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PARALLEL MULTISPLITTING AOR METHOD FOR SOLVING A CLASS OF SYSTEM OF NONLINEAR ALGEBRAIC EQUATIONS

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Abstract

A class of parallel, multisplitting accelerated overrelaxation (AOR) method is set up for solving large-scale system of nonlinear algebraic equations $A\varphi(x)+B\psi(x)=b$*. Under certain conditions, we prove the existence and uniqueness of the solution of this system of nonlinear equations and set up the global convergence theory of the new method.*

Key words system of nonlinear algebraic equations, parallel method, relaxation, H-matrix

I. Introduction

Many weakly nonlinear elliptic partial differential equations such as the Stefan problem (see [2, 5, 6]), when being discretized by the finite element method or the difference method, can often generate the following large-scale nonlinear system of algebraic equations

$$A\varphi(\boldsymbol{x})+B\psi(\boldsymbol{x})=\boldsymbol{b},\quad A,B\in L(R^n),\quad \boldsymbol{x},\boldsymbol{b}\in R^n \tag{1.1}$$

where

$$\varphi(\boldsymbol{x})=(\varphi_i(x_i)),\quad \psi(\boldsymbol{x})=(\psi_i(x_i))\in R^n$$

are continuous functions, but may have discontinuous derivatives, $\boldsymbol{x}$ is the unknown vector while $\boldsymbol{b}$ is a constant vector.

By making use of the matrix multisplitting methodology (see [1]), White (see [2]) designed a kind of paralled nonlinear Gauss-Seidel method in 1986 for solving this class of special problem being of important value in practice. This method has shown good numerical effect in concrete implementation.

In this paper, by introducing relaxation parameters in the method proposed in [2], we set up a class of parallel multisplitting accelerated overrelaxation (AOR) method for solving the large-scale nonlinear system of algebraic equations (1.1). Since there are two parameters can be arbitrarily chosen in the new method, it is then much more flexible and practical. Moreover, faster convergence rate can be resulted. Corresponding to particular choices of the relaxation parameters, the new method not only can include the parallel nonlinear Gauss-Seidel method given in [2], but also can generate a lot of practical and efficient parallel methods such as parallel multisplitting extrapolated Gauss-Seidel method, parallel multisplitting successive overrelaxation (SOR) method and so on, for solving the nonlinear system of algebraic equations (1.1). Under suitable conditions, the existence and uniqueness of the solution of nonlinear system of algebraic equations (1.1) are proven, and the global convergence theory of the new method is established thoroughly.

II. Establishment of the Method

Denote

$$A=(a_{ij}),\quad B=(b_{ij}),\quad D=\mathrm{diag}(A),\quad E=\mathrm{diag}(B)$$

Given a positive integer $\alpha\,(\alpha\leqslant n)$, for $k=1, 2, \cdots, \alpha$, let $L_k=(l_{ij}^{(k)})$, $M_k=(m_{ij}^{(k)})\in L(R^n)$ be strictly lower triangular matrices, $U_k=(u_{ij}^{(k)})$, $V_k=(V_{ij}^{(k)})\in L(R^n)$ be zero-diagonal matrices and $E_k=\mathrm{diag}(e_1^{(k)}, e_2^{(k)}, \cdots, e_n^{(k)})\in L(R^n)$ be nonnegative matrices. If there hold

(i) $A=D-L_k-U_k \qquad (k=1,2,\cdots,\alpha)$;

(ii) $B=E-M_k-V_k \qquad (k=1,2,\cdots,\alpha)$;

(iii) $\det(D)\neq 0$, $\det(E)\neq 0$;

(iv) $\sum\limits_k E_k=I \qquad$ ($I\in L(R^n)$ is identity matrix),

we call the collection $(D-L_k, U_k;\ E-M_k,\ V_k;\ E_k)\ (k=1,2,\cdots,\alpha)$ a multisplitting of the matrix pair $(A;\ B)$.

