MATHEMATICS OF COMPUTATION
Volume 83, Number 288, July 2014, Pages 1617–1644
S 0025-5718(2014)02779-8
Article electronically published on February 3, 2014

PARALLEL, NON-ITERATIVE, MULTI-PHYSICS DOMAIN DECOMPOSITION METHODS FOR TIME-DEPENDENT STOKES-DARCY SYSTEMS

YANZHAO CAO, MAX GUNZBURGER, XIAOMING HE, AND XIAOMING WANG

ABSTRACT. Two parallel, non-iterative, multi-physics, domain decomposition methods are proposed to solve a coupled time-dependent Stokes-Darcy system with the Beavers-Joseph-Saffman-Jones interface condition. For both methods, spatial discretization is effected using finite element methods. The backward Euler method and a three-step backward differentiation method are used for the temporal discretization. Results obtained at previous time steps are used to approximate the coupling information on the interface between the Darcy and Stokes subdomains at the current time step. Hence, at each time step, only a single Stokes and a single Darcy problem need be solved; as these are uncoupled, they can be solved in parallel. The unconditional stability and convergence of the first method is proved and also illustrated through numerical experiments. The improved temporal convergence and unconditional stability of the second method is also illustrated through numerical experiments.

1. INTRODUCTION

There exist many important applications that involve a free flow and a porous medium flow occurring in separate but abutting domains, with the two flows coupled at the interface between the two domains. Such flows arise in surface water flows, subsurface oil and groundwater flows such as karst aquifers, and flows in a vuggy porous medium; see, e.g., [2, 20, 21, 25, 40, 60, 65, 76, 83, 98] and the references cited therein. An important model describing such coupled flows is the Stokes-Darcy system in which the Stokes and Darcy systems are used to model the free and porous medium flows, respectively. The two systems of partial differential equations are coupled through interface conditions applied at the interface between the two flows, enabling a better description of the physics compared to that possible with a single-system model. It is not surprising, then, that a great deal of effort has been devoted to the development of methods for the approximate solution of the Stokes-Darcy system, including coupled finite element methods [2, 4, 20–22, 26, 32, 49, 51, 72, 93, 98], domain decomposition methods [19, 27, 40–45, 64, 67], Lagrange multiplier methods [3, 56, 57, 76], two-grid methods [16, 83], decoupled marching schemes [78, 84], discontinuous Galerkin methods [23, 30, 34, 60, 71, 91, 92], mortar

Received by the editor June 7, 2010 and, in revised form, July 22, 2012.
2010 *Mathematics Subject Classification.* Primary 65M55, 65M12, 65M15, 65M60, 35M10, 35Q35, 76D07, 76S05.
Key words and phrases. Stokes-Darcy model, domain decomposition methods, parallel algorithms, finite element methods.

©2014 American Mathematical Society

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

discretizations [9, 50, 52–54], boundary integral methods [10, 97], and others [17, 58, 73, 75, 85, 87]. Stokes-Brinkman and other models have also been studied and compared; see, e.g., [1, 5, 7–9, 14, 15, 28, 36, 61, 81, 82, 86, 88, 95, 100, 102, 103] and the references cited therein.

Physics-based domain decomposition approaches are especially attractive for solving coupled Stokes-Darcy systems because of the obvious possibility of breaking up the problem into two single-physics problems that might be solved in parallel, each possibly using a legacy code. This possibility has motivated the development of several efficient methods for solving the discretized Stokes-Darcy systems; see, e.g., [19, 27, 42, 44, 45, 67]. However, most existing work addresses steady-state Stokes-Darcy models instead of the more interesting and more useful time-dependent models considered in this paper.

In the single-physics setting, domain decomposition has provided a natural and efficient means for solving discretized partial differential equations in parallel.[1] For non-overlapping domain decomposition methods, the major difficulty faced is how to define the values on the interface between subdomains. For elliptic equations, convergent iterations are used to predict the values needed on the interfaces; see, e.g., [12, 55, 80, 90, 101]. For time-dependent problems, there are two popular ways to effect domain decomposition. The traditional way is to apply an iterative domain decomposition method at each time step; see, e.g., [18, 33, 35, 47, 74, 79, 89]. Alternately, one can take advantage of information obtained in previous time steps to construct a non-iterative domain decomposition method. Based on an implicit discretization in time, such methods make use of results from previous time steps to predict the values on the interface between the two subdomains at the current time step. Obviously, the second approach saves on computation and communication costs because it is non-iterative. The key issue encountered in non-iterative domain decomposition is how to obtain optimal accuracy and good stability properties because interface values are obtained from results at previous time steps, i.e., in an explicit manner, instead of using iterations to predict those values. Example methods in the second framework are the explicit/implicit domain decomposition method (EIDD) [37–39, 105, 106], the stabilized EIDD method [107, 108], IPIC methods [70], ADI methods [69], and others [48, 104]. The EIDD method first uses an explicit scheme as a predictor to obtain the information on the interface and then applies an implicit scheme to the interior of each subdomain. Because of the explicit nature of the predictor, the EIDD method is conditionally stable. The IPIC and ADI methods are stabilized EIDD methods developed to achieve better stability with some additional cost.

In this paper, we develop and analyze two parallel non-iterative domain decomposition methods for a time-dependent Stokes-Darcy model with the Beavers-Joseph-Saffman-Jones (BJSJ) interface condition [66, 68, 94]. The central advantages of our approach are as follows.

- The methods are *non-iterative*, that is, at each time step, a *single* Stokes solve and a *single* Darcy solve are needed and, because those solves are

[1] In the single-physics setting, the solution domain is usually artificially subdivided into many subdomains as opposed to the obvious domain subdivision according to relevant physics in the multi-physics setting. Of course, in the latter case, one can also further divide each physics-based subdomain in order to achieve greater parallelism. However, in this paper, we only consider physics-based domain decomposition.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

uncoupled, they may be done in parallel; this is the least one can hope to have to do at each time step.[2]

- Even though the coupling terms in the interface conditions are treated in an explicit manner, the methods are *unconditionally stable*.
- The methods yield *optimally accurate* approximations.

For example, compared to the EIDD method, the new methods feature no stability requirement and optimal convergence and are also simpler and less costly to implement.

The rest of the paper is organized as follows. In Section 2, we introduce the Stokes-Darcy system we study. In Section 3, that system is decoupled using appropriate Robin boundary conditions. In Section 4, the semi-discretization of the decoupled system is studied. The first parallel non-iterative domain decomposition method is proposed in Section 5 and analyzed in Section 6. In Section 7, we propose the second parallel non-iterative domain decomposition method that improves the accuracy of the first. Finally, in Section 8, we present numerical examples illustrating the convergence and stability properties of the two methods.

2. The Stokes-Darcy model

We consider a coupled Stokes-Darcy system on a bounded domain $\Omega = \Omega_D \cup \Omega_S \subset \mathbb{R}^d$, $d = 2, 3$, where Ω_D and Ω_S denote disjoint open regions with common boundary $\Gamma = \overline{\Omega}_D \cap \overline{\Omega}_S$; see Figure 1.

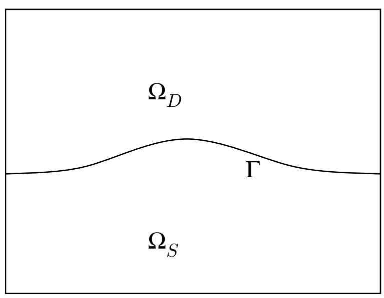

Figure 1. A sketch of the porous medium domain Ω_D, the free flow domain Ω_S, and the interface Γ.

In the porous medium region Ω_D, let $\vec{u}_D$ denote the fluid discharge rate, $\mathbb{K}$ the hydraulic conductivity tensor, and f_D the sink/source term; $\phi_D = z + \frac{p_D}{\rho g}$ denotes the hydraulic head, where p_D denotes the dynamic pressure, z the height, ρ the density, and g is the gravity constant. Then, the flow in the porous medium is assumed to satisfy, for $t \in (0, T]$, the Darcy system

$$\begin{cases} \vec{u}_D = -\mathbb{K}\nabla\phi_D, \\ \dfrac{\partial \phi_D}{\partial t} + \nabla \cdot \vec{u}_D = f_D. \end{cases} \tag{2.1}$$

[2]Alternative approaches either require, *for each time step*, a single **coupled** Stokes-Darcy solve or require an iterative procedure involving **multiple** uncoupled Stokes and Darcy solves; both such approaches are substantially more costly compared to our non-iterative domain decomposition approach which merely requires, *for each time step*, **single, uncoupled** Stokes and Darcy solves. A third alternative is to use a space-time discretization instead of a straightforward marching scheme; in general, such an approach is more costly compared to ours.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Eliminating $\vec{u}_D$, we obtain the second-order form of the Darcy system

$$\frac{\partial \phi_D}{\partial t} - \nabla \cdot (\mathbb{K}\nabla\phi_D) \;=\; f_D. \tag{2.2}$$

With a modest amount of work and with appropriate modifications, most of what follows remains valid if we use the first-order formulation (2.1) instead of (2.2). We choose to use the latter because it is simpler to implement and its analysis and numerical analysis are also simpler to present.

