PARALLEL FRAMES OF NON-LIGHTLIKE CURVES

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Abstract. The parallel frame of a curve is an alternative approach to defining a moving frame. In this paper, we express the parallel frames of timelike and spacelike curves in a Minkowski space.

1. Introduction. The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. Curvature may vanish at some points on the curve. That is, the second derivative of the curve may be zero. In this situation, we can use an alternative frame. We use the tangent vector $\vec{T}$ and two relatively parallel vector fields to construct this alternative frame such that the normal vector field $\vec{N}$ along the curve is relatively parallel if its derivative is tangential. We call this frame a parallel frame along $\vec{T}$. The reason for the name parallel is because the normal component of the derivatives of the normal vector field is zero. The advantages of the parallel frame and the comparable parallel frame with the Frenet frame in Euclidean 3-space was given and studied by Bishop [1] and Hanson [3, 4].

The basic concept of this frame is to take the unique tangent vector and choose any convenient basis $\{\vec{N}_1(s), \vec{N}_2(s)\}$ in the plane perpendicular to $\vec{T}(s)$ at each point such that the derivatives of $\{\vec{N}_1(s), \vec{N}_2(s)\}$ depend on only $\vec{T}(s)$ and not each other. A parallel frame is not unique, in contrast to a Frenet frame.

In three dimensional Euclidean space the alternative parallel frame equations are

$$\begin{bmatrix} \vec{T}'(s) \\ \vec{N}_1'(s) \\ \vec{N}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(s) \\ \vec{N}_1(s) \\ \vec{N}_2(s) \end{bmatrix}$$

for a parametrized unit length curve. The relation between Frenet frames and parallel frames is as follows:

$$\begin{bmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & \sin\theta(s) \\ 0 & -\sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} \vec{T}(s) \\ \vec{N}_1(s) \\ \vec{N}_2(s) \end{bmatrix}$$

such that $\sin\theta = \frac{k_1}{\kappa}$ and $\cos\theta = \frac{k_2}{\kappa}$. Also, $\kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)}$, $\theta(s) = \arctan\dfrac{k_2}{k_1}$, and $\tau(s) = \theta'(s)$ so that $k_1(s)$ and $k_2(s)$ correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(s)\, ds$.

The functions $k_1(s)$, $k_2(s)$ are called the principal curvature along $\vec{N}_1(s)$, $\vec{N}_2(s)$, respectively. Also, $(k_1(s), k_2(s))$ can be seen as a sort of invariant of the curve. This is more difficult to conceive than in the case (κ, τ), since parallel frames are not unique. We call $(k_1(s), k_2(s))$ the normal development of the curve. If two regular curves in Euclidean space have the same normal development, then the curves are congruent [1].

2. Preliminaries. The Minkowski space $\mathbb{E}_1^3$ is the Euclidean space $\mathbb{E}^3$ provided with the Lorentzian inner product $\langle \vec{u}, \vec{v} \rangle_L = -u_1v_1 + u_2v_2 + u_1v_2$, where $\vec{u} = (u_1, u_1, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$. We say that a vector $\vec{u}$ in E_1^3 is spacelike, lightlike, or timelike if $\langle \vec{u}, \vec{u} \rangle_L > 0$, $\langle \vec{u}, \vec{u} \rangle_L = 0$, or $\langle \vec{u}, \vec{u} \rangle_L < 0$, respectively. The norm of the vector $\vec{u} \in \mathbb{E}_1^3$ is defined by $\|\vec{u}\| = \sqrt{|\langle \vec{u}, \vec{u} \rangle_L|}$. For any $\vec{u} = (u_1, u_1, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{E}_1^3$, the Lorentzian vector product $\vec{u} \times_L \vec{v}$ of $\vec{u}$ and $\vec{v}$ is defined as follows:

$$\vec{u} \times_L \vec{v} = (-u_2v_3 + u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

An arbitrary curve $\alpha = \alpha(s) : I \to \mathbb{E}_1^3$ is spacelike, timelike, or null, if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike, or null, for each $s \in I \subset R$. Throughout this paper we shall assume all curves are parametrized by their arc length.