Based on this novel concept, we presently construct the following parallel multisplitting AOR (accelerated overrelaxation) method for solving the large-scale nonlinear system of algebraic equations (1.1):

Method Given initial vector $x^0\in R^n$, for $p=0, 1, 2, \cdots$, compute

$$x_i^{p+1}=\sum_k e_i^{(k)}x_i^{p,k} \qquad (i=1(1)n) \tag{2.1}$$

where

$$\left.\begin{aligned} x_i^{p,k}&=\frac{\omega}{r}\,\bar{x}_i^{p,k}+\left(1-\frac{\omega}{r}\right)x_i^p\\ \bar{x}_i^{p,k}&=r\hat{x}_i^{p,k}+(1-r)x_i^p\end{aligned}\qquad (i=1(1)n;\ k=1,2,\cdots,\alpha)\right\} \tag{2.2}$$

while $\hat{x}_i^{p,k}\,(i=1(1)n, k=1,2,\cdots,\alpha)$ is successively determined by the system of equations

$$\begin{aligned} a_{ii}\varphi_i(\hat{x}_i^{p,k})+b_{ii}\psi_i(\hat{x}_i^{p,k})-\sum_{j<i}\left[l_{ij}^{(k)}\varphi_j(\bar{x}_j^{p,k})+m_{ij}^{(k)}\psi_j(\bar{x}_j^{p,k})\right]&\\ -\sum_{j\neq i}\left[u_{ij}^{(k)}\varphi_j(x_j^p)+v_{ij}^{(k)}\psi_j(x_j^p)\right]&=b_i\\ (i=1(1)n;\ k=1,2,\cdots,\alpha)& \end{aligned} \tag{2.3}$$

Here, $r\in(0,\infty)$ is called a relaxation factor while $\omega\in(0,\infty)$ an acceleration one.

Clearly, (2.2) can be equivalently expressed as

$$\left.\begin{aligned} x_i^{p,k}&=\omega\hat{x}_i^{p,k}+(1-\omega)x_i^p\\ \bar{x}_i^{p,k}&=r\hat{x}_i^{p,k}+(1-r)x_i^p\end{aligned}\qquad (i=1(1)n, k=1,2,\cdots,\alpha)\right\} \tag{2.4}$$

so the new method is practically proceeded with the formulae (2.1), (2.3) and (2.4), too.

In the parallel method defined by (2.1), (2.3) and (2.4), corresponding to particular choices $(0,1)$, $(0,\omega)$, $(1,1)$, $(1,\omega)$ and (ω,ω) of the parameter pair (r,ω), practical and efficient parallel multisplitting relaxed methods, that is, Jacobi method, extrapolated Jacobi method, Gauss-Seidel method (see [2]), extrapolated Gauss-Seidel method (see [2]) and SOR method, for solving the large-scale nonlinear system of algebraic equations (1.1) can be respectively obtained. Particularly, as $\varphi(\boldsymbol{x})=\boldsymbol{x}$, $B=I$, the new method naturally reduces to the parallel multisplitting AOR method set up in [3] for solving the large-scale nonlinear system of algebraic equations $A\boldsymbol{x}+\psi(\boldsymbol{x})=\boldsymbol{b}$.

III. Preliminary Knowledge

In the subsequent discussions, we will carry on the notations and concepts used in [1] and [2]. Moreover, $\langle\cdot\rangle$ and $\rho(\cdot)$ are used to denote the comparison matrix and the spectral radius of the corresponding matrix, respectively. Besides, the following basic assumptions on the nonlinear system of algebraic equations (1.1) are necessary:

(A_1) $A\in L(R^n)$ is and H-matrix;

(A_2) φ,ψ: $R^n\to R^n$ are continuously diagonal mappings, and for $\forall \boldsymbol{x},\boldsymbol{y}\in R^n$, there have

$$\begin{cases}|\varphi(\boldsymbol{x})-\varphi(\boldsymbol{y})|\geqslant\varepsilon|\boldsymbol{x}-\boldsymbol{y}|, & \varepsilon=\mathrm{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\geqslant 0\\ |\psi(\boldsymbol{x})-\psi(\boldsymbol{y})|\geqslant\eta|\boldsymbol{x}-\boldsymbol{y}|, & \eta=\mathrm{diag}(\eta_1,\eta_2,\cdots,\eta_n)\geqslant 0\end{cases};$$

