Paraconsistent Reasoning for Semantic Web Agents

Linh Anh Nguyen[1] and Andrzej Szałas[1,2]

[1] Institute of Informatics, University of Warsaw
Banacha 2, 02-097 Warsaw, Poland
nguyen@mimuw.edu.pl, andsz@mimuw.edu.pl

[2] Dept. of Computer and Information Science, Linköping University
SE-581 83 Linköping, Sweden

Abstract. Description logics refer to a family of formalisms concentrated around concepts, roles and individuals. They are used in many multiagent and Semantic Web applications as a foundation for specifying knowledge bases and reasoning about them. Among them, one of the most important logics is $\mathcal{SROIQ}$, providing the logical foundation for the OWL 2 Web Ontology Language recommended by W3C in October 2009.

In the current paper we address the problem of inconsistent knowledge. Inconsistencies may naturally appear in the considered application domains, for example as a result of fusing knowledge from distributed sources. We introduce a number of paraconsistent semantics for $\mathcal{SROIQ}$, including three-valued and four-valued semantics. The four-valued semantics reflects the well-known approach introduced in [5, 4] and is considered here for comparison reasons only. We also study the relationship between the semantics and paraconsistent reasoning in $\mathcal{SROIQ}$ through a translation into the traditional two-valued semantics. Such a translation allows one to use existing tools and reasoners to deal with inconsistent knowledge.

1 Introduction

The Web Ontology Language (OWL) is a family of knowledge representation languages for designing ontologies. It is considered one of the fundamental technologies underpinning the Semantic Web, and has attracted both academic and commercial interest. OWL has a formal semantics based on description logics (DLs), which are formalisms concentrated around concepts (classes of individuals) and roles (binary relations between individuals), and aim to specify concepts and concept hierarchies and to reason about them.[3] DLs belong to the most frequently used knowledge representation formalisms and provide a logical basis to a variety of well known paradigms, including frame-based systems, semantic

[3] There is a rich literature on DLs. For good surveys consult [2], in particular papers [20, 3] as well as the bibliography provided there.

networks and Semantic Web ontologies and reasoners. The extension OWL 2 of OWL, based on the DL $\mathcal{SROIQ}$ [10], became a W3C recommendation in October 2009.

Description logics have usually been considered as syntactic variants of restricted versions of classical first-order logic. On the other hand, in Semantic Web and multiagent applications, knowledge/ontology fusion frequently leads to inconsistencies. When inconsistencies occur in facts provided by different sites of a distributed system (e.g., in the ABox of a combined ontology), the consensus-based method proposed by N.T. Nguyen [23–26] is an advanced approach that can be used to solve conflicts. When inconsistencies are caused through ontological knowledge (e.g., a TBox) rather than by direct conflicts in facts, one can adapt paraconsistent reasoning approaches. For example, consider an ontology KB_1 reflecting the typical relationship between concepts $Bird$ and Fly, $Bird \sqsubseteq Fly$, an ontology KB_2 extending KB_1 with axioms $Penguin \sqsubseteq Bird$ and $Penguin \sqsubseteq \neg Fly$, and an ontology KB_3 extending KB_2 with facts $Bird(a)$ and $Penguin(tweety)$. Then, using paraconsistent reasoning we would like to draw from KB_3 facts $Fly(a)$ and $Bird(tweety)$. Also, both $Fly(tweety)$ and $\neg Fly(tweety)$ can be derived, so $Fly(tweety)$ is inconsistent w.r.t. KB_3. However, we do not want to draw $\neg Fly(a)$ from KB_3. This example will be continued in Section 6.

There is a rich literature on paraconsistent logics (see, e.g., [7] and references there). In general, paraconsistent reasoning relies on weakening the traditional reasoning methods in order to avoid trivialization (which allows to draw any conclusion from an inconsistent knowledge base). In [11] Hunter listed a few approaches for dealing with paraconsistent reasoning in classical propositional logic:

1. restricting to a consistent subset of the knowledge base
2. forbidding some inference rules
3. using four-valued semantics
4. using quasi-classical semantics
5. using argumentation-based reasoning.

All of these approaches can be applied for paraconsistent reasoning in DLs. The first approach involves knowledge maintenance and will not be addressed in this paper. The second approach lacks semantics and usually leads to non-intuitive consequences [11], and hence not received much attention from the DL community. The fifth approach has recently been applied for the basic DL $\mathcal{ALC}$ by Zhang et al. [32, 33]. However, they did not provide adequate reasoning methods. The third and fourth approaches will be addressed in more detail for DLs and compared with our approaches presented in this paper.

A number of researchers have extended description logics with paraconsistent semantics and adequate reasoning methods [19, 28, 27, 15, 14, 31, 30, 22, 21]. The work [27] studies a constructive version of the basic description logic $\mathcal{ALC}$, but it is not clear how to extend the semantics provided in this work to other description logics. Papers [19, 28, 15, 14] are based on the well-known Belnap's four-valued logic [5, 4]. Truth values in this logic represent truth (t), falsity (f), the lack of knowledge ($\mathfrak{u}$) and inconsistency ($\mathfrak{i}$). However, there are serious problems with using Belnap's logic for the Semantic Web. Some of these problems are considered in the general context, e.g., in [18, 29]. We give here some others, more directly related to description logics (see also Sections 3 and 4):

- According to the semantics considered in [19, 28, 15, 14], if $(x \in C^{\mathcal{I}}) = \mathfrak{i}$ and $(x \in D^{\mathcal{I}}) = \mathfrak{u}$ then $(x \in (C \sqcap D)^{\mathcal{I}}) = \mathfrak{f}$ and $(x \in (C \sqcup D)^{\mathcal{I}}) = \mathfrak{t}$ which, in our opinion, is not intuitive.
- A knowledge base, as a theory, may be incomplete in the sense that truth value of a formula given as a query to the knowledge base may be not determined using the theory. In such cases, we have a meta-unknown value. If the semantics uses the truth value $\mathfrak{u}$, one can raise the question about the relationship between $\mathfrak{u}$ and the meta-unknown value. This problem was not addressed in the mentioned works.
- One of the most common approaches is to use paraconsistent reasoning for knowledge bases specified in the traditional way without explicit truth values $\mathfrak{t}$, $\mathfrak{f}$, $\mathfrak{i}$, $\mathfrak{u}$. The reason is that, if we allow explicit uses of $\mathfrak{t}$, $\mathfrak{f}$, $\mathfrak{i}$, $\mathfrak{u}$ then, for example, two facts $C(a) : \mathfrak{t}$ and $C(a) : \mathfrak{u}$ in a knowledge base form a clash. With this approach, as used in [19, 28, 15, 14], $\mathfrak{u}$ is not used for knowledge representation but only for the semantics. On the other hand, in many cases allowing the value $\mathfrak{u}$ by excluding the axioms $\top \sqsubseteq A \sqcup \neg A$ weakens the logic too much.

In [31] Zhang et al. gave a quasi-classical semantics for the DL $\mathcal{SHIQ}$, which is a sublogic of $\mathcal{SHOIQ}$ used for OWL 1. The semantics is based on both Belnap's four-valued logic and the quasi-classical logic of Besnard and Hunter [6, 12]. In [30] Zhang et al. also gave a paradoxical semantics for the basic DL $\mathcal{ALC}$, which is based on a three-valued semantics.[4]

Independently from [30], in the conference paper [22] we modeled inconsistency using only three truth values $\mathfrak{t}$, $\mathfrak{f}$, $\mathfrak{i}$ (as in Kleene's three-valued logic [13, 8]) for the DL $\mathcal{SHIQ}$, which is more expressive than $\mathcal{ALC}$. In a sense, we identified inconsistency with the lack of knowledge. There are many good reasons for such an identification (see, e.g., [9]). Assuming that the objective reality is consistent, the value $\mathfrak{i}$ reflects a sort of lack of knowledge. Namely, inconsistent information often reflects differences in subjective realities of agents resulting, for example, from their different perceptual capabilities. Inconsistency appears, when different information sources do not agree with one another and one cannot decide which of them is right. Also, in many multiagent and Semantic Web scenarios one has contradictory evidence as to a given fact. In [22] we also gave a faithful translation of our formalism into a suitable version of a two-valued description logic. Such a translation allows one to use existing tools and reasoners to deal with inconsistent knowledge in $\mathcal{SHIQ}$.

In [21] Nguyen extended the method and results of [22] for the expressive DL $\mathcal{SROIQ}$. He introduced a number of different paraconsistent semantics for

[4] Theorems 3, 5, 6 of [30] are wrong. For Theorem 3 of [30], take $\top \sqsubseteq \bot$ as an ontological axiom. For Theorems 5 and 6 of [30], take $\phi = (A \sqcap \neg A \sqsubseteq \bot)$.

$\mathcal{SROIQ}$ and studied the relationship between them. He also addressed paraconsistent reasoning in $\mathcal{SROIQ}$ w.r.t. some of such semantics through a translation into the traditional semantics. His paraconsistent semantics for $\mathcal{SROIQ}$ are characterized by four parameters for:

- using two-, three-, or four-valued semantics for concept names
- using two-, three-, or four-valued semantics for role names
- considering two kinds of interpretation of concepts of the form $\forall R.C$ or $\exists R.C$
- using weak, moderate, or strong semantics for terminological axioms.

Due to the lack of space, the results of [21] were given without proofs. Furthermore, the definition of $\mathfrak{s} \sqsubseteq \mathfrak{s}'$ given in Section 5 of that paper is not entirely correct.

This work is a revised and extended version of the conference papers [22, 21]. The main contributions of the current paper comparing to [22, 21] are a correction for the mentioned definition of [21] and full proofs for the results listed in that paper. Also, new discussions and examples are provided. As $\mathcal{SROIQ}$ and its simpler versions are used for specifying Web ontologies, our semantics and method are useful for paraconsistent reasoning for Semantic Web agents.

Let us emphasize that the four-valued semantics is considered in our paper for comparisons only. It reflects the approach based on Belnap's logic which we found inadequate for Semantic Web applications (see Sections 3 and 4). A logic which seems to behave much better in this context is provided in [18, 29]. We do not consider it here as its adaptation to Semantic Web applications is not obvious and requires further investigations, especially in the light of new developments in the field of paraconsistent rule languages [17, 16].

Note that, in the context of description logics, three-valued semantics has been studied earlier only for $\mathcal{ALC}$ [30] and $\mathcal{SHIQ}$ [22]. Also note that, studying four-valued semantics for DLs, Ma and Hitzler [14] did not consider all features of $\mathcal{SROIQ}$. For example, they did not consider concepts of the form $\exists R.\mathtt{Self}$ and individual assertions of the form $\neg S(a, b)$.

The rest of this paper is structured as follows. In Section 2 we recall notations and semantics of $\mathcal{SROIQ}$. We present our paraconsistent semantics for $\mathcal{SROIQ}$ in Section 3 and study the relationship between them in Section 4. Comparison with other authors' paraconsistent semantics of $\mathcal{SROIQ}$ and $\mathcal{SHIQ}$ is given in Section 5. In Section 6 we give a faithful translation of the problem of conjunctive query answering w.r.t. some of the considered paraconsistent semantics into a version that uses the traditional semantics. Section 7 concludes this work.