Let $\alpha(s)$ be a non-lightlike curve and $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$ are Frenet vector fields. Then Frenet formulas are as follows:

$$\begin{bmatrix} \vec{T}'(s) \\ \vec{N}'(s) \\ \vec{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ (\epsilon_{\vec{B}})\,\kappa(s) & 0 & \tau(s) \\ 0 & (\epsilon_{\vec{T}})\,\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{bmatrix}, \tag{1}$$

where $\epsilon_{\vec{X}} =< \vec{X}, \vec{X} >_L$ and $\kappa(s)$ and $\tau(s)$ are the curvature and torsion function, respectively [5, 6].

3. Parallel Frame of a Non-Lightlike Curve. In this section, we describe parallel frames for timelike and spacelike curves in Minkowski 3-space using methods similar to the methods in Euclidean 3-space. The parallel frame is an alternative frame for curves and can be more useful compared to the Frenet frame. First, we state the following known proposition.

Proposition 1. Let $e_1, e_2, \ldots, e_n : [a, b] \to \mathbb{E}_1^n$ be smooth maps and $\{e_1(s), e_2(s), \ldots, e_n(s)\}$ be an orthonormal basis of $\mathbb{E}_1^n$ for all $s \in [a, b]$. Then, there exists a matrix valued function $A(s) = (a_{ij}(s))$ so that

i) $e_i'(s) = \sum_{j=1}^{n} a_{ji}(s)\, e_j(s)$,

ii) $a_{ij} = \epsilon_{e_i} < e_j', e_i >_L$,

iii) $\epsilon_{ej} a_{ij} + \epsilon_{e_i} a_{ji} = 0$ (that is $A(t)$ is semi-skew symmetric), where $\epsilon_{e_i} = < e_i, e_i >_L$.

First, we construct the parallel frame for the timelike curves. Let $\{\vec{T}, \vec{Z}_1, \vec{Z}_2\}$ be a smooth orthonormal frame along a unit speed timelike curve $\alpha(s) : I \rightarrow \mathbb{E}_1^3$. According to the proposition above, there exist smooth functions f_1, f_2, and f_3 such that

$$\begin{bmatrix} \vec{T}' \\ \vec{Z}_1 \\ \vec{Z}_2 \end{bmatrix} = \begin{bmatrix} 0 & f_1 & f_2 \\ f_1 & 0 & f_3 \\ f_2 & -f_3 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{Z}_1 \\ \vec{Z}_2 \end{bmatrix}.$$

Now, let's change the orthonormal frame $\{\vec{T}, \vec{Z}_1, \vec{Z}_2\}$ to $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$ such that entries 2,3 and 3,2 of the resulting matrix

$$\begin{bmatrix} \vec{T}' \\ \vec{N}_1' \\ \vec{N}_2' \end{bmatrix}$$

are zero. So, we rotate the spacelike normal vectors $\{\vec{Z}_1, \vec{Z}_2\}$ by angle $\theta(s)$. That is,

$$\vec{N}_1 = \cos\theta(s)\, \vec{Z}_1 - \sin\theta(s)\, \vec{Z}_2,$$
$$\vec{N}_2 = \sin\theta(s)\, \vec{Z}_1 + \cos\theta(s)\, \vec{Z}_2.$$

We want the normal component of the derivative of $\vec{N}_1$ and $\vec{N}_2$ to be zero. Namely, $< \vec{N}_1', \vec{N}_2 >_L = 0$ and $< \vec{N}_1, \vec{N}_2' >_L = 0$. So,

$$\begin{aligned} & < \vec{N}_1', \vec{N}_2 >_L \\ & = < -\theta' \sin\theta \vec{Z}_1 - \theta' \cos\theta \vec{Z}_2 + \cos\theta(s)\, \vec{Z}_1 - \sin\theta(s)\, \vec{Z}_2, \vec{N}_2 >_L \\ & = -\theta' \left(\sin^2\theta + \cos^2\theta\right) + \cos\theta\,(f_3 \cos\theta) + \sin\theta\,(f_3 \sin\theta) \\ & = -\theta' + f_3. \end{aligned}$$