(A_3) $P:=|D|\varepsilon+|E|\eta$ is a positive diagonal matrix;

(A_4) $\mathrm{sgn}(a_{ii}b_{ii})(\varphi_i(s)-\varphi_i(t))(\psi_i(s)-\psi_i(t))\geqslant 0$ $(\forall s,t\in R^1;\ i=1(1)n)$;

(A_5) $\langle A\rangle|D|^{-1}|E|\leqslant\langle B\rangle$;

(A_6) For $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in R^n$, $t\in R^1$, there have

$$\begin{cases}|\varphi(t\boldsymbol{x}+(1-t)\boldsymbol{y})-\varphi(\boldsymbol{z})|\leqslant|t|\,|\varphi(\boldsymbol{x})-\varphi(\boldsymbol{z})|+|1-t|\,|\varphi(\boldsymbol{y})-\varphi(\boldsymbol{z})|\\ |\psi(t\boldsymbol{x}+(1-t)\boldsymbol{y})-\psi(\boldsymbol{z})|\leqslant|t|\,|\psi(\boldsymbol{x})-\psi(\boldsymbol{z})|+|1-t|\,|\psi(\boldsymbol{y})-\psi(\boldsymbol{z})|.\end{cases}$$

To obtain the existence and uniqueness of the solution of the nonlinear system of algebraic equation (1.1), as well as to establish the global convergence theorem about the new method, the following lemmas are indispensable.

Lemma 1 Let $g: R^n\to R^n$ be defined by

$$g(x)=D\varphi(x)+E\psi(x) \tag{3.1}$$

Then,

(1) if the basic assumptions (A_2) —(A_4) are satisfied, g $R^n\to R^n$ is a homomorphism and there holds

$$\begin{aligned}|g(\boldsymbol{x})-g(\boldsymbol{y})|&=|D|\,|\varphi(\boldsymbol{x})-\varphi(\boldsymbol{y})|+|E|\,|\psi(\boldsymbol{x})-\psi(\boldsymbol{y})|\\&\geqslant P|\boldsymbol{x}-\boldsymbol{y}|,\quad\forall\boldsymbol{x},\boldsymbol{y}\in R^n\end{aligned} \tag{3.2}$$

(2) if the basic assumptions (A_4) and (A_6) are satisfied, there holds for any positive integer N. vectors $\boldsymbol{x}$, $\boldsymbol{x}^{(i)}\in R^n$ and real numbers $t_i\in R^1$, $i=1(1)N$ that

$$\left|g\left(\sum_{i=1}^{N}t_i\boldsymbol{x}^{(i)}\right)-g(\boldsymbol{x})\right|\leqslant\sum_{i=1}^{N}|t_i|\,|g(\boldsymbol{x}^{(i)})-g(\boldsymbol{x})| \tag{3.3}$$

provided $\sum_{i=1}^{N} t_i = 1$.

Proof Conclusion (1) can be easily got through direct deduction. Now, we verify conclusion (2) by the induction.

When $N=1$, (3.3) holds obviously. Suppose (3.3) be true for $N=m$. then as $N=m+1$, by denoting

$$T^{(m)} = \sum_{i=1}^{m} t_i, \quad X^{(m)} = \sum_{i=1}^{m} \frac{t_i}{T(m)} x^{(i)}$$

and observing

$$\sum_{i=1}^{m+1} t_i x^{(i)} = T^{(m)} X^{(m)} + t_{m+1} x^{(m+1)},$$

$$T^{(m)} + t_{m+1} = 1, \quad \sum_{i=1}^{m} \frac{t_i}{T^{(m)}} = 1$$

we can inductively obtain

$$\left| g\left(\sum_{i=1}^{m} t_i x^{(i)}\right) - g(x) \right| = | g(T^{(m)} X^{(m)} + t_{m+1} x^{(m+1)}) - g(x) |$$

$$\leqslant |T^{(m)}| \, | g(X^{(m)}) - g(x) | + |t_{m+1}| \, | g(x^{(m+1)}) - g(x) |$$

$$\leqslant |T^{(m)}| \sum_{i=1}^{m} \frac{|t_i|}{|T^{(m)}|} | g(x^{(i)}) - g(x) | + |t_{m+1}| \, | g(x^{(m+1)}) - g(x) |$$

$$= \sum_{i=1}^{m+1} |t_i| \, | g(x^{(i)}) - g(x) |$$

This shows conclusion (3.3) holds for $N=m+1$. Therefore, (3.3) is proved by the induction till now.

