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Paraconsistency properties in degree-preserving fuzzy logics

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Abstract Paraconsistent logics are specially tailored to deal with inconsistency, while fuzzy logics primarily deal with graded truth and vagueness. Aiming to find logics that can handle inconsistency and graded truth at once, in this paper we explore the notion of paraconsistent fuzzy logic. We show that degree-preserving fuzzy logics have paraconsistency features and study them as logics of formal inconsistency. We also consider their expansions with additional negation connectives and first-order formalisms and study their paraconsistency properties. Finally, we compare our approach to other paraconsistent logics in the literature.

Keywords Mathematical fuzzy logic, degree-preserving fuzzy logics, paraconsistent logics, logics of formal inconsistency.

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1 Introduction

Non-classical logics aim to formalize reasoning in a wide variety of different contexts in which the classical approach might be inadequate or not sufficiently flexible. This is typically the case when the information to reason about is not perfect, e.g. because it is incomplete, imprecise or contradictory.

On the one hand, fuzzy logics have been proposed as a powerful tool for reasoning with imprecise information, in particular for reasoning with propositions containing vague predicates. Their main feature is that they allow to interpret formulas in a linearly ordered scale of truth values which makes them specially suited for representing the gradual aspects of vagueness. Originating from fuzzy set theory [47] they have given rise to the deeply developed area of mathematical fuzzy logic [12] (MFL). Particular deductive systems in MFL have been usually studied under the paradigm of *truth-preservation* which, generalizing the classical notion of consequence, postulates that a formula follows from a set of premises if every algebraic evaluation that interprets the premises as true also interprets the conclusion as true. Despite of the fact that the semantics is given by algebras with many truth values (or truth degrees), the only values relevant as regards to consequence (those that have to be *preserved*) are only those in a designated set of values in the algebras (often just one designated value), which are regarded as the *full* or *complete* truth-degrees. In other words, the defining requirement in the *truth-preservation* paradigm for an inference to be valid is, actually, that every algebraic evaluation that interprets the premises as completely true, will also interpret the conclusion as completely true. An alternative approach that has recently received some attention is based on the *degree-preservation* paradigm (see [25,6]), in which a conclusion follows from a set of premises if, for all evaluations, the truth degree of the conclusion is not lower than that of the premises. It has been argued that this approach is more coherent with the commitment of many-valued logics to truth-degree semantics because all values play an equally important rôle in the corresponding notion of consequence (see e.g. [26]).

On the other hand, paraconsistent logics have been introduced, among other approaches (see e.g. [5]), as deductive systems able to cope with contradictions. As much as vagueness, inconsistency is ubiquitious in many contexts in which, regardless of the information being contradictory, one is still expected to extract inferences "in a sensible way". Classical logic, and in general any logic validating the *ex contradictione quodlibet* principle (ECQ), does not allow to reason in any interesting way in the presence of contradictions, since they trivialize deduction allowing to extract any conclusion from an inconsistent theory. They are explosive, in this sense. In contrast, paraconsistent logics are deductive systems where ECQ does not hold, so they allow to tackle contradictions without trivializing the logic. This kind of systems can be found, for example, in the realm of relevant logics, whose paraconsistent features are not central, but a by-product of the general principle that one should not infer conclusions which do not bear a "relevant connection" with their premises. Besides those, there have been many studies purposefully focused on paraconsistency giving rise to a variety of logical systems: non-adjunctive systems like Jáskowski's discussive logic, non-truth-functional logics like da Costa's C_1 and C_ω, adaptive logics, Priest's logic of paradox and similar many-valued paraconsistent systems, logics with relational valuations, paraconsistent logics with an algebraic semantics, etc. (see e.g. [42] for a, slightly dated, survey on these systems, and [35] for a more recent one). Yet another approach to paraconsistency that, stemming from da Costa's approach [16,9], has recently attracted interest is that of *logics of formal inconsistency* (**LFI**s), mainly studied by the Brazilian school [8] but also by other scholars [3,2]. The main merit of **LFI**s is that they are paraconsistent logics that manage to internalize the notions of consistency and inconsistency at the object-language level.[1]

Obviously, those phenomena of imperfect information are not mutually independent, but very often found together in many particular examples. Therefore, one might wish for logical systems to be able to cope with several of them at once. In particular, it would be desirable to have logics for vague and inconsistent information. In this paper we take the first steps towards an approach to this problem in the context of MFL which, to the best of our knowledge, has not been considered yet. We want to study paraconsistent fuzzy logics, hoping to have the best of both worlds, i.e. a good tool for reasoning with gradual predicates in possibly contradictory theories. We will argue that the appropriate paradigm for that is not the usual truth-preserving approach, but the degree-preserving one, setting the stage for future development.

After this introduction, Section 2 briefly introduces the necessary basic notions on both paraconsistent and fuzzy logics. Then Section 3 shows that truth-preserving fuzzy logics are explosive, while under some conditions degree-preserving logics are not, and hence they can be seen as paraconsistent systems; we explore their paraconsistency features, give particular examples to illustrate them and characterize a family of **LFI**s inside fuzzy logics. Since paraconsistency is always defined with respect to a particular negation connective (responsible for the contradictions in inconsistent theories), Section 4 explores alternative negations in fuzzy logics and their interplay with paraconsistency. Section 5 studies first-order predicate degree-preserving fuzzy logics and their paraconsistency properties. Finally, in Section 6 we add some concluding remarks in which we briefly compare our proposed paraconsistent fuzzy logics with other paraconsistent logics.

2 Preliminaries

In this section we introduce the necessary notation and results that will support our investigation. In particular, we briefly present the basic notions on paraconsistent logics (focusing on logics of formal inconsistency) and fuzzy logics (focusing on degree-preserving fuzzy logics) that will be used in the paper. We invite the reader to consult [8] and [6] respectively, for more exhaustive treatments of both kinds of logics.

2.1 About paraconsistency and logics of formal inconsistency

As already mentioned above, paraconsistent logics are systems that allow to deal with contradictions without trivializing the logic. In what follows we will always assume each logic to be finitary, monotonic and to have at least one negation connective that we will denote, as usual, by $\neg$.[2]

[1] Notice here that in the frame of **LFI**s the term *consistent* refers to formulas that basically exhibit a classical logic behaviour, so in particular an *explosive* behaviour.

[2] In a very general setting, one could argue what properties should be required for a unary connective to be properly

Definition 1 A logic L is *explosive (with respect to $\neg$)* if $\alpha, \neg\alpha \vdash_L \beta$, for every formula α and β. L is *paraconsistent (with respect to $\neg$)* if it is not explosive (with respect to $\neg$).

Whenever clear from the context, we will omit to write with respect to which negation a given logic is explosive or paraconsistent. Following [8], paraconsistent logics can be further classified according to several features they exhibit. We provide here the main definitions from [8] (remember we assume the logic L to be monotonic).

Definition 2 Let L be a logic and let $\sigma(p_0, \ldots, p_n)$ be a formula. The logic L is said to be:

1. *partially explosive with respect to σ* (or *σ-partially explosive*), provided that
   (a) there are formulas $\psi_0, \ldots, \psi_n$ such that
   $\nvdash_L \sigma(\psi_0, \ldots, \psi_n)$, and
   (b) for all formulas $\psi_0, \ldots, \psi_n, \varphi$, it holds
   $\varphi, \neg\varphi \vdash_L \sigma(\psi_0, \ldots, \psi_n)$.
2. *boldly paraconsistent* if there is no σ such that L is σ-partially explosive,
3. *controllably explosive in contact with σ*, if
   (a) there are formulas $\alpha, \alpha_0, \ldots, \alpha_n, \beta$ such that
   $\sigma(\alpha_0, \ldots, \alpha_n) \nvdash_L \alpha$,
   $\neg\sigma(\alpha_0, \ldots, \alpha_n) \nvdash_L \beta$,
   and
   (b) for all formulas $\psi_0, \ldots, \psi_n, \varphi$, it holds
   $\sigma(\psi_0, \ldots, \psi_n), \neg\sigma(\psi_0, \ldots, \psi_n) \vdash_L \varphi$.

Johansson's minimal logic [34], where from a contradiction every negation follows, is an example of a logic that is paraconsistent but not boldly paraconsistent, since from a contradiction every negation follows. In Section 3 we will provide both examples of paraconsistent fuzzy logics (related to finitely-valued Łukasiewicz logics) that are controllably explosive and examples (related to the infinitely-valued Łukasiewicz logic) that are not controllably explosive.

As a notation, let us write $\bigcirc(p)$ to denote a (possibly empty) set of formulas which only depends on the propositional variable p.

Definition 3 Let L be a logic and $\bigcirc(p)$ a set of formulas. L is *gently explosive with respect to $\bigcirc(p)$* if

(a) there are formulas φ and ψ such that

$$\bigcirc(\varphi), \varphi \nvdash_L \psi,$$
$$\bigcirc(\varphi), \neg\varphi \nvdash_L \psi,$$

and

(b) for all formulas φ and ψ, it holds

$$\bigcirc(\varphi), \varphi, \neg\varphi \vdash_L \psi.$$

If furthermore $\bigcirc(p)$ is finite, we say that L is *finitely gently explosive.*

Observe that if L is finitary and gently explosive, then it is also finitely gently explosive.

Following [8], given a negation $\neg$, we say that a paraconsistent logic L is a *Logic of Formal Inconsistency (with respect to $\neg$)*, ($\neg$-**LFI** in symbols), if there exists a set of formulas $\bigcirc(p)$ such that L is $\neg$-gently explosive w.r.t. $\bigcirc(p)$.

2.2 About truth-preserving and degree-preserving fuzzy logics

For the sake of not making this paper excessively long, in the following we only introduce the main needed notions of some classes of fuzzy logics, though not in full details. However, any unexplained notion mentioned in the paper can be found e.g. in [12].

Truth-preserving fuzzy logics. The most well known and studied systems of mathematical fuzzy logic are the so-called *t-norm based fuzzy logics*, corresponding to formal many-valued calculi with truth-values in the real unit interval $[0, 1]$ and with a conjunction and an implication interpreted respectively by a (left-) continuous t-norm and its residuum. For instance, the well-known Łukasiewicz and Gödel infinitely-valued logics, correspond to the calculi defined by Łukasiewicz and min t-norms respectively. The weakest t-norm based fuzzy logic is the logic MTL (monoidal t-norm based logic) introduced in [21], whose theorems correspond to the common tautologies of all many-valued calculi defined by a left-continuous t-norm and its residuated implication [33].

The language of MTL consists of denumerably many propositional variables $p_1, p_2, \ldots$, binary connectives $\wedge$, $\&$, $\to$, and the truth constant $\overline{0}$. Formulas, which will be denoted by lower case greek letters $\varphi, \psi, \chi, \ldots$, are defined by induction as usual. Further connectives and constants are definable, in particular: $\neg\varphi$ stands for $\varphi \to \overline{0}$, $\varphi \vee \psi$ stands for $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$, and $\overline{1}$ stands for $\neg\overline{0}$. A Hilbert-style calculus for MTL was introduced in [21] with the following set of axioms:

called a *negation*. However, in the context of the fuzzy logic systems considered later in this paper, all the negation connectives that we will deal with are indeed proper negations, in the sense that their truth-tables always revert to the classical negation truth-table as soon as we restrict ourselves to the classical 0 and 1 truth-values.