2 The Two-Valued Description Logic $\mathcal{SROIQ}$

In this section we recall notations and semantics of the DL $\mathcal{SROIQ}$ [10]. Assume that our language uses a finite set $\mathbf{C}$ of *concept names*, a subset $\mathbf{N} \subseteq \mathbf{C}$ of *nominals*, a finite set $\mathbf{R}$ of role names including the universal role U, and a finite set $\mathbf{I}$ of individual names. Let $\mathbf{R}^- \stackrel{\text{def}}{=} \{r^- \mid r \in \mathbf{R} \setminus \{U\}\}$ be the set of *inverse roles*. A *role* is any member of $\mathbf{R} \cup \mathbf{R}^-$. We use letters like R and S for roles.

An *interpretation* $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ consists of a non-empty set $\Delta^{\mathcal{I}}$, called the *domain* of $\mathcal{I}$, and a function $\cdot^{\mathcal{I}}$, called the *interpretation function* of $\mathcal{I}$, which maps every concept name A to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, where $A^{\mathcal{I}}$ is a singleton set if $A \in \mathbf{N}$, and maps every role name r to a binary relation $r^{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$, with $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and maps every individual name a to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. Inverse roles are interpreted as usual, i.e., for $r \in \mathbf{R}$, we define

$$(r^-)^{\mathcal{I}} \stackrel{\text{def}}{=} (r^{\mathcal{I}})^{-1} = \{\langle x, y\rangle \mid \langle y, x\rangle \in r^{\mathcal{I}}\}.$$

A *role inclusion axiom* is an expression of the form $R_1 \circ \ldots \circ R_k \sqsubseteq S$. A *role assertion* is an expression of the form $\mathtt{Ref}(R)$, $\mathtt{Irr}(R)$, $\mathtt{Sym}(R)$, $\mathtt{Tra}(R)$, or $\mathtt{Dis}(R, S)$, where $R, S \neq U$. Given an interpretation $\mathcal{I}$, define that:

$$\begin{array}{lll}
\mathcal{I} \models R_1 \circ \ldots \circ R_k \sqsubseteq S & \text{if} & R_1^{\mathcal{I}} \circ \ldots \circ R_k^{\mathcal{I}} \subseteq S^{\mathcal{I}} \\
\mathcal{I} \models \mathtt{Ref}(R) & \text{if} & R^{\mathcal{I}} \text{ is reflexive} \\
\mathcal{I} \models \mathtt{Irr}(R) & \text{if} & R^{\mathcal{I}} \text{ is irreflexive} \\
\mathcal{I} \models \mathtt{Sym}(R) & \text{if} & R^{\mathcal{I}} \text{ is symmetric} \\
\mathcal{I} \models \mathtt{Tra}(R) & \text{if} & R^{\mathcal{I}} \text{ is transitive} \\
\mathcal{I} \models \mathtt{Dis}(R, S) & \text{if} & R^{\mathcal{I}} \text{ and } S^{\mathcal{I}} \text{ are disjoint,}
\end{array}$$

where the operator $\circ$ stands for the composition of relations. By a *role axiom* we mean either a role inclusion axiom or a role assertion. We say that a role axiom φ is *valid* in $\mathcal{I}$ (or $\mathcal{I}$ *validates* φ) if $\mathcal{I} \models \varphi$.

An *RBox* is a set $\mathcal{R} = \mathcal{R}_h \cup \mathcal{R}_a$, where $\mathcal{R}_h$ is a finite set of role inclusion axioms and $\mathcal{R}_a$ is a finite set of role assertions. It is required that $\mathcal{R}_h$ is *regular* and $\mathcal{R}_a$ is *simple*. In particular, $\mathcal{R}_a$ is simple if all roles R, S appearing in role assertions of the form $\mathtt{Irr}(R)$ or $\mathtt{Dis}(R, S)$ are *simple roles* w.r.t. $\mathcal{R}_h$. These notions (of regularity and simplicity) will not be exploited in this paper and we refer the reader to [10] for their definitions. An interpretation $\mathcal{I}$ is a *model* of an RBox $\mathcal{R}$, denoted by $\mathcal{I} \models \mathcal{R}$, if it validates all role axioms of $\mathcal{R}$.

The set of *concepts* is the smallest set such that:

- all concept names (including nominals) and $\top$, $\bot$ are concepts
- if C, D are concepts, R is a role, S is a simple role and n is a non-negative integer, then $\neg C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$, $\exists R.C$, $\exists S.\mathtt{Self}$, $\geq n S.C$, and $\leq n S.C$ are also concepts.

We use letters like A, B to denote concept names, and letters like C, D to denote concepts.

Given an interpretation $\mathcal{I}$, the interpretation function $\cdot^{\mathcal{I}}$ is extended to complex concepts as follows, where $\#\Gamma$ stands for the number of elements in the set Γ:

$$\begin{aligned}
\top^{\mathcal{I}} &\stackrel{\text{def}}{=} \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &\stackrel{\text{def}}{=} \emptyset \\
(\neg C)^{\mathcal{I}} &\stackrel{\text{def}}{=} \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &\stackrel{\text{def}}{=} C^{\mathcal{I}} \cap D^{\mathcal{I}}
\end{aligned}$$

$$(C \sqcup D)^{\mathcal{I}} \stackrel{\text{def}}{=} C^{\mathcal{I}} \cup D^{\mathcal{I}}$$
$$(\forall R.C)^{\mathcal{I}} \stackrel{\text{def}}{=} \{x \in \Delta^{\mathcal{I}} \mid \forall y[\langle x, y\rangle \in R^{\mathcal{I}} \text{ implies } y \in C^{\mathcal{I}}]\}$$
$$(\exists R.C)^{\mathcal{I}} \stackrel{\text{def}}{=} \{x \in \Delta^{\mathcal{I}} \mid \exists y[\langle x, y\rangle \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}]\}$$
$$(\exists S.\mathtt{Self})^{\mathcal{I}} \stackrel{\text{def}}{=} \{x \in \Delta^{\mathcal{I}} \mid \langle x, x\rangle \in S^{\mathcal{I}}\}$$
$$(\geq n\, S.C)^{\mathcal{I}} \stackrel{\text{def}}{=} \{x \in \Delta^{\mathcal{I}} \mid \#\{y \mid \langle x, y\rangle \in S^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\} \geq n\}$$
$$(\leq n\, S.C)^{\mathcal{I}} \stackrel{\text{def}}{=} \{x \in \Delta^{\mathcal{I}} \mid \#\{y \mid \langle x, y\rangle \in S^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\} \leq n\}.$$

A *terminological axiom*, also called a *general concept inclusion* (GCI), is an expression of the form $C \sqsubseteq D$. A *TBox* is a finite set of terminological axioms. An interpretation $\mathcal{I}$ validates an axiom $C \sqsubseteq D$, denoted by $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. We say that $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, denoted by $\mathcal{I} \models \mathcal{T}$, if it validates all axioms of $\mathcal{T}$.

We use letters like a and b to denote individual names. An *individual assertion* is an expression of the form $a \neq b$, $C(a)$, $R(a, b)$, or $\neg S(a, b)$, where S is a simple role and $R, S \neq U$. Given an interpretation $\mathcal{I}$, define that:

$$\begin{array}{ll} \mathcal{I} \models a \neq b & \text{if } a^{\mathcal{I}} \neq b^{\mathcal{I}} \\ \mathcal{I} \models C(a) & \text{if } a^{\mathcal{I}} \in C^{\mathcal{I}} \\ \mathcal{I} \models R(a,b) & \text{if } \langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in R^{\mathcal{I}} \\ \mathcal{I} \models \neg S(a,b) & \text{if } \langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \notin S^{\mathcal{I}}. \end{array}$$

We say that $\mathcal{I}$ *satisfies* an individual assertion φ if $\mathcal{I} \models \varphi$. An *ABox* is a finite set of individual assertions. An interpretation $\mathcal{I}$ is a *model* of an ABox $\mathcal{A}$, denoted by $\mathcal{I} \models \mathcal{A}$, if it satisfies all assertions of $\mathcal{A}$.

A *knowledge base* is a tuple $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$, where $\mathcal{R}$ is an RBox, $\mathcal{T}$ is a TBox, and $\mathcal{A}$ is an ABox. An interpretation $\mathcal{I}$ is a *model* of a knowledge base $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ if it is a model of all $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{A}$. A knowledge base is *satisfiable* if it has a model.

A (*conjunctive*) *query* is an expression of the form $\varphi_1 \wedge \ldots \wedge \varphi_k$, where each φ_i is an individual assertion. An interpretation $\mathcal{I}$ satisfies a query $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_k$, denoted by $\mathcal{I} \models \varphi$, if $\mathcal{I} \models \varphi_i$ for all $1 \leq i \leq k$. We say that a query φ is a *logical consequence* of a knowledge base $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$, denoted by $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle \models \varphi$, if every model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ satisfies φ.

Note that, queries are defined to be “ground”. In a more general context, queries may contain variables for individuals. However, one of the approaches to deal with such queries is to instantiate variables by individuals occurring in the knowledge base or the query.

3 Paraconsistent Semantics for $\mathcal{SROIQ}$

3.1 Discussion and Definitions

Recall that, using the traditional semantics, every query is a logical consequence of an inconsistent knowledge base. A knowledge base may be inconsistent, for example, when it contains both individual assertions $A(a)$ and $\neg A(a)$ for some

$A \in \mathbf{C}$ and $a \in \mathbf{I}$. Paraconsistent reasoning is inconsistency-tolerant and aims to derive (only) meaningful logical consequences even when the knowledge base is inconsistent. Following the recommendation of W3C for OWL, we use the traditional syntax of DLs and only change its semantics to cover paraconsistency. The general approach is to define a semantics $\mathfrak{s}$ such that, given a knowledge base KB, the set $Cons_{\mathfrak{s}}(KB)$ of logical consequences of KB w.r.t. semantics $\mathfrak{s}$ is a subset of the set $Cons(KB)$ of logical consequences of KB w.r.t. the traditional semantics, with the property that $Cons_{\mathfrak{s}}(KB)$ contains mainly only meaningful logical consequences of KB and $Cons_{\mathfrak{s}}(KB)$ approximates $Cons(KB)$ as much as possible.

In this paper, we introduce a number of paraconsistent semantics for the DL $\mathcal{SROIQ}$. Each of them, let's say $\mathfrak{s}$, is characterized by four parameters, denoted by $\mathfrak{s}_{\mathtt{C}}$, $\mathfrak{s}_{\mathtt{R}}$, $\mathfrak{s}_{\forall\exists}$, $\mathfrak{s}_{\mathtt{GCI}}$, with the following intuitive meanings:

- $\mathfrak{s}_{\mathtt{C}} \in \{2, 3, 4\}$ specifies the number of possible truth values of assertions of the form $x \in A^{\mathcal{I}}$, where A is a concept name not being a nominal and $\mathcal{I}$ is an interpretation. In the case $\mathfrak{s}_{\mathtt{C}} = 2$, the truth values are $\mathfrak{t}$ (true) and $\mathfrak{f}$ (false). In the case $\mathfrak{s}_{\mathtt{C}} = 3$, the third truth value is $\mathfrak{i}$ (inconsistent). In the case $\mathfrak{s}_{\mathtt{C}} = 4$, the additional truth value is $\mathfrak{u}$ (unknown). When $\mathfrak{s}_{\mathtt{C}} = 3$, one can identify inconsistency with the lack of knowledge, and the third value $\mathfrak{i}$ can be read either as inconsistent or as unknown.
- $\mathfrak{s}_{\mathtt{R}} \in \{2, 3, 4\}$ specifies the number of possible truth values of assertions of the form $\langle x, y\rangle \in r^{\mathcal{I}}$, where r is a role name different from the universal role U and $\mathcal{I}$ is an interpretation. The truth values are as in the case of $\mathfrak{s}_{\mathtt{C}}$.
- $\mathfrak{s}_{\forall\exists} \in \{+, \pm\}$ specifies which of the two semantics studied by Straccia [28] for concepts of the form $\forall R.C$ or $\exists R.C$ is used.
- $\mathfrak{s}_{\mathtt{GCI}} \in \{w, m, s\}$ specifies one of the three semantics for general concept inclusions: weak (w), moderate (m), strong (s).

