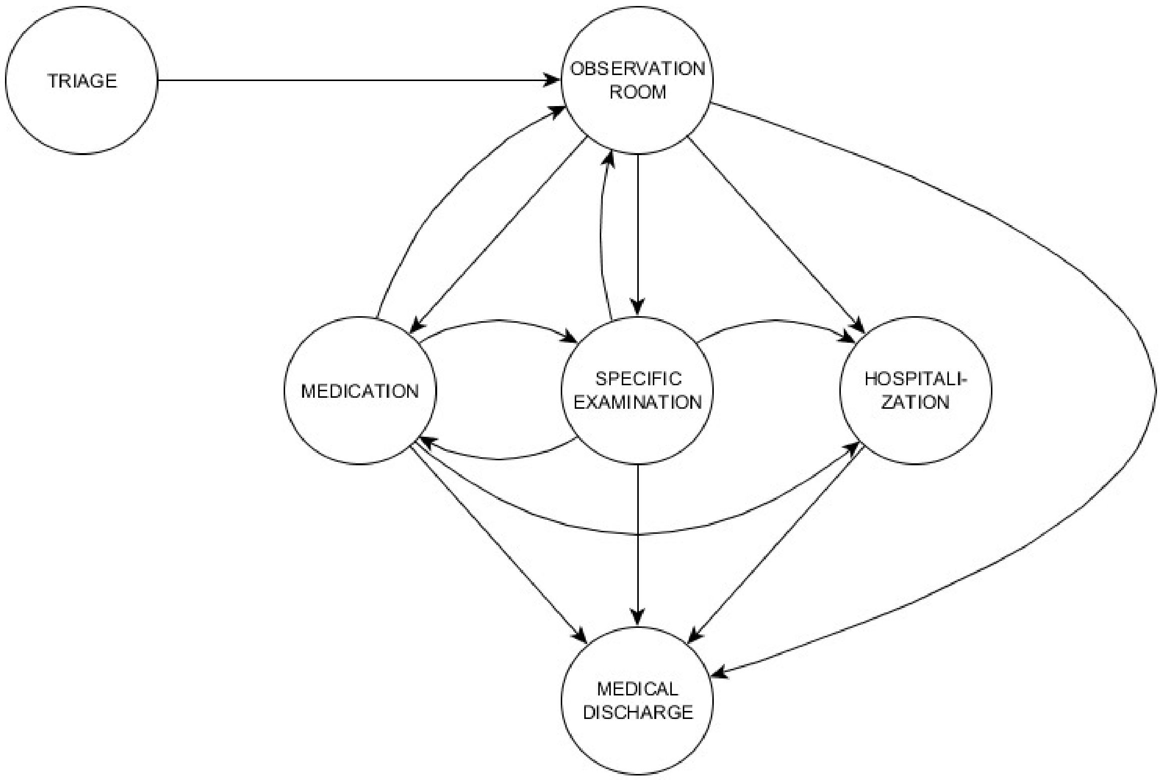

Figure 5.8: The care delivery process.

A patient coming into a hospital is consulted at the triage station. The next step in the care delivery process is the observation of the patient by physicians at an observation room. From this stage, several things can happen: (1) the patient may need to take some medication, (2) the patient may need to take an examination, (3) the patient may need to be hospitalized, or (4) the patient may be discharged. If the medication has no effect, or the examination is inconclusive, the patient returns to the previous state. The following situations may also occur: (i) the patient takes medication and after takes an examination or vice-versa, (ii) the patient after being medicated or examined needs to be hospitalized, (iii) the patient only needed one of the following — medication/examination/hospitalization — and is discharged after that treatment. This representation corresponds to the set Δ, which must be satisfied in every model, and that comes in the form of:

$$\Delta = \{@_{Triage}(\Box Obs.room \wedge \Diamond Obs.room), @_{Obs.room}\Diamond Medication \vee @_{Obs.room}\Diamond Examination \vee$$

$$@_{Obs.room}\Diamond Hospitalization \vee @_{Obs.room}\Diamond Medical\ discharge,$$

$$@_{Medication}\Diamond Examination \vee @_{Medication}\Diamond Hospitalization \vee @_{Medication}\Diamond Medical\ discharge,$$

$$@_{Examination}\Diamond Medication \vee @_{Examination}\Diamond Hospitalization \vee$$

$$@_{Examination}\Diamond Medical\ discharge, @_{Hospitalization}\Box Medical\ discharge.\}$$

The pathway of cares of the patient can be represented by a Kripke frame and the reports made at each stage are represented resorting to a decoupled valuation. Note that the decouple of the valuation is mandatory since very often the diagnosis is not deterministic and we have to allow inconsistencies; actually a team of physicians may not agree in the diagnosis of a specific disease or even an exam can be inconclusive (for example a CT Screening for lung cancer may hold inconclusive evidence).