In the free-flow region Ω_S, let $\vec{u}_S$ denote the fluid velocity, p_S the kinematic pressure, $\vec{f}_S$ the external body force density, and ν the kinematic viscosity of the fluid. Additionally, $\mathbb{T}(\vec{u}_S, p_S) = 2\nu\mathbb{D}(\vec{u}_S) - p_S\mathbb{I}$ denotes the stress tensor, where $\mathbb{D}(\vec{u}_S) = \frac{1}{2}(\nabla\vec{u}_S + \nabla^T\vec{u}_S)$ denotes the rate of deformation tensor and $\mathbb{I}$ the identity tensor. The free flow in Ω_S is assumed to satisfy, for $t \in (0, T]$, the Stokes system

$$\begin{cases} \dfrac{\partial \vec{u}_S}{\partial t} - \nabla \cdot \mathbb{T}(\vec{u}_S, p_S) = \vec{f}_S, \\ \nabla \cdot \vec{u}_S = 0. \end{cases} \tag{2.3}$$

Along the interface Γ, we first impose the two well-accepted interface conditions

$$\vec{u}_S \cdot \vec{n}_S = -\vec{u}_D \cdot \vec{n}_D \qquad \text{and} \qquad -\vec{n}_S \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = g(\phi_D - z), \tag{2.4}$$

where $\vec{n}_S$ and $\vec{n}_D$ denote the unit outer normal to the free flow and porous medium regions at the interface Γ, respectively, and z denotes the vertical Cartesian coordinate. These interface conditions imply the continuity of the normal components of the velocity and the balance of forces normal to the interface. In the tangential direction along the interface, the Beavers-Joseph-Saffman-Jones (BJSJ) interface condition [66, 68, 94]

$$-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}}\,\boldsymbol{\tau}_j \cdot \vec{u}_S \tag{2.5}$$

is imposed, where $\boldsymbol{\tau}_j$ $(j = 1, \ldots, d-1)$ denote mutually orthogonal unit tangential vectors along the interface Γ and Π denotes the permeability of the porous media.

Remark 2.1. The permeability Π is a property of the porous media whereas ν denotes the kinematic viscosity of the fluid. On the other hand, the hydraulic conductivity $\mathbb{K} = \frac{g}{\nu}\Pi$ relies on properties of both the fluid and the porous media. In a simulation, one can specify any two of these after which the third is determined.

For simplicity, except on Γ, we impose homogeneous Dirichlet boundary conditions for the hydraulic head ϕ_D and the free flow velocity $\vec{u}_S$ on the boundaries $\partial\Omega_D$ and $\partial\Omega_S$ of Ω_S and Ω_D, respectively, i.e., we have

$$\phi_D = 0 \qquad \text{on } \partial\Omega_D\backslash\Gamma, \tag{2.6}$$

$$\vec{u}_S = 0 \qquad \text{on } \partial\Omega_S\backslash\Gamma. \tag{2.7}$$

Finally, we impose the initial conditions

$$\phi_D(0, x, y) = \phi_0(x, y), \tag{2.8}$$

$$\vec{u}_S(0, x, y) = \vec{u}_0(x, y). \tag{2.9}$$

The Stokes-Darcy system we consider is given by (2.2)–(2.9).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

The finite element methods we consider are based on a variational formulation of (2.2)–(2.9) that is defined with respect to the function spaces

$$\begin{aligned} X_S &= \{\vec{v} \in [H^1(\Omega_S)]^d \mid \vec{v} = 0 \text{ on } \partial\Omega_S\backslash\Gamma\}, \\ Q_S &= L^2(\Omega_S), \\ X_D &= \{\psi \in H^1(\Omega_D) \mid \psi = 0 \text{ on } \partial\Omega_D\backslash\Gamma\}, \\ L^2(0,T;Q_S) &= \{\phi : \phi(t,\cdot) \in Q_S, \ \forall\, t \in [0,T]\}, \\ H^1(0,T;X_D,X_D') &= \{\phi : \phi \in L^2(0,T;X_D) \text{ and } \frac{\partial \phi}{\partial t} \in L^2(0,T;X_D')\}, \\ H^1(0,T;X_S,X_S') &= \{\phi : \phi \in L^2(0,T;X_S) \text{ and } \frac{\partial \phi}{\partial t} \in L^2(0,T;X_S')\}, \end{aligned}$$

where X_D' and X_S' denote the dual spaces of X_D and X_S, respectively. For the domain D ($D = \Omega_S$ or Ω_D), $(\cdot,\cdot)_D$ denotes the L^2 inner product and $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product on the interface Γ or the duality pairing between $(H_{00}^{1/2}(\Gamma))'$ and $H_{00}^{1/2}(\Gamma)$. P_τ denotes the projection onto the tangent space on Γ, i.e.,

$$P_\tau \vec{u} = \sum_{j=1}^{d-1} (\vec{u}\cdot\boldsymbol{\tau}_j)\boldsymbol{\tau}_j.$$

We also define the bilinear forms

$$\begin{aligned} &a_D(\phi_D,\psi) = (\mathbb{K}\nabla\phi_D,\nabla\psi)_{\Omega_D}, \\ &a_S(\vec{u}_S,\vec{v}) = 2\nu(\mathbb{D}(\vec{u}_S),\mathbb{D}(\vec{v}))_{\Omega_S}, \qquad \text{and} \qquad b_S(\vec{v},q) = -(\nabla\cdot\vec{v},q)_{\Omega_S}. \end{aligned}$$

With these notations, a weak formulation of the coupled Stokes-Darcy system (2.2)–(2.9) is given as follows [20, 21, 27, 45]: find $(\vec{u}_S,p_S) \in H^1(0,T;X_S,X_S') \times L^2(0,T;Q_S)$ and $\phi_D \in H^1(0,T;X_D,X_D')$ such that

$$\begin{aligned} &(\frac{\partial \vec{u}_S}{\partial t},\vec{v})_{\Omega_S} + (\frac{\partial \phi_D}{\partial t},\psi)_{\Omega_D} + a_S(\vec{u}_S,\vec{v}) + b_S(\vec{v},p_S) + a_D(\phi_D,\psi) \\ &\quad + \langle g\phi_D,\vec{v}\cdot\vec{n}_S\rangle - \langle\vec{u}_S\cdot\vec{n}_S,\psi\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}}\langle P_\tau\vec{u}_S,P_\tau\vec{v}\rangle \\ &= (f_D,\psi)_{\Omega_D} + (\vec{f}_S,\vec{v})_{\Omega_S} + \langle gz,\vec{v}\cdot\vec{n}_S\rangle \quad \forall\,\vec{v}\in X_S, \ \psi\in X_D, \end{aligned} \tag{2.10}$$

$$b_S(\vec{u}_S,q) = 0, \quad \forall\, q \in Q_S, \tag{2.11}$$

$$\vec{u}_S(0) = u_0, \qquad \phi_D(0) = \phi_0. \tag{2.12}$$

The system of (2.10)–(2.12) is well posed for $\vec{f}_S \in [L^2(\Omega_S)]^d$ and $f_D \in L^2(\Omega_D)$; see, e.g., [21, 27, 45].

3. Robin boundary conditions and the decoupled system

In order to solve the coupled Stokes-Darcy problem utilizing domain decomposition, following [27], we introduce Robin boundary conditions for the Darcy and Stokes systems.

First, we introduce the Robin condition

$$\mathbb{K}\nabla\widehat{\phi}_D\cdot\vec{n}_D + g\widehat{\phi}_D = \xi_D \qquad \text{on } \Gamma \tag{3.1}$$

for the Darcy system, where ξ_D denotes a function defined on Γ. Then, the corresponding weak formulation for the *uncoupled* Darcy system (2.2), (2.6), (2.8), and

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

(3.1) is given by: for $\xi_D \in L^2(0,T;L^2(\Gamma))$, find $\widehat{\phi}_D \in H^1(0,T;X_D,X_D')$ such that

$$(\frac{\partial\widehat{\phi}_D}{\partial t},\psi)_{\Omega_D} + a_D(\widehat{\phi}_D,\psi) + \langle g\widehat{\phi}_D,\psi\rangle = (f_D,\psi)_{\Omega_D} + \langle\xi_D,\psi\rangle \quad \forall\,\psi \in X_D, \tag{3.2}$$

$$\widehat{\phi}_D(0) = \phi_0. \tag{3.3}$$

For the Stokes system, we introduce the Robin-type condition

$$\vec{n}_S\cdot(\mathbb{T}(\widehat{\vec{u}}_S,\widehat{p}_S)\cdot\vec{n}_S) + \widehat{\vec{u}}_S\cdot\vec{n}_S = \xi_S \qquad \text{on } \Gamma, \tag{3.4}$$

where ξ_S denotes a function defined on Γ. Then, the corresponding weak formulation for the *uncoupled* Stokes system (2.3), (2.5), (2.7), (2.9), and (3.4) is given by: for $\xi_S \in L^2(0,T;L^2(\Gamma))$, find $\widehat{\vec{u}}_S \in H^1(0,T;X_S,X_S')$ and $\widehat{p}_S \in L^2(0,T;Q_S)$ such that

$$\begin{aligned} &(\frac{\partial\widehat{\vec{u}}_S}{\partial t},\vec{v})_{\Omega_S} + a_S(\widehat{\vec{u}}_S,\vec{v}) + b_S(\vec{v},\widehat{p}_S) + \langle\widehat{\vec{u}}_S\cdot\vec{n}_S,\vec{v}\cdot\vec{n}_S\rangle \\ &\quad + \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\Pi)}}\langle P_\tau\widehat{\vec{u}}_S,P_\tau\vec{v}\rangle \;=\; (\vec{f}_S,\vec{v})_{\Omega_S} + \langle\xi_S,\vec{v}\cdot\vec{n}_S\rangle \quad \forall\,\vec{v}\in X_S, \end{aligned} \tag{3.5}$$

$$b_S(\widehat{\vec{u}}_S,q) \;=\; 0 \quad \forall\, q\in Q_S, \tag{3.6}$$

$$\widehat{\vec{u}}_S(0) \;=\; \vec{u}_0. \tag{3.7}$$

It is not difficult to show that there exists a unique solution of the systems (3.2)–(3.7); see, e.g., [21] for the case of coupled Stokes-Darcy system.