For simplicity, we use the same value of $\mathfrak{s}_{\mathtt{C}}$ for all concept names of $\mathbf{C} \setminus \mathbf{N}$ and use the same value of $\mathfrak{s}_{\mathtt{R}}$ for all role names of $\mathbf{R} \setminus \{U\}$. One may want to consider different values of $\mathfrak{s}_{\mathtt{C}}$ for different concept names, and different values of $\mathfrak{s}_{\mathtt{R}}$ for different role names. The methods and results of this paper can be generalized for that case in a straightforward way.

We identify $\mathfrak{s}$ with the tuple $\langle \mathfrak{s}_{\mathtt{C}}, \mathfrak{s}_{\mathtt{R}}, \mathfrak{s}_{\forall\exists}, \mathfrak{s}_{\mathtt{GCI}}\rangle$. The set $\mathfrak{S}$ of considered paraconsistent semantics is thus $\{2, 3, 4\} \times \{2, 3, 4\} \times \{+, \pm\} \times \{w, m, s\}$.

For $\mathfrak{s} \in \mathfrak{S}$, an *$\mathfrak{s}$-interpretation* $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\rangle$ is similar to a traditional interpretation except that the interpretation function maps every concept name A to a pair $A^{\mathcal{I}} = \langle A^{\mathcal{I}}_{+}, A^{\mathcal{I}}_{-}\rangle$ of subsets of $\Delta^{\mathcal{I}}$ and maps every role name r to a pair $r^{\mathcal{I}} = \langle r^{\mathcal{I}}_{+}, r^{\mathcal{I}}_{-}\rangle$ of binary relations on $\Delta^{\mathcal{I}}$ such that:

- if $\mathfrak{s}_{\mathtt{C}} = 2$ then $A^{\mathcal{I}}_{+} = \Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}_{-}$
- if $\mathfrak{s}_{\mathtt{C}} = 3$ then $A^{\mathcal{I}}_{+} \cup A^{\mathcal{I}}_{-} = \Delta^{\mathcal{I}}$
- if $\mathfrak{s}_{\mathtt{R}} = 2$ then $r^{\mathcal{I}}_{+} = (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus r^{\mathcal{I}}_{-}$
- if $\mathfrak{s}_{\mathtt{R}} = 3$ then $r^{\mathcal{I}}_{+} \cup r^{\mathcal{I}}_{-} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
- if A is a nominal then $A^{\mathcal{I}}_{+}$ is a singleton set and $A^{\mathcal{I}}_{-} = \Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}_{+}$

– $U_+^\mathcal{I} = \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ and $U_-^\mathcal{I} = \emptyset$.

Remark 3.1. The intuition behind $A^\mathcal{I} = \langle A_+^\mathcal{I}, A_-^\mathcal{I}\rangle$ is that $A_+^\mathcal{I}$ gathers positive evidence about A, while $A_-^\mathcal{I}$ gathers negative evidence about A. Thus, $A^\mathcal{I}$ can be treated as the function from $\Delta^\mathcal{I}$ to $\{\mathfrak{t}, \mathfrak{f}, \mathfrak{i}, \mathfrak{u}\}$ defined below:

$$A^\mathcal{I}(x) \stackrel{\text{def}}{=} \begin{cases} \mathfrak{t} \text{ for } x \in A_+^\mathcal{I} \text{ and } x \notin A_-^\mathcal{I} \\ \mathfrak{f} \text{ for } x \in A_-^\mathcal{I} \text{ and } x \notin A_+^\mathcal{I} \\ \mathfrak{i} \text{ for } x \in A_+^\mathcal{I} \text{ and } x \in A_-^\mathcal{I} \\ \mathfrak{u} \text{ for } x \notin A_+^\mathcal{I} \text{ and } x \notin A_-^\mathcal{I} \end{cases} \tag{1}$$

Informally, $A^\mathcal{I}(x)$ can be thought of as the truth value of $x \in A^\mathcal{I}$. Note that $A^\mathcal{I}(x) \in \{\mathfrak{t}, \mathfrak{f}\}$ if $\mathfrak{s}_\mathtt{C} = 2$ or A is a nominal, and $A^\mathcal{I}(x) \in \{\mathfrak{t}, \mathfrak{f}, \mathfrak{i}\}$ if $\mathfrak{s}_\mathtt{C} = 3$. The intuition behind $r^\mathcal{I} = \langle r_+^\mathcal{I}, r_-^\mathcal{I}\rangle$ is similar, and under which $r^\mathcal{I}(x,y) \in \{\mathfrak{t}, \mathfrak{f}\}$ if $\mathfrak{s}_\mathtt{R} = 2$ or $r = U$, and $r^\mathcal{I}(x,y) \in \{\mathfrak{t}, \mathfrak{f}, \mathfrak{i}\}$ if $\mathfrak{s}_\mathtt{R} = 3$. ◁

The interpretation function $\cdot^\mathcal{I}$ maps an inverse role R to a pair $R^\mathcal{I} = \langle R_+^\mathcal{I}, R_-^\mathcal{I}\rangle$ defined by $(r^-)^\mathcal{I} \stackrel{\text{def}}{=} \langle (r_+^\mathcal{I})^{-1}, (r_-^\mathcal{I})^{-1}\rangle$. It maps a complex concept C to a pair $C^\mathcal{I} = \langle C_+^\mathcal{I}, C_-^\mathcal{I}\rangle$ of subsets of $\Delta^\mathcal{I}$ defined as follows:

$$\begin{aligned}
\top^\mathcal{I} &\stackrel{\text{def}}{=} \langle \Delta^\mathcal{I}, \emptyset\rangle \\
\bot^\mathcal{I} &\stackrel{\text{def}}{=} \langle \emptyset, \Delta^\mathcal{I}\rangle \\
(\neg C)^\mathcal{I} &\stackrel{\text{def}}{=} \langle C_-^\mathcal{I}, C_+^\mathcal{I}\rangle \\
(C \sqcap D)^\mathcal{I} &\stackrel{\text{def}}{=} \langle C_+^\mathcal{I} \cap D_+^\mathcal{I}, C_-^\mathcal{I} \cup D_-^\mathcal{I}\rangle \\
(C \sqcup D)^\mathcal{I} &\stackrel{\text{def}}{=} \langle C_+^\mathcal{I} \cup D_+^\mathcal{I}, C_-^\mathcal{I} \cap D_-^\mathcal{I}\rangle \\
(\exists R.\mathtt{Self})^\mathcal{I} &\stackrel{\text{def}}{=} \big\langle \{x \in \Delta^\mathcal{I} \mid \langle x,x\rangle \in R_+^\mathcal{I}\}, \{x \in \Delta^\mathcal{I} \mid \langle x,x\rangle \in R_-^\mathcal{I}\}\big\rangle \\
(\geq n\, R.C)^\mathcal{I} &\stackrel{\text{def}}{=} \big\langle \{x \in \Delta^\mathcal{I} \mid \#\{y \mid \langle x,y\rangle \in R_+^\mathcal{I} \text{ and } y \in C_+^\mathcal{I}\} \geq n\}, \\
&\qquad \{x \in \Delta^\mathcal{I} \mid \#\{y \mid \langle x,y\rangle \in R_+^\mathcal{I} \text{ and } y \notin C_-^\mathcal{I}\} < n\}\big\rangle \\
(\leq n\, R.C)^\mathcal{I} &\stackrel{\text{def}}{=} \big\langle \{x \in \Delta^\mathcal{I} \mid \#\{y \mid \langle x,y\rangle \in R_+^\mathcal{I} \text{ and } y \notin C_-^\mathcal{I}\} \leq n\}, \\
&\qquad \{x \in \Delta^\mathcal{I} \mid \#\{y \mid \langle x,y\rangle \in R_+^\mathcal{I} \text{ and } y \in C_+^\mathcal{I}\} > n\}\big\rangle;
\end{aligned}$$

if $\mathfrak{s}_{\forall\exists} = +$ then

$$\begin{aligned}
(\forall R.C)^\mathcal{I} &\stackrel{\text{def}}{=} \big\langle \{x \in \Delta^\mathcal{I} \mid \forall y(\langle x,y\rangle \in R_+^\mathcal{I} \text{ implies } y \in C_+^\mathcal{I})\}, \\
&\qquad \{x \in \Delta^\mathcal{I} \mid \exists y(\langle x,y\rangle \in R_+^\mathcal{I} \text{ and } y \in C_-^\mathcal{I})\}\big\rangle \\
(\exists R.C)^\mathcal{I} &\stackrel{\text{def}}{=} \big\langle \{x \in \Delta^\mathcal{I} \mid \exists y(\langle x,y\rangle \in R_+^\mathcal{I} \text{ and } y \in C_+^\mathcal{I})\}, \\
&\qquad \{x \in \Delta^\mathcal{I} \mid \forall y(\langle x,y\rangle \in R_+^\mathcal{I} \text{ implies } y \in C_-^\mathcal{I})\}\big\rangle;
\end{aligned}$$

if $\mathfrak{s}_{\forall\exists} = \pm$ then

$$\begin{aligned}
(\forall R.C)^\mathcal{I} &\stackrel{\text{def}}{=} \big\langle \{x \in \Delta^\mathcal{I} \mid \forall y(\langle x,y\rangle \in R_-^\mathcal{I} \text{ or } y \in C_+^\mathcal{I})\}, \\
&\qquad \{x \in \Delta^\mathcal{I} \mid \exists y(\langle x,y\rangle \in R_+^\mathcal{I} \text{ and } y \in C_-^\mathcal{I})\}\big\rangle
\end{aligned}$$

$$(\exists R.C)^{\mathcal{I}} \stackrel{\text{def}}{=} \big\langle \{x \in \Delta^{\mathcal{I}} \mid \exists y(\langle x, y\rangle \in R^{\mathcal{I}}_{+} \text{ and } y \in C^{\mathcal{I}}_{+})\},$$
$$\{x \in \Delta^{\mathcal{I}} \mid \forall y(\langle x, y\rangle \in R^{\mathcal{I}}_{-} \text{ or } y \in C^{\mathcal{I}}_{-})\}\big\rangle.$$

Remark 3.1 applies also to complex concepts. For example, we say that $C^{\mathcal{I}}(x) = \mathfrak{i}$ if $x \in C^{\mathcal{I}}_{+}$ and $x \in C^{\mathcal{I}}_{-}$. Note that $C^{\mathcal{I}}$ is computed in the standard way [15, 14, 31, 22] for the case C is of the form $\top$, $\bot$, $\neg D$, $D \sqcap D'$, $D \sqcup D'$, $\geq n\, R.D$ or $\leq n\, R.D$. When $\mathfrak{s}_{\forall\exists} = +$, $(\forall R.C)^{\mathcal{I}}$ and $(\exists R.C)^{\mathcal{I}}$ are computed as in [15, 14, 31, 22] and as using semantics A of [28]. When $\mathfrak{s}_{\forall\exists} = \pm$, $(\forall R.C)^{\mathcal{I}}$ and $(\exists R.C)^{\mathcal{I}}$ are computed as using semantics B of [28].