Thus, if we choose $\theta(s)$ to satisfy the equality

$$\theta'(s) = f_3, \tag{2}$$

we obtain a new orthonormal frame $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$ satisfying the derivative formulas

$$\begin{bmatrix} \vec{T}' \\ \vec{N}_1' \\ \vec{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \end{bmatrix},$$

such that $k_1 = < \vec{T}', \vec{N}_1 >_L$ and $k_2 = < \vec{T}', \vec{N}_2 >_l$. Observe that the parallel frame is not unique, since the choice only depends on (2).

Now, let's construct parallel frames for spacelike curves using a similar method. But, in this case we must consider that one of $\vec{N}_1$ or $\vec{N}_2$ is timelike.

At first, suppose that $\{\vec{T}, \vec{Z}_1, \vec{Z}_2\}$ is a smooth orthonormal frame along a unit speed spacelike curve $\alpha(s) : I \to \mathbb{E}_1^3$ such that $\vec{Z}_1$ is timelike. Then there exist smooth functions f_1, f_2 and f_3 satisfying

$$\begin{bmatrix} \vec{T}' \\ \vec{Z}_1 \\ \vec{Z}_2 \end{bmatrix} = \begin{bmatrix} 0 & f_1 & f_2 \\ f_1 & 0 & f_3 \\ -f_2 & f_3 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{Z}_1 \\ \vec{Z}_2 \end{bmatrix},$$

where the matrix on the right side is semi-skew symmetric with respect to the signature $(+, -, +)$, that is in the order of the causal characters of $\{\vec{T}, \vec{Z}_1, \vec{Z}_2\}$.

We wish to make entries 2,3 and 3,2 vanish in the resulting matrix of

$$\begin{bmatrix} \vec{T}' \\ \vec{N}_1' \\ \vec{N}_2' \end{bmatrix}$$

that rotates $\{\vec{T}, \vec{Z}_1, \vec{Z}_2\}$ to $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$. So, we rotate the normal vectors $\{\vec{Z}_1, \vec{Z}_2\}$ by angle $\theta(s)$. But, in this case, we express the rotation matrix using hyperbolic functions since $\vec{Z}_1$ is timelike. So we can write

$$\begin{aligned} \vec{N}_1 &= \cosh\theta(s)\, \vec{Z}_1 + \sinh\theta(s)\, \vec{Z}_2, \\ \vec{N}_2 &= \sinh\theta(s)\, \vec{Z}_1 + \cosh\theta(s)\, \vec{Z}_2. \end{aligned}$$

Similarly, we want the normal component of the derivative of $\vec{N}_1$ and $\vec{N}_2$ to vanish. Therefore, the equalities $< \vec{N}_1', \vec{N}_2 >_L = 0$ and $< \vec{N}_1, \vec{N}_2' >_L = 0$ must be satisfied. Thus, considering also that the normal vector M_1 is timelike,

$$\begin{aligned} & < \vec{N}_1', \vec{N}_2 >_L \\ &= < \theta' \sinh\theta \vec{Z}_1 + \theta' \cosh\theta \vec{Z}_2 + \cosh\theta(s)\, \vec{Z}_1 + \sinh\theta(s)\, \vec{Z}_2, \vec{N}_2 >_L \\ &= \theta' \left(-\sinh^2\theta + \cosh^2\theta\right) + \cosh\theta\,(f_3 \cosh\theta) - \sinh\theta\,(f_3 \sinh\theta) \\ &= \theta' + f_3. \end{aligned}$$

Thus, if we choose $\theta(s)$ to satisfy the equality

$$\theta'(s) = -f_3, \tag{3}$$

we obtain a new orthonormal frame $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$ satisfying the derivative formulas

$$\begin{bmatrix} \vec{T}' \\ \vec{N}_1' \\ \vec{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \end{bmatrix}$$

such that $k_1 = - < \vec{T}', N_1 >_L$ and $k_2 =< \vec{T}', N_2 >_L$.