Propositional variables are used to represent data in the patient report that may vary from one observation to another. More specifically, a propositional variable can be seen as a health feature observed in the patient (for example fever, cancer, cough, pallor). Some of them are classical, however some others are paraconsistent. Nominals are used to name referential states (*i.e.* important moments of diagnosis), while modalities are used to label transitions in the flow, for example transitions induced by the administration of a certain medicine or by a specific examination.

Practical Example

In this practical example, transitions are not labeled – they only mean the displacement of the patient between states.

Figure 5.9 represents the pathway of care of patient A that appears at the triage with cough. At the observation room, he is diagnosed with disease X. Another physician disagrees and diagnoses him not with disease X and with disease Y. The patient takes an examination that is conclusive: the patient has disease X but not disease Y, then the patient takes some medication, however the cough does not disappear so the patient is hospitalized until it does and then discharged. The medical pathway of the patient A can be seen as a paraconsistent model, where Prop $= \{cough, disease\ X, disease\ Y\}$ and $W = \{Triage, Obs.\ room, Medication, Examination,\ Hospitalization, MedicalDischarge\}$ and the valuations are given in Figure 5.9.

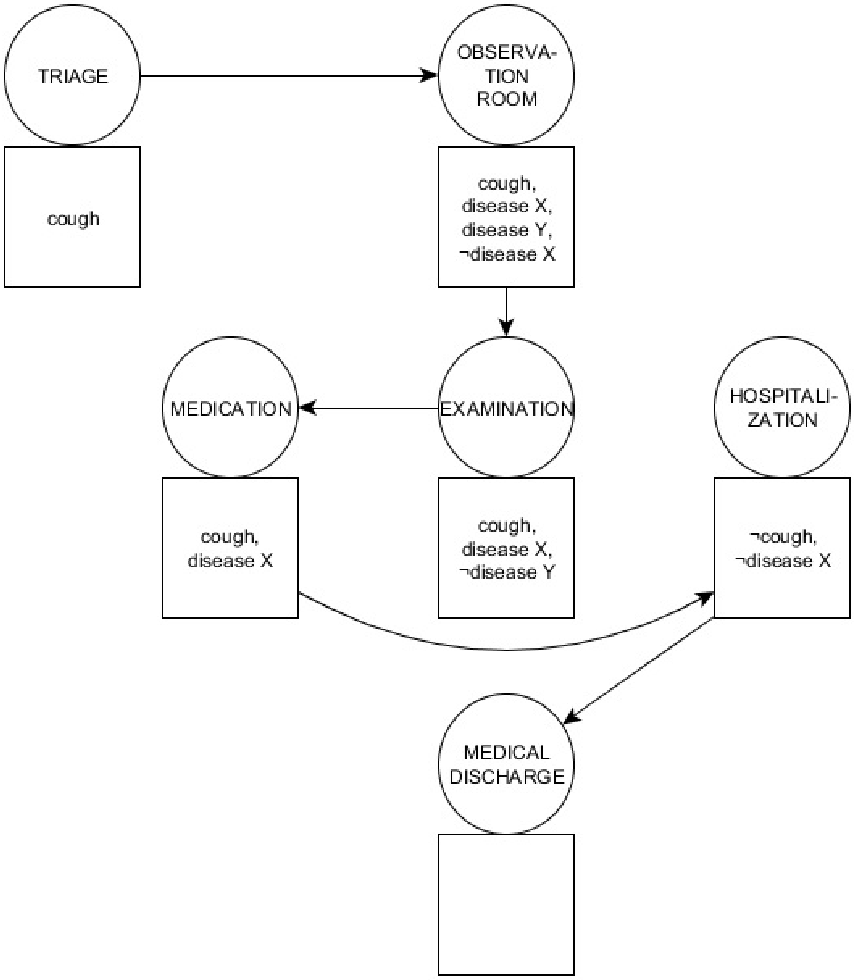

Figure 5.9: The QH model $\mathcal{M}$ of a patient A.