3.2 Example

The following example illustrates the above definitions.

Consider a Semantic Web service supplying information about stocks. Assume that a web agent looks for low risk stocks, promising big gain. The agent's query can be expressed by

$$(LR \sqcap BG)(x), \tag{2}$$

where LR and BG stand for "low risk" and "big gain", respectively.

For simplicity, assume that the service has a knowledge base consisting only of the following concept assertions (perhaps provided by different experts/agents):

$$LR(s_1), \neg LR(s_1), \neg LR(s_2), \neg BG(s_2), LR(s_3), BG(s_3).$$

We then consider the interpretation $\mathcal{I}$ with:

$$LR^{\mathcal{I}} = \langle\{s_1, s_3\}, \{s_1, s_2\}\rangle \text{ and } BG^{\mathcal{I}} = \langle\{s_1, s_3\}, \{s_2\}\rangle. \tag{3}$$

The query (2) looks for stocks x that are instances of $LR \sqcap BG$ w.r.t. $\mathcal{I}$.

In the case of the traditional (two-valued) semantics, the knowledge base has no models, and hence all of s_1, s_2, s_3 are answers to the query, despite the fact that s_2 is of high risk and low gain.

Using any semantics $\mathfrak{s} \in \mathfrak{S}$ with $\mathfrak{s}_{\mathsf{C}} = 3$, we have that

$$(LR \sqcap BG)^{\mathcal{I}} = \langle LR^{\mathcal{I}}_{+} \cap BG^{\mathcal{I}}_{+}, LR^{\mathcal{I}}_{-} \cup BG^{\mathcal{I}}_{-}\rangle = \langle\{s_1, s_3\}, \{s_1, s_2\}\rangle,$$

meaning that (according to (1)):

$$(LR \sqcap BG)^{\mathcal{I}}(s_1) = \mathfrak{i}, \;\; (LR \sqcap BG)^{\mathcal{I}}(s_2) = \mathfrak{f} \text{ and } (LR \sqcap BG)^{\mathcal{I}}(s_3) = \mathfrak{t},$$

which is well-justified. Namely, there is both positive and negative evidence that s_1 satisfies (2), there is only negative evidence that s_2 satisfies (2) and there is only positive evidence that s_3 satisfies (2).

Consider now the four-valued semantics and let $\mathcal{I}'$ differ from $\mathcal{I}$ given by (3) in that $BG^{\mathcal{I}'} = \langle\{s_3\}, \{s_2\}\rangle$. Notice that $BG^{\mathcal{I}'}_{+} \cup BG^{\mathcal{I}'}_{-} \neq \Delta^{\mathcal{I}'}$. In this case, according to (1), we have that $BG^{\mathcal{I}'}(s_1) = \mathfrak{u}$. Now

$$(LR \sqcap BG)^{\mathcal{I}'} = \langle\{s_3\}, \{s_1, s_2\}\rangle,$$

that is,

$$(LR \sqcap BG)^{\mathcal{I}'}(s_1) = \mathfrak{f},\ \ (LR \sqcap BG)^{\mathcal{I}'}(s_2) = \mathfrak{f}\ \text{ and }\ (LR \sqcap BG)^{\mathcal{I}'}(s_3) = \mathfrak{t}.$$

The result that $(LR \sqcap BG)^{\mathcal{I}'}(s_1) = \mathfrak{f}$ is not intuitive. Namely, we have inconsistent information that s_1 is low risk and have no information whether it promises big gain and still we have the result that the conjunction of both is false.

Observe also that

$$(LR \sqcup BG)^{\mathcal{I}'} = \langle LR_+^{\mathcal{I}'} \cup BG_+^{\mathcal{I}'}, LR_-^{\mathcal{I}'} \cap BG_-^{\mathcal{I}'} \rangle = \langle \{s_1, s_3\}, \{s_2\} \rangle.$$

This means that the disjunction

$$\underbrace{s_1 \text{ is of low risk}}_{\mathfrak{i}} \text{ or } \underbrace{s_1 \text{ promises big gain}}_{\mathfrak{u}}$$

is $\mathfrak{t}$, which is again not intuitive.

In fact, the definitions of $\sqcap$ and $\sqcup$ in the four-valued context reflect the truth ordering proposed by Belnap [5, 4] and used in the Semantic Web context, e.g., in [14, 15]. The use of Belnap's knowledge ordering also provides non-intuitive results in many other cases. Therefore we advocate for using three-valued logic, as proposed in [22] in the case of complete knowledge. In the case of incomplete knowledge the use of truth ordering proposed independently in [1] and [18, 29] provides much more intuitive results ($\mathfrak{i}$ for the disjunction and $\mathfrak{u}$ for the conjunction).

3.3 Properties of Paraconsistent Semantics

We write $C \equiv_{\mathfrak{s}} D$ and say that C and D are *equivalent w.r.t.* $\mathfrak{s}$ if $C^{\mathcal{I}} = D^{\mathcal{I}}$ for every $\mathfrak{s}$-interpretation $\mathcal{I}$. The following proposition states that De Morgan laws hold for our constructors w.r.t. any semantics from $\mathfrak{S}$. Its proof is straightforward.

Proposition 3.2. *The following equivalences hold for every* $\mathfrak{s} \in \mathfrak{S}$ *:*

$$\begin{aligned}
(\neg\neg C)^{\mathcal{I}} &\equiv_{\mathfrak{s}} C^{\mathcal{I}} \\
(\neg\top)^{\mathcal{I}} &\equiv_{\mathfrak{s}} \bot^{\mathcal{I}} \\
(\neg(C \sqcap D))^{\mathcal{I}} &\equiv_{\mathfrak{s}} (\neg C \sqcup \neg D)^{\mathcal{I}} \\
(\neg\forall R.C)^{\mathcal{I}} &\equiv_{\mathfrak{s}} (\exists R.\neg C)^{\mathcal{I}} \\
(\neg(\geq 0\, R.C))^{\mathcal{I}} &\equiv_{\mathfrak{s}} \bot^{\mathcal{I}} \\
(\neg(\geq (n+1)\, R.C))^{\mathcal{I}} &\equiv_{\mathfrak{s}} (\leq n\, R.C)^{\mathcal{I}} \\
(\neg(\leq n\, R.C))^{\mathcal{I}} &\equiv_{\mathfrak{s}} (\geq (n+1)\, R.C)^{\mathcal{I}}
\end{aligned}$$

◁

The following proposition means that: if $\mathfrak{s}_C \in \{2, 3\}$ and $\mathfrak{s}_R \in \{2, 3\}$ then $\mathfrak{s}$ is a three-valued semantics; if $\mathfrak{s}_C = 2$ and $\mathfrak{s}_R = 2$ then $\mathfrak{s}$ is a two-valued semantics. Its proof is straightforward via induction on the structure of C and R.

Proposition 3.3. *Let $\mathfrak{s} \in \mathfrak{S}$ be a semantics such that $\mathfrak{s}_{\mathtt{C}} \in \{2, 3\}$ and $\mathfrak{s}_{\mathtt{R}} \in \{2, 3\}$. Let $\mathcal{I}$ be an $\mathfrak{s}$-interpretation, C be a concept, and R be a role. Then $C_+^{\mathcal{I}} \cup C_-^{\mathcal{I}} = \Delta^{\mathcal{I}}$ and $R_+^{\mathcal{I}} \cup R_-^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Furthermore, if $\mathfrak{s}_{\mathtt{C}} = 2$ and $\mathfrak{s}_{\mathtt{R}} = 2$ then $C_+^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C_-^{\mathcal{I}}$ and $R_+^{\mathcal{I}} = (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus R_-^{\mathcal{I}}$.* ◁

Let $\mathfrak{s} \in \mathfrak{S}$ and let $\mathcal{I}$ be an $\mathfrak{s}$-interpretation. We say that:

- $\mathcal{I}$ $\mathfrak{s}$-*validates* a role axiom $R_1 \circ \ldots \circ R_k \sqsubseteq S$ if $R_{1+}^{\mathcal{I}} \circ \ldots \circ R_{k+}^{\mathcal{I}} \subseteq S_+^{\mathcal{I}}$
- $\mathcal{I}$ $\mathfrak{s}$-*validates* a role assertion $\mathtt{Ref}(R)$ (resp. $\mathtt{Irr}(R)$, $\mathtt{Sym}(R)$, $\mathtt{Tra}(R)$) if $R_+^{\mathcal{I}}$ is reflexive (resp. irreflexive, symmetric, transitive)
- $\mathcal{I}$ $\mathfrak{s}$-*validates* a role assertion $\mathtt{Dis}(R, S)$ if $R_+^{\mathcal{I}}$ and $S_+^{\mathcal{I}}$ are disjoint
- $\mathcal{I}$ is an $\mathfrak{s}$-*model* of an RBox $\mathcal{R}$, denoted by $\mathcal{I} \models_{\mathfrak{s}} \mathcal{R}$, if it $\mathfrak{s}$-validates all axioms of $\mathcal{R}$

- $\mathcal{I}$ $\mathfrak{s}$-*validates* $C \sqsubseteq D$, denoted by $\mathcal{I} \models_{\mathfrak{s}} C \sqsubseteq D$, if:
  - case $\mathfrak{s}_{\mathtt{GCI}} = w$: $C_-^{\mathcal{I}} \cup D_+^{\mathcal{I}} = \Delta^{\mathcal{I}}$
  - case $\mathfrak{s}_{\mathtt{GCI}} = m$: $C_+^{\mathcal{I}} \subseteq D_+^{\mathcal{I}}$
  - case $\mathfrak{s}_{\mathtt{GCI}} = s$: $C_+^{\mathcal{I}} \subseteq D_+^{\mathcal{I}}$ and $D_-^{\mathcal{I}} \subseteq C_-^{\mathcal{I}}$
- $\mathcal{I}$ is an $\mathfrak{s}$-*model* of a TBox $\mathcal{T}$, denoted by $\mathcal{I} \models_{\mathfrak{s}} \mathcal{T}$, if it $\mathfrak{s}$-validates all axioms of $\mathcal{T}$

- $\mathcal{I}$ $\mathfrak{s}$-*satisfies* an individual assertion φ if $\mathcal{I} \models_{\mathfrak{s}} \varphi$, where
  $\mathcal{I} \models_{\mathfrak{s}} a \neq b$ if $a^{\mathcal{I}} \neq b^{\mathcal{I}}$
  $\mathcal{I} \models_{\mathfrak{s}} C(a)$ if $a^{\mathcal{I}} \in C_+^{\mathcal{I}}$
  $\mathcal{I} \models_{\mathfrak{s}} R(a, b)$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R_+^{\mathcal{I}}$
  $\mathcal{I} \models_{\mathfrak{s}} \neg S(a, b)$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in S_-^{\mathcal{I}}$
- $\mathcal{I}$ is an $\mathfrak{s}$-*model* of an ABox $\mathcal{A}$, denoted by $\mathcal{I} \models_{\mathfrak{s}} \mathcal{A}$, if it $\mathfrak{s}$-satisfies all assertions of $\mathcal{A}$

- $\mathcal{I}$ is an $\mathfrak{s}$-*model* of a knowledge base $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ if it is an $\mathfrak{s}$-model of all $\mathcal{R}$, $\mathcal{T}$ and $\mathcal{A}$
- a knowledge base $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ is $\mathfrak{s}$-*satisfiable* if it has an $\mathfrak{s}$-model
- $\mathcal{I}$ $\mathfrak{s}$-*satisfies* a query $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_k$, denoted by $\mathcal{I} \models_{\mathfrak{s}} \varphi$, if $\mathcal{I} \models_{\mathfrak{s}} \varphi_i$ for all $1 \leq i \leq k$
- φ is an $\mathfrak{s}$-*logical consequence* of a knowledge base $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$, denoted by $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models_{\mathfrak{s}} \varphi$, if every $\mathfrak{s}$-model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ $\mathfrak{s}$-satisfies φ.