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \,\&\, \psi \to \varphi$
(A3)	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$
(A4)	$\varphi \wedge \psi \to \varphi$
(A5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(A6)	$\varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$
(A7b)	$(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$\overline{0} \to \varphi$

and *modus ponens* as its unique inference rule: from φ and $\varphi \to \psi$ derive ψ.

MTL is an algebraizable logic in the sense of Blok and Pigozzi [7] and its equivalent algebraic semantics is given by the class of MTL-algebras, that is indeed a variety; call it $\mathbb{MTL}$. MTL-algebras can be equivalently introduced as commutative, bounded, integral residuated lattices $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ further satisfying the following prelinearity condition: $(x \to^{\boldsymbol{A}} y) \vee^{\boldsymbol{A}} (y \to^{\boldsymbol{A}} x) = \overline{1}^{\boldsymbol{A}}$ for every $x, y \in A$.

Given an MTL-algebra $\boldsymbol{A}$, an $\boldsymbol{A}$-evaluation is any function mapping each propositional variable into A, $e(\overline{0}) = \overline{0}^{\boldsymbol{A}}$ and such that, for each formulas φ and ψ, we have $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{A}} e(\psi)$; $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{A}} e(\psi)$; $e(\varphi \,\&\, \psi) = e(\varphi) \,\&^{\boldsymbol{A}}\, e(\psi)$; $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{A}} e(\psi)$. An evaluation e is said to be a *model* for a set of formulas Γ, if $e(\gamma) = \overline{1}^{\boldsymbol{A}}$ for each $\gamma \in \Gamma$.

We shall henceforth adopt a lighter notation dropping the superscript $\boldsymbol{A}$ when no confusion is possible.

Algebraizability gives the following strong completeness theorem:

> *For every set $\Gamma \cup \{\varphi\}$ of formulae, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ iff, for every $\boldsymbol{A} \in \mathbb{MTL}$ and every $\boldsymbol{A}$-evaluation e, if e is a model of Γ then e is a model of φ as well.*

For this reason, since the consequence relation amounts to preservation of the truth-constant $\overline{1}$, MTL can be called a *truth-preserving* logic.

In Tables 1 and 2 one can find the definitions of the main axiomatic extensions of MTL that will be referred to along the paper. Observe that the extension of any of these systems with the excluded middle, $\varphi \vee \neg\varphi$, is already classical logic.

Actually, the algebraizability is preserved for any logic L that is an axiomatic expansion of MTL satisfying the following congruence property

$$\text{(Cng)}\quad \varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \varphi, \ldots, \chi_n) \to c(\chi_1, \ldots, \psi, \ldots, \chi_n)$$

Axiom schema	Name
$\neg\neg\varphi \to \varphi$	(Inv)
$\neg\varphi \vee ((\varphi \to \varphi \,\&\, \psi) \to \psi)$	(C)
$\varphi \to \varphi \,\&\, \varphi$	(Con)
$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)$	(Div)
$\varphi \wedge \neg\varphi \to \overline{0}$	(PC)
$(\varphi \,\&\, \psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi \,\&\, \psi)$	(WNM)
$\varphi \vee \neg\varphi$	(EM)

Table 1 Some usual axiom schemata in fuzzy logics.

for any possible new n-ary connective c.[3] This is due to the fact that such axiomatic expansions, also called *core fuzzy logics*, are in fact *Rasiowa-implicative logics* (cf. [44]) and, as proved in [14], every Rasiowa-implicative logic L is algebraizable. Moreover, if it is finitary, then its equivalent algebraic semantics, the class $\mathbb{L}$ of L-algebras, is a quasivariety (a variety in the case of a core fuzzy logic).

Logic	Additional axioms
Strict MTL (SMTL)	(PC)
Involutive MTL (IMTL)	(Inv)
Weak Nilpotent Minimum (WNM)	(WNM)
Nilpotent Minimum (NM)	(Inv) and (WNM)
Basic Logic (BL)	(Div)
Strict Basic Logic (SBL)	(Div) and (PC)
Łukasiewicz Logic (Ł)	(Div) and (Inv)
Product Logic (Π)	(Div) and (C)
Gödel Logic (G)	(Con)
Classical Logic (CL)	(EM)

Table 2 Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.

As a consequence, any core fuzzy logic L enjoys the same kind of completeness theorem with respect to the corresponding L-algebras. However, more than that, the variety of L-algebras can also be shown to be generated by the subclass of all its linearly ordered members [14].[4] This means that any core fuzzy logic L is strongly complete with respect to the class of L-chains, that is, core fuzzy logics are *semilinear*.

The logic $\mathrm{MTL}_\triangle$ is the (non-axiomatic) expansion of MTL with the Monteiro-Baaz projection connective

[3] $c(\chi_1, \ldots, \varphi, \ldots, \chi_n)$ and $c(\chi_1, \ldots, \psi, \ldots, \chi_n)$ denote two instances of the n-ary connective c where φ and ψ appear in a same (arbitrary) i-th place in c (for $1 \leq i \leq n$), while keeping the same formulas χ_j's (with $j \neq i$) in the other places.

[4] Moreover, for a number of core fuzzy logics, including MTL, it has been shown that their corresponding varieties are also generated by the subclass of MTL-chains defined on the real unit interval, called *standard* algebras. For instance, MTL is also complete wrt standard MTL-chains, that are of the form $[0,1]_* = \langle [0,1], \min, \max, *, \to_*, 1, 0\rangle$ of type $\langle 2,2,2,2,0,0\rangle$, where $*$ denotes a left-continuous t-norm and $\to_*$ is its residuum [33].

$\triangle$, which turns out to be a finitary Rasiowa-implicative semilinear logic as well. Then, one analogously defines $\triangle$-core fuzzy logics as axiomatic expansions of $\mathrm{MTL}_\triangle$ satisfying (Cng) for any possible new connective.

Semilinearity is also inherited by many expansions of ($\triangle$-)core fuzzy logics with new (finitary) inference rules. Indeed, in [14] it is shown that an expansion L of a core fuzzy logic is semilinear iff it is closed under $\vee$-forms of each newly added finitary inference rule, i.e. for each such rule

(R) from Γ derive φ,

its corresponding $\vee$-form

($\mathrm{R}^\vee$) from $\Gamma \vee p$ derive $\varphi \vee p$

is derivable in L as well, where p is an arbitrary propositional variable not appearing in $\Gamma \cup \{\varphi\}$ and $\Gamma \vee p = \{\psi \vee p \mid \psi \in \Gamma\}$.

Degree-preserving fuzzy logics. Clearly, core fuzzy logics and their Rasiowa-implicative semilinear expansions are truth-preserving fuzzy logics. However, besides this paradigm that we have so far considered, one can find an alternative approach in the literature. Given a (finitary Rasiowa-implicative semilinear expansion of a) core fuzzy logic L, and based on the definitions in [6], we introduce a variant of L that we will denote by $\mathrm{L}^\leq$, whose associated deducibility relation has the following semantics, where $\mathbb{K}$ is the class of L-chains:

> For every set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\mathrm{L}^\leq} \varphi$ iff there exists a finite $\Gamma_0 \subseteq \Gamma$ such that for every $\boldsymbol{A} \in \mathbb{K}$, every $a \in A$, and every $\boldsymbol{A}$-evaluation v, if $a \leq v(\psi)$ for every $\psi \in \Gamma_0$, then $a \leq v(\varphi)$.

For this reason $\mathrm{L}^\leq$ is known as a fuzzy logic *preserving degrees of truth*, or the *degree-preserving companion* of L. As it is clear from the definition, $\mathrm{L}^\leq$ is a finitary logic.[5] Actually it is very easy to check that if L is complete with respect to a subclass of L-chains $\mathbb{K}' \subseteq \mathbb{K}$, one can safely replace $\mathbb{K}$ by $\mathbb{K}'$ in the above definition of $\vdash_{\mathrm{L}^\leq}$. Notice that there are many ($\triangle$-)core fuzzy logics that are indeed complete with respect to a single L-chain.

In this paper, we will often use generic statements about "every logic $\mathrm{L}^\leq$" referring to "the degree-preserving companion of any finitary Rasiowa-implicative semilinear expansion of a ($\triangle$-)core fuzzy logic L".

Let L be a core fuzzy logic. We know it has a Hilbert-style axiomatization with *modus ponens* as the only inference rule. It is not difficult to obtain an axiomatic system for $\mathrm{L}^\leq$, taking the axioms of L and the following deduction rules [6]:

(Adj-$\wedge$) from φ and ψ derive $\varphi \wedge \psi$,
(MP-r) if $\vdash_{\mathrm{L}} \varphi \to \psi$ (i.e. if $\varphi \to \psi$ is a theorem of L), then from φ and $\varphi \to \psi$ derive ψ .

Note that if the set of theorems of L is decidable, then the above is in fact a recursive Hilbert-style axiomatization of $\mathrm{L}^\leq$. The notion of proof, denoted $\vdash_{\mathrm{L}^\leq}$, is defined as usual from the above set of axioms and rules.

In general, let L be a finitary Rasiowa-implicative semilinear expansion of MTL with a set of new inference rules

(R_i) from Γ_i derive φ_i,

for $i \in I$. Then, following the same idea of the proof of [6, Th. 2.12], we have the following generalised result.

Proposition 1 $\mathrm{L}^\leq$ *is axiomatized by adding to the axioms of* L *the above two inference rules plus the following restricted rules*

(R_i-r) If $\vdash_{\mathrm{L}} \Gamma_i$, then from Γ_i derive φ_i

for each $i \in I$.

Proof: First of all, notice that each rule (R_i-r) is sound with respect to the semantics of $\mathrm{L}^\leq$. W.l.o.g. assume $\Sigma \vdash_{\mathrm{L}^\leq} \psi$, where $\Sigma = \{\delta_1, \ldots, \delta_n\}$ is a finite set of formulas. By the semantics of $\vdash_{\mathrm{L}^\leq}$, this means that $\vdash_{\mathrm{L}} \Sigma^\wedge \to \psi$, where $\Sigma^\wedge = \bigwedge\{\delta_i \mid i = 1, \ldots, n\}$. In other words, $\Sigma^\wedge \to \psi$ is a theorem of L, and hence there is a proof Φ in L from its axioms and rules. Then we can easily convert Φ into a proof Φ' in $\mathrm{L}^\leq$ of ψ from Σ. Indeed, all we have to do is to replace every application of an inference rule (R) from L (including *modus ponens*) by its corresponding restricted form (R-r),[6] followed by $n-1$ applications of the rule (Adj-$\wedge$) to obtain $\Sigma^\wedge$, and a last application of the rule (MP-r) to $\Sigma^\wedge$ and $\Sigma^\wedge \to \psi$ to finally obtain ψ. ⊣

In particular, if L is a $\triangle$-core fuzzy logic, then the only rule one should add is the following restricted necessitation rule for $\triangle$:

($\triangle$-r) if $\vdash_{\mathrm{L}} \varphi$, then from φ derive $\triangle\varphi$.