The following proposition shows that, for appropriate choices of ξ_S and ξ_D, (smooth) solutions of the coupled Stokes-Darcy system (2.10)–(2.12) are equivalent to solutions of the uncoupled systems (3.2)–(3.3) and (3.5)–(3.7), and hence we may solve the latter systems instead of the former. Even though we have the additional term gz in the second interface condition in (2.4) and the problem we consider is time-dependent, the proof is similar to that of [27, Lemma 2.2] for the steady-state case.

Proposition 3.1. *Let* $(\phi_D,\vec{u}_S,p_S)$ *denote the solution of the coupled Stokes-Darcy system* (2.10)*–*(2.12) *and let* $(\widehat{\phi}_D,\widehat{\vec{u}}_S,\widehat{p}_S)$ *denote the solution of the decoupled Darcy and Stokes systems* (3.2)*–*(3.3) *and* (3.5)*–*(3.7)*, respectively, with Robin boundary conditions at the interface. Suppose* $(\phi_D,\vec{u}_S,p_S)$ *is regular in the sense that* $\phi_D \in L^2(0,T;H^2(\Omega_D))$, $u_S \in L^2(0,T;H^2(\Omega_S))$, *and* $p_S \in L^2(0,T;H^1(\Omega_S))$, *then* $(\widehat{\phi}_D,\widehat{\vec{u}}_S,\widehat{p}_S) = (\phi_D,\vec{u}_S,p_S)$ *if and only if* ξ_S *and* ξ_D *satisfy the compatibility conditions*

$$\xi_D \;=\; \widehat{\vec{u}}_S\cdot\vec{n}_S + g\widehat{\phi}_D, \tag{3.8}$$

$$\xi_S \;=\; \widehat{\vec{u}}_S\cdot\vec{n}_S - g\widehat{\phi}_D + gz. \tag{3.9}$$

Remark 3.2. The terms in the Robin conditions (3.1) and (3.4) appear to be dimensionally inconsistent; however, they contain "hidden" dimensional parameters. In fact, two parameters γ_p and γ_f are utilized in [19,27] to define the Robin boundary conditions

$$\gamma_p\mathbb{K}\nabla\widehat{\phi}_D\cdot\vec{n}_D + g\widehat{\phi}_D = \xi_D \qquad \text{on } \Gamma,$$
$$\vec{n}_S\cdot(\mathbb{T}(\widehat{\vec{u}}_S,\widehat{p}_S)\cdot\vec{n}_S) + \gamma_f\widehat{\vec{u}}_S\cdot\vec{n}_S = \xi_S \qquad \text{on } \Gamma.$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

In this article, we set, in appropriate units, $\gamma_f = \gamma_p = 1$ for simplicity. The proof of Proposition 3.1 remains valid for arbitrary choices of positive $\gamma_f = \gamma_p$. This is consistent with the results of [19, 27] because convergence is guaranteed as long as we set $\gamma_p = \gamma_f > 0$.

4. Semi-discretization of the decoupled system

In this section, we present spatial semi-discretizations of (3.2)–(3.3) and (3.5)–(3.6). For simplicity of notation, let $\|\phi\|_k$ denote $\|\phi\|_{H^k(\Omega_D)}$ $(k = 1, 2)$, $\|\vec{u}\|_k$ denote $\|\vec{u}\|_{[H^k(\Omega_S)]^d}$ $(k = 1, 2)$, $\|p\|_1$ denote $\|p\|_{H^1(\Omega_S)}$, $\|\phi\|_0$ denote $\|\phi\|_{L^2(\Omega_D)}$, $\|\vec{u}\|_0$ denote $\|\vec{u}\|_{[L^2(\Omega_S)]^d}$, and $\|p\|_0$ denote $\|p\|_{L^2(\Omega_S)}$. We assume that we have in hand regular subdivisions $\mathcal{T}_D$ and $\mathcal{T}_S$ of Ω_D and Ω_S, respectively, into finite elements; we further assume that the interface Γ consists of the union of faces of a subset of the elements of each of the two subdivisions and that the two subdivisions exactly match along Γ. Based on the subdivisions $\mathcal{T}_D$ and $\mathcal{T}_S$, one can define finite element spaces $X_{Dh} \subset X_D$, $X_{Sh} \subset X_S$, and $Q_{Sh} \subset Q_S$; these spaces are parameterized by the mesh size h.

We assume that X_{Sh} and Q_{Sh} consist of first or higher order of piecewise polynomials and satisfy the inf-sup condition [59, 63, 76],

$$\inf_{0 \neq q \in Q_{sh}} \sup_{0 \neq \vec{v} \in X_{Sh}} \frac{b_S(\vec{v}, q)}{\|\vec{v}\|_1 \, \|q\|_0} > \beta, \tag{4.1}$$

where $\beta > 0$ is a constant independent of the mesh size h; this condition is needed to ensure that the spatial discretizations of the Stokes system are stable. Moreover, we need inverse inequalities in both X_{Dh} and X_{Sh}: there exist C_1 and C_2, depending on Ω_D and Ω_S, respectively, such that

$$\|\phi_h\|_1 \leq C_1 h^{-1} \|\phi_h\|_0 \, , \ \forall \ \phi_h \in X_{Dh}, \tag{4.2}$$

$$\|\vec{u}_h\|_1 \leq C_2 h^{-1} \|\vec{u}_h\|_0 \, , \ \forall \ \vec{u}_h \in X_{Sh}. \tag{4.3}$$

See [13, 31, 59, 63] for details and for many examples of pairs of finite element spaces X_{Dh}, X_{Sh}, and Q_{Sh} that satisfy (4.1)–(4.3). One example is the Taylor-Hood element pair that we use in the numerical experiments; for that pair, X_{Dh} and X_{Sh} consist of continuous piecewise quadratic polynomials and Q_{Sh} consists of continuous piecewise linear polynomials.

Remark 4.1. The requirement that the finite elements exactly conform to the interface Γ is, of course, restrictive as it is tantamount to assuming that the interface is piecewise planar. Of course, this assumption enables substantial simplifications in the analysis. In the more general case, the two finite element grids would match along a piecewise planar surface Γ_h that approximates a possibly curved interface Γ; one would then have to account for errors due to the interface approximation throughout the analysis; see [11, 29, 77] and the references therein for related works. Furthermore, if the two grids do not match on the interface, then one has to interpolate data from one side of the interface onto the grid on the other side which introduces additional errors that must be accounted for. All of these topics are of interest for future work. The key foci in this article are the non-iterative aspects of our domain decomposition algorithms and the unconditional stability for the algorithms which treat the interface term in an explicit manner.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Define $P_h : X_D \to X_{Dh}$ and $\mathbb{P}_h : X_S \to X_{Sh}$ to be the regular orthogonal projections. We are now in a position to define spatial semi-discretizations of the Darcy system (3.2)–(3.3) and the Stokes system (3.5)–(3.7): find $\widehat{\phi}_h \in H^1(0,T;X_{Dh})$, $\widehat{\vec{u}}_h \in H^1(0,T;X_{Sh})$ and $\widehat{p}_h \in L^2(0,T;Q_{Sh})$ such that

$$(\frac{\partial \widehat{\phi}_h}{\partial t}, \psi_h)_{\Omega_D} + a_D(\widehat{\phi}_h, \psi_h) + \langle g\widehat{\phi}_h, \psi_h \rangle = (f_D, \psi_h)_{\Omega_D} + \langle \xi_{Dh}, \psi_h \rangle \quad \forall \psi_h \in X_{Dh}, \tag{4.4}$$

$$(\frac{\partial \widehat{\vec{u}}_h}{\partial t}, \vec{v}_h)_{\Omega_S} + a_S(\widehat{\vec{u}}_h, \vec{v}_h) + b_S(\vec{v}_h, \widehat{p}_h) + \langle \widehat{\vec{u}}_h \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S \rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\Pi)}} \langle P_\tau \widehat{\vec{u}}_h, P_\tau \vec{v}_h \rangle = (\vec{f}_S, \vec{v}_h)_{\Omega_S} + \langle \xi_{Sh}, \vec{v}_h \cdot \vec{n}_S \rangle \quad \forall \vec{v}_h \in X_{Sh}, \tag{4.5}$$

$$b_S(\widehat{\vec{u}}_h, q_h) = 0 \quad \forall q_h \in Q_{Sh}, \tag{4.6}$$

$$\widehat{\phi}_h(0) = P_h \phi_0, \qquad \widehat{\vec{u}}_h(0) = \mathbb{P}_h \vec{u}_0, \tag{4.7}$$

where

$$\xi_{Dh} = \widehat{\vec{u}}_h \cdot \vec{n}_S + g\widehat{\phi}_h \qquad \text{on } \Gamma, \tag{4.8}$$

$$\xi_{Sh} = \widehat{\vec{u}}_h \cdot \vec{n}_S - g\widehat{\phi}_h + gz \qquad \text{on } \Gamma. \tag{4.9}$$

There still remains temporal discretizations of (4.4)–(4.9). In Sections 5 and 7, we define two such discretizations that lead to non-iterative domain decomposition methods for the Stokes-Darcy system.