Lemma 2[1] Let $A \in L(R^n)$ be an M-matrix.

$$A = B_k - C_k \, (k=1,2,\cdots,\alpha)$$

be weak regular splittings of matrix A. Then there holds

$$\rho\left(\sum_k E_k B_k^{-1} C_k\right) < 1.$$

IV. Global Convergence Analysis of the Parallel Method

First of all, we prove the existence and uniqueness of the solution of the nonlinear system of algebraic equations (1.1) in R^n.

Theorem 1 Assume the basic assumptions (A_1) — (A_5) be satisfied, then for any right-hand side vector $\boldsymbol{b} \in R^n$, the nonlinear system of algebraic equations (1.1) has unique solution x^* in R^n.

Proof Denote

$$G = D - A, \quad H = E - B$$

As $A \in L(R^n)$ is an H-matrix.

$$\rho(|D|^{-1}|G|)=\rho(|G||D|^{-1})<1$$

In light of the Perron-Frobinius theorem in the nonnegative matrix theory and the continuity of the spectral radius, we know that for sufficiently small $\delta>0$ there holds

$$\rho_\delta=\rho(|G||D|^{-1}+\delta ee^T)<1,\quad e=(1,1,\cdots,1)^T\in R^n \tag{4.1}$$

and there exists a positive vector $x_\delta\in R^n$ such that

$$\rho(|G||D|^{-1}+\delta ee^T)x_\delta=\rho_\delta x_\delta \tag{4.2}$$

Let $g: R^n\to R^n$ be defined by (3.1). For arbitrarily given initial vector $x^0\in R^n$, we construct an iterative sequence $\{x^p\}$ according to

$$g(x^{p+1})=b+G\varphi(x^p)+H\psi(x^p)\quad (p=0,1,2,\cdots) \tag{4.3}$$

Remembering Lemma 1 (1), it is easy to see that $\{x^p\}$ is uniquely determined in R^n and there exists $\sigma>0$ such that

$$|g(x^1)-g(x^0)|\leqslant\sigma x_\delta \tag{4.4}$$

Noticing the basic assumption (A_3) being also equivalent to

$$|H|\leqslant|G||D|^{-1}|E| \tag{4.5}$$

by making use of (3.2) and (4.2), it can be got that

$$\begin{aligned}|g(x^{p+1})-g(x^p)|&\leqslant|G||\varphi(x^p)-\varphi(x^{p-1})|+|H||\psi(x^p)-\psi(x^{p-1})|\\&=|G||D|^{-1}|g(x^p)-g(x^{p-1})|+(|H|-|G||D|^{-1}|E|)|\psi(x^p)-\psi(x^{p-1})|\\&\leqslant|G||D|^{-1}|g(x^p)-g(x^{p-1})|\\&\leqslant(|G||D|^{-1}+\delta ee^T)^p|g(x^1)-g(x^0)|\\&\leqslant\rho_\delta^p\rho x_\delta,\end{aligned}$$

Hence, for any positive integer q, it holds

$$|g(x^{p+q+1})-g(x^p)|\leqslant\frac{\sigma}{1-\rho_\delta}\rho_\delta^p x_\delta \tag{4.6}$$

Inequality (4.6) shows that $\{g(x^p)\}$ is a Cauchy sequence in R^n. By Lemma 1(1) again, we see that $g: R^n\to R^n$ is a homomorphism. Therefore, $\{x^p\}$ is a Cauchy sequence in R^n, too. The above demonstration shows that $\lim\limits_{p\to\infty}x^p$ exists. Write $\lim\limits_{p\to\infty}x^p=x^*$

Now, taking limits in both sides of (4.3), we immediately know that $x^*\in R^n$ is a solution of the nonlinear system of algebraic equations (1.1).