4 The Relationship between the Semantics

The following proposition states that if $\mathfrak{s} \in \mathfrak{S}$ is a semantics such that $\mathfrak{s}_{\mathtt{C}} = 2$ and $\mathfrak{s}_{\mathtt{R}} = 2$ then $\mathfrak{s}$ coincides with the traditional semantics.

Proposition 4.1. *Let $\mathfrak{s} \in \mathfrak{S}$ be a semantics such that $\mathfrak{s}_{\mathtt{C}} = 2$ and $\mathfrak{s}_{\mathtt{R}} = 2$, let $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ be a knowledge base, and φ be a query. Then $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models_{\mathfrak{s}} \varphi$ iff $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models \varphi$.*

Proof. Consider the "if" direction. Suppose that $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models \varphi$. We show that $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models_{\mathfrak{s}} \varphi$. Let $\mathcal{I}$ be an $\mathfrak{s}$-model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$. We show that $\mathcal{I} \models_{\mathfrak{s}} \varphi$.

Let $\mathcal{I}'$ be the traditional interpretation specified by $\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}}$, $A^{\mathcal{I}'} = A^{\mathcal{I}}_+$ for $A \in \mathbf{C}$, $r^{\mathcal{I}'} = r^{\mathcal{I}}_+$ for $r \in \mathbf{R}$, and $a^{\mathcal{I}'} = a^{\mathcal{I}}$ for $a \in \mathbf{I}$. It can be proved by induction (on the structure of C) that, for any concept C, $C^{\mathcal{I}'} = C^{\mathcal{I}}_+$. Clearly, we also have that $R^{\mathcal{I}'} = R^{\mathcal{I}}_+$ for any role R.

Since $\mathcal{I} \models_{\mathfrak{s}} \mathcal{R}$, it follows that $\mathcal{I}' \models \mathcal{R}$. By Proposition 3.3, for any concept C, $C^{\mathcal{I}}_- = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}_+$. Hence, for any terminological axiom $C \sqsubseteq D$, $\mathcal{I} \models_{\mathfrak{s}} C \sqsubseteq D$ iff $C^{\mathcal{I}}_+ \subseteq D^{\mathcal{I}}_+$. Since $\mathcal{I} \models_{\mathfrak{s}} \mathcal{T}$, it follows that $\mathcal{I}' \models \mathcal{T}$. By Proposition 3.3, we also have that $R^{\mathcal{I}}_- = (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus R^{\mathcal{I}}_+$ for any role R. Hence, for any individual assertion ψ, $\mathcal{I} \models_{\mathfrak{s}} \psi$ iff $\mathcal{I}' \models \psi$. Since $\mathcal{I} \models_{\mathfrak{s}} \mathcal{A}$, it follows that $\mathcal{I}' \models \mathcal{A}$.

Therefore, $\mathcal{I}'$ is a model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$. Since $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models \varphi$, it follows that $\mathcal{I}' \models \varphi$, which implies that $\mathcal{I} \models_{\mathfrak{s}} \varphi$. This completes the proof of the "if" direction. The "only if" direction can be proved analogously. $\triangleleft$

Proposition 4.2. *Let $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S}$ be semantics such that $\mathfrak{s}_{\mathtt{R}} = \mathfrak{s}'_{\mathtt{R}} = 2$, $\mathfrak{s}_{\mathtt{C}} = \mathfrak{s}'_{\mathtt{C}}$, $\mathfrak{s}_{\mathtt{GCI}} = \mathfrak{s}'_{\mathtt{GCI}}$, but $\mathfrak{s}_{\forall\exists} \neq \mathfrak{s}'_{\forall\exists}$. Then $\mathfrak{s}$ and $\mathfrak{s}'$ are equivalent in the sense that, for every knowledge base $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ and every query φ, $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models_{\mathfrak{s}} \varphi$ iff $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models_{\mathfrak{s}'} \varphi$.* $\triangleleft$

The proof of this lemma is straightforward.

Let $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S}$. We say that $\mathfrak{s}$ is *weaker than or equal to* $\mathfrak{s}'$ (and $\mathfrak{s}'$ is *stronger than or equal to* $\mathfrak{s}$) if for any knowledge base KB, $Cons_{\mathfrak{s}}(KB) \subseteq Cons_{\mathfrak{s}'}(KB)$. (Recall that $Cons_{\mathfrak{s}}(KB)$ stands for the set of $\mathfrak{s}$-logical consequences of KB.)

Define $\mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}}$ according to $w \sqsubseteq m \sqsubseteq s$, where $\sqsubseteq$ is transitive. Define that $\mathfrak{s} \sqsubseteq \mathfrak{s}'$ if:[5]

$$\mathfrak{s}'_{\mathtt{C}} \leq \mathfrak{s}_{\mathtt{C}} \leq 3,\ \mathfrak{s}'_{\mathtt{R}} \leq \mathfrak{s}_{\mathtt{R}} \leq 3,\ \mathfrak{s}_{\forall\exists} = \mathfrak{s}'_{\forall\exists},\ \text{and } \mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}};\ \text{or} \tag{4}$$

$$\mathfrak{s}'_{\mathtt{C}} \leq \mathfrak{s}_{\mathtt{C}},\ \mathfrak{s}'_{\mathtt{R}} \leq \mathfrak{s}_{\mathtt{R}},\ \mathfrak{s}_{\forall\exists} = \mathfrak{s}'_{\forall\exists},\ \text{and } m \sqsubseteq \mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}};\ \text{or} \tag{5}$$

$$\mathfrak{s}'_{\mathtt{C}} \leq \mathfrak{s}_{\mathtt{C}} \leq 3,\ \mathfrak{s}_{\mathtt{R}} = \mathfrak{s}'_{\mathtt{R}} = 2,\ \text{and } \mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}};\ \text{or} \tag{6}$$

$$\mathfrak{s}'_{\mathtt{C}} \leq \mathfrak{s}_{\mathtt{C}},\ \mathfrak{s}_{\mathtt{R}} = \mathfrak{s}'_{\mathtt{R}} = 2,\ \text{and } m \sqsubseteq \mathfrak{s}_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}};\ \text{or} \tag{7}$$

$$\mathfrak{s}_{\mathtt{C}} = \mathfrak{s}'_{\mathtt{C}} = 2 \text{ and } \mathfrak{s}_{\mathtt{R}} = \mathfrak{s}'_{\mathtt{R}} = 2. \tag{8}$$

Theorem 4.3. *Let $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S}$ be semantics such that $\mathfrak{s} \sqsubseteq \mathfrak{s}'$. Then $\mathfrak{s}$ is weaker than or equal to $\mathfrak{s}'$ (i.e., for any knowledge base KB, $Cons_{\mathfrak{s}}(KB) \subseteq Cons_{\mathfrak{s}'}(KB)$).*

Proof. The assertion for the case (8) follows from Proposition 4.1. By using Proposition 4.2, the cases (6) and (7) are reduced to the cases (4) and (5), respectively. Consider the cases (4) and (5), and assume that one of them holds. Let $\mathfrak{s}'' = \langle \mathfrak{s}'_{\mathtt{C}}, \mathfrak{s}'_{\mathtt{R}}, \mathfrak{s}_{\forall\exists}, \mathfrak{s}_{\mathtt{GCI}} \rangle$. We show that $\mathfrak{s}$ is weaker than or equal to $\mathfrak{s}''$, and $\mathfrak{s}''$ is weaker than or equal to $\mathfrak{s}'$, which together imply the assertion of the theorem.

Observe that every $\mathfrak{s}''$-interpretation is an $\mathfrak{s}$-interpretation. Furthermore, since $\mathfrak{s}''_{\forall\exists} = \mathfrak{s}_{\forall\exists}$ and $\mathfrak{s}''_{\mathtt{GCI}} = \mathfrak{s}_{\mathtt{GCI}}$, if $\mathcal{I}$ is an $\mathfrak{s}''$-interpretation then, for every

[5] This corrects the corresponding definition given in [21].

knowledge base KB and every query φ, $\mathcal{I} \models_{\mathfrak{s}''} KB$ iff $\mathcal{I} \models_{\mathfrak{s}} KB$, and $\mathcal{I} \models_{\mathfrak{s}''} \varphi$ iff $\mathcal{I} \models_{\mathfrak{s}} \varphi$. Hence, for every knowledge base KB and every query φ, $KB \models_{\mathfrak{s}} \varphi$ implies $KB \models_{\mathfrak{s}''} \varphi$. That is, $\mathfrak{s}$ is weaker than or equal to $\mathfrak{s}''$.

Semantics $\mathfrak{s}''$ may differ from $\mathfrak{s}'$ only by the pair $\mathfrak{s}''_{\mathtt{GCI}}$ and $\mathfrak{s}'_{\mathtt{GCI}}$, with $\mathfrak{s}''_{\mathtt{GCI}} \sqsubseteq \mathfrak{s}'_{\mathtt{GCI}}$. Every $\mathfrak{s}''$-interpretation is an $\mathfrak{s}'$-interpretation, and vice versa. Let $\mathcal{I}$ be an arbitrary $\mathfrak{s}''$-interpretation. Observe that, for any terminological axiom $C \sqsubseteq D$, if $\mathcal{I} \models_{\mathfrak{s}'} C \sqsubseteq D$ then $\mathcal{I} \models_{\mathfrak{s}''} C \sqsubseteq D$ (for the case (4), note that $C^{\mathcal{I}}_{+} \subseteq D^{\mathcal{I}}_{+}$ implies $C^{\mathcal{I}}_{-} \cup C^{\mathcal{I}}_{+} \subseteq C^{\mathcal{I}}_{-} \cup D^{\mathcal{I}}_{+}$ and hence $C^{\mathcal{I}}_{-} \cup D^{\mathcal{I}}_{+} = \Delta^{\mathcal{I}}$). Hence, for every knowledge base KB, if $\mathcal{I} \models_{\mathfrak{s}'} KB$ then $\mathcal{I} \models_{\mathfrak{s}''} KB$. Clearly, for every query φ, $\mathcal{I} \models_{\mathfrak{s}'} \varphi$ iff $\mathcal{I} \models_{\mathfrak{s}''} \varphi$. Hence, for every knowledge base KB and every query φ, $KB \models_{\mathfrak{s}''} \varphi$ implies $KB \models_{\mathfrak{s}'} \varphi$. That is, $\mathfrak{s}''$ is weaker than or equal to $\mathfrak{s}'$. ◁

We give below a revised version of a corollary of [21] stating which semantics from $\mathfrak{S}$ give only correct answers. It follows immediately from the above theorem and Proposition 4.1.