For the third case, that is, for the spacelike curves such that $\vec{Z}_2$ is timelike, we find

$$\begin{bmatrix} \vec{T}' \\ \vec{N}_1' \\ \vec{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \end{bmatrix}$$

such that $k_1 =< \vec{T}', \vec{N}_1 >_L$ and $k_2 = - < \vec{T}', \vec{N}_2 >_L$.

So, we have described parallel frames of spacelike and timelike curves in Minkowski 3 space. We can summarize the derivative formulas of parallel frames for timelike and spacelike curves as follows:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}_1' \\ \vec{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -\epsilon_{\vec{N}_1} k_1 & 0 & 0 \\ -\epsilon_{\vec{N}_2} k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \end{bmatrix}, \tag{4}$$

where $k_1 = \epsilon_{\vec{N}_1} < \vec{T}', \vec{N}_1 >_L$, $k_2 = \epsilon_{\vec{N}_1} < \vec{T}', \vec{N}_2 >_L$ with $\epsilon_{\vec{X}} =< \vec{X}, \vec{X} >_L$. The functions k_1 and k_2 are called the principal curvatures along $\vec{N}_1$ and $\vec{N}_2$. Also, parallel frames are not unique, since the choice of the parallel frame only depends on (2) and (3).

Also, we assume that $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$ is positively oriented and the vector products of these vectors are defined as follows:

$$\vec{T} \times_L \vec{N}_1 = \epsilon_{\vec{N}_2} \vec{N}_2, \vec{N}_1 \times_L \vec{N}_2 = \epsilon_{\vec{T}} \vec{T} \text{ and } \vec{N}_2 \times_L \vec{T} = \epsilon_{\vec{N}_1} \vec{N}_1.$$

In Minkowski 3-space, the Darboux vector $\vec{w}$ of the Frenet frame of a curve α is defined as follows:

1) If α is timelike, $\vec{w}_f = \tau \vec{T} + \kappa \vec{B}$;
2) If α is spacelike with timelike normal, $\vec{w}_f = \tau \vec{T} - \kappa \vec{B}$;
3) If α is spacelike with timelike binormal, $\vec{w}_f = -\tau T + \kappa B$.

For the parallel frame of a timelike curve, considering the parallel frame derivative formulas and properties of the vector products of $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$, it can be seen that the equalities

$$\frac{d\overrightarrow{T}}{ds} = \vec{w}_p \times_L \overrightarrow{T}, \ \frac{d\vec{N}_1}{ds} = \vec{w}_p \times_L \overrightarrow{N_1}, \ \frac{d\vec{N}_2}{ds} = \vec{w}_p \times_L \overrightarrow{N_2}$$

are satisfied in the case of the vector $\vec{w}_p = -k_2 \vec{N}_1 + k_1 \vec{N}_2$. So, the Darboux vector of the parallel frame of a timelike curve is $\vec{w}_p = -k_2 \vec{N}_1 + k_1 \vec{N}_2$. In a similar way, we find the Darboux vector of a spacelike curve in the Minkowski 3-space as follows:

1) If $\vec{N}_1$ is timelike, then $\vec{w}_p = -k_2 \vec{N}_1 - k_1 \vec{N}_2$.
2) If $\vec{N}_2$ is timelike, then $\vec{w}_p = k_2 \vec{N}_1 + k_1 \vec{N}_2$.