Let $y^*\in R^n$ be another solution of the nonlinear system of algebraic equations (1.1). By using (3.2) and (4.5), and through simple calculations, we get

$$|g(x^*)-g(y^*)|\leqslant|G||D|^{-1}|g(x^*)-g(y^*)|$$

or equivalently,

$$(I-|G||D|^{-1})|g(x^*)-g(y)|\leqslant0 \tag{4.7}$$

Considering $\rho(|G||D|^{-1})<1$, there holds $(I-|G||D|^{-1})^{-1}\geqslant0$. Combining this with (4.7) there has

$$g(x^*)=g(y^*)$$

Now, the homomorphism property $g: R^n \to R^n$ implies

$$y^*=x^*$$

This shows that $x^* \in R^n$ is the unique solution of the nonlinear system of algebraic equations (1.1) in R^n.

Presently, we discuss the global convergence of the new method.

Theorem 2 Let the basic assumptions (A_1) – (A_4) and (A_6) hold, and $(D-L_k, U_k; E-M_k, V_k; E_k)$ $(k=1,2,\cdots,\alpha)$ be a multisplitting of the matrix pair $(A; B)$ satisfying

$$\langle A\rangle=|D|-|L_k|-|U_k|=|D|-|G| \qquad (k=1,2,\cdots,\alpha) \tag{4.8}$$

$$|M_k|\leqslant|L_k||D|^{-1}|E|, \quad |V_k|\leqslant|U_k||D|^{-1}|E| \qquad (k=1,2,\cdots,\alpha) \tag{4.9}$$

Then, the sequence $\{x^p\}$ starting from any initial vector $x^0\in R^n$ and generated by the parallel multisplitting AOR method converges to the unique solution $x^*\in R^n$ of the nonlinear system of algebraic equations (1.1) provided the relaxation parameters r and ω satisfy

$$0\leqslant r\leqslant\omega, \quad 0<\omega<2/(1+\rho(|D|^{-1}|G|)) \tag{4.10}$$

Proof (4.9) obviously implies the validity of the basic assumption (A_5), so in light of Lemma 1 the nonlinear system of algebraic equations (1.1) has unique solution x^* in R^n.

Noticing (2.3), $x^*\in R^n$ evidently obeys

$$a_{ii}\varphi_i(x_i^*)+b_{ii}\psi_i(x_i^*)-\sum_{j<i}[l_{ij}^{(k)}\varphi_j(x_j^*)+m_{ij}^{(k)}\psi_j(x_j^*)]$$

$$-\sum_{j\neq i}[u_{ij}^{(k)}\varphi_j(x_j^*)+v_{ij}^{(k)}\psi_j(x_j^*)]=b_i$$

$$(i=1(1)n; \ k=1,2,\cdots,\alpha) \tag{4.11}$$

By subtracting (4.11) from (2.3) and making use of (3.1), there has

$$\begin{aligned}|g(\hat{x}^{p,k})-g(x^*)|\leqslant&|L_k||\varphi(\bar{x}^{p,k})-\varphi(x^*)|+|M_k||\psi(\bar{x}^{p,k})-\psi(x^*)|\\&+|U_k||\varphi(x^p)-\varphi(x^*)|+|V_k||\psi(x^p)-\psi(x^*)|\end{aligned} \tag{4.12}$$

From (3.2) we have

$$\left.\begin{aligned}&|\varphi(\bar{x}^{p,k})-\varphi(x^*)|=|D|^{-1}(|g(\bar{x}^{p,k})-g(x^*)|-|E||\psi(\bar{x}^{p,k})-\psi(x^*)|)\\&|\varphi(x^p)-\varphi(x^*)|=|D|^{-1}(|g|(x^p)-g(x^*)|-|E||\psi(x^p)-\psi(x^*)|)\end{aligned}\right\} \tag{4.13}$$