5. The first parallel non-iterative domain decomposition method

The first parallel domain decomposition algorithm is based on, but is not exactly, the backward-Euler method for temporal discretization. All terms in (4.4)–(4.6) are treated implicitly except for the evaluation of ξ_{Dh} and ξ_{Sh} which are treated explictly so that the Darcy and Stokes parts of (4.4)–(4.7) uncouple at each time step. The algorithm proceeds as follows, assuming that the interval $[0,T]$ is partitioned into N equal intervals of length $\triangle t = T/N$; the uniformity of the partition is not essential to the algorithm.

Algorithm 1 – First non-iterative domain decomposition method. Set $\widehat{\phi}_h^0 = P_h\phi_0$ and $\widehat{\vec{u}}_h^0 = \mathbb{P}_h \vec{u}_0$. Then, for $n = 0, 1, 2, \ldots, N-1$,
1. set

$$\xi_{Dh}^n = \widehat{\vec{u}}_h^n \cdot \vec{n}_S + g\widehat{\phi}_h^n \qquad \text{on } \Gamma, \tag{5.1}$$

$$\xi_{Sh}^n = \widehat{\vec{u}}_h^n \cdot \vec{n}_S - g\widehat{\phi}_h^n + gz \qquad \text{on } \Gamma. \tag{5.2}$$

2. Independently solve

$$(\frac{\widehat{\phi}_h^{n+1} - \widehat{\phi}_h^n}{\triangle t}, \psi_h)_{\Omega_D} + a_D(\widehat{\phi}_h^{n+1}, \psi_h) + \langle g\widehat{\phi}_h^{n+1}, \psi_h \rangle = (f_D^{n+1}, \psi_h)_{\Omega_D} + \langle \xi_{Dh}^n, \psi_h \rangle \quad \forall \psi_h \in X_{Dh} \tag{5.3}$$

for $\widehat{\phi}_h^{n+1}$ and

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$
\begin{aligned}
&(\frac{\widehat{\vec{u}}_h^{n+1} - \widehat{\vec{u}}_h^{n}}{\triangle t}, \vec{v}_h)_{\Omega_S} + a_S(\widehat{\vec{u}}_h^{n+1}, \vec{v}_h) + b_S(\vec{v}_h, \widehat{p}_h^{n+1}) \\
&\quad + \langle \widehat{\vec{u}}_h^{n+1} \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S \rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}} \langle P_\tau \widehat{\vec{u}}_h^{n+1}, P_\tau \vec{v}_h \rangle, \\
&= (\vec{f}_S^{n+1}, \vec{v}_h)_{\Omega_S} + \langle \xi_{Sh}^n, \vec{v}_h \cdot \vec{n}_S \rangle \quad \forall \vec{v}_h \in X_{Sh},
\end{aligned} \tag{5.4}
$$

$$
b_S(\widehat{\vec{u}}_h^{n+1}, q_h) = 0 \quad \forall q_h \in Q_{Sh}, \tag{5.5}
$$

for $\widehat{\vec{u}}_h^{n+1}$ and $\widehat{p}_h^{n+1}$.

6. Convergence analysis for the first parallel non-iterative domain decomposition method

We use the well-known energy method framework to analyze the convergence properties of the first non-iterative domain decomposition method. Using this framework, we separate the analysis of errors for the spatially semi-discrete approximation of Section 4 and the fully discrete approximation of Section 5.

In what follows, $C > 0$ denotes a generic constant whose value may be different from place to place, but which is independent of the spatial and temporal grid sizes h and $\triangle t$, respectively.

6.1. **Error of the semi-discrete approximate solution.** We will follow the regular convergence analysis framework for finite element semi-discretizations [46, 96, 99]. For the analysis of the porous medium flow, we define R_h to be the Ritz projection satifying

$$
a_D(R_h\phi, \psi_h) = a_D(\phi, \psi_h) \quad \forall \psi_h \in X_{Dh},\ \phi \in X_D. \tag{6.1}
$$

For the analysis in the Stokes system, we introduce the projection operator $\mathbb{P} = (\mathbb{P}_s, \mathbb{P}_p) : X_S \times Q_S \to X_{Sh} \times Q_{Sh}$ such that, for any $\vec{u} \in X_S$ and $p \in Q_S$,

$$
\begin{aligned}
&a_S(\mathbb{P}_s\vec{u}, \vec{v}_h) + b_S(\vec{v}_h, \mathbb{P}_p p) + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}} \langle P_\tau(\mathbb{P}_s\vec{u}), P_\tau\vec{v}_h \rangle \\
&= a_S(\vec{u}, \vec{v}_h) + b_S(\vec{v}_h, p) + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}} \langle P_\tau\vec{u}, P_\tau\vec{v}_h \rangle \quad \forall \vec{v}_h \in X_{Sh},
\end{aligned} \tag{6.2}
$$

$$
b_S(\mathbb{P}_s\vec{u}, q_h) \;=\; b_S(\vec{u}, q_h) \quad \forall q_h \in Q_{Sh}. \tag{6.3}
$$

Under certain smoothness assumptions we have the following error estimates (see [78, page 8] and [84, pages 712 and 715]). For $\vec{u} \in [H^2(\Omega_S)]^d$ and $\phi \in H^2(\Omega_D)$,

$$
\|P_h\phi - \phi\|_0 \;\le\; Ch^2 \|\phi\|_2\,, \tag{6.4}
$$

$$
\|\mathbb{P}_h\vec{u} - \vec{u}\|_0 \;\le\; Ch^2 \|\vec{u}\|_2\,, \tag{6.5}
$$

$$
\|R_h\phi - \phi\|_0 \;\le\; Ch^2 \|\phi\|_2\,, \tag{6.6}
$$

$$
\|R_h\phi - \phi\|_1 \;\le\; Ch \|\phi\|_2\,, \tag{6.7}
$$

and for $(\vec{u}, p) \in [H^2(\Omega_S)]^d \times H^1(\Omega_S)$,

$$
\|\mathbb{P}_s\vec{u} - \vec{u}\|_0 \;\le\; Ch^2(\|\vec{u}\|_2 + \|p\|_1), \tag{6.8}
$$

$$
\|\mathbb{P}_s\vec{u} - \vec{u}\|_1 \;\le\; Ch(\|\vec{u}\|_2 + \|p\|_1), \tag{6.9}
$$

$$
\|\mathbb{P}_p p - p\|_0 \;\le\; Ch(\|\vec{u}\|_2 + \|p\|_1). \tag{6.10}
$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Remark 6.1. We refer to [11, 24, 29, 96] and the references cited therein for further discussions about relaxing the above assumptions.

Then, the error estimates for the semi-discrete approximations are given as follows.

Theorem 6.2. *Assume that* $\widehat{\phi}_D \in H^1(0,T;H^2(\Omega_D))$ *and* $\widehat{\vec{u}}_S \in H^1(0,T;[H^2(\Omega_S)]^d)$. *Then,*

$$\begin{aligned}&\left\|\widehat{\phi}_h - \widehat{\phi}_D\right\|_1 + \left\|\widehat{\vec{u}}_h - \widehat{\vec{u}}_S\right\|_1 \\ &\quad \leq Ch\Big(\left\|\widehat{\phi}_D\right\|_{H^1(0,T;H^2(\Omega_D))} + \left\|\widehat{\vec{u}}_S\right\|_{H^1(0,T;[H^2(\Omega_S)]^d)} \Big).\end{aligned} \tag{6.11}$$

Proof. Define

$$\theta = \widehat{\phi}_h - R_h\widehat{\phi}_D \qquad \text{and} \qquad \rho = R_h\widehat{\phi}_D - \widehat{\phi}_D. \tag{6.12}$$

Then,

$$\widehat{\phi}_h - \widehat{\phi}_D = \theta + \rho. \tag{6.13}$$

Taking $\psi = \psi_h \in X_{Dh}$ in (3.2), substituting (3.8) into (3.2), and subtracting (3.2) from (4.4), we have

$$\begin{aligned}&\Big(\frac{\partial\widehat{\phi}_h}{\partial t} - \frac{\partial\widehat{\phi}_D}{\partial t}, \psi_h\Big)_{\Omega_D} + a_D(\widehat{\phi}_h - \widehat{\phi}_D, \psi_h) + \langle g(\widehat{\phi}_h - \widehat{\phi}_D), \psi_h\rangle \\ &\quad = \langle (\widehat{\vec{u}}_h - \widehat{\vec{u}}_S)\cdot\vec{n}_S + g(\widehat{\phi}_h - \widehat{\phi}_D), \psi_h\rangle \quad \forall\,\psi_h \in X_{Dh}.\end{aligned} \tag{6.14}$$