The following proposition points out some key analogies and differences between L and $\mathrm{L}^\leq$ that we will use in the rest of this paper.

Proposition 2 *[cf. [6]] The following facts hold:*

(1) The two logics L *and* $\mathrm{L}^\leq$ *have the same theorems:* $\vdash_{\mathrm{L}} \varphi$ *iff* $\vdash_{\mathrm{L}^\leq} \varphi$.
(2) For all formulas φ, ψ one has:
  (i) $\varphi, \psi \vdash_{\mathrm{L}} \varphi \,\&\, \psi$; $\varphi, \psi \vdash_{\mathrm{L}} \varphi \wedge \psi$;
  (ii) $\varphi, \psi \vdash_{\mathrm{L}^\leq} \varphi \wedge \psi$.

[5] It is worth noticing that, even if we drop in the above definition the condition of the existence of a finite $\Gamma_0 \subseteq \Gamma$, the logic $\mathrm{L}^\leq$ remains finitary [30].

[6] Note that applications of inference rules in Φ are only to theorems of L.

(3) $\varphi_1, \ldots, \varphi_n \vdash_{L^\leq} \psi$ *iff* $\vdash_L (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$.

The last item (3) interestingly points out that, indeed, deductions in $L^\leq$ exactly correspond to theorems in L. Moreover, it makes clear that the logic $L^\leq$ is monotone.

3 Paraconsistent fuzzy logics

The first important observation is that (△-)core fuzzy logics as studied in the truth-preservation paradigm do not have any paraconsistency feature regarding their residual negation ¬.

Proposition 3 *(△-)Core fuzzy logics are explosive with respect to ¬.*

Proof: It is easy to see that in these logics the following derivations hold: $\varphi, \neg\varphi \vdash \varphi \& \neg\varphi$, and $\varphi \& \neg\varphi \vdash \overline{0}$. ⊣

Thus, (△-)core fuzzy logics are not paraconsistent. In contrast, their degree-preserving companions are paraconsistent provided that they do not prove the pseudo-complementation law (PC): $(\varphi \wedge \neg\varphi) \to \overline{0}$.[7]

Proposition 4 *Let* L *be a (△-)core fuzzy logic. Then* $L^\leq$ *is paraconsistent iff* L *is not an expansion of* SMTL*, i.e. iff (PC) does not hold in* L.

Proof: $L^\leq$ is explosive iff $\varphi, \neg\varphi \vdash_{L^\leq} \overline{0}$ iff (by the third item of Proposition 2) $\vdash_L \varphi \wedge \neg\varphi \to \overline{0}$ iff L is an expansion of SMTL. ⊣

Next, we study what kinds of paraconsistency properties those logics enjoy. The first obvious question is whether they are boldly paraconsistent or partially explosive with respect to some formula.

Proposition 5 *Every paraconsistent logic* $L^\leq$ *is partially explosive with respect to* $\sigma(p) = p \vee \neg p$.

Proof: L proves Kleene's axiom $(\varphi \wedge \neg\varphi) \to (\psi \vee \neg\psi)$ (as it can be easily checked over chains of the corresponding variety, which, as we know, give a complete semantics for the logic). Therefore, we have $\varphi, \neg\varphi \vdash_{L^\leq} \psi \vee \neg\psi$. On the other hand, if L is consistent and is not classical logic, $\psi \vee \neg\psi$ is not a theorem of $L^\leq$ (if L is classical logic, then so is $L^\leq$, and thus it is explosive; if L is inconsistent, then so is $L^\leq$ and thus also explosive). ⊣

Therefore, the logics $L^\leq$ may be paraconsistent, but they are never boldly paraconsistent. When it comes to controllable explosion, we can characterize the class of such logics which are controllably explosive in terms of the following notion of locally Boolean logic.

[7] It is worth noticing that Priest already noticed in [43] that the degree-preserving Łukasiewicz logic $Ł^\leq$ was paraconsistent.

Definition 4 A logic $L^\leq$ is *locally Boolean* if there exists a formula σ such that $\nvdash_{L^\leq} \neg\sigma$, $\nvdash_{L^\leq} \neg\neg\sigma$, and for every L-chain $\boldsymbol{A}$ and every $\boldsymbol{A}$-evaluation v, $v(\neg\sigma) \in \{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$.

Proposition 6 *A paraconsistent logic* $L^\leq$ *is controllably explosive iff it is locally Boolean.*

Proof: Assume that $L^\leq$ is controllably explosive w.r.t. a formula $\sigma(p_0, \ldots, p_n)$. This means that there are formulas $\alpha, \alpha_0, \ldots, \alpha_n, \beta$ such that $\sigma(\alpha_0, \ldots, \alpha_n) \nvdash_{L^\leq} \alpha$, and $\neg\sigma(\alpha_0, \ldots, \alpha_n) \nvdash_{L^\leq} \beta$; moreover, for every $\gamma_0, \ldots, \gamma_n, \gamma$, it holds

$$\sigma(\gamma_0, \ldots, \gamma_n), \neg\sigma(\gamma_0, \ldots, \gamma_n) \vdash_{L^\leq} \gamma.$$

Therefore, by completeness w.r.t. chains, the above holds iff for every L-chain $\boldsymbol{A}$, and every $\boldsymbol{A}$-evaluation v,

$$v(\sigma(\gamma_0, \ldots, \gamma_n) \wedge \neg\sigma(\gamma_0, \ldots, \gamma_n)) = \overline{0}^{\boldsymbol{A}}.$$

Then either $v(\sigma(\gamma_0, \ldots, \gamma_n)) = \overline{0}^{\boldsymbol{A}}$, and hence we have $v(\neg\sigma(\gamma_0, \ldots, \gamma_n)) = \overline{1}^{\boldsymbol{A}}$, or $v(\neg\sigma(\gamma_0, \ldots, \gamma_n)) = \overline{0}^{\boldsymbol{A}}$ otherwise. Moreover, from the existence of formulas $\alpha, \alpha_0, \ldots, \alpha_n$ such that $\sigma(\alpha_0, \ldots, \alpha_n) \nvdash_{L^\leq} \alpha$, we infer that there must exist an L-chain $\boldsymbol{B}$ and a $\boldsymbol{B}$-evaluation e such that $e(\sigma) \neq \overline{0}^{\boldsymbol{B}}$, and hence $\nvdash_{L^\leq} \neg\sigma$. Similarly, from the fact that $\neg\sigma(\alpha_0, \ldots, \alpha_n) \nvdash_{L^\leq} \beta$, we know that there is an L-chain $\boldsymbol{C}$ and a $\boldsymbol{C}$-evaluation e' such that $e'(\neg\sigma) \neq \overline{0}^{\boldsymbol{C}}$; therefore we have $e'(\sigma) = \overline{0}^{\boldsymbol{C}}$ and thus $e'(\neg\neg\sigma) = \overline{0}^{\boldsymbol{C}}$ and $\nvdash_{L^\leq} \neg\neg\sigma$. Therefore $L^\leq$ is locally Boolean.

Now assume that $L^\leq$ is locally Boolean, i.e. there is a formula σ such that $\nvdash_{L^\leq} \neg\sigma$, $\nvdash_{L^\leq} \neg\neg\sigma$, and for every L-chain $\boldsymbol{A}$ and every $\boldsymbol{A}$-evaluation v, we have $v(\neg\sigma) \in \{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$. Let $p_0, \ldots, p_n$ be the variables occurring in σ. Thus, for every substitution of $p_0, \ldots, p_n$ by arbitrary formulas $\gamma_0, \ldots, \gamma_n$, we have $v(\neg\sigma(\gamma_0, \ldots, \gamma_n)) \in \{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$. Thus, either it holds that $v(\neg\sigma(\gamma_0, \ldots, \gamma_n)) = \overline{0}^{\boldsymbol{A}}$, or $v(\neg\sigma(\gamma_0, \ldots, \gamma_n)) = \overline{1}^{\boldsymbol{A}}$ and hence, in the latter case, $v(\sigma(\gamma_0, \ldots, \gamma_n)) = \overline{0}^{\boldsymbol{A}}$. Therefore, for every $\gamma_0, \ldots, \gamma_n, \gamma$,

$$\overline{0}^{\boldsymbol{A}} = v(\sigma(\gamma_0, \ldots, \gamma_n) \wedge \neg\sigma(\gamma_0, \ldots, \gamma_n)) \leq v(\gamma),$$

that is,

$$\sigma(\gamma_0, \ldots, \gamma_n), \neg\sigma(\gamma_0, \ldots, \gamma_n) \vdash_{L^\leq} \gamma.$$

On the other hand, since $\nvdash_{L^\leq} \neg\sigma$, there is an L-chain $\boldsymbol{B}$ and a $\boldsymbol{B}$-evaluation e such that $e(\neg\sigma) \neq \overline{1}^{\boldsymbol{B}}$ and hence $e(\sigma) \neq \overline{0}^{\boldsymbol{B}}$. Similarly, since $\nvdash_{L^\leq} \neg\neg\sigma$, there is an L-chain $\boldsymbol{C}$ and a $\boldsymbol{C}$-evaluation e' such that $e(\neg\neg\sigma) \neq \overline{1}^{\boldsymbol{C}}$ and hence $e(\neg\sigma) \neq \overline{0}^{\boldsymbol{C}}$. Hence $L^\leq$ is controllably explosive. ⊣

Next we give some examples of families of paraconsistent fuzzy logics that are locally Boolean and some that are not. In these examples, given an MTL-chain $\boldsymbol{C}$, $\mathrm{L}^{\leq}$ denotes the degree-preserving companion of the extension of MTL whose equivalent algebraic semantics is $\mathbf{V}(\boldsymbol{C})$, i.e. the variety generated by $\boldsymbol{C}$.

Example 1 Let $\boldsymbol{C}$ be an MTL-chain. Suppose that the set of its positive and negative elements are respectively defined as $C_+ = \{a \in C \mid a > \neg a\}$ and $C_- = \{a \in C \mid a \leq \neg a\}$. Assume that C_+ is an MTL-filter, i.e. a non-empty upset w.r.t. the order and closed under &. This means that C_+ coincides with the radical of $\boldsymbol{C}$ (i.e. the intersection of all maximal filters of $\boldsymbol{C}$; see e.g. [37]).[8] Then $\boldsymbol{C}$ is either bipartite or bipartite with a fixpoint. In either case, the quotient algebra $\boldsymbol{C}/C_+$ is the two-element Boolean algebra $\boldsymbol{B}_2$, if $\boldsymbol{C}$ has no negation fixpoint, or the three-element MV-algebra $\mathbf{Ł}_3$ otherwise. In both cases the logic of $\boldsymbol{C}$ is locally Boolean with the formula[9] $\sigma(p) = (\neg(p^2))^2$. Indeed, it is easy to see that $\sigma^{\boldsymbol{C}}(x) = \overline{1}^{\boldsymbol{C}}$ if $x \in C_-$ and $\sigma^{\boldsymbol{C}}(x) = \overline{0}^{\boldsymbol{C}}$ if $x \in C_+$. Examples of MTL-chains satisfying this condition are the Chang MV-algebra, and any WNM-chain (thus including NM-chains).