Corollary 4.4. *Let $\mathfrak{s} \in \mathfrak{S}$ be a semantics such that $\mathfrak{s}_{\mathtt{GCI}} \neq w$ or $\mathfrak{s}_{\mathtt{C}} \leq 3$ and $\mathfrak{s}_{\mathtt{R}} \leq 3$, and let $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ be a knowledge base and φ be a query. Then $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models_{\mathfrak{s}} \varphi$ implies $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \models \varphi$.* ◁

5 Comparison with Existing Paraconsistent Semantics

Here, we restrict only to many-valued semantics and quasi-classical semantics for DLs. Other paraconsistent semantics have been discussed in the introduction.

In [14, 15] Ma et al. use non-traditional inclusion axioms $C \mapsto D$, $C \sqsubset D$ and $C \to D$, which correspond to our inclusion $C \sqsubseteq D$ w.r.t. semantics $\mathfrak{s}$ with $\mathfrak{s}_{\mathtt{GCI}} = w, m, s$, respectively. The work [15] concerns paraconsistent reasoning in the DL $\mathcal{SHIQ}$, which is later extended in [14] for paraconsistent reasoning in the DL $\mathcal{SROIQ}$. Defining a four-valued semantics for $\mathcal{SROIQ}$, Ma and Hitzler [14] did not consider all features of $\mathcal{SROIQ}$. For example, they did not consider concepts of the form $\exists R.\mathtt{Self}$ and individual assertions of the form $\neg S(a, b)$. Ignoring such detailed differences, their four-valued semantics for $\mathcal{SROIQ}$ can be characterized using the traditional language (with $\sqsubseteq$ instead of $\mapsto$, $\sqsubset$, $\to$), when $\sqsubseteq$ is interpreted as:

$\mapsto$ **:** their semantics is equivalent to our semantics $\langle 4, 2, +, w \rangle$
$\sqsubset$ **:** their semantics is equivalent to our semantics $\langle 4, 2, +, m \rangle$
$\to$ **:** their semantics is equivalent to our semantics $\langle 4, 2, +, s \rangle$.

By Theorem 4.3, their semantics is weaker than our semantics $\langle 3, 2, +, \mathfrak{s}_{\mathtt{GCI}} \rangle$, where $\mathfrak{s}_{\mathtt{GCI}} \in \{w, m, s\}$ when $\sqsubseteq$ is interpreted as $\mapsto$, $\sqsubset$, $\to$, respectively.

Recall also that in Section 3.2 we have shown that approaches based on Belnap's four-valued logic, like [14, 15], sometimes lead to counter-intuitive results.

In [31], Zhang et al. define weak and strong quasi-classical semantics for $\mathcal{SHIQ}$, which will be denoted here by $\models^{w}_{4}$ and $\models^{s}_{4}$. These semantics are four-valued semantics and are based on the quasi-classical semantics of Besnard and

Hunter [6, 12]. For the conjunctive query answering in $\mathcal{SHIQ}$, the weak quasi-classical semantics is weaker than our semantics $\langle 3, 2, +, w\rangle$. Comparing our semantics $\langle 3, 2, +, s\rangle$ with $\models^s_4$, neither of them is stronger than the other. The relationship between these semantics (for $\mathcal{SHIQ}$) can be characterized as follows. First, axioms of the form $C \sqsubseteq C$ is not valid w.r.t. the semantics $\models^s_4$, which is quite unusual. Second, extending $\models^s_4$ with axioms $A \sqsubseteq A$ for all atomic concepts A results in a three-valued semantics that differs from our three-valued semantics in that it assumes $\mathfrak{t} \wedge \mathfrak{i} = \mathfrak{t}$ and $\mathfrak{f} \vee \mathfrak{i} = \mathfrak{f}$, while our three-valued semantics assume $\mathfrak{t} \wedge \mathfrak{i} = \mathfrak{i}$ and $\mathfrak{f} \vee \mathfrak{i} = \mathfrak{i}$. Moreover, the same problem as indicated in Section 3.2, applies to both weak and strong semantics proposed in [31].

Recall that the paradoxical semantics for $\mathcal{ALC}$ by Zhang et al. [30] is based on a three-valued semantics and was developed independently from our three-valued semantics for $\mathcal{SHIQ}$ [22]. It is equivalent to our semantics $\langle 3, 2, +, w\rangle$ when restricted to $\mathcal{ALC}$ (and the case without negative individual assertions of the form $\neg S(a, b)$).

6 A Translation into the Traditional Semantics

In this section we give a linear translation $\pi_\mathfrak{s}$, for $\mathfrak{s} \in \mathfrak{S}$ with $\mathfrak{s}_\mathtt{C} \in \{3, 4\}$, $\mathfrak{s}_\mathtt{R} \in \{2, 4\}$ and $\mathfrak{s}_{\forall\exists} = +$, such that, for every knowledge base KB and every query φ, $KB \models_\mathfrak{s} \varphi$ iff $\pi_\mathfrak{s}(KB) \models \pi_\mathfrak{s}(\varphi)$. In this section, if not otherwise stated, we assume that $\mathfrak{s}$ satisfies the mentioned conditions.

For $A \in \mathbf{C} \setminus \mathbf{N}$, let A_+ and A_- be new concept names. For $r \in \mathbf{R} \setminus \{U\}$, let r_+ and r_- be new role names. In accordance to the semantics $\mathfrak{s}$, let $\mathbf{C}' = \{A_+, A_- \mid A \in \mathbf{C} \setminus \mathbf{N}\} \cup \mathbf{N}$, and

$$\mathbf{R}' = \begin{cases} \mathbf{R} & \text{for } \mathfrak{s}_\mathtt{R} = 2 \\ \{r_+, r_- \mid r \in \mathbf{R} \setminus \{U\}\} \cup \{U\} & \text{for } \mathfrak{s}_\mathtt{R} = 4. \end{cases}$$

We also define two auxiliary translations $\pi_{\mathfrak{s}+}$ and $\pi_{\mathfrak{s}-}$. In the following, if not otherwise stated, r, R, S, A, C, D, a, b, $\mathcal{R}$, $\mathcal{T}$, $\mathcal{A}$ are arbitrary elements of their appropriate types (according to the used convention) in the language using $\mathbf{C}$ and $\mathbf{R}$.

If $\mathfrak{s}_\mathtt{R} = 2$ then:

- $\pi_{\mathfrak{s}+}(R) \stackrel{\text{def}}{=} R$ and $\pi_\mathfrak{s}(\mathcal{R}) \stackrel{\text{def}}{=} \mathcal{R}$
- $\pi_\mathfrak{s}(R(a, b)) \stackrel{\text{def}}{=} R(a, b)$ and $\pi_\mathfrak{s}(\neg S(a, b)) \stackrel{\text{def}}{=} \neg S(a, b)$
- $\pi_{\mathfrak{s}+}(\exists R.\mathtt{Self}) \stackrel{\text{def}}{=} \exists R.\mathtt{Self}$ and $\pi_{\mathfrak{s}-}(\exists R.\mathtt{Self}) \stackrel{\text{def}}{=} \neg\exists R.\mathtt{Self}$.

If $\mathfrak{s}_\mathtt{R} = 4$ then:

- $\pi_{\mathfrak{s}+}(U) \stackrel{\text{def}}{=} U$
- $\pi_{\mathfrak{s}+}(r) \stackrel{\text{def}}{=} r_+$ and $\pi_{\mathfrak{s}-}(r) \stackrel{\text{def}}{=} r_-$, where $r \neq U$
- $\pi_{\mathfrak{s}+}(r^-) \stackrel{\text{def}}{=} (r_+)^-$ and $\pi_{\mathfrak{s}-}(r^-) \stackrel{\text{def}}{=} (r_-)^-$, where $r \neq U$

$$\begin{array}{rlrl}
\pi_{\mathfrak{s}+}(\top) & \stackrel{\text{def}}{=} \top & \pi_{\mathfrak{s}-}(\top) & \stackrel{\text{def}}{=} \bot \\
\pi_{\mathfrak{s}+}(\bot) & \stackrel{\text{def}}{=} \bot & \pi_{\mathfrak{s}-}(\bot) & \stackrel{\text{def}}{=} \top \\
\pi_{\mathfrak{s}+}(\neg C) & \stackrel{\text{def}}{=} \pi_{\mathfrak{s}-}(C) & \pi_{\mathfrak{s}-}(\neg C) & \stackrel{\text{def}}{=} \pi_{\mathfrak{s}+}(C) \\
\pi_{\mathfrak{s}+}(C \sqcap D) & \stackrel{\text{def}}{=} \pi_{\mathfrak{s}+}(C) \sqcap \pi_{\mathfrak{s}+}(D) & \pi_{\mathfrak{s}-}(C \sqcap D) & \stackrel{\text{def}}{=} \pi_{\mathfrak{s}-}(C) \sqcup \pi_{\mathfrak{s}-}(D) \\
\pi_{\mathfrak{s}+}(C \sqcup D) & \stackrel{\text{def}}{=} \pi_{\mathfrak{s}+}(C) \sqcup \pi_{\mathfrak{s}+}(D) & \pi_{\mathfrak{s}-}(C \sqcup D) & \stackrel{\text{def}}{=} \pi_{\mathfrak{s}-}(C) \sqcap \pi_{\mathfrak{s}-}(D) \\
\pi_{\mathfrak{s}+}(\forall R.C) & \stackrel{\text{def}}{=} \forall \pi_{\mathfrak{s}+}(R).\pi_{\mathfrak{s}+}(C) & \pi_{\mathfrak{s}-}(\forall R.C) & \stackrel{\text{def}}{=} \exists \pi_{\mathfrak{s}+}(R).\pi_{\mathfrak{s}-}(C) \\
\pi_{\mathfrak{s}+}(\exists R.C) & \stackrel{\text{def}}{=} \exists \pi_{\mathfrak{s}+}(R).\pi_{\mathfrak{s}+}(C) & \pi_{\mathfrak{s}-}(\exists R.C) & \stackrel{\text{def}}{=} \forall \pi_{\mathfrak{s}+}(R).\pi_{\mathfrak{s}-}(C)
\end{array}$$

$$\begin{array}{rl}
\pi_{\mathfrak{s}+}(\geq n\, R.C) & \stackrel{\text{def}}{=} \geq n\, \pi_{\mathfrak{s}+}(R).\pi_{\mathfrak{s}+}(C) \\
\pi_{\mathfrak{s}-}(\geq (n+1)\, R.C) & \stackrel{\text{def}}{=} \leq n\, \pi_{\mathfrak{s}+}(R).\neg\pi_{\mathfrak{s}-}(C) \\
\pi_{\mathfrak{s}-}(\geq 0\, R.C) & \stackrel{\text{def}}{=} \bot \\
\pi_{\mathfrak{s}+}(\leq n\, R.C) & \stackrel{\text{def}}{=} \leq n\, \pi_{\mathfrak{s}+}(R).\neg\pi_{\mathfrak{s}-}(C) \\
\pi_{\mathfrak{s}-}(\leq n\, R.C) & \stackrel{\text{def}}{=} \geq (n+1)\, \pi_{\mathfrak{s}+}(R).\pi_{\mathfrak{s}+}(C)
\end{array}$$

Fig. 1. A partial specification of $\pi_{\mathfrak{s}+}$ and $\pi_{\mathfrak{s}-}$.