Now let's find the relation between Frenet frames and parallel frames of a non-lightlike curve. Suppose $\alpha(s): I \to \mathbb{E}_1^3$ is a non-lightlike curve in Minkowski 3-space. Also, let the Frenet frame and the parallel frame of this curve be $\{\vec{T}, \vec{N}, \vec{B}\}$ and $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$, respectively.

From the Frenet formulas, we know that $\vec{T}' = \kappa \vec{N}$. So, using this equality and the parallel frame derivative formulas, we find that

$$\vec{N} = \frac{\vec{T}'}{\kappa} = \frac{k_1 \vec{N}_1 + k_2 \vec{N}_2}{\kappa}. \tag{5}$$

Taking the vector product of $\vec{T}$ on the left with both sides of equation (5), we obtain

$$B = \frac{\epsilon_{\vec{N}_2} k_1 \vec{N}_2 - \epsilon_{\vec{N}_1} k_2 \vec{N}_1}{\epsilon_{\vec{B}} \kappa}. \tag{6}$$

Now, we examine equations (5) and (6) with respect to the causal character of the curve.

Case 1. Assume the curve is timelike.

In this case, if we write $\cos\theta = \frac{k_1}{\kappa}$ and $\sin\theta = \frac{k_2}{\kappa}$ in (5) and (6), we obtain the equation

$$\begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \end{bmatrix}.$$

This means that the parallel frame of a timelike curve is obtained by rotating the Frenet frame about the timelike vector $\vec{T}$ with angle θ.

Case 2. Assume the curve is spacelike with timelike normal $\vec{N}_1$.

If we write $\epsilon_{\vec{N}_1} = -1$, $\cosh\theta = \frac{k_1}{\kappa}$ and $\sinh\theta = \frac{k_2}{\kappa}$ in equations (5) and (6), we obtain the equation

$$\begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \end{bmatrix}.$$

Case 3. Assume the curve is spacelike with timelike normal $\vec{N}_2$.

In this case, B is also timelike and if we write again $\epsilon_{\vec{N}_2} = \epsilon_{\vec{B}} = -1$, $\cosh\theta = \frac{k_1}{\kappa}$ and $\sinh\theta = \frac{k_2}{\kappa}$ in equations (5) and (6), we obtain

$$\begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \end{bmatrix}.$$

Corollary 2. The parallel frame of a non-lightlike curve in Minkowski 3-space is obtained by rotating the Frenet frame of the curve about the speed vector $\vec{T}$ with angle $\theta(s) = \arctan \frac{k_2}{k_1}(s)$.

Corollary 3. Causal characters of $\vec{N}$ and $\vec{N}_1$ or $\vec{B}$ and $\vec{N}_2$ are the same, respectively.

Lemma 4. Let α be a unit speed timelike curve. Then

$$\kappa(s) = \sqrt{k_1^2 + k_2^2} \text{ and } \tau(s) = \theta'(s), \; \theta(s) = \arctan \frac{k_2}{k_1},$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion functions of the curve and k_1, k_2 are the principal curvatures along the $\vec{N}_1$ and $\vec{N}_2$, respectively.

Proof. Using the equation $\kappa = \|T'\|$, we obtain $\kappa(s) = \sqrt{k_1^2 + k_2^2}$. For the second equation in the lemma, we write

$$\begin{aligned} \alpha''' &= k_1'\vec{N}_1 + k_2'\vec{N}_2 + k_1^2\vec{T} + k_2^2\vec{T}, \\ \alpha' \times \alpha'' &= T \times \left(k_1\vec{N}_1 + k_2\vec{N}_2\right) = k_1\vec{N}_2 - k_2\vec{N}_1, \end{aligned}$$

in the formula

$$\tau = \frac{< \alpha''', \alpha' \times \alpha'' >_L}{\|\alpha' \times \alpha''\|^2}.$$

Therefore,

$$\tau = \frac{-k_1'k_2 + k_1k_2'}{\left\|k_1\vec{N}_2 + k_2\vec{N}_1\right\|^2} = \frac{k_1k_2' - k_2k_1'}{k_1^2 + k_2^2} = \left(\arctan \frac{k_2}{k_1}\right)'.$$

Also, we can state the following lemma for a spacelike curve.