Define

$$\vec{\theta}_1 = \widehat{\vec{u}}_h - \mathbb{P}_s\widehat{\vec{u}}_S, \qquad \vec{\rho}_1 = \mathbb{P}_s\widehat{\vec{u}}_S - \widehat{\vec{u}}_S, \tag{6.15}$$

$$\theta_2 = \widehat{p}_h - \mathbb{P}_p\widehat{p}_S, \qquad \rho_2^n = \mathbb{P}_p\widehat{p}_S - \widehat{p}_S. \tag{6.16}$$

Then,

$$\widehat{\vec{u}}_h - \widehat{\vec{u}}_S = \vec{\theta}_1 + \vec{\rho}_1 \qquad \text{and} \qquad \widehat{p}_h - \widehat{p}_S = \theta_2 + \rho_2. \tag{6.17}$$

Taking $\vec{v} = \vec{v}_h \in X_{Sh}$ in (3.5), substituting (3.9) into (3.5), and subtracting (3.5) from (4.5), we obtain

$$\begin{aligned}&\Big(\frac{\partial(\widehat{\vec{u}}_S - \widehat{\vec{u}}_h)}{\partial t}, \vec{v}_h\Big)_{\Omega_S} + a_S(\widehat{\vec{u}}_S - \widehat{\vec{u}}_h, \vec{v}_h) + b_S(\vec{v}_h, \widehat{p}_h - \widehat{p}_S) \\ &\qquad + \langle(\widehat{\vec{u}}_S - \widehat{\vec{u}}_h)\cdot\vec{n}_S, \vec{v}_h\cdot\vec{n}_S\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}}\langle P_\tau\left(\widehat{\vec{u}}_S - \widehat{\vec{u}}_h\right), P_\tau\vec{v}_h\rangle \\ &\quad = \langle(\widehat{\vec{u}}_h - \widehat{\vec{u}}_S)\cdot\vec{n}_S - g(\widehat{\phi}_h - \widehat{\phi}_D), \vec{v}_h\cdot\vec{n}_S\rangle \quad \forall\,\vec{v}_h \in X_{Sh}.\end{aligned} \tag{6.18}$$

Taking $q = q_h \in Q_{Sh}$ in (3.6) and subtracting the result from (4.6), we have

$$b_S(\widehat{\vec{u}}_h - \widehat{\vec{u}}_S, q_h) \;=\; 0 \qquad \forall\, q_h \in Q_{Sh}. \tag{6.19}$$

Then, substituing (6.13) and (6.17) into (6.14), (6.18), and (6.19), choosing $\psi_h = \theta$, $\vec{v}_h = \vec{\theta}_1$, and $q_h = \theta_2$, and using (6.1), (6.2), and (6.3), we obtain

$$\Big(\frac{\partial\theta}{\partial t}, \theta\Big)_{\Omega_D} + a_D(\theta,\theta) = -\Big(\frac{\partial\rho}{\partial t}, \theta\Big)_{\Omega_D} + \langle(\vec{\theta}_1 + \vec{\rho}_1)\cdot\vec{n}_S, \theta\rangle$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

and

$$
\begin{aligned}
&(\frac{\partial \vec{\theta}_1}{\partial t}, \vec{\theta}_1)_{\Omega_S} + a_S(\vec{\theta}_1, \vec{\theta}_1) + b_S(\vec{\theta}_1, \theta_2) + \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\Pi)}}\langle P_\tau \vec{\theta}_1, P_\tau \vec{\theta}_1\rangle \\
&\quad = -(\frac{\partial \vec{\rho}_1}{\partial t}, \vec{\theta}_1)_{\Omega_S} - \langle g(\theta+\rho), \vec{\theta}_1 \cdot \vec{n}_S\rangle, \\
&b_S(\vec{\theta}_1, \theta_2) \;=\; 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(\frac{\partial \theta}{\partial t}, \theta)_{\Omega_D} + (\frac{\partial \vec{\theta}_1}{\partial t}, \vec{\theta}_1)_{\Omega_S} + a_D(\theta,\theta) + a_S(\vec{\theta}_1, \vec{\theta}_1) \\
&\quad = -(\frac{\partial \rho}{\partial t}, \theta)_{\Omega_D} + \langle (\vec{\theta}_1 + \vec{\rho}_1)\cdot \vec{n}_S, \theta\rangle - (\frac{\partial \vec{\rho}_1}{\partial t}, \vec{\theta}_1)_{\Omega_S} - \langle g(\theta+\rho), \vec{\theta}_1 \cdot \vec{n}_S\rangle.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{d\left\|\theta\right\|_0^2}{dt} + \frac{d\left\|\vec{\theta}_1\right\|_0^2}{dt} + 2a_D(\theta,\theta) + 2a_S(\vec{\theta}_1,\vec{\theta}_1) \\
&\leq \left\|\frac{\partial \rho}{\partial t}\right\|_0^2 + \left\|\theta\right\|_0^2 + \left\|\frac{\partial \vec{\rho}_1}{\partial t}\right\|_0^2 + \left\|\vec{\theta}_1\right\|_0^2 + \left\|\vec{\rho}_1\right\|_{0,\Gamma}^2 + g\left\|\rho\right\|_{0,\Gamma}^2 + (1+2g)\left\|\vec{\theta}_1\right\|_{0,\Gamma}^2 \\
&\quad + (2+g)\left\|\theta\right\|_{0,\Gamma}^2.
\end{aligned}
$$

Using trace theory, we have

$$
\left\|\theta\right\|_{0,\Gamma}^2 \leq C\left\|\theta\right\|_0 \left\|\theta\right\|_1 \;\leq\; \frac{C}{2}\left[\frac{1}{\varepsilon}\left\|\theta\right\|_0^2 + \varepsilon\left\|\theta\right\|_1^2\right], \tag{6.20}
$$

$$
\left\|\vec{\theta}_1\right\|_{0,\Gamma}^2 \leq C\left\|\vec{\theta}_1\right\|_0 \left\|\vec{\theta}_1\right\|_1 \;\leq\; \frac{C}{2}\left[\frac{1}{\varepsilon}\left\|\vec{\theta}_1\right\|_0^2 + \varepsilon\left\|\vec{\theta}_1\right\|_1^2\right]. \tag{6.21}
$$

Hence,

$$
\begin{aligned}
&\frac{d\left\|\theta\right\|_0^2}{dt} + \frac{d\left\|\vec{\theta}_1\right\|_0^2}{dt} + 2a_D(\theta,\theta) + 2a_S(\vec{\theta}_1,\vec{\theta}_1) \\
&\leq \left\|\frac{\partial \rho}{\partial t}\right\|_0^2 + \left\|\theta\right\|_0^2 + \left\|\frac{\partial \vec{\rho}_1}{\partial t}\right\|_0^2 + \left\|\vec{\theta}_1\right\|_0^2 + \left\|\vec{\rho}_1\right\|_1^2 + \left\|\rho\right\|_1^2 \\
&\quad + (1+2g)\frac{C}{2}\left[\frac{1}{\varepsilon}\left\|\vec{\theta}_1\right\|_0^2 + \varepsilon\left\|\vec{\theta}_1\right\|_1^2\right] + (2+g)\frac{C}{2}\left[\frac{1}{\varepsilon}\left\|\theta\right\|_0^2 + \varepsilon\left\|\theta\right\|_1^2\right].
\end{aligned} \tag{6.22}
$$

By the Korn and Poincaré inequalities, we can choose ε such that

$$
(1+2g)\frac{C}{2}\varepsilon\left\|\vec{\theta}_1\right\|_1^2 + (2+g)\frac{C}{2}\varepsilon\left\|\theta\right\|_1^2 \leq C_1\left(\left\|\vec{\theta}_1\right\|_1^2 + \left\|\theta\right\|_1^2\right) \leq a_D(\theta,\theta) + a_S(\vec{\theta}_1,\vec{\theta}_1),
$$

where $C_1 > 0$ is a constant. Hence, we have

$$
\begin{aligned}
&\frac{d\left\|\theta\right\|_0^2}{dt} + \frac{d\left\|\vec{\theta}_1\right\|_0^2}{dt} + C_1\left(\left\|\vec{\theta}_1\right\|_1^2 + \left\|\theta\right\|_1^2\right) \\
&\leq C(\left\|\theta\right\|_0^2 + \left\|\vec{\theta}_1\right\|_0^2) + \left\|\frac{\partial \rho}{\partial t}\right\|_0^2 + \left\|\frac{\partial \vec{\rho}_1}{\partial t}\right\|_0^2 + C(\left\|\vec{\rho}_1\right\|_1^2 + \left\|\rho\right\|_1^2).
\end{aligned} \tag{6.23}
$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Then,

$$\frac{d\|\theta\|_0^2}{dt}+\frac{d\left\|\vec{\theta}_1\right\|_0^2}{dt}$$

$$\le C(\|\theta\|_0^2+\left\|\vec{\theta}_1\right\|_0^2)+\left\|\frac{\partial\rho}{\partial t}\right\|_0^2+\left\|\frac{\partial\vec{\rho}_1}{\partial t}\right\|_0^2+C(\|\vec{\rho}_1\|_1^2+\|\rho\|_1^2). \tag{6.24}$$

Integrating (6.24) from 0 to t and applying Gronwall's inequaltiy, we obtain

$$\|\theta(t)\|_0^2+\left\|\vec{\theta}_1(t)\right\|_0^2\le C\Big[\|\theta(0)\|_0^2+\left\|\vec{\theta}_1(0)\right\|_0^2$$
$$+\int_0^t\left(\left\|\frac{\partial\rho(s)}{\partial t}\right\|_0^2+\left\|\frac{\partial\vec{\rho}_1}{\partial t}\right\|_0^2+C\|\vec{\rho}_1\|_1^2+C\|\rho\|_1^2\right)ds\Big].$$

Then by (6.4)–(6.10), the proof is completed. □

Remark 6.3. Higher convergence rates are possible if the projection errors (6.4)–(6.10) are of higher order. However, higher order projection errors are not generally valid assumptions unless the interface is convex with respect to the domains on both sides, i.e., the interface is a straight line for $d=2$, which is the case in our numerical experiments.