- for every role axiom φ, $\pi_{\mathfrak{s}}(\varphi) \stackrel{\text{def}}{=} \varphi'$, where φ' is the role axiom obtained from φ by replacing each role R by $\pi_{\mathfrak{s}+}(R)$
- $\pi_{\mathfrak{s}}(\mathcal{R}) \stackrel{\text{def}}{=} \{\pi_{\mathfrak{s}}(\varphi) \mid \varphi \in \mathcal{R}\}$
- $\pi_{\mathfrak{s}}(R(a,b)) \stackrel{\text{def}}{=} \pi_{\mathfrak{s}+}(R)(a,b)$ and $\pi_{\mathfrak{s}}(\neg S(a,b)) \stackrel{\text{def}}{=} \pi_{\mathfrak{s}-}(S)(a,b)$, for $R, S \neq U$
- $\pi_{\mathfrak{s}+}(\exists R.\mathtt{Self}) \stackrel{\text{def}}{=} \exists \pi_{\mathfrak{s}+}(R).\mathtt{Self}$ and $\pi_{\mathfrak{s}-}(\exists R.\mathtt{Self}) \stackrel{\text{def}}{=} \exists \pi_{\mathfrak{s}-}(R).\mathtt{Self}$.

If A is a nominal then $\pi_{\mathfrak{s}+}(A) \stackrel{\text{def}}{=} A$ and $\pi_{\mathfrak{s}-}(A) \stackrel{\text{def}}{=} \neg A$. If A is a concept name but not a nominal then $\pi_{\mathfrak{s}+}(A) \stackrel{\text{def}}{=} A_+$ and $\pi_{\mathfrak{s}-}(A) \stackrel{\text{def}}{=} A_-$.

The translations $\pi_{\mathfrak{s}+}(C)$ and $\pi_{\mathfrak{s}-}(C)$ for the case C is not of the form A or $\exists R.\mathtt{Self}$ are defined as in Figure 1.

Define $\pi_{\mathfrak{s}}(C \sqsubseteq D)$ and $\pi_{\mathfrak{s}}(\mathcal{T})$ as follows:

- case $\mathfrak{s}_{\mathtt{GCI}} = w$: $\pi_{\mathfrak{s}}(C \sqsubseteq D) \stackrel{\text{def}}{=} \{\top \sqsubseteq \pi_{\mathfrak{s}-}(C) \sqcup \pi_{\mathfrak{s}+}(D)\}$
- case $\mathfrak{s}_{\mathtt{GCI}} = m$: $\pi_{\mathfrak{s}}(C \sqsubseteq D) \stackrel{\text{def}}{=} \{\pi_{\mathfrak{s}+}(C) \sqsubseteq \pi_{\mathfrak{s}+}(D)\}$
- case $\mathfrak{s}_{\mathtt{GCI}} = s$: $\pi_{\mathfrak{s}}(C \sqsubseteq D) \stackrel{\text{def}}{=} \{\pi_{\mathfrak{s}+}(C) \sqsubseteq \pi_{\mathfrak{s}+}(D),\ \pi_{\mathfrak{s}-}(D) \sqsubseteq \pi_{\mathfrak{s}-}(C)\}$
- case $\mathfrak{s}_{\mathtt{C}} = 3$: $\pi_{\mathfrak{s}}(\mathcal{T}) \stackrel{\text{def}}{=} \bigcup_{\varphi \in \mathcal{T}} \pi_{\mathfrak{s}}(\varphi) \cup \{\top \sqsubseteq A_+ \sqcup A_- \mid A \in \mathbf{C} \setminus \mathbf{N}\}$
- case $\mathfrak{s}_{\mathtt{C}} = 4$: $\pi_{\mathfrak{s}}(\mathcal{T}) \stackrel{\text{def}}{=} \bigcup_{\varphi \in \mathcal{T}} \pi_{\mathfrak{s}}(\varphi)$.

Define that:

- $\pi_{\mathfrak{s}}(a \neq b) \stackrel{\text{def}}{=} a \neq b$ and $\pi_{\mathfrak{s}}(C(a)) \stackrel{\text{def}}{=} \pi_{\mathfrak{s}+}(C)(a)$
- $\pi_{\mathfrak{s}}(\mathcal{A}) \stackrel{\text{def}}{=} \{\pi_{\mathfrak{s}}(\varphi) \mid \varphi \in \mathcal{A}\}$

– $\pi_{\mathfrak{s}}(\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle) \stackrel{\text{def}}{=} \langle \pi_{\mathfrak{s}}(\mathcal{R}), \pi_{\mathfrak{s}}(\mathcal{T}), \pi_{\mathfrak{s}}(\mathcal{A})\rangle$
– for a query $\varphi = \varphi_1 \land \ldots \land \varphi_k$, define $\pi_{\mathfrak{s}}(\varphi) \stackrel{\text{def}}{=} \pi_{\mathfrak{s}}(\varphi_1) \land \ldots \land \pi_{\mathfrak{s}}(\varphi_k)$.

Note that, if $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ is a knowledge base and φ is a query in $\mathcal{SROIQ}$ using $\mathbf{C}$ and $\mathbf{R}$, then $\pi_{\mathfrak{s}}(\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle)$ is a knowledge base and $\pi_{\mathfrak{s}}(\varphi)$ is a query in $\mathcal{SROIQ}$ using $\mathbf{C}'$ and $\mathbf{R}'$, with the property that:

– the length of $\pi_{\mathfrak{s}}(\varphi)$ is linear in the length of φ
– the size of $\pi_{\mathfrak{s}}(\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle)$ is linear in the size of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ in the case $\mathfrak{s}_{\mathtt{C}} = 4$, and linear in the sizes of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ and $\mathbf{C} \setminus \mathbf{N}$ in the case $\mathfrak{s}_{\mathtt{C}} = 3$.[6]

To have a translation for the case $\mathfrak{s}_{\mathtt{R}} = 3$ one would have to allow role axioms of the form $U \sqsubseteq r \cup r'$ (for expressing $U \sqsubseteq s_+ \cup s_-$). To have a translation for the case $\mathfrak{s}_{\forall\exists} = \pm$ one would have to allow concepts of the form $\forall(\neg r).C$ (for expressing $\forall(\neg s_-).D_+$). These features fall out of $\mathcal{SROIQ}$ and that is why we do not present translation for the case $\mathfrak{s}_{\mathtt{R}} = 3$ or $\mathfrak{s}_{\forall\exists} = \pm$.

Theorem 6.1. *Let $\mathfrak{s} \in \mathfrak{S}$ be a semantics such that $\mathfrak{s}_{\mathtt{C}} \in \{3, 4\}$, $\mathfrak{s}_{\mathtt{R}} \in \{2, 4\}$ and $\mathfrak{s}_{\forall\exists} = +$. Let $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ be a knowledge base and φ be a query in the language using $\mathbf{C}$ and $\mathbf{R}$. Then $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle \models_{\mathfrak{s}} \varphi$ iff $\pi_{\mathfrak{s}}(\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle) \models \pi_{\mathfrak{s}}(\varphi)$.*

Proof. Consider the left to right implication and suppose that $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle \models_{\mathfrak{s}} \varphi$. Let $\mathcal{I}'$ be a traditional model of $\pi_{\mathfrak{s}}(\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle)$ in the language using $\mathbf{C}'$ and $\mathbf{R}'$. We show that $\mathcal{I}' \models \pi_{\mathfrak{s}}(\varphi)$. Let $\mathcal{I}$ be the $\mathfrak{s}$-interpretation in the language using $\mathbf{C}$ and $\mathbf{R}$ specified as follows:

– $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}'}$
– for $A \in \mathbf{C} \setminus \mathbf{N}$, $A_+^{\mathcal{I}} = (A_+)^{\mathcal{I}'}$ and $A_-^{\mathcal{I}} = (A_-)^{\mathcal{I}'}$
– for $A \in \mathbf{N}$, $A_+^{\mathcal{I}} = A^{\mathcal{I}'}$ and $A_-^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus A_+^{\mathcal{I}}$
– if $\mathfrak{s}_{\mathtt{R}} = 2$ then, for $r \in \mathbf{R}$, $r_+^{\mathcal{I}} = r^{\mathcal{I}'}$ and $r_-^{\mathcal{I}} = (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus r_+^{\mathcal{I}}$
– if $\mathfrak{s}_{\mathtt{R}} = 4$ then
  • for $r \in \mathbf{R} \setminus \{U\}$, $r_+^{\mathcal{I}} = (r_+)^{\mathcal{I}'}$ and $r_-^{\mathcal{I}} = (r_-)^{\mathcal{I}'}$
  • $U_+^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and $U_-^{\mathcal{I}} = \emptyset$
– for $a \in \mathbf{I}$, $a^{\mathcal{I}} = a^{\mathcal{I}'}$.

Observe that $\mathcal{I}$ is indeed an $\mathfrak{s}$-interpretation. It can be proved by induction on the structure of C and R that, for any concept C and role R :

– $C^{\mathcal{I}} = \langle (\pi_{\mathfrak{s}+}(C))^{\mathcal{I}'}, (\pi_{\mathfrak{s}-}(C))^{\mathcal{I}'}\rangle$
– if $\mathfrak{s}_{\mathtt{R}} = 2$ then $R^{\mathcal{I}} = \langle R^{\mathcal{I}'}, (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus R^{\mathcal{I}'}\rangle$
– if $\mathfrak{s}_{\mathtt{R}} = 4$ and $R \neq U$ then $R^{\mathcal{I}} = \langle (\pi_{\mathfrak{s}+}(R))^{\mathcal{I}'}, (\pi_{\mathfrak{s}-}(R))^{\mathcal{I}'}\rangle$.

Using this and the assumption that $\mathcal{I}' \models \pi_{\mathfrak{s}}(\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle)$, we derive that $\mathcal{I} \models_{\mathfrak{s}} \langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$. Hence $\mathcal{I} \models_{\mathfrak{s}} \varphi$, and it follows that $\mathcal{I}' \models \pi_{\mathfrak{s}}(\varphi)$.

The right to left implication can be proved analogously. ◁

[6] where the notions of length and size are defined as usual

To check whether $\pi_{\mathfrak{s}}(\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle) \models \pi_{\mathfrak{s}}(\varphi)$ one can use, e.g., the tableau method given in [10]. We have the following corollary of Theorem 6.1 by taking $\varphi = \bot$.