Lemma 5. Let α be a unit speed spacelike curve. Then

$$\kappa(s) = \sqrt{\epsilon_{\vec{N}_1} k_1^2 + \epsilon_{\vec{N}_2} k_2^2}$$

and

$$\tau(s) = \epsilon_{\vec{N}_1}\theta'(s), \theta(s) = \arctan h \frac{k_2}{k_1},$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion functions of the curve and k_1, k_2 are the principal curvatures along the $\vec{N}_1$ and $\vec{N}_2$, respectively.

Corollary 6. k_1 and k_2 correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(s)\, ds$.

Corollary 7. A unit speed non-lightlike curve in Minkowski 3-space lies in a plane if and only if its normal development lies on a line through the origin.

Proof. It is well-known that a curve lies in a plane if and only if the torsion function of the curve is zero ($\tau(s) = 0$). This means that $\theta' = 0$. On the other hand,

$$\kappa(s) = \sqrt{\epsilon_{\vec{N}_1} k_1^2 + \epsilon_{\vec{N}_2} k_2^2} = \left| k_1 \vec{N}_1 + k_2 \vec{N}_2 \right|.$$

Thus, the normal development (k_1, k_2) has $\theta =$ constant. That is, its normal development lies on a line through the origin.

In addition, the parallel frame of a timelike curve in Minkowski n-space can be expressed as

$$\begin{bmatrix} \vec{T}' \\ \vec{N}_1' \\ \vec{N}_2' \\ \vdots \\ \vec{N}_{n-1}' \\ \vec{N}_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 & \dots & k_n \\ k_1 & 0 & 0 & 0 & \dots & 0 \\ k_2 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ k_{r-1} & 0 & 0 & 0 & \dots & 0 \\ k_n & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \\ \vdots \\ \vec{N}_{n-1} \\ \vec{N}_n \end{bmatrix}$$

and the parallel frame of a spacelike curve in Minkowski n-space can be expressed as

$$\begin{bmatrix} \vec{T}' \\ \vec{N}_1' \\ \vec{N}_2' \\ \vdots \\ \vec{N}_{n-1}' \\ \vec{N}_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 & ... & k_n \\ -(\epsilon_{N_1}) k_1 & 0 & 0 & 0 & \dots & 0 \\ -(\epsilon_{N_2}) k_2 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ -\left(\epsilon_{N_{n-1}}\right) k_{r-1} & 0 & 0 & 0 & \dots & 0 \\ -(\epsilon_{N_n}) k_n & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \\ \vdots \\ \vec{N}_{n-1} \\ \vec{N}_n \end{bmatrix}.$$

Now, we will determine the osculating sphere of a non-lightlike curve using its parallel frame. Let α be a non-lightlike curve in $\mathbb{E}_1^3$. Take the function $F(t)$ as

$$F(t) = < \alpha(t) - c, \alpha(t) - c >_L \pm r^2,$$

where r and c are the radius and center of the osculating sphere, respectively. Since, the osculating sphere must have 4-point contact with α, then we have

$$F(t) = F'(t) = F''(t) = F'''(t) = 0.$$

If we use $\vec{T}' = k_1 \vec{N}_1 + k_2 \vec{N}_2$ for a non-lightlike curve and

$$\left(< \vec{N}_1, \alpha(t) - c >_L\right)' = \left(< \vec{N}_2, \alpha(t) - c >_L\right)' = 0$$

we find that

$$\begin{cases} F'(t) = 0 \Longrightarrow \vec{T} \perp \alpha(t) - c \\ F''(t) = 0 \Longrightarrow k_1 < \vec{N}_1, \alpha(t) - c >_L + k_2 < \vec{N}_2, \alpha(t) - c >_L = -\epsilon_{\vec{T}} \\ F'''(t) = 0 \Longrightarrow k_1' < \vec{N}_1, \alpha(t) - c >_L + k_2' < \vec{N}_2, \alpha(t) - c >_L = 0. \end{cases} \tag{7}$$