Example 2 Let $\boldsymbol{C}$ be the standard MV-chain $[0,1]_Ł$. Then the degree-preserving companion $\mathrm{Ł}^{\leq}$ of Łukasiewicz logic – the logic of $\boldsymbol{C}$ – is not locally Boolean. The result is obvious because, for every $m \geq 1$, any function in the free m-generated algebra $Free_m(\boldsymbol{C})$ of the variety $\mathbb{MV}$ generated by $\boldsymbol{C}$, is piecewise linear and continuous from $[0,1]^m$ into $[0,1]$ (a McNaughton function [11], in particular). Hence, the unique Boolean functions of $Free_m(\boldsymbol{C})$ are f_1 (the function constantly equal to 1) and f_0 (the function constantly equal to 0). Therefore, recalling that $\vdash_Ł \varphi \leftrightarrow \neg\neg\varphi$, if σ is any formula of $\mathrm{Ł}^{\leq}$ such that, for every $\boldsymbol{C}$-evaluation v, $v(\neg\sigma) \in \{\overline{0}^{\boldsymbol{C}}, \overline{1}^{\boldsymbol{C}}\}$, then either $f_\sigma = f_{\neg\neg\sigma} = f_1$, and hence $\vdash_{Ł^{\leq}} \neg\neg\sigma$, or $f_\sigma = f_0$, and hence $f_{\neg\sigma} = f_1$, that is $\vdash_{Ł^{\leq}} \neg\sigma$.

The following two propositions are more general characterizations of families of paraconsistent fuzzy logics that are either locally Boolean or not in the setting of logics of BL-chains. Using the same notation as in the previous examples we have the following results.

Proposition 7 *Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be* MTL*-chains, assume that the Monteiro-Baaz operator $\triangle$ is definable in $\boldsymbol{A}_1$ and take $\boldsymbol{C} = \boldsymbol{A}_1 \oplus \boldsymbol{A}_2$. Then $\mathrm{L}^{\leq}$ is locally Boolean.*

[8] This type of chains are studied in [38].

[9] Given a natural number n, φ^n is an abbreviation for $\varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi$, that is the fomula obtained as conjunction of n times φ.

Proof: Let $\delta(p)$ be the term defining the Monteiro-Baaz operator $\triangle$ in $\boldsymbol{A}_1$, and hence in all chains of the variety generated by $\boldsymbol{A}_1$, since $\triangle$ is well-known to be defined by a set of equations. Then $\mathrm{L}^{\leq}$ is locally Boolean with $\sigma(p) = \neg\delta(p)$. Indeed, observe that for any evaluation v on $\boldsymbol{C} = \boldsymbol{A}_1 \oplus \boldsymbol{A}_2$, $v(\sigma(p)) = \overline{1}^{\boldsymbol{C}}$ if $v(p) \in A_1$ and $v(\sigma(p)) = \overline{0}^{\boldsymbol{C}}$ otherwise. Finally, the result follows from the fact that any chain of the variety generated by $\boldsymbol{C}$ is of the form $\boldsymbol{B}_1 \oplus \boldsymbol{B}_2$, where $\boldsymbol{B}_1$ belongs to the variety generated by $\boldsymbol{A}_1$. ⊣

Remark 1 In the proof of the following proposition, we will use tools from [1] related to the functional representation of free BL-algebras with m generators. Let us recall some facts from that paper that are needed in the proposition below. Let $\boldsymbol{C}$ be a BL-chain which can be displayed as the ordinal sum $[0,1]_L \oplus \boldsymbol{A}$ (where $\boldsymbol{A}$ is any BL-chain). The free m-generated algebra $Free_m(\boldsymbol{C})$ in the variety $\mathbf{V}(\boldsymbol{C})$ has as elements functions from the hypercube C^m into C. Each function $f \in Free_m(\boldsymbol{C})$ satisfies the two following properties:

1. The restriction $\hat{f}$ of f to $([0,1]_L)^m$, takes value in $[0,1]_L$ and it is a McNaughton function. Therefore, in particular, $\hat{f}$ is continuous.
2. For any $c = (c_1, \ldots, c_m) \in C^m \setminus ([0,1]_L)^m$, $f(c) = 0$ iff there exists an $x = (x_1, \ldots, x_m) \in ([0,1]_L)^m$ such that:
   - there exists at least one coordinate $j \in \{1, \ldots, m\}$ such that $x_j = 1$,
   - $c_k = x_k$ for all $k \in \{1, \ldots, m\}$ such that $x_k \neq 1$,
   - $\hat{f}(x) = 0$.

   In other words, if for some x in the border of $(0,1]^m$ (say $x = (1, x_2, \ldots, x_m)$ and $x_k \neq 1$ for all $2 \leq k \leq m$) we have $f(x) = 0$, then $\hat{f}(c) = 0$ for all c of the form $(c_1, x_2, \ldots, x_m) \in C^m \setminus ([0,1]_L)^m$.

Proposition 8 *Let $\boldsymbol{C}$ be a* BL*-chain such that the logic $\mathrm{L}^{\leq}$ is paraconsistent. Then:*

1. If $\boldsymbol{C}$ is defined by an ordinal sum whose first component is a finite BL*-chain, then $\mathrm{L}^{\leq}$ is locally Boolean,*
2. if $\boldsymbol{C}$ is defined by an ordinal sum whose first component is the Łukasiewicz t-norm, then $\mathrm{L}^{\leq}$ is not locally Boolean.

Proof: Consider the decomposition of $\boldsymbol{C}$ as ordinal sum of irreductible BL-chains $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots$ Since $\boldsymbol{C}$ is not pseudo-complemented (because $\mathrm{L}^{\leq}$ is paraconsistent) we know that the first component $\boldsymbol{A}_1$ has to be an MV-algebra. Then, we consider two cases:

(1) $\boldsymbol{A}_1$ is a finite MV-chain. Then the claim follows by the above Proposition 7 and reminding that in every finite MV-chain $Ł_k$, the operator $\triangle$ is definable as $\triangle\varphi := \varphi^k$.

(2) $\boldsymbol{A}_1$ is the standard MV-chain $[0,1]_L$. We prove that for $\mathrm{L}^{\leq}$ we cannot find any formula σ such that $\not\vdash_{\mathrm{L}^{\leq}} \neg\sigma$, $\not\vdash_{\mathrm{L}^{\leq}} \neg\neg\sigma$ and for every valuation v, $v(\neg\sigma) \in \{0,1\}$. Assume by way of contradiction that such a σ exists and assume, without loss of generality, that σ has m propositional variables. Then, in the m-generated free algebra $Free_m(\boldsymbol{C})$ of $\mathbf{V}(\boldsymbol{C})$, there is a function f_σ (corresponding to the equivalence class $[\sigma]$ modulo logical equivalence) such that $f_\sigma : C^m \to C$, $f_\sigma \neq \overline{1}$, $f_\sigma \neq \overline{0}$, but f_σ is Boolean. Recalling Remark 1, if such an f_σ exists, then the restriction $\hat{f}_\sigma$ of f_σ to $([0,1]_L)^m$ is a McNaughton function (and hence it is continuous), and we have three subcases:

1. If f_σ restricted to $([0,1]_L)^m$ is $\overline{0}$, then, in particular, $f_\sigma(x) = 0$, for all $x = (x_1, \ldots, x_m)$ for which at least one index j is such that $x_j = 1$. Then f_σ is also $\overline{0}$ in the second component $\boldsymbol{A}_2$ of the ordinal sum defining $\boldsymbol{C}$, i.e. f_σ is the map on C^m which is constantly 0. This contradicts the hypothesis that $\not\vdash_{\mathrm{L}^{\leq}} \neg\sigma$.
2. If f_σ restricted to $([0,1]_L)^m$ is $\overline{1}$, then for no other element $c \in C^m \setminus ([0,1]_L)^m$, $f_\sigma(c) = 0$ because, in particular, $\hat{f}_\sigma(x) \neq 0$, for all $(x_1, \ldots, x_m) \in ([0,1]_L)^m$ with $x_j = 1$ for some j. This contradicts the hypothesis that $\not\vdash_{\mathrm{L}^{\leq}} \neg\neg\sigma$.
3. $\hat{f}_\sigma$ is a Boolean function different from the map which is constantly 1 or 0. This is absurd since $\hat{f}_\sigma$ is a McNaughton function and hence, as we already recalled in Example 2, it is continuous.

Hence a contradiction has been reached. $\dashv$

Corollary 1 *Let $\boldsymbol{C}$ be a* BL*-chain defined by a continuous t-norm and such that the logic* $\mathrm{L}^{\leq}$ *is paraconsistent. Then* $\mathrm{L}^{\leq}$ *is not locally Boolean.*

The corollary is an easy consequence of the fact that any non pseudo-complemented continuous t-norm is decomposable as an ordinal sum which has the Łukasiewicz t-norm as the first component.

Finally, let us consider the notion of gently explosive logic with respect to a set of formulas $\bigcirc(p)$. Recall Definition 3 and assume that L is a ($\triangle$-)core fuzzy logic complete with respect to a class $\mathbb{K}$ of L-chains. Then, thanks to the fact that $\mathrm{L}^{\leq}$ is finitary and the presence of the adjunction rule (**Adj**$-\wedge$), we can assume that $\bigcirc(p)$ is just one formula and the definition of $\mathrm{L}^{\leq}$ being gently explosive can be reformulated in semantical terms as follows:

– (GE-a) there are formulas φ, ψ such that:
 - there is a chain $\boldsymbol{A} \in \mathbb{K}$ and an $\boldsymbol{A}$-evaluation e_1 such that $e_1(\bigcirc(\varphi) \wedge \varphi) > e_1(\psi)$,
 - there is a chain $\boldsymbol{B} \in \mathbb{K}$ and an $\boldsymbol{B}$-evaluation e_2 such that $e_2(\bigcirc(\varphi) \wedge \neg\varphi) > e_2(\psi)$,
– (GE-b) for every formula φ, every chain $\boldsymbol{A} \in \mathbb{K}$ and every $\boldsymbol{A}$-evaluation e, $e(\bigcirc(\varphi) \wedge \varphi \wedge \neg\varphi) = \overline{0}^{\boldsymbol{A}}$.

In the case L is complete with respect to a single L-chain $\boldsymbol{A}$, these conditions imply that the unary operation $\bigcirc^{\boldsymbol{A}}$ (the interpretation of $\bigcirc$ on the algebra $\boldsymbol{A}$) has to satisfy the properties given in the next proposition.