Corollary 6.2. *Let $\mathfrak{s} \in \mathfrak{S}$ be a semantics such that $\mathfrak{s}_{\mathtt{C}} \in \{3, 4\}$, $\mathfrak{s}_{\mathtt{R}} \in \{2, 4\}$ and $\mathfrak{s}_{\forall\exists} = +$, and let $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ be a knowledge base in the language using* **C** *and* **R**. *Then $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ is $\mathfrak{s}$-satisfiable iff $\pi_{\mathfrak{s}}(\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle)$ is satisfiable (w.r.t. the traditional semantics).* ◁

Example 6.3. Consider the knowledge base $KB = \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$, where

$$\begin{aligned}
\mathcal{R} &= \emptyset \\
\mathcal{T} &= \{Bird \sqsubseteq Fly, \\
&\qquad Penguin \sqsubseteq Bird, \\
&\qquad Penguin \sqsubseteq \neg Fly\} \\
\mathcal{A} &= \{Bird(a),\ Penguin(tweety)\}.
\end{aligned}$$

Let $\mathfrak{s}$ be any semantics from $\mathfrak{S}$ with $\mathfrak{s}_{\mathtt{C}} = 3$ and $\mathfrak{s}_{\mathtt{GCI}} = m$.
We have that $\pi_{\mathfrak{s}}(KB) = KB' = \langle \emptyset, \mathcal{T}', \mathcal{A}' \rangle$, where

$$\begin{aligned}
\mathcal{T}' &= \{Bird_+ \sqsubseteq Fly_+, \\
&\qquad Penguin_+ \sqsubseteq Bird_+, \\
&\qquad Penguin_+ \sqsubseteq Fly_-\} \\
\mathcal{A}' &= \{Bird_+(a),\ Penguin_+(tweety)\}.
\end{aligned}$$

We also have that

$$\begin{aligned}
\pi_{\mathfrak{s}}(Bird(tweety)) &= Bird_+(tweety) \\
\pi_{\mathfrak{s}}(Fly(tweety)) &= Fly_+(tweety) \\
\pi_{\mathfrak{s}}(\neg Fly(tweety)) &= Fly_-(tweety) \\
\pi_{\mathfrak{s}}(Fly(a)) &= Fly_+(a) \\
\pi_{\mathfrak{s}}(\neg Fly(a)) &= Fly_-(a).
\end{aligned}$$

Observe that $\pi_{\mathfrak{s}}(Bird(tweety))$, $\pi_{\mathfrak{s}}(Fly(tweety))$, $\pi_{\mathfrak{s}}(\neg Fly(tweety))$ and $\pi_{\mathfrak{s}}(Fly(a))$ are logical consequences of KB' using the the traditional two-valued semantics, but $\pi_{\mathfrak{s}}(\neg Fly(a))$ is not. This implies that $Bird(tweety)$, $Fly(tweety)$, $\neg Fly(tweety)$ and $Fly(a)$ are $\mathfrak{s}$-logical consequences of KB, but $\neg Fly(a)$ is not. ◁

7 Conclusions

$\mathcal{SROIQ}$ is a powerful DL used as the logical foundation of OWL 2. In this work, we introduced and studied a number of different paraconsistent semantics for $\mathcal{SROIQ}$ in a uniform way. We gave a translation of the problem of conjunctive query answering w.r.t. some of the considered paraconsistent semantics into a version that uses the traditional semantics. This allows one to directly use existing tools and reasoners of $\mathcal{SROIQ}$ for paraconsistent reasoning.

Note that answering queries that contain negative individual assertions of the form $\neg S(a,b)$ using a paraconsistent semantics is first studied in this work. Also note that only a four-valued paraconsistent semantics has previously been introduced for $\mathcal{SROIQ}$ [14] (without considering some important features of $\mathcal{SROIQ}$ and having conceptual problems, as shown in Section 3.2). If $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S}$ are semantics such that $\mathfrak{s} \sqsubseteq \mathfrak{s}'$ and $\mathfrak{s}'$ is weaker than the traditional semantics then, by Theorem 4.3, for the conjunctive query answering problem, $KB \models_{\mathfrak{s}'} \varphi$ approximates $KB \models \varphi$ better than $KB \models_{\mathfrak{s}} \varphi$ does. Our postulate is that, if $\mathfrak{s} \sqsubseteq \mathfrak{s}'$ and KB is $\mathfrak{s}'$-satisfiable, then it is better to use $\mathfrak{s}'$ than $\mathfrak{s}$. In particular, one should use a four-valued semantics only when the considered knowledge base is $\mathfrak{s}'$-unsatisfiable in semantics $\mathfrak{s}'$ with $\mathfrak{s}'_{\mathsf{C}} = 3$. In such cases the four-valued semantics based on truth ordering proposed in [1, 18, 29] appears to be a better choice than the four valued semantics based on Belnap's logic [5, 4]. Its adaptation to paraconsistent reasoning in the Semantic Web is, however, left for future work.

The approach of this work and [19, 28, 15, 14, 31, 22] does not guarantee that all knowledge bases are satisfiable in the considered paraconsistent logic. The reason is that axioms like $\top \sqsubseteq \bot$ are not valid in any $\mathfrak{s}$-interpretation, where $\mathfrak{s} \in \mathfrak{S}$. Due to the specific meanings of the universal role U and nominals, we do not propose three- and four-semantics for them.[7] This may also cause a knowledge base KB $\mathfrak{s}$-unsatisfiable, e.g., when KB contains both individual assertions $A(a)$ and $\neg A(a)$ with $A \in \mathbf{N}$. In [22], we provided a quite general syntactic condition of safeness guaranteeing satisfiability of a knowledge base in $\mathcal{SHIQ}$ w.r.t. three-valued semantics.

To overcome the above mentioned problems one may want to define and use constructive DLs in a similar way as Odintsov and Wansing did for their constructive version of the basic DL $\mathcal{ALC}$. Extending such an approach to dealing with number restrictions $\geq nS.C$ and $\leq nS.C$ is not obvious. We leave this for future work.

Acknowledgements

This work was supported by the Polish MNiSW under Grant No. N N206 399334 and by the National Centre for Research and Development (NCBiR) under Grant No. SP/I/1/77065/10 by the strategic scientific research and experimental development program: "Interdisciplinary System for Interactive Scientific and Scientific-Technical Information".

References

1. S. Amo and M.S. Pais. A paraconsistent logic approach for querying inconsistent databases. *International Journal of Approximate Reasoning*, 46:366–386, 2007.

[7] The way of dealing with nominals in [14] is not appropriate.

2. F. Baader, D. Calvanese, D.L. McGuinness, D. Nardi, and P.F. Patel-Schneider, editors. *Description Logic Handbook.* Cambridge University Press, 2002.
3. F. Baader and W. Nutt. Basic description logics. In Baader et al. [2], pages 47–100.
4. N.D. Belnap. How a computer should think. In G. Ryle, editor, *Contemporary Aspects of Philosophy*, pages 30–55, Stocksfield, 1977. Oriel Press.
5. N.D. Belnap. A useful four-valued logic. In G. Eptein and J.M. Dunn, editors, *Modern Uses of Many Valued Logic*, pages 8–37. Reidel, 1977.
6. P. Besnard and A. Hunter. Quasi-classical logic: Non-trivializable classical reasoning from incosistent information. In *Proceedings of ECSQARU'95*, volume 946 of *LNCS*, pages 44–51. Springer, 1995.
7. J.-Y. Béziau, W. Carnielli, and D.M. Gabbay, editors. *Handbook of Paraconsistency*, volume 9 of *Logic and cognitive systems.* College Publications, 2007.
8. A. Bloesch. A tableau style proof system for two paraconsistent logics. *Notre Dame Journal of Formal Logic*, 34(2):295–301, 1993.
9. P. Doherty, W. Łukaszewicz, A. Skowron, and A. Szałas. *Knowledge Representation Techniques. A Rough Set Approach*, volume 202 of *Studies in Fuziness and Soft Computing.* Springer Verlag, 2006.
10. I. Horrocks, O. Kutz, and U. Sattler. The even more irresistible $\mathcal{SROIQ}$. In P. Doherty, J. Mylopoulos, and C.A. Welty, editors, *Proceedings of KR'2006*, pages 57–67. AAAI Press, 2006.
11. A. Hunter. Paraconsistent logics. In D. Gabbay and P. Smets, editors, *Handbook of Defeasible Reasoning and Uncertain Information*, pages 11–36. Kluwer, 1998.
12. A. Hunter. Reasoning with contradictory information using quasi-classical logic. *J. Log. Comput.*, 10(5):677–703, 2000.
13. S.C. Kleene. *Introduction to Metamathematics.* D. Van Nostrand, Princeton, 1952.
14. Y. Ma and P. Hitzler. Paraconsistent reasoning for OWL 2. In A. Polleres and T. Swift, editors, *Proceedings of Web Reasoning and Rule Systems*, volume 5837 of *LNCS*, pages 197–211. Springer, 2009.
15. Y. Ma, P. Hitzler, and Z. Lin. Paraconsistent reasoning for expressive and tractable description logics. In *Proceedings of Description Logics*, 2008.
16. J. Małuszyński and A. Szałas. Computational aspects of paraconsistent query language 4QL. *Journal of Applied Non-classical Logics*, 21(2):211–232, 2011.
17. J. Małuszyński and A. Szałas. Living with inconsistency and taming nonmonotonicity. In *Datalog 2010*, volume 6702 of *LNCS*, pages 334–398. Springer, 2011.
18. J. Maluszyński, A. Szałas, and A. Vitória. Paraconsistent logic programs with four-valued rough sets. In C.-C. Chan, J. Grzymala-Busse, and W. Ziarko, editors, *Proceedings of RSCTC'2008*, volume 5306 of *LNAI*, pages 41–51, 2008.
19. C. Meghini and U. Straccia. A relevance terminological logic for information retrieval. In *Proceedings of SIGIR'96*, pages 197–205. ACM, 1996.
20. D. Nardi and R. J. Brachman. An introduction to description logics. In Baader et al. [2], pages 5–44.
21. L.A. Nguyen. Paraconsistent and approximate semantics for the OWL 2 Web Ontology Language. In M. Szczuka et al., editor, *Proceedings of RSCTC'2010*, volume 6086 of *LNAI*, pages 710–720. Springer, 2010.
22. L.A. Nguyen and A. Szałas. Three-valued paraconsistent reasoning for Semantic Web agents. In P. Jędrzejowicz et al., editor, *Proceedings of KES-AMSTA 2010, Part I*, volume 6070 of *LNAI*, pages 152–162. Springer, 2010.
23. N.T. Nguyen. Using distance functions to solve representation choice problems. *Fundam. Inform.*, 48(4):295–314, 2001.
24. N.T. Nguyen. Consensus system for solving conflicts in distributed systems. *Inf. Sci.*, 147(1-4):91–122, 2002.

25. N.T. Nguyen. Inconsistency of knowledge and collective intelligence. *Cybernetics and Systems*, 39(6):542–562, 2008.
26. N.T. Nguyen and H.B. Truong. A consensus-based method for fuzzy ontology integration. In J.-S. Pan, S.-M. Chen, and N.T. Nguyen, editors, *Proceedings of ICCCI'2010, Part II*, volume 6422 of *LNAI*, pages 480–489. Springer, 2010.
27. S.P. Odintsov and H. Wansing. Inconsistency-tolerant description logic. part II: A tableau algorithm for $\mathrm{CACL}^{\mathrm{C}}$. *Journal of Applied Logic*, 6(3):343–360, 2008.
28. U. Straccia. A sequent calculus for reasoning in four-valued description logics. In D. Galmiche, editor, *Proceedings of TABLEAUX'97*, volume 1227 of *LNCS*, pages 343–357. Springer, 1997.
29. A. Vitória, J. Maluszyński, and A. Szałas. Modeling and reasoning in paraconsistent rough sets. *Fundamenta Informaticae*, 97(4):405–438, 2009.
30. X. Zhang, Z. Lin, and K. Wang. Towards a paradoxical description logic for the semantic web. In S. Link and H. Prade, editors, *Proceedings of FoIKS'2010*, volume 5956 of *LNCS*, pages 306–325. Springer, 2010.
31. X. Zhang, G. Qi, Y. Ma, and Z. Lin. Quasi-classical semantics for expressive description logics. In *Proceedings of Description Logics*, 2009.
32. X. Zhang, Z. Zhang, and Z. Lin. An argumentative semantics for paraconsistent reasoning in description logic ALC. In *Proceedings of Description Logics*, 2009.
33. X. Zhang, Z. Zhang, D. Xu, and Z. Lin. Argumentation-based reasoning with inconsistent knowledge bases. In A. Farzindar and V. Keselj, editors, *Canadian Conference on AI*, volume 6085 of *LNCS*, pages 87–99. Springer, 2010.