We can measure the inconsistency of this model in order to compare it with others. Assuming that the propositional variable *cough* can not be paraconsistent, and that the paraconsistency relies only on the medical diagnoses, and also that at triage there is no paraconsistency as well as at medical discharge because there are not diagnoses to make, the measure of inconsistency for this model is:

$$ModelInc(\mathcal{M}, L, W) = \frac{|Conflictbase(\mathcal{M})|}{|IL(L, W)|} = \frac{1}{8}$$

Figure 5.10 represents the pathway of care of patient B that enters triage with fever and elevated heart rate. At observation room, two physicians disagree with the diagnose, one keeping that the patient has disease X but not disease Y and the other stating the converse. The patient takes an examination where it is concluded that he has disease X

but not disease Y. The patient takes a certain medicine that cures his fever and elevated heart rate and is discharged. Again, the patient's medical pathway can be seen as a paraconsistent model, where Prop $= \{fever, elevated\ heart\ rate, disease\ X, disease\ Y\}$ and $W = \{Triage, Obs.\ room, Medication, Examination, Hospitalization,$ $MedicalDischarge\}$ and the valuations are given in Figure 5.10.

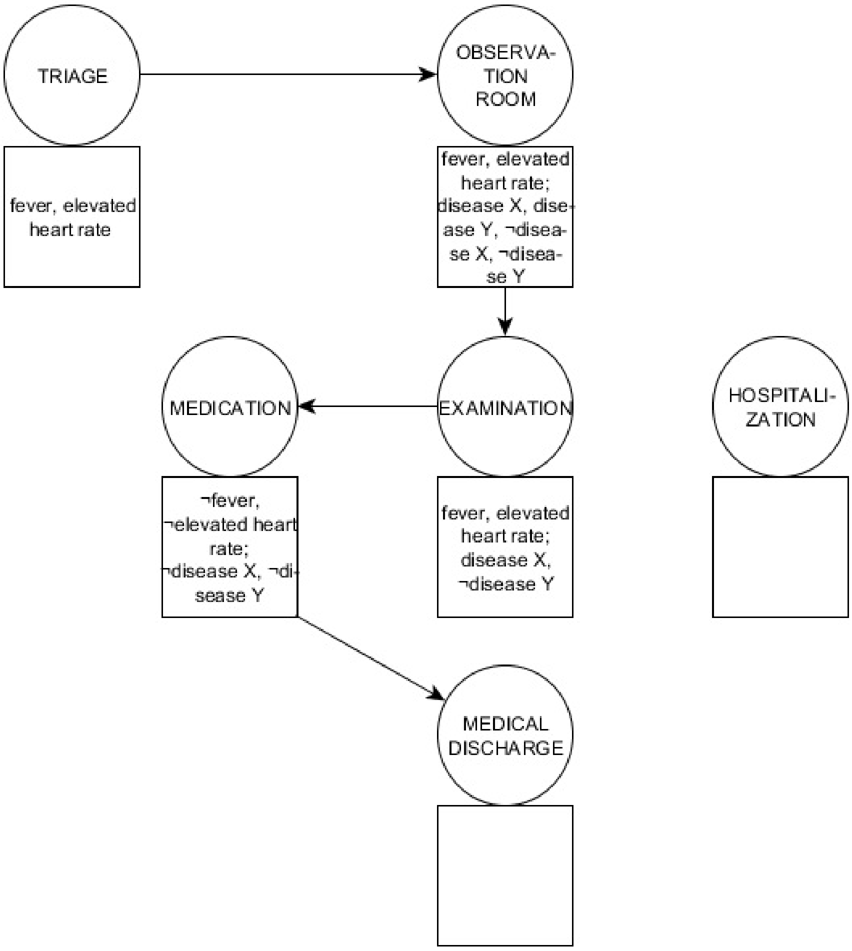

Figure 5.10: A QH model $\mathcal{M}'$ of the patient B.

Assuming that the propositional variables $fever$ and $elevated\ heart\ rate$ are not paraconsistent, and that the paraconsistency relies only on the medical diagnoses, and also that at triage there is no paraconsistency, as well as at medical discharge because there are not diagnoses to make, the measure of inconsistency for this model is:

$$ModelInc(\mathcal{M}', L, W) = \frac{|Conflictbase(\mathcal{M}')|}{|IL(L, W)|} = \frac{2}{8}$$

Comparing the two situations, we conclude that $\mathcal{M}$ is less inconsistent than $\mathcal{M}'$.