Proposition 9 *Let* L *be the logic of a chain $\boldsymbol{A}$. Then the following are equivalent:*

1. $\mathrm{L}^{\leq}$ *is gently explosive;*
2. There exists a term $\bigcirc(p)$ such that
 - $\bigcirc^{\boldsymbol{A}}(\overline{0}^{\boldsymbol{A}}) > \overline{0}^{\boldsymbol{A}}$,
 - *there is an $x \in A$ with $\neg x = \overline{0}^{\boldsymbol{A}}$ and $\bigcirc^{\boldsymbol{A}}(x) > \overline{0}^{\boldsymbol{A}}$,*
 - $\bigcirc^{\boldsymbol{A}}(t) = \overline{0}^{\boldsymbol{A}}$*, for each $t \in A$ such that $t, \neg t > \overline{0}^{\boldsymbol{A}}$.*

Proof: In the proof we use $\bigcirc$ for both the term and its corresponding operation on the chain $\boldsymbol{A}$ as the context will avoid any possible confusion. Assume that $\mathrm{L}^{\leq}$ is gently explosive. Then there exists a formula $\bigcirc(p)$ satisfying the reformulation mentioned above. Thus there are $x, y \in A$, such that $x \wedge \bigcirc(x) > \overline{0}^{\boldsymbol{A}}$ and $\bigcirc(y) \wedge \neg y > \overline{0}^{\boldsymbol{A}}$,[10] and for every $z \in A$, $z \wedge \neg z \wedge \bigcirc(z) = \overline{0}^{\boldsymbol{A}}$. It is clear that the latter equality for every $z \in A$ implies the last condition from 2. From the properties of x and y it follows that $x, \neg y > \overline{0}^{\boldsymbol{A}}$, $\neg x = \overline{0}^{\boldsymbol{A}}$, and $y = \overline{0}^{\boldsymbol{A}}$, and hence $\bigcirc(\overline{0}^{\boldsymbol{A}}) > \overline{0}^{\boldsymbol{A}}$, so the remaining two conditions are satisfied.

Reciprocally, if 2 is satisfied, let $x \in A$ be such that $\neg x = \overline{0}^{\boldsymbol{A}}$ and $\bigcirc(x) > \overline{0}^{\boldsymbol{A}}$, that exists by hypothesis. Obviously, such an x has to be greater than $\overline{0}^{\boldsymbol{A}}$. Now take $\varphi = p$ and $\psi = \overline{0}$, where p is a propositional variable and let e_1 be an $\boldsymbol{A}$-evaluation such that $e_1(p) = x$. It is clear that $e_1(\bigcirc(\varphi) \wedge \varphi) = \min\{\bigcirc^{\boldsymbol{A}}(x), x\} > \overline{0}^{\boldsymbol{A}} = e_1(\psi)$. Now let e_2 be an $\boldsymbol{A}$-evaluation such that $e_2(p) = \overline{0}^{\boldsymbol{A}}$. Then, since by hypothesis $\bigcirc^{\boldsymbol{A}}(\overline{0}^{\boldsymbol{A}}) > \overline{0}^{\boldsymbol{A}}$, it is also clear that $e_2(\bigcirc(\varphi) \wedge \neg\varphi) = \min\{\bigcirc^{\boldsymbol{A}}(\overline{0}^{\boldsymbol{A}}), \overline{1}^{\boldsymbol{A}}\} > \overline{0}^{\boldsymbol{A}} = e_2(\psi)$. Thus the proposition is proved. $\dashv$

We have, therefore, identified the conditions for the degree-preserving fuzzy logic of an (expansion of an) MTL-chain to be gently explosive. The following examples show that the degree-preserving version of the $[0,1]$-valued Łukasiewicz logic is not gently explosive, while finitely-valued Łukasiewicz logics are gently explosive.

Example 3 The logic $\mathrm{Ł}^{\leq}$, i.e. the degree-preserving companion of Łukasiewicz logic, is not gently explosive. In fact, as we recalled in Example 2, every definable term

[10] Note that x and y correspond respectively to $e_1(\varphi)$ and $e_2(\varphi)$.

of such logic corresponds to a McNaughton function [11], and McNaughton functions, being continuous, cannot satisfy the conditions of the previous proposition.

Example 4 If L has the Monteiro-Baaz's $\triangle$ connective (as primitive or definable), then $L^{\leq}$ is gently explosive with $\bigcirc(\alpha) = \triangle(\alpha \vee \neg\alpha)$, as one can easily check using the conditions of the previous proposition. This is the case of the logic of a finite MV-chain $Ł_n$ (where $\triangle\varphi = \varphi^n$) or, more in general, the logic of an S_nMTL-chain[11] (where $\triangle\varphi = \neg\varphi^n \vee \varphi$) [29].

As an immediate corollary of the preceding proposition, we have the following characterization of when a degree-preserving fuzzy logic of an (expansion of an) MTL-chain is an **LFI** with respect to the residual negation $\neg$.

Corollary 2 *Let* L *be the logic of a chain* $\boldsymbol{A}$ *that is not an* SMTL*-algebra, i.e. such that there exists* $x \in A$ *with* $x \wedge \neg x > \overline{0}^{\boldsymbol{A}}$. *Then the following are equivalent:*

1. $L^{\leq}$ *is an* **LFI** *with respect to* $\neg$*;*
2. *There exists a term* $\bigcirc(p)$ *such that*
   - $\bigcirc^{\boldsymbol{A}}(\overline{0}^{\boldsymbol{A}}) > \overline{0}^{\boldsymbol{A}}$,
   - *there is an* $x \in A$ *with* $\neg x = \overline{0}^{\boldsymbol{A}}$ *and* $\bigcirc^{\boldsymbol{A}}(x) > \overline{0}^{\boldsymbol{A}}$,
   - $\bigcirc^{\boldsymbol{A}}(t) = \overline{0}^{\boldsymbol{A}}$, *for each* $t \in A$ *such that* $t, \neg t > \overline{0}^{\boldsymbol{A}}$.

4 Paraconsistency of fuzzy logics expanded with further negations

In this section we consider fuzzy logics expanded with negations different from the residuated one and explore their paraconsistency properties with respect to these new negations. To remain in the realm of fuzzy logics, whose algebraic semantics are given by classes of linearly ordered algebras, we only consider expansions of fuzzy logics with negations defined in such a way that semilinearity is preserved, i.e. they remain core fuzzy logics. In such a case, since the negation on a chain $\boldsymbol{A}$ is a generalization of classical negation, the truth function **neg** of any such negation *neg* satisfies $\mathbf{neg}(\overline{0}^{\boldsymbol{A}}) = \overline{1}^{\boldsymbol{A}}$ and $\mathbf{neg}(\overline{1}^{\boldsymbol{A}}) = \overline{0}^{\boldsymbol{A}}$. Therefore, although a (truth-preserving) fuzzy logic L expanded with a negation *neg* will also not be paraconsistent, its degree preserving companion will be paraconsistent provided that $\varphi \wedge neg(\varphi)$ is not equivalent to $\overline{0}$ (as in the case of a non pseudo-complemented residuated negation).

The next two subsections are devoted to the study of expansions of a core fuzzy logic L and of its degree preserving companion $L^{\leq}$ obtained by adding either the dual intuitionistic negation D, or an involutive negation $\sim$. In the last case, the expansion has sense only if the residuated negation of L is not already involutive. In what follows we will denote by L_D and $L^{\leq}_D$, and by $L_\sim$ and $L^{\leq}_\sim$, the expansions of L and $L^{\leq}$ with D and $\sim$ respectively.

4.1 Adding the dual intuitionistic negation

In his 1919 paper [46], Skolem studied lattices expanded with the relative pseudo-complement and its dual. This dual operation, which he called *Subtraktion* and for which we use the notation $\dot{-}$, satisfies the following condition:

$$a \dot{-} b \leq c \text{ iff } a \leq b \vee c \ .$$

He noted that it follows the existence of both top 1 and bottom. He also briefly considered the associated negation $1 \dot{-} b$ of b, for which we will use the notation Db. It follows that $Db \leq c$ iff $b \vee c = 1$.

Afterwards, in his 1942 paper [36] and independently of [46], Moisil provided an axiomatization of the expansion of positive intuitionistic logic with the dual of the intuitionistic conditional. In particular, in the case of the dual of intuitionistic negation, for which we use again D, he obtained the following derivable formula and rule:

(D1) $\varphi \vee D\varphi$,
(DR) from $\varphi \vee \psi$ derive $D\varphi \to \psi$.

Note that the given axiom and rule define D univocally, in the sense that, duplicating (D1) and (DR) for a connective D', it follows that $D'\varphi$ and $D\varphi$ are interderivable.

Later on, in her 1974 paper [45] and independently of [46] and [36], Rauszer presented a logico-algebraic study of what she called *semi-Boolean algebras*. These are expansions of Heyting algebras with the mentioned dual operator $\dot{-}$ already used by Skolem. She also provided an axiomatization that, though being different, has the same consequences as the one by Moisil.

More recently, Priest [41] provided a natural deduction version of the logic we are considering. However, in the case of D, instead of using a rule equivalent to (DR), he used a rule that in the context of a Hilbert style axiomatization can be given as follows:

(DR-r) If $\vdash \varphi \vee \psi$, then from $\varphi \vee \psi$ derive $D\varphi \to \psi$.

Honoring da Costa, he called his logic *da Costa Logic* and used the notation **daC**.

Further investigations have been provided by Castiglioni and Ertola in [10], where they proved that **daC** is boldly paraconsistent, and by Ferguson in [24], where

[11] An S_nMTL-chain **A** is a MTL-chain satisfying the equation $x \vee \neg x^{n-1} = \overline{1}^{\mathbf{A}}$.

he proved that **daC** is an **LFI**. The operator D had also been discussed by Ertola in [20].

In this section we study D-paraconsistency properties in the setting of (semilinear) fuzzy logics. First of all, we need to specify the behaviour of this D operator. We start from the Hilbert style axiomatization of D consisting of any axiomatization of intuitionistic positive logic (for example, as given in [10]) with *modus ponens* as only rule, and add (D1) as axiom and (DR) as a new rule. Honoring Moisil, let us call this logic **M**.

Intuitively, given any core fuzzy logic L, if we want the axiom (D1) to be always evaluated to $\overline{1}^{\boldsymbol{A}}$ in any expanded L-chain $\boldsymbol{A}$ with an operator D, it has to satisfy $Dx = \overline{1}^{\boldsymbol{A}}$, for every $x \in A$ such that $x < \overline{1}^{\boldsymbol{A}}$, while the validity of the rule (DR) implies that $D(\overline{1}^{\boldsymbol{A}}) = \overline{0}^{\boldsymbol{A}}$. Therefore the axiom and rule of **M** totally determine the algebraic counterpart of the D operator on chains, but not on arbitrary algebras. Since we aim at defining a semilinear logic, this is not a problem. Indeed, the semilinearity of the expanded logic can be enforced in different ways. One possibility, as will be shown later, is to replace the rule (DR) by a somewhat stronger rule, leading us to the following definition.

Definition 5 For each core fuzzy logic L, the logic L_D is defined by expanding the language of L with the unary connective D and adding the following axiom and rule:

(D1) $\quad \varphi \vee D\varphi$
(DN) $\quad$ from $\varphi \vee \psi$ derive $\neg D\varphi \vee \psi$.

It can be easily checked that, in contrast to the logic **M**, L_D proves the theorem $D\varphi \vee \neg D\varphi$, forcing formulas of the form $D\varphi$ to be classical. Moreover, as expected, one has that $\varphi, D\varphi \vdash_{\mathrm{L}_D} \overline{0}$, and hence L_D is explosive with respect to D. Note that the latter is also true in the case of **M**.

Next we show that L_D satisfies the congruence property (Cng) for D, which is true also in the case of **M**.