Substituting (4.13) into (4.12) and through simple manipulations, we obtain

$$\begin{aligned}|g(\hat{x}^{p,k})-g(x^*)|\leqslant&|L_k||D|^{-1}|g(\bar{x}^{p,k})-g(x^*)|+|U_k||D|^{-1}|g(x^p)-g(x^*)|\\&+(|M_k|-|L_k||D|^{-1}|E|)|\psi(\bar{x}^{p,k})-\psi(x^*)|\\&+(|V_k|-|U_k||D|^{-1}|E|)|\psi(x^p)-\psi(x^*)|.\end{aligned}$$

Now, using (4.9), there immediately have

$$|g(\hat{x}^{p,k})-g(x^*)|\leqslant|L_k||D|^{-1}|g(\bar{x}^{p,k})-g(x^*)|+|U_k||D|^{-1}|g(x^p)-g(x^*)| \tag{4.14}$$

By (2.2) we see that

$$\bar{x}^{p,k}=\frac{r}{\omega}x^{p,k}\left(1-\frac{r}{\omega}\right)x^{p} \tag{4.15}$$

Considering (2.4) and making use of Lemma 1(2), there hold

$$\left.\begin{aligned} &|g(x^{p,k})-g(x^*)|\leqslant\omega|g(\hat{x}^{p,k})-g(x^*)|+|1-\omega||g(x^p)-g(x^*)| \\ &|g(\bar{x}^{p,k})-g(x^*)|\leqslant\frac{r}{\omega}|g(x^{p,k})-g(x^*)|+\left(1-\frac{r}{\omega}\right)|g(x^p)-g(x^*)| \end{aligned}\right\} \tag{4.16}$$

Combine (4.14) and (4.16), we can get

$$\begin{aligned} |g(x^{p,k})-g(x^*)|\leqslant&|D|-r|L_k|)^{-1}[|1-\omega||D|+(\omega-r)|L_k|+\omega|U_k|] \\ &\cdot|D|^{-1}|g(x^p)-g(x^*)| \end{aligned} \tag{4.17}$$

Write

$$\mathscr{L}(r,\omega)=\sum_k E_k(|D|-r|L_k|)^{-1}[|1-\omega||D|+(\omega-r)|L_k|+\omega|U_k|]$$

From (4.17) there holds

$$\sum_k E_k|g(x^{p,k})-g(x^*)|\leqslant|D|\mathscr{L}(r,\omega)|D|^{-1}\, g(x^p)-g(x^*)| \tag{4.18}$$

Presently, for each $i\in\{1,2,\cdots,n\}$, by Lemma 1(2) we has

$$\begin{aligned} |g_i(x_i^{p+1})-g_i(x_i^*)|&=|g_i\left(\sum_k e_i^{k)}\, x_i^{p,k}\right)-g_i(x_i^*)| \\ &\leqslant\sum_k e_i^{k)}|g_i(x_i^{p,k})-g_i(x_i^*)| \end{aligned}$$

Making use of (4.18), we can eventually obtain

$$|g(x^{p+1})-g(x)|\leqslant|D|\mathscr{L}(r,\omega)|D|^{-1}|g(x^p)-g(x^*)| \tag{4.19}$$

Let

$$A(\omega)=\frac{1}{\omega}(1-|1-\omega|)|D|-|G|,$$

$$B_k(r,\omega)=\frac{1}{\omega}(|D|-r|L_k|),$$

$$C_k(r,\omega)=\frac{1}{\omega}[|1-\omega||D|+(\omega-r)|L_k|+\omega|U_k|],$$

$$(k=1,2,\cdots,a)$$

Clearly,

$$A(\omega)=B_k(r,\omega)-C_k(r,\omega) \qquad (k=1,2,\cdots,a)$$

are all weak regular splittings of $A(\omega)\in L(R^n$, and $A(\omega)$ is an M-matrix provided r,ω are within the region determined by (4.10). According to Lemma 2, there holds

$$\rho(|D|\mathscr{L}(r,\omega)|D|^{-1})=\rho(\mathscr{L}(r,\omega))<1$$

Therefore,

$$\lim_{p\to\infty} g(x^p)=g(x^*)$$

The homomorphism property of the mapping $g: R^n \to R^n$ then guarantees

$$\lim_{p\to\infty} x^p = x^*$$

Up to now, the proof of this theorem is thoroughly fulfilled.

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