Further considerations. Transitions between states can also be labeled with modalities that might correspond to specific medications or examinations. If we had the chance to fully axiomatize the medical guideline, we would have the perfect case. The axiomatization of the medical guideline would include: (1) the complete (not a fragment) clinical flow of patients in a central hospital, *i.e.*, all the possible transitions between different stages (formulas of the form $@_i\langle\pi\rangle j$, i, j nominals, π a modality), (2) the action of specific medication in the problems verified in the patient, for example $p \rightarrow \langle A\rangle\neg p$ means that a patient with problem p would take medicine A and get cured, and furthermore, (3) the diagnose of a disease by means of a specific examination, *i.e.*, for any disease there would be an examination conclusive — the disease is present or not.

Chapter 6

Conclusion and Further Work

Inconsistency is a pervasive, and unavoidable, topic in data and knowledge management. One must consider it natural, since inconsistent information can appear everywhere, and for many reasons. Namely, contradictory information may arise in systems which are safety critical, such as health systems, aviation systems and many others. As a means of increasing the reliability of systems, especially those where extra care is required, formal methods, *i.e.* mathematical tools, have been advocated. This dissertation aims to provide a logic capable of handling with those systems.

Just by allowing inconsistent data to coexist in a knowledgebase, some kind of paraconsistent logic is required. So, being a fulcral theme for this dissertation, paraconsistent logics were introduced, as well as some applications of paraconsistent logics, fields where their presence is mandatory, but before, several schools of paraconsistent logic were discussed, and one formal kind of paraconsistent logic was particularly addressed: the Quasi-Classical Logic, the version introduced by Hunter and Grant in [58]. By reading this article, it was clear to me that a *powerful* version should exist. Since hybrid logics are a precious asset for description logics, and furthermore they are so useful to model relational structures (or multigraphs), by means of formulating the relations and equalities between states, plus the ability to refer to specific ones, it seemed possible and even challenging to find a paraconsistent version for them. This way, hybrid logics were introduced in its weakest and strongest form, along with the concepts of structure, model, diagram, bisimulation, standard and hybrid translations.

The new development on this dissertation is the adaptation of the basic hybrid logic to a paraconsistent version so that it can accommodate inconsistencies. Models are described as sets of atomic and negation of atomic formulas which is possible due to the existence of diagrams and it is because of this arrangement that measuring inconsistency is possible. The measure of inconsistency, in addition to interesting, has several applications. As the examples presented showed, it is worth to integrate the method mentioned in Chapter 4 as part of the solution for problems in many areas. And the work that we can develop in the future seems quite exciting. In particular,

- one can investigate inconsistency not only in propositional symbols, but also in what concerns to nominals, by allowing two nominations: a positive one for nominals, and a negative one for the negation of nominals. This way, we are able to handle the possibility of receiving information of the form $@_i j$ and $@_i \neg j$, meaning that someone was confused about the state that they were referring to;

- this work can be easily translated into multimodal basic hybrid logic, *i.e.*, considering different modalities, however it is not so straightforward to account inconsistency in modalities, for example by allowing both $@_i \Diamond j$ and $@_i \Box \neg j$ to be true, so this would be an interesting study, maybe by considering the coexistence of different interpretations for modalities, and having formulas in conjunctive normal form as in [9] rather than in negation normal form;

- studying paraconsistency in the context of strong Priorean logic is a main topic for future research – for example the need of satisfying formulas of the form $\forall s.@_s \varphi$ requires models to include $@_{s_i} \varphi$ for all $s_i \in \mathrm{WVar}$. Another interesting subject for investigation is the introduction of paraconsistency in first-order hybrid logic, ([30]). The existence of diagrams is crucial and it seems possible to define them;

- as seen in Chapter 3, hybrid logics are provided with a standard translation which allows working with a fragment of first-order logic, and it is even possible to define a hybrid translation such that *any* first-order expression in a language with a binary relation R (for expressing relations between states), and with an unary relation P (for making assertions at states), can be translated into a formula of $\mathcal{H}(@, \forall)$. It seems likely that similar translations can be accomplished for paraconsistent versions, namely, defining a quasi-standard translation from formulas in this quasi-hybrid basic logic into formulas in the quasi-classical logic designed by Grant and Hunter in [58] so it would be possible to reason on quasi-classical logic instead of quasi-hybrid basic logic.

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