Lemma 1 *If* L *is a core fuzzy logic, in* L_D *the following deduction holds:*

$$\varphi \to \psi \vdash_{\mathrm{L}_D} D\psi \to D\varphi.$$

Proof: From $\varphi \to \psi$ and $\varphi \vee D\varphi$ one can easily derive $\psi \vee D\varphi$, and using (DN) one obtains $\neg D\psi \vee D\varphi$, and hence, $D\psi \to D\varphi$ holds as well. $\dashv$

Therefore, the congruence condition (Cng) holds for D and thus L_D is a Rasiowa-implicative logic. Since the rule (DN) is closed under $\vee$-forms, it follows that L_D is semilinear as well.

The corresponding algebraic semantics for the logic L_D is given by the class of L_D-algebras. Those are structures $\langle A, \wedge, \vee, \&, \to, D, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$, where D is a unary operation, such that their D-free reduct is an L-algebra and the two following properties hold for each $x, y \in A$:

- $x \vee Dx = \overline{1}^{\boldsymbol{A}}$,
- if $x \vee y = \overline{1}^{\boldsymbol{A}}$, then $\neg Dx \vee y = \overline{1}^{\boldsymbol{A}}$.

From this definition, it is clear that the class of L_D-algebras is a quasivariety. We shall show shortly that the class of L_D-algebras is indeed a variety. Since L_D is semilinear, it is complete with respect to the class of L_D-chains. Moreover, it is easy to check that if L is standard complete, then so is L_D. Furthermore, let us remark that, as already announced, in any L_D-chain, the two conditions above univocally determine the D operator to be defined in the following manner:

$$Dx = \begin{cases} \overline{1}^{\boldsymbol{A}}, \text{ if } x < \overline{1}^{\boldsymbol{A}}, \\ \overline{0}^{\boldsymbol{A}}, \text{ if } x = \overline{1}^{\boldsymbol{A}}. \end{cases}$$

It follows that D is indeed the dual intuitionistic negation (it satisfies $Dx = \min\{y \mid x \vee y = \overline{1}^{\boldsymbol{A}}\}$).

Regarding the interaction between the two negations $\neg$ and D, it is clear that in any L_D-chain we have the following *negative* combinations:

$$\neg D \neg x \le \neg x \le D \neg\neg x \le Dx.$$

Note that $\neg D \neg$ is in fact the intuitionistic (Gödel) negation, and hence the smallest (strongest) negation definable in a chain, while D is the greatest (weakest) definable negation in a chain. On the other hand, we have the following *positive* combinations:

$$\neg Dx \le x \le \neg\neg x \le D\neg x,$$

also having that $\neg Dx \le DD\neg\neg x \le \neg\neg x$, with $DD\neg\neg x$ being not comparable with x. Note that if $\neg$ is Gödel negation, then $DD\neg\neg x = \neg\neg x = D\neg x$.

It is also straightforward to observe that on every L_D-chain, D behaves exactly as the residual negation composed with the Monteiro-Baaz operator $\triangle$. Actually, one can check that the logic L_D is equivalent to $\mathrm{L}_\triangle$, since in $\mathrm{L}_\triangle$ the connective D is definable as

$$D\varphi := \neg \triangle \varphi$$

and, vice versa, in L_D the connective $\triangle$ is indeed definable as

$$\triangle \varphi := \neg D \varphi.$$

Thus, L_D is equivalent to $\mathrm{L}_\triangle$ and therefore L_D-algebras are termwise equivalent to $\mathrm{L}_\triangle$, whence they form a variety.

Concerning paraconsistency properties related to D, as already noticed above, for any core fuzzy logic L, the logic L_D is not D-paraconsistent. Therefore, let us turn our attention to their degree-preserving companions $L_D^{\leq}$. As usual, the logic $L_D^{\leq}$ is defined from $L^{\leq}$ by adding the axiom (D1) and the following restriction of the rule (DN):

(DN-r) If $\vdash_{L_D} \varphi \vee \psi$, then from $\varphi \vee \psi$ derive $\neg D\varphi \vee \psi$.

These logics are D-paraconsistent.

Proposition 10 *For any core fuzzy logic* L*, the logic* $L_D^{\leq}$ *is* D*-paraconsistent.*

Proof: It is clear that, in any L_D-chain $\boldsymbol{A}$, $x \wedge Dx > \overline{0}^{\boldsymbol{A}}$ for $\overline{0}^{\boldsymbol{A}} < x < \overline{1}^{\boldsymbol{A}}$. Hence, it is clear that, if p and q are two different propositional variables, then $p, Dp \nvdash_{L_D^{\leq}} q$. Therefore, the logic $L_D^{\leq}$ is D-paraconsistent. ⊣

Moreover, we can show that every logic $L_D^{\leq}$ is gently D-paraconsistent and, in some cases, even boldly paraconsistent. Namely, bold paraconsistency is obtained provided that L_D is complete with respect to chains without coatom (the coatom of a chain $\boldsymbol{A}$ is the element $\max(A \setminus \{\overline{1}^{\boldsymbol{A}}\})$, which need not exist); such requirement is met by many fuzzy logics, e.g. by logics complete w.r.t. densely ordered chains (in particular, logics satisfying standard completeness).

Proposition 11 $L_D^{\leq}$ *is boldly* D*-paraconsistent if* L_D *is complete with respect to chains without coatom.*

Proof: Suppose now that $\psi(p_1, \ldots, p_n)$ is a formula such that $\nvdash_{L_D} \psi$. By assumption, there exists an evaluation v on an L_D-chain $\boldsymbol{A}$ without coatom such that $v(\psi) < \overline{1}^{\boldsymbol{A}}$. In order to prove that $L_D^{\leq}$ is boldly D-paraconsistent it is enough to show that there exists a formula φ such that $\varphi, D\varphi \nvdash_{L_D^{\leq}} \psi$. Let hence φ be a variable q not occurring in ψ. Then, define an $\boldsymbol{A}$-evaluation v' such that $v'(p_i) = v(p_i)$ for each $i = 1, \ldots, n$ and $v'(q) = \beta$, where $\beta \in A$ is such that $\overline{1}^{\boldsymbol{A}} > \beta > v'(\psi) = v(\psi)$. Observe that the fact that $\boldsymbol{A}$ has no coatom guarantees the existence of such a β. Then, we clearly have $v'(q \wedge Dq) = v'(q) > v'(\psi)$, and hence $p, Dp \nvdash_{L_D^{\leq}} \psi$, that is to say, the logic $L_D^{\leq}$ is not partially explosive with respect to any σ. ⊣

We leave as an open problem whether the condition of being complete with respect to chains without coatom is also necessary. All we can say is that there are logics $L_D^{\leq}$ that are not boldly paraconsistent with L_D being complete with respect to chains with a coatom. Namely, for instance if L is the three-valued Gödel or Łukasiewicz logic, it is easy to check that $L_D^{\leq}$ is partially D-explosive with respect to $\sigma(p) = p \vee \neg p$. Indeed we have that $\varphi, D\varphi \vdash_{L_D^{\leq}} \psi \vee \neg\psi$, for all φ and ψ.

Proposition 12 *For any core fuzzy logic* L*, the logic* $L_D^{\leq}$ *is gently* D*-paraconsistent, and hence it is a* D*-***LFI**.

Proof: In order to prove that the logic is gently D-paraconsistent, consider

$$\bigcirc(p) = \triangle(p \vee \neg p) = \neg D(p \vee \neg p).$$

An easy computation shows that the formula $\bigcirc(p)$ satisfies the required conditions. In fact, it is obvious that over any chain $\boldsymbol{C}$ of the variety, we have:

(1) there is an evaluation e such that $e(\bigcirc(p) \wedge p) = \overline{1}^{\boldsymbol{C}}$ (take $e(p) = \overline{1}^{\boldsymbol{C}}$);
(2) there is an evaluation v such that $v(\bigcirc(p) \wedge Dp) = \overline{1}^{\boldsymbol{C}}$ (take $e(p) = \overline{0}^{\boldsymbol{C}}$); and
(3) for each evaluation e, $e(\bigcirc(\varphi) \wedge \varphi \wedge D\varphi) = \overline{0}^{\boldsymbol{C}}$. Indeed, take into account that if $e(\varphi) \in \{\overline{0}^{\boldsymbol{C}}, \overline{1}^{\boldsymbol{C}}\}$ the result is obvious, and if $\overline{0}^{\boldsymbol{C}} < e(\varphi) < \overline{1}^{\boldsymbol{C}}$, then $e(\bigcirc(p)) = e(\triangle(p \vee \neg p)) = \overline{0}^{\boldsymbol{C}}$).

Therefore, conditions (GE-a) and (GE-b) are satisfied. Thus, $L_D^{\leq}$ is gently D-paraconsistent, and thus is an **LFI** as well. ⊣

Notice that the argument involving Kleene axiom we used in Proposition 5 to show that a degree-preserving fuzzy logic $L^{\leq}$ is partially explosive (and hence not boldly paraconsistent), cannot be applied when the considered negation is the dual intuitionistic negation D. In fact, although Kleene axiom $(\varphi \wedge D\varphi) \to (\psi \vee D\psi)$ trivially holds, the argument used above cannot be applied in this framework since the formula $\psi \vee D\psi$ is a theorem of $L_D^{\leq}$. On the other hand, the condition that the logic is complete with respect to chains without coatom cannot be removed, since there are examples of $L_D^{\leq}$ that are partially explosive. Take for instance the degree-preserving companion $Ł_n^{\leq}$ of the logic $Ł_n$ which is complete with respect to evaluations over the finite chain $\boldsymbol{Ł}_n$ (the Łukasiewicz chain of $n+1$ elements) and consider a formula $\psi = \sigma(p_1, \ldots, p_k)$ such that for any evaluation e, $e(\psi) \geq r_n$, where r_n is the coatom of $\boldsymbol{Ł}_n$. Therefore, in the logic $Ł_{n,D}^{\leq}$, it holds that, for each formula φ and each evaluation e, $e(\varphi \wedge D\varphi) \leq r_n$ and thus $\varphi, D\varphi \vdash_{Ł_{n,D}^{\leq}} \psi$, i.e. the logic is partially explosive.

4.2 Adding an involutive negation

Another kind of negations very relevant in fuzzy logics are involutive negations. There is a whole class of extensions of MTL whose residual negation is not involutive (all those logics that are not IMTL), among them Gödel and Product logic. Therefore for all these logics it makes sense to consider expansions with a new involutive negation.

As far as we know, expansions of fuzzy logics with an involutive negation have only been studied in the literature together with the Monteiro-Baaz $\triangle$ operator [22,13,23]. Here we define an expansion of a core fuzzy logic L by an involutive negation without using $\triangle$.[12] We hence define the logic $\mathrm{L}_\sim$ as the expansion of L by a new unary connective $\sim$ with the following additional axiom and rule:

($\sim$) $\sim\sim\varphi \leftrightarrow \varphi$,
(OR) from $(\varphi \to \psi) \vee \chi$ derive $(\sim\psi \to \sim\varphi) \vee \chi$.

Note that, using ($\sim$) and (OR), one can show that $\sim\overline{1} \leftrightarrow \overline{0}$ and $\sim\overline{0} \leftrightarrow \overline{1}$. Also notice that rule (OR) implies that the congruence condition (Cng) holds for $\sim$ and thus $\mathrm{L}_\sim$ is a Rasiowa-implicative logic. Moreover, the rule (OR) is closed under $\vee$-forms, implying that $\mathrm{L}_\sim$ is semilinear as well (see [14]).