6.2. **Error of the fully discrete approximate solution.** Standard analyses of the backward Euler method for the heat and Stokes equations yield that the temporal error is $O(\triangle t)$. The first domain decomposition algorithm consists of a heat equation and a Stokes system which have been decoupled by using the Robin boundary conditions which, in turn, introduce two coupling functions ξ_D and ξ_S. Therefore, compared with the standard analysis of the backward Euler method, the additional work here is to analyze the error caused by using ξ_D^n and ξ_S^n to replace $\xi_D(t_{n+1})$ and $\xi_S(t_{n+1})$. However, because this replacement also has accuracy order of $O(\triangle t)$, we expect the same order of accuracy for the error of the full discretization.

The following theorem states that the first parallel non-iterative domain decomposition method is unconditionally stable and has optimal rates of convergence.

Theorem 6.4. *If* $\widehat{\phi}_D\in H^1(0,T;H^2(\Omega_D))\cap L^\infty(0,T;H^2(\Omega_D))\cap H^2(0,T;L^2(\Omega_D))$, $\widehat{\vec{u}}_S\in H^1(0,T;H^2(\Omega_S))\cap L^\infty(0,T;H^2(\Omega_S))\cap H^2(0,T;L^2(\Omega_S))$, $\xi_D\in H^1(0,T;L^2(\Gamma))$, *and* $\xi_S\in H^1(0,T;L^2(\Gamma))$, *then*

$$\left\|\widehat{\phi}_h^n-\widehat{\phi}_D(t_n)\right\|_0+\left\|\widehat{\vec{u}}_h^n-\widehat{\vec{u}}_S(t_n)\right\|_0$$
$$\le Ce^{CT}\triangle t\Big[\int_0^{t_n}\left\|\frac{\partial^2\widehat{\phi}_D}{\partial t^2}\right\|_0 dt+\int_0^{t_n}\left\|\frac{\partial\xi_D}{\partial t}\right\|_{0,\Gamma}dt$$
$$+\int_0^{t_n}\left\|\frac{\partial^2\widehat{\vec{u}}_S}{\partial t^2}\right\|_0 dt+\int_0^{t_n}\left\|\frac{\partial\xi_S(s)}{\partial t}\right\|_{0,\Gamma}dt\Big]$$
$$+Ce^{CT}h^2\Big[\int_0^{t_n}\left\|\frac{\partial\widehat{\phi}_D}{\partial t}\right\|_2 dt+\int_0^{t_n}\left\|\frac{\partial\widehat{\vec{u}}_S}{\partial t}\right\|_2 dt$$
$$+\max_{0\le s\le t_n}\left\|\widehat{\phi}_D(s)\right\|_2+\max_{0\le s\le t_n}\left(\left\|\widehat{\vec{u}}_S(s)\right\|_2+\|\widehat{p}_S(s)\|_1\right)\Big]. \tag{6.25}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Proof. We follow the standard energy method framework [46, 96, 99] to analyze the error of fully discrete approximations. For the Darcy part, taking $\psi = \psi_h \in X_{Dh}$ in (3.2) and subtracting (3.2) from (5.3), we have

$$\begin{aligned}&(\frac{\widehat{\phi}_h^{n+1}-\widehat{\phi}_h^n}{\triangle t}-\frac{\partial\widehat{\phi}_D(t_{n+1})}{\partial t},\psi_h)_{\Omega_D}+a_D(\widehat{\phi}_h^{n+1}-\widehat{\phi}_D(t_{n+1}),\psi_h)\\&+\langle g(\widehat{\phi}_h^{n+1}-\widehat{\phi}_D(t_{n+1})),\psi_h\rangle = \langle \xi_D^n-\xi_D(t_{n+1}),\psi_h\rangle \quad \forall\, \psi_h\in X_{Dh}.\end{aligned} \tag{6.26}$$

Define

$$\theta^n = \widehat{\phi}_h^n - R_h\widehat{\phi}_D(t_n) \qquad \text{and} \qquad \rho^n = R_h\widehat{\phi}_D(t_n) - \widehat{\phi}_D(t_n). \tag{6.27}$$

Then,

$$\widehat{\phi}_h^n - \widehat{\phi}_D(t_n) = \theta^n + \rho^n. \tag{6.28}$$

Here, ρ^n is bounded because of (6.6) and (6.7), i.e.,

$$\|\rho^n\|_0 \le Ch^2\left\|\widehat{\phi}_D(t_n)\right\|_2. \tag{6.29}$$

For the Stokes part, take $\vec{v} = \vec{v}_h \in X_{Sh}$ in (3.5) and $q = q_h \in Q_{Sh}$ in (3.6). Then, subtracting (3.5) and (3.6) from (5.4), and (5.5) separately, we obtain

$$\begin{aligned}&(\frac{\widehat{\vec{u}}_h^{n+1}-\widehat{\vec{u}}_h^n}{\triangle t}-\frac{\partial\widehat{\vec{u}}_S(t_{n+1})}{\partial t},\vec{v}_h)_{\Omega_S}+a_S(\widehat{\vec{u}}_h^{n+1}-\widehat{\vec{u}}_S(t_{n+1}),\vec{v}_h)\\&\quad+b_S(\vec{v}_h,\widehat{p}_h^{n+1}-\widehat{p}_S(t_{n+1}))+\langle\left(\widehat{\vec{u}}_h^{n+1}-\widehat{\vec{u}}_S(t_{n+1})\right)\cdot\vec{n}_S,\vec{v}_h\cdot\vec{n}_S\rangle\\&\quad+\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}}\langle P_\tau\left(\widehat{\vec{u}}_h^{n+1}-\widehat{\vec{u}}_S(t_{n+1})\right),P_\tau\vec{v}_h\rangle\\&= \langle \xi_S^n-\xi_S(t_{n+1}),\vec{v}_h\cdot\vec{n}_S\rangle \quad \forall\,\vec{v}_h\in X_{Sh},\end{aligned} \tag{6.30}$$

$$b_S(\widehat{\vec{u}}_h^{n+1}-\widehat{\vec{u}}_S(t_{n+1}),q_h)=0 \quad \forall\, q_h\in Q_{Sh}. \tag{6.31}$$

Define

$$\vec{\theta}_1^n = \widehat{\vec{u}}_h^n - \mathbb{P}_s\widehat{\vec{u}}_S(t_n), \qquad \vec{\rho}_1^{\,n} = \mathbb{P}_s\widehat{\vec{u}}_S(t_n) - \widehat{\vec{u}}_S(t_n), \tag{6.32}$$

$$\theta_2^n = \widehat{p}_h^n - \mathbb{P}_p\widehat{p}_S(t_n), \qquad \rho_2^n = \mathbb{P}_p\widehat{p}_S(t_n) - \widehat{p}_S(t_n). \tag{6.33}$$

Then,

$$\widehat{\vec{u}}_h^n - \widehat{\vec{u}}_S(t_n) = \vec{\theta}_1^n + \vec{\rho}_1^{\,n} \qquad \text{and} \qquad \widehat{p}_h^n - \widehat{p}_S(t_n) = \theta_2^n + \rho_2^n. \tag{6.34}$$

Here (6.8) and (6.9) give us the estimates

$$\|\vec{\rho}_1^{\,n}\|_0 + h\,\|\vec{\rho}_1^{\,n}\|_1 \le Ch^2(\left\|\widehat{\vec{u}}_S(t_n)\right\|_2 + \|\widehat{p}_S(t_n)\|_1), \tag{6.35}$$

$$\|\rho_2^n\|_0 \le Ch^2(\left\|\widehat{\vec{u}}_S(t_n)\right\|_2 + \|\widehat{p}_S(t_n)\|_1). \tag{6.36}$$

Also, we have the following relations for the approximations of the coupling functions. Subtracting (3.8) and (3.9) from (5.1) and (5.2) separately, we have

$$\begin{aligned}\xi_D^n - \xi_D(t_n) &= \left(\widehat{\vec{u}}_h^n\cdot\vec{n}_S + g\widehat{\phi}_h^n\right) - \left(\widehat{\vec{u}}_S(t_n)\cdot\vec{n}_S + g\widehat{\phi}_D(t_n)\right)\\ &= \left(\vec{\theta}_1^n + \vec{\rho}_1^{\,n}\right)\cdot\vec{n}_S + g\left(\theta^n+\rho^n\right)\end{aligned} \tag{6.37}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

and

$$\begin{aligned} \xi_S^n - \xi_S(t_n) &= \left(\widehat{\vec{u}}_h^n \cdot \vec{n}_S - g\widehat{\phi}_h^n + gz\right) - \left(\widehat{\vec{u}}_S(t_n) \cdot \vec{n}_S - g\widehat{\phi}_D(t_n) + gz\right) \\ &= \left(\vec{\theta}_1^n + \vec{\rho}_1^{\,n}\right) \cdot \vec{n}_S - g\left(\theta^n + \rho^n\right). \end{aligned} \tag{6.38}$$