If we say $\left(\epsilon_{\vec{N}_1}\right) a =< \vec{N}_1, \alpha(t) - c >_L$ and $\left(\epsilon_{\vec{N}_2}\right) b =< \vec{N}_2, \alpha(t) - c >_L$ in the equations in (7) such that a and b are constant from above, we obtain

$$\begin{aligned} k_1 \left(\epsilon_{\vec{N}_1}\right) a + \left(\epsilon_{\vec{N}_2}\right) k_2 b &= -\epsilon_T \\ k_1' \left(\epsilon_{\vec{N}_1}\right) a + \left(\epsilon_{\vec{N}_2}\right) k_2' b &= 0. \end{aligned} \tag{8}$$

From the solution of this linear equation system, we find that

$$a = -\frac{\left(\epsilon_{\vec{T}}\right)\left(\epsilon_{\vec{N}_1}\right) k_2'}{k_1 k_2' - k_2 k_1'}$$

and

$$b = \frac{\left(\epsilon_{\vec{T}}\right)\left(\epsilon_{\vec{N}_2}\right) k_1'}{k_1 k_2' - k_2 k_1'}.$$

So, from $\alpha(s) - c = a\vec{N}_1 + b\vec{N}_2$, the center and radius of the osculating sphere is

$$c(s) = \alpha(s) + \frac{\left(\epsilon_{\vec{T}}\right)\left(\epsilon_{\vec{N}_1}\right) k_2'}{k_1 k_2' - k_2 k_1'} \vec{N}_1 - \frac{\left(\epsilon_{\vec{T}}\right)\left(\epsilon_{\vec{N}_2}\right) k_1'}{k_1 k_2' - k_2 k_1'} \vec{N}_2 \tag{9}$$

and

$$r = \sqrt{\frac{\left(\epsilon_{\vec{N}_1}\right) k_1'^2 + \left(\epsilon_{\vec{N}_2}\right) k_2'^2}{\left(k_1 k_2' - k_2 k_1'\right)^2}}, \tag{10}$$

respectively. Thus, we have proved the following theorem.

Theorem 8. Let α be a unit speed non-lightlike curve. Then the radius and the center of the osculating sphere of α at $\alpha(t)$ are

$$r = \sqrt{\frac{\left(\epsilon_{\vec{N}_1}\right) k_1'^2 + \left(\epsilon_{\vec{N}_2}\right) k_2'^2}{\left(k_1 k_2' - k_2 k_1'\right)^2}}$$

and

$$c = \alpha(t) - \frac{\left(\epsilon_{\vec{T}}\right)\left(\epsilon_{\vec{N}_1}\right) k_2'}{k_1 k_2' - k_2 k_1'} \vec{N}_1 - \frac{\left(\epsilon_{N_2}\right) k_1'}{k_1 k_2' - k_2 k_1'} \vec{N}_2,$$

where $\vec{N}_1$ and $\vec{N}_2$ are parallel vector fields along the curve α.

Corollary 9. Let α be a unit speed non-lightlike curve in the $\mathbb{E}_1^3$. α has an osculating sphere at $\alpha(s)$ if and only if $k_1 k_2' \neq k_2 k_1'$.

Also, we can state the following corollary from (8) related to the normal development (k_1, k_2) of the curve.

Corollary 10. A unit speed non-lightlike curve is spherical if and only if its normal development (k_1, k_2) lies on a line not through the origin. Also, the distance of this line from the origin is radius of the sphere since

$$r^2 =< \alpha(s) - c, \alpha(s) - c >_L = \left(\epsilon_{\vec{N}_1}\right) a^2 + \left(\epsilon_{\vec{N}_2}\right) b^2.$$

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