An $\mathrm{L}_\sim$-algebra is a structure $\langle A, \wedge, \vee, \&, \to, \sim, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ such that the $\sim$-free reduct is an L-algebra and the two following properties hold for each $x, y, z \in A$:

- $\sim\sim x = x$,
- if $(x \to y) \vee z = \overline{1}^{\boldsymbol{A}}$, then $(\sim y \to \sim x) \vee z = \overline{1}^{\boldsymbol{A}}$.

As for the interaction between the residual negation $\neg$ and the involutive negation $\sim$, let us remark that they are incomparable in general. However, when $\neg$ is Gödel negation, then for any $x \in A$ we clearly have the following *negative* combinations

$$\neg x \leq \sim x \leq \neg\neg\sim x.$$

Note that $\neg\neg\sim = D$. As for the *positive* combinations we have:

$$\neg\sim x \leq x = \sim\sim x \leq \neg\neg x = \sim\neg x.$$

Given the axiomatization of $\mathrm{L}_\sim$, we can easily obtain an axiomatization of $\mathrm{L}^{\leq}_{\sim}$ just by replacing the (OR) rule by its restriction to theorems:

(OR-r) if $\vdash_{\mathrm{L}_\sim} (\varphi \to \psi) \vee \chi$,
from $(\varphi \to \psi) \vee \chi$ derive $(\sim\psi \to \sim\varphi) \vee \chi$.

Now we turn our attention to paraconsistency with respect to $\sim$.

Proposition 13 $\mathrm{L}_\sim$ *is not $\sim$-paraconsistent, but* $\mathrm{L}^{\leq}_{\sim}$ *is always $\sim$-paraconsistent.*

Proof: Observe that there is no evaluation e such that $e(\varphi) = e(\sim\varphi) = \overline{1}$, and hence, for all formulas φ and ψ, we have $\{\varphi, \sim\varphi\} \vdash_{\mathrm{L}_\sim} \psi$, and thus $\mathrm{L}_\sim$ is $\sim$-explosive. Moreover, the same argument used in the proof of Proposition 4 easily shows that $\mathrm{L}^{\leq}_{\sim}$ is $\sim$-paraconsistent. Notice that the logic $\mathrm{L}^{\leq}_{\sim}$ is $\sim$-paraconsistent for any axiomatic extension L of MTL, and not only for non pseudo-complemented extensions, because $\sim$ is involutive. Indeed, if $\boldsymbol{A}$ is an $\mathrm{L}_\sim$-chain with more than two elements, one can always find an $\boldsymbol{A}$-evaluation e such that $e(p \wedge \sim p) > \overline{0}^{\boldsymbol{A}}$. ⊣

Proposition 14 *The logic* $\mathrm{L}^{\leq}_{\sim}$ *is not $\sim$-boldly paraconsistent. Indeed, it is partially $\sim$-explosive with respect to $\sigma(p) = p \vee \sim p$.*

Proof: It is obvious that Kleene's axiom is also valid for the negation $\sim$, and, if φ is not a theorem of $\mathrm{L}_\sim$, then $\varphi \vee \sim\varphi$ is not a theorem as well. Then the proof of Proposition 5 is also valid and therefore the logic $\mathrm{L}^{\leq}_{\sim}$ is partially $\sim$-explosive. ⊣

Finally, whether $\mathrm{L}^{\leq}_{\sim}$ is gently $\sim$-explosive (and hence a $\sim$-**LFI**) depends on the initial logic L. For example, if $\triangle$ is a definable connective[13] in $\mathrm{L}_\sim$, then it is immediate that $\mathrm{L}^{\leq}_{\sim}$ is gently $\sim$-explosive. Indeed, consider $\bigcirc(\varphi) = \triangle(\varphi \vee \neg\varphi)$ and an obvious computation proves that the operator $\bigcirc$ satisfies the required conditions. Observe that in the logics where $\triangle$ is definable, the dual intuitionistic negation is also definable (remember that $D\varphi \leftrightarrow \neg\triangle\varphi$) and therefore, in this setting, both D and $\sim$ appear together.

Remark 2 In this subsection we have discussed the paraconsistent properties of degree-preserving fuzzy logics when expanded by an involutive negation. In particular, it is worth noticing that Proposition 13 also applies to the $\sim$-expansions of those logics which, with respect to their residual negation $\neg$, are explosive. This is the case, for instance, of the degree-preserving companion of any pseudo-complemented expansion of MTL (i.e. expansion of SMTL). Nevertheless, there are other techniques which can be used to introduce an *involutive* variant of these logics –and hence a paraconsistent degree-preserving companion of these logics– which uses the so called *connected* and *disconnected rotation* constructions [32]. As shown in [39,37], in fact, for each SMTL-chain $\boldsymbol{A}$, its connected rotation is a *perfect* IMTL-chain with negation fixpoint (cf. [37, Theorem 6.40]), while the disconnected rotation of $\boldsymbol{A}$ is an IMTL-chain without fixpoint [39, Theorem 2].

[12] Of course, the interesting case is when the negation $\neg$ of L is not involutive.

[13] As it occurs either in any pseudo-complemented logic where $\triangle$ is definable as $\triangle\varphi := \neg\sim\varphi$ or in a finitely-valued Łukasiewicz logic $Ł_n$ where $\triangle$ is definable as $\triangle\varphi := \varphi^n$.

5 First-order degree-preserving fuzzy logics

In this final section we will consider first-order fuzzy logics with paraconsistency properties. First we need to recall the usual presentation of first-order formalisms for fuzzy logics.[14]

Let us fix a finitary semilinear expansion of a core fuzzy logic L satisfying (Cng) in order to define its truth-preserving first-order extension L∀. The predicate language $\mathcal{P}$ of L∀ is built in the standard classical way with a set of predicate symbols $Pred$, a set of function symbols $Funct$, and a set of object variables Var, together with the quantifiers ∀ and ∃. The set of terms $Term$ is the minimum set containing the elements of Var and closed under the functions. Atomic formulas are expressions of the form $P(t^1, \ldots, t^n)$, where $P \in Pred$ and $t^1, \ldots, t^n \in Term$. The set of all formulas is obtained by closing the set of atomic formulas under combination by propositional connectives and quantification, i.e. if φ is a formula and x is an object variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas as well.

In first-order fuzzy logics the semantics is based on chains only. Given an L-chain $\boldsymbol{A}$, an $\boldsymbol{A}$-structure is $\mathbf{M} = \langle M, \langle P_\mathbf{M}\rangle_{P\in Pred}, \langle f_\mathbf{M}\rangle_{f\in Funct}\rangle$ where $M \neq \emptyset$, $f_\mathbf{M} : M^{ar(f)} \to M$, and $P_\mathbf{M} : M^{ar(P)} \to A$ for each $f \in Funct$ and $P \in Pred$ (where ar is the function that gives the arity of function and predicate symbols). For each $\mathbf{M}$-evaluation of variables $v : Var \to M$, the interpretation of a $t \in Term$, denoted $t_{\mathbf{M},v}$, is defined as in classical first-order logic. The truth-value $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v}$ of a formula is defined inductively from

$$\|P(t^1, \ldots, t^n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = P_\mathbf{M}(t^1_{\mathbf{M},v}, \ldots, t^n_{\mathbf{M},v}),$$

taking into account that the value commutes with connectives, and defining

$$\|(\forall x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables } y \text{, except } x\}$$
$$\|(\exists x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \sup\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables } y \text{, except } x\}$$

if the infimum and supremum exist in $\boldsymbol{A}$, otherwise the truth-value(s) remain undefined. An $\boldsymbol{A}$-structure $\mathbf{M}$ is called *safe* if all infima and suprema needed for the definition of the truth-value of any formula exist in $\boldsymbol{A}$.

The axioms for L∀ are obtained from those of L by substitution of propositional variables with first-order formulas plus the following axioms for quantifiers:

(∀1)	$(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)
(∃1)	$\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
(∀2)	$(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)
(∃2)	$(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$ (x not free in ν)
(∀3)	$(\forall x)(\varphi \vee \nu) \to ((\forall x)\varphi \vee \nu)$ (x not free in ν)

[14] For more details and proofs see e.g [12].

The rules of inference of L∀ are the rules of L (again by substituting propositional variables with first-order formulas) plus generalization: from φ infer $(\forall x)\varphi$. Note that *modus ponens* is already in L.

This axiomatic system captures the intended truth-preserving semantical consequence in the following way: for any set of formulas T and each formula φ, we have that $T \vdash_{\text{L}\forall} \varphi$ iff for each L-chain $\boldsymbol{A}$ and each safe $\boldsymbol{A}$-structure, if $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \bar{1}^{\boldsymbol{A}}$ for each $\psi \in T$ and each $\mathbf{M}$-evaluation v, then also $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \bar{1}^{\boldsymbol{A}}$ for each $\mathbf{M}$-evaluation v.

Degree-preserving first-order fuzzy logics have not been considered in the literature yet. However, it is not difficult to extend the definitions from [6] to first-order logics.

Definition 6 Given a first-order fuzzy logic L∀, its degree-preserving companion is denoted as $\text{L}\forall^{\leq}$ and it is semantically defined in the following way: for every set of predicate formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\text{L}\forall^{\leq}} \varphi$ iff there is a finite $\Gamma_0 \subseteq \Gamma$ such that for every L-chain $\boldsymbol{A}$, every $a \in A$, every $\boldsymbol{A}$-structure $\mathbf{M}$ and every $\mathbf{M}$-evaluation v, if $a \leq \|\psi\|^{\boldsymbol{A}}_{\mathbf{M},v}$ for every $\psi \in \Gamma_0$, then $a \leq \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v}$.

The relations between the truth-preserving logic and its degree-preserving companion are analogous to those described in the propositional case:

Proposition 15 *The following facts hold:*

(1) The two logics L∀ *and* $\text{L}\forall^{\leq}$ *have the same tautologies.*
(2) For all formulas φ, ψ *one has:* $\varphi, \psi \vdash_{\text{L}\forall^{\leq}} \varphi \wedge \psi$.
(3) $\varphi_1, \ldots, \varphi_n \vdash_{\text{L}\forall^{\leq}} \psi$ *iff* $\vdash_{\text{L}\forall} (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$.