Now we begin to analyze the Darcy part of the Stokes-Darcy system. Define

$$w_1^{n+1} = R_h\left(\frac{\widehat{\phi}_D(t_{n+1}) - \widehat{\phi}_D(t_n)}{\triangle t}\right) - \frac{\partial \widehat{\phi}_D(t_{n+1})}{\partial t}, \tag{6.39}$$

$$w_2^{n+1} = \xi_D(t_{n+1}) - \xi_D(t_n). \tag{6.40}$$

By (6.1) and (6.27), we have

$$a_D(\rho^{n+1}, \psi_h) = 0 \quad \forall\, \psi_h \in X_{Dh}. \tag{6.41}$$

Choose $\psi_h = \theta^{n+1}$. Then, substituting (6.28), (6.37), (6.39), (6.40), and (6.41) into (6.26), we obtain

$$\begin{aligned} &(\frac{\theta^{n+1} - \theta^n}{\triangle t}, \theta^{n+1})_{\Omega_D} + a_D(\theta^{n+1}, \theta^{n+1}) + \langle g\theta^{n+1}, \theta^{n+1}\rangle \\ &= -(w_1^{n+1}, \theta^{n+1})_{\Omega_D} - \langle w_2^{n+1}, \theta^{n+1}\rangle - \langle g\rho^{n+1}, \theta^{n+1}\rangle \\ &\quad + \langle \vec{\rho}_1^{\,n} \cdot \vec{n}_S + g\rho^n, \theta^{n+1}\rangle + \langle \vec{\theta}_1^n \cdot \vec{n}_S + g\theta^n, \theta^{n+1}\rangle. \end{aligned} \tag{6.42}$$

Now we turn to the analysis of the Stokes part. Define

$$\vec{w}_3^{n+1} = \mathbb{P}_s\left(\frac{\widehat{\vec{u}}_S(t_{n+1}) - \widehat{\vec{u}}_S(t_n)}{\triangle t}\right) - \frac{\partial \widehat{\vec{u}}_S(t_{n+1})}{\partial t}, \tag{6.43}$$

$$w_4^{n+1} = \xi_S(t_{n+1}) - \xi_S(t_n). \tag{6.44}$$

By (6.2) and (6.3), we have

$$\begin{aligned} &a_S(\vec{\rho}_1^{\,n+1}, \vec{v}_h) + b_S(\vec{v}_h, \rho_2^{n+1}) + \langle \vec{\rho}_1^{\,n+1} \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S\rangle \\ &\quad + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}}\langle P_\tau \vec{\rho}_1^{\,n+1}, P_\tau \vec{v}_h\rangle = 0 \quad \forall\, \vec{v}_h \in X_{Sh}, \end{aligned} \tag{6.45}$$

$$b_S(\vec{\rho}_1^{\,n+1}, q_h) = 0 \quad \forall\, q_h \in Q_{Sh}. \tag{6.46}$$

Choose $\vec{v}_h = \vec{\theta}_1^{n+1}$ and $q_h = \theta_2^{n+1}$. Then, substituting (6.34), (6.43), (6.44), (6.45), and (6.46) into (6.30) and (6.31), we obtain

$$\begin{aligned} &(\frac{\vec{\theta}_1^{n+1} - \vec{\theta}_1^n}{\triangle t}, \vec{\theta}_1^{n+1})_{\Omega_S} + a_S(\vec{\theta}_1^{n+1}, \vec{\theta}_1^{n+1}) + b_S(\vec{\theta}_1^{n+1}, \theta_2^{n+1}) \\ &\quad + \langle \vec{\theta}_1^{n+1} \cdot \vec{n}_S, \vec{\theta}_1^{n+1} \cdot \vec{n}_S\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}}\langle P_\tau \vec{\theta}_1^{n+1}, P_\tau \vec{\theta}_1^{n+1}\rangle \\ &= -(\vec{w}_3^{n+1}, \vec{\theta}_1^{n+1})_{\Omega_S} - \langle w_4^{n+1}, \vec{\theta}_1^{n+1} \cdot \vec{n}_S\rangle - \langle \vec{\rho}_1^{\,n+1} \cdot \vec{n}_S, \vec{\theta}_1^{n+1} \cdot \vec{n}_S\rangle \\ &\quad + \langle \vec{\rho}_1^{\,n} \cdot \vec{n}_S - g\rho^n, \vec{\theta}_1^{n+1} \cdot \vec{n}_S\rangle + \langle \vec{\theta}_1^n \cdot \vec{n}_S - g\theta^n, \vec{\theta}_1^{n+1} \cdot \vec{n}_S\rangle \\ &b_S(\vec{\theta}_1^{n+1}, \theta_2^{n+1}) = 0. \end{aligned}$$

Hence,

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$
\begin{aligned}
&(\frac{\vec{\theta}_1^{n+1}-\vec{\theta}_1^n}{\triangle t},\vec{\theta}_1^{n+1})_{\Omega_S}+a_S(\vec{\theta}_1^{n+1},\vec{\theta}_1^{n+1})+\langle\vec{\theta}_1^{n+1}\cdot\vec{n}_S,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle\\
&\le-(\vec{w}_3^{n+1},\vec{\theta}_1^{n+1})_{\Omega_S}-\langle w_4^{n+1},\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle-\langle\vec{\rho}_1^{\,n+1}\cdot\vec{n}_S,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle\\
&\quad+\langle\vec{\rho}_1^{\,n}\cdot\vec{n}_S-g\rho^n,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle+\langle\vec{\theta}_1^n\cdot\vec{n}_S-g\theta^n,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle.
\end{aligned}\tag{6.47}
$$

Adding (6.47) to (6.42), we have

$$
\begin{aligned}
&(\frac{\theta^{n+1}-\theta^n}{\triangle t},\theta^{n+1})_{\Omega_D}+(\frac{\vec{\theta}_1^{n+1}-\vec{\theta}_1^n}{\triangle t},\vec{\theta}_1^{n+1})_{\Omega_S}+a_D(\theta^{n+1},\theta^{n+1})\\
&\quad+a_S(\vec{\theta}_1^{n+1},\vec{\theta}_1^{n+1})+\langle\vec{\theta}_1^{n+1}\cdot\vec{n}_S,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle+\langle g\theta^{n+1},\theta^{n+1}\rangle\\
&\le-(w_1^{n+1},\theta^{n+1})_{\Omega_D}-\langle w_2^{n+1},\theta^{n+1}\rangle-(\vec{w}_3^{n+1},\vec{\theta}_1^{n+1})_{\Omega_S}-\langle w_4^{n+1},\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle\\
&\quad-\langle g\rho^{\,n+1},\theta^{n+1}\rangle-\langle\vec{\rho}_1^{n+1}\cdot\vec{n}_S,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle\\
&\quad+\langle\vec{\rho}_1 n\cdot\vec{n}_S+g\rho^n,\theta^{n+1}\rangle+\langle\vec{\rho}_1^{\,n}\cdot\vec{n}_S-g\rho^n,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle\\
&\quad+\langle\vec{\theta}_1^n\cdot\vec{n}_S+g\theta^n,\theta^{n+1}\rangle+\langle\vec{\theta}_1^n\cdot\vec{n}_S-g\theta^n,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle.
\end{aligned}\tag{6.48}
$$

Using the Schwarz and Young inequalities and trace theory, we have

$$
\begin{aligned}
&\left\|\theta^{n+1}\right\|_0^2+\left\|\vec{\theta}_1^{n+1}\right\|_0^2+\triangle t a_D(\theta^{n+1},\theta^{n+1})+\triangle t a_S(\vec{\theta}_1^{n+1},\vec{\theta}_1^{n+1})\\
&\quad+g\triangle t\left\|\theta^{n+1}\right\|_{0,\Gamma}^2+\triangle t\left\|\vec{\theta}_1^{n+1}\cdot\vec{n}_S\right\|_{0,\Gamma}^2\\
&\le\frac{1}{2}\left\|\theta^n\right\|_0^2+\frac{1}{2}\left\|\theta^{n+1}\right\|_0^2+\frac{1}{2}\left\|\vec{\theta}_1^n\right\|_0^2+\frac{1}{2}\left\|\vec{\theta}_1^{n+1}\right\|_0^2+\frac{\triangle t}{2}\Big[\frac{1}{\varepsilon_1}\left\|w_1^{n+1}\right\|_0^2\\
&\quad+\varepsilon_1\left\|\theta^{n+1}\right\|_0^2+\frac{1}{\varepsilon_1}\left\|\vec{w}_3^{n+1}\right\|_0^2+\varepsilon_1\left\|\vec{\theta}_1^{n+1}\right\|_0^2\Big]+C\triangle t\Big[\frac{1}{\varepsilon_2}\left\|w_2^{n+1}\right\|_{0,\Gamma}^2\\
&\quad+\varepsilon_2\left\|\theta^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_2}\left\|w_4^{n+1}\right\|_{0,\Gamma}^2+\varepsilon_2\left\|\vec{\theta}_1^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_2}\left\|\rho^{n+1}\right\|_{0,\Gamma}^2\\
&\quad+\varepsilon_2\left\|\theta^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_2}\left\|\vec{\rho}_1^{\,n+1}\right\|_{0,\Gamma}^2+\varepsilon_2\left\|\vec{\theta}_1^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_2}\left\|\vec{\rho}_1^{\,n}\right\|_{0,\Gamma}^2\\
&\quad+\varepsilon_2\left\|\theta^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_2}\left\|\rho^n\right\|_{0,\Gamma}^2+\varepsilon_2\left\|\theta^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_2}\left\|\vec{\rho}_1^{\,n}\right\|_{0,\Gamma}^2\\
&\quad+\varepsilon_2\left\|\vec{\theta}_1^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_2}\left\|\rho^n\right\|_{0,\Gamma}^2+\varepsilon_2\left\|\vec{\theta}_1^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_3}\left\|\vec{\theta}_1^n\right\|_{0,\Gamma}^2\\
&\quad+\varepsilon_3\left\|\theta^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_3}\left\|\theta^n\right\|_{0,\Gamma}^2+\varepsilon_3\left\|\theta^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_3}\left\|\vec{\theta}_1^n\right\|_{0,\Gamma}^2\\
&\quad+\varepsilon_3\left\|\vec{\theta}_1^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_3}\left\|\theta^n\right\|_{0,\Gamma}^2+\varepsilon_3\left\|\vec{\theta}_1^{n+1}\right\|_{0,\Gamma}^2\Big]\\
&\le\frac{1}{2}\left\|\theta^n\right\|_0^2+\frac{1}{2}\left\|\theta^{n+1}\right\|_0^2+\frac{1}{2}\left\|\vec{\theta}_1^n\right\|_0^2+\frac{1}{2}\left\|\vec{\theta}_1^{n+1}\right\|_0^2+\frac{\triangle t}{2}\Big[\frac{1}{\varepsilon_1}\left\|w_1^{n+1}\right\|_0^2\\
&\quad+\varepsilon_1\left\|\theta^{n+1}\right\|_0^2+\frac{1}{\varepsilon_1}\left\|\vec{w}_3^{n+1}\right\|_0^2+\varepsilon_1\left\|\vec{\theta}_1^{n+1}\right\|_0^2\Big]+\frac{C\triangle t}{\varepsilon_2}\Big[\left\|w_2^{n+1}\right\|_{0,\Gamma}^2\\
&\quad+\left\|w_4^{n+1}\right\|_{0,\Gamma}^2+\left\|\vec{\rho}_1^{\,n}\right\|_{0,\Gamma}^2+\left\|\rho^n\right\|_{0,\Gamma}^2+\left\|\rho^{n+1}\right\|_{0,\Gamma}^2+\left\|\vec{\rho}_1^{\,n+1}\right\|_{0,\Gamma}^2\Big]\\
&\quad+C\varepsilon_2\triangle t\Big[\left\|\theta^{n+1}\right\|_{0,\Gamma}^2+\left\|\vec{\theta}_1^{n+1}\right\|_{0,\Gamma}^2\Big]+C\triangle t\Big[\varepsilon_3\left\|\theta^{n+1}\right\|_{0,\Gamma}^2\\
&\quad+\varepsilon_3\left\|\vec{\theta}_1^{n+1}\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_3}\left\|\vec{\theta}_1^n\right\|_{0,\Gamma}^2+\frac{1}{\varepsilon_3}\left\|\theta^n\right\|_{0,\Gamma}^2\Big].
\end{aligned}\tag{6.49}
$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

By the coercivity of $a_D(\cdot,\cdot)$ and $a_S(\cdot,\cdot)$ and the Korn and Poincaré inequalities, we can choose ε_1, ε_2, and ε_3 such that

$$\frac{\varepsilon_1}{2}\left\|\theta^{n+1}\right\|_0^2 + C(\varepsilon_2+\varepsilon_3)\left\|\theta^{n+1}\right\|_{0,\Gamma}^2 \;\le\; \frac{1}{2}a_D(\theta^{n+1},\theta^{n+1}) + g\left\|\theta^{n+1}\right\|_{0,\Gamma}^2 \tag{6.50}$$

and

$$\frac{\varepsilon_1}{2}\left\|\vec{\theta}_1^{n+1}\right\|_0^2 + C(\varepsilon_2+\varepsilon_3)\left\|\vec{\theta}_1^{n+1}\right\|_{0,\Gamma}^2 \le \frac{1}{2}a_S(\vec{\theta}_1^{n+1},\vec{\theta}_1^{n+1}) + \left\|\vec{\theta}_1^{n+1}\cdot\vec{n}_S\right\|_{0,\Gamma}^2. \tag{6.51}$$

Also, by trace theory, we have

$$\|\theta^n\|_{0,\Gamma}^2 \;\le\; C\|\theta^n\|_0\|\theta^n\|_1 \;\le\; \frac{C}{2}\left[\frac{1}{\varepsilon_4}\|\theta^n\|_0^2 + \varepsilon_4\|\theta^n\|_1^2\right], \tag{6.52}$$

$$\left\|\vec{\theta}_1^n\right\|_{0,\Gamma}^2 \;\le\; C\left\|\vec{\theta}_1^n\right\|_0\left\|\vec{\theta}_1^n\right\|_1 \;\le\; \frac{C}{2}\left[\frac{1}{\varepsilon_4}\left\|\vec{\theta}_1^n\right\|_0^2 + \varepsilon_4\left\|\vec{\theta}_1^n\right\|_1^2\right]. \tag{6.53}$$

Then, substituting (6.50)–(6.53) into (6.49), we have

$$\begin{aligned}
&\frac{1}{2}\left\|\theta^{n+1}\right\|_0^2 + \frac{1}{2}\left\|\vec{\theta}_1^{n+1}\right\|_0^2 + \frac{\triangle t}{2}a_D(\theta^{n+1},\theta^{n+1}) + \frac{\triangle t}{2}a_S(\vec{\theta}_1^{n+1},\vec{\theta}_1^{n+1})\\
&\quad\le \frac{1}{2}\|\theta^n\|_0^2 + \frac{1}{2}\left\|\vec{\theta}_1^n\right\|_0^2 + \frac{\triangle t}{2\varepsilon_1}\left\|w_1^{n+1}\right\|_0^2 + \frac{\triangle t}{2\varepsilon_1}\left\|\vec{w}_3^{n+1}\right\|_0^2\\
&\qquad + \frac{C}{\varepsilon_2}\triangle t\Big[\left\|w_2^{n+1}\right\|_{0,\Gamma}^2 + \left\|w_4^{n+1}\right\|_{0,\Gamma}^2 + \left\|\vec{\rho}_1^{\,n}\right\|_{0,\Gamma}^2 + \|\rho^n\|_{0,\Gamma}^2\\
&\qquad + \left\|\rho^{n+1}\right\|_{0,\Gamma}^2 + \left\|\vec{\rho}_1^{\,n+1}\right\|_{0,\Gamma}^2\Big] + C\triangle t\left[\frac{1}{\varepsilon_3\varepsilon_4}\left\|\vec{\theta}_1^n\right\|_0^2 + \frac{1}{\varepsilon_3\varepsilon_4}\|\theta^n\|_0^2\right]\\
&\qquad + C\triangle t\left[\frac{\varepsilon_4}{\varepsilon_3}\left\|\vec{\theta}_1^n\right\|_1^2 + \frac{\varepsilon_4}{\varepsilon_3}\|\theta^n\|_1^2\right].
\end{aligned} \tag{6.54}$$

Based on the coercivity of $a_D(\cdot,\cdot)$ and $a_S(\cdot,\cdot)$, we may choose ε_4 such that

$$\frac{C\varepsilon_4}{\varepsilon_3}\left\|\theta^{n+1}\right\|_1^2 \;\le\; \frac{1}{2}a_D(\theta^{n+1},\theta^{n+1}), \tag{6.55}$$

$$\frac{C\varepsilon_4}{\varepsilon_3}\left\|\vec{\theta}_1^{n+1}\right\|_1^2 \;\le\; \frac{1}{2}a_S(\vec{\theta}_1^{n+1},\vec{\theta}_1^{n+1}). \tag{6.56}$$

Then,

$$\begin{aligned}
&\left\|\theta^{n+1}\right\|_0^2 + \left\|\vec{\theta}_1^{n+1}\right\|_0^2 + \frac{2C\varepsilon_4\triangle t}{\varepsilon_3}\left\|\theta^{n+1}\right\|_1^2 + \frac{2C\varepsilon_4\triangle t}{\varepsilon_3}\left\|\vec{\theta}_1^{n+1}\right\|_1^2\\
&\quad\le \|\theta^n\|_0^2 + \left\|\vec{\theta}_1^n\right\|_0^2 + \frac{\triangle t}{\varepsilon_1}\left\|w_1^{n+1}\right\|_0^2 + \frac{\triangle t}{\varepsilon_1}