Proof All the claims are straightforward; let us prove the last one as an example. Assume first that we have $\varphi_1, \ldots, \varphi_n \vdash_{\text{L}\forall^{\leq}} \psi$. Let $\boldsymbol{A}$ be an L-chain, $\mathbf{M}$ an $\boldsymbol{A}$-structure, and v an $\mathbf{M}$-evaluation. For each $a \in A$, we know that if $a \leq \|\varphi_i\|^{\boldsymbol{A}}_{\mathbf{M},v}$ for each i, then $a \leq \|\psi\|^{\boldsymbol{A}}_{\mathbf{M},v}$. So, taking $a = \min\{\|\varphi_1\|^{\boldsymbol{A}}_{\mathbf{M},v}, \ldots, \|\varphi_n\|^{\boldsymbol{A}}_{\mathbf{M},v}\}$, we obtain that $a = \|\varphi_1 \wedge \ldots \wedge \varphi_n\|^{\boldsymbol{A}}_{\mathbf{M},v} \leq \|\psi\|^{\boldsymbol{A}}_{\mathbf{M},v}$, hence $\|(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \bar{1}^{\boldsymbol{A}}$, and so $\vdash_{\text{L}\forall} (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$. Conversely, assume that $\vdash_{\text{L}\forall} (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$ and take an L-chain $\boldsymbol{A}$, an $\boldsymbol{A}$-structure $\mathbf{M}$, an $\mathbf{M}$-evaluation v, and $a \in A$ such that $a \leq \|\varphi_i\|^{\boldsymbol{A}}_{\mathbf{M},v}$ for each i. Then, since $\|(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \bar{1}^{\boldsymbol{A}}$, we have that $\|\varphi_1 \wedge \ldots \wedge \varphi_n\|^{\boldsymbol{A}}_{\mathbf{M},v} \leq \|\psi_i\|^{\boldsymbol{A}}_{\mathbf{M},v}$, and hence $a \leq \|\psi\|^{\boldsymbol{A}}_{\mathbf{M},v}$.

Moreover, it is quite straightforward to obtain a Hilbert-style presentation for $\text{L}\forall^{\leq}$:

Proposition 16 *The logic* $\text{L}\forall^{\leq}$ *can be presented by a Hilbert-style proof system with the same axioms as* L∀ *and the following inference rules:*

(Adj-$\wedge$) from φ and ψ derive $\varphi \wedge \psi$,
(MP-r) if $\vdash_{L\forall} \varphi \to \psi$, then from φ and $\varphi \to \psi$ derive ψ,
(gen-r) if $\vdash_{L\forall} \varphi$, then from φ derive $(\forall x)\varphi$,
(R-r) if (R) is a rule of L$\forall$ *(obtained from a propositional rule of* L *different from* modus ponens*) whose premises are theorems of* L$\forall$*, then from the premises one can derive the conclusion.*

Proof: Let us denote the provability relation induced by the Hilbert-style system as $\vdash_S$. We have to show that for every set of formulas $T \cup \{\varphi\}$, it holds that $T \vdash_S \varphi$ iff $T \vdash_{L\forall^{\leq}} \varphi$. Soundness is obvious. Suppose that $T \vdash_{L\forall^{\leq}} \varphi$. We can assume that T is finite, say $T = \{\varphi_1, \ldots, \varphi_k\}$. We obtain $\vdash_{L\forall} \bigwedge_{i=1}^{k} \varphi_i \to \varphi$. Let $\langle \psi_1, \ldots, \psi_{n-1}, \bigwedge_{i=1}^{k} \varphi_i \to \varphi \rangle$ be a proof of $\bigwedge_{i=1}^{k} \varphi_i \to \varphi$ in $\vdash_{L\forall}$. Then $\langle \psi_1, \ldots, \psi_{n-1}, \bigwedge_{i=1}^{k} \varphi_i \to \varphi, \varphi_1, \ldots, \varphi_k, \bigwedge_{i=1}^{k} \varphi_i, \varphi \rangle$ is indeed a proof in $\vdash_S$ of φ from T, using:

(i) (MP-r), (gen-r) and (R-r) instead of each application of modus ponens, generalization and rules (R) in the original proof, and
(ii) (Adj-$\wedge$) and (MP-*r*) in the last two steps.

$\dashv$

The notions of paraconsistency considered in this paper are essentially propositional because they refer to the behaviour of a negation connective and their characterizations refer to propositional conditions (pseudo-complementation, existence of certain propositional formulas $\sigma(p)$ or $\bigcirc(p)$). Therefore, regarding their paraconsistency, we can obtain for first-order fuzzy logics the same results as for the propositional ones; to sum it up:

- Truth-preserving logics L$\forall$ are explosive with respect to $\neg$.
- L$\forall^{\leq}$ is paraconsistent iff L is not pseudo-complemented.
- L$\forall^{\leq}$ is partially explosive with respect to $\sigma(p) = p \vee \neg p$.
- L$\forall^{\leq}$ is controllably explosive iff it is locally Boolean.
- The notion of gently explosive and its characterization in Proposition 9.
- $L_D\forall$ is D-explosive, but $L_D\forall^{\leq}$ is D-paraconsistent.
- $L_D\forall^{\leq}$ is gently D-paraconsistent, and so, an **LFI**.
- $L_D\forall^{\leq}$ is boldly paraconsistent if it is complete with respect to models over chains without coatom.
- $L_{\sim}\forall^{\leq}$ is $\sim$-paraconsistent.
- $L_{\sim}\forall^{\leq}$ is partially $\sim$-explosive with respect to $\sigma(p) = p \vee {\sim}p$.

6 Final remarks

In this paper we have been concerned with exploring paraconsistency properties of different kinds of formal systems of fuzzy logic. It has been shown that, while truth-preserving fuzzy logics are not paraconsistent, a class of degree-preserving fuzzy logics are indeed paraconsistent, and some of them can be even considered as proper **LFI**s, so the fuzzy logic paradigm provides brand new examples of well-behaved paraconsistent logics. In this final section we want to briefly comment on their distinctive features and similarities with respect to other paraconsistent systems.

- Our paraconsistent fuzzy logics satisfy the adjunction rule (Adj-$\wedge$), i.e. from φ and ψ one can derive $\varphi \wedge \psi$. This is not the case in other paraconsistent logics such as Jaśkowski's discussive logic [31] (defined as a modification of the modal logic S5: $\Gamma \vdash_J \varphi$ iff $\diamond\Gamma \vdash_{S5} \diamond\varphi$). It is clear that $p, \neg p \nvdash_J q$, while $p \wedge \neg p \vdash_J q$. In $L^{\leq}$ logics, both derivations fail, i.e. $p, \neg p \nvdash_{L^{\leq}} q$ and $p \wedge \neg p \nvdash_{L^{\leq}} q$, which shows a more robust non-explosive character.
- Unlike paraconsistent systems obtained by requiring only some conditions on classical evaluations (like da Costa's C_1 and C_ω [17] or De Batens' PI [4]), $L^{\leq}$ logics are completely truth-functional, i.e. the value of any complex formula can be computed from the truth value of its atomic parts. Moreover, we do not consider only evaluations over the classical truth values $\{0, 1\}$, but also over MTL-chains and their expansions.
- $L^{\leq}$ logics are genuine many-valued logics, directly introduced in terms of consequence relation with respect to an intended algebraic semantics. In this aspect, they are similar to other paraconsistent logics such as Priest's logic of paradox LP [40] which has also been defended as "a candidate for a paraconsistent fuzzy logic" (see e.g. [42]). LP is a three-valued logic with truth values 0, 1 a third value b for *both true and false*; the connectives $\wedge, \vee, \neg$ are defined as in the three-valued Kleene logic and the set of designated values is $\{b, 1\}$, instead of just $\{1\}$ as in Kleene logic. The tautologies of LP coincide with those of classical logic. One could define analogous systems over richer sets of truth values, even the continuous interval, but they would still be equivalent to LP. We argue that $L^{\leq}$ logics are more suitable as paraconsistent fuzzy logics, since they do not validate all classical tautologies.
A more interesting many-valued paraconsistent logic is Pac, obtained as the conservative expansion of LP with classical implication. Pac is boldly paraconsistent, but not controllably explosive and not an **LFI**; however it can be expanded to the system J_3 (also known as LFI1) which is boldly paraconsistent and an **LFI** (see [8] and references thereof for more information about these systems). Again, regardless

of their interest as very expressive paraconsistent logics, the fact that these many-valued logics prove the excluded middle law sets them apart from the fuzzy logic paradigm we have followed here.

- The degree-preserving algebraic semantics we have proposed was not alien to the paraconsistent world. For instance, Dunn's system FDE [18] can be presented (see e.g. [42]) as the degree-preserving consequence relation given by the four-element De Morgan algebra. If a and b are the two non-classical elements of the algebra, the paraconsistency of the logic follows from the fact that $a \wedge \neg a = a \wedge a = a \not\leq b$. Another interesting example is Goodman's logic [27] defined as degree-preserving consequence on dual Heyting algebras; it has the same tautologies as classical logic. Also, as already mentioned in Section 3, Priest already noticed in [43] that the degree-preserving Łukasiewicz logic $Ł^{\leq}$ was paraconsistent.
- All degree-preserving fuzzy logics studied in this paper satisfy the weakening law (i.e. they prove the theorem $\varphi \to (\psi \to \varphi)$, or equivalently in their algebraic semantics, the neutral element $\overline{1}$ is the maximum element in the lattice order), because they are based on ($\triangle$-)core fuzzy logics that already satisfy this law. Moreover, with the exception of Gödel-Dummett logic (for which $G = G^{\leq}$), they do not satisfy the contraction law ($\varphi \to \varphi \,\&\, \varphi$ or, algebraically, idempotence of &). This separates our approach from studies of paraconsistency in the framework of relevant logics, that cannot satisfy weakening (whereas many of them satisfy contraction). An interesting topic for further research would be to consider a systematic study of weakening-free semilinear substructural logics which, even in the truth-preserving paradigm, would display a paraconsistent behavior. This should take into account, as a prominent example, the relevance logic with mingle RM (see e.g. [19]).
- Many paraconsistent logics, such as da Costa's logics C_n $(1 \leq n < \omega)$ can be axiomatized as expansions of classical positive logic. This is not the case for $Ł^{\leq}$, which, already in the fragment without $\neg$ and $\overline{0}$ have a strictly subclassical behavior.
- A usual matter of concern in paraconsistent systems is whether they can have a material implication like classical logic (see e.g. [42]). Our approach does not consider material implication. Instead of that we are based on a residuated implication $\to$ which plays an essential rôle from the very notion of semilinearity. Indeed, the algebraic semantincs of our logics is ordered by $\to$ (i.e. in every algebra $\boldsymbol{A}$, for each $a, b \in A$ $a \leq b$ iff $a \to^{\boldsymbol{A}} b = \overline{1}^{\boldsymbol{A}}$) and this order relation determines the chains with respect to which the logic is required to be complete.

As regards decidability and complexity issues, it is worth mentioning that our proposed logics have a nice behaviour or, at least, no worse than that of their truth-preserving counterparts. Indeed, the theorems of L and $L^{\leq}$ coincide, and for most well known fuzzy logics this set is decidable and even **coNP**-complete (see e.g. [28]). As for derivations, just recall that $\varphi_1, \ldots, \varphi_n \vdash_{L^{\leq}} \psi$ iff $\vdash_{L} (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$.

An important issue for the study of degree-preserving fuzzy logics as **LFI**s is that of understanding their consistency operators from an algebraic semantical point of view. This is the topic of the recent work [15], which follows the proposal of the present paper.

As a last remark, we would like to point out that the kind of inconsistencies that our paraconsistent fuzzy logics can deal with only arise from the very reason of dealing with intermediate degrees of truth, that is, all these systems immediately become explosive as soon as one forces propositions to be two-valued. Practical inconsistency handling mechanisms using these paraconsistent fuzzy logics remain to be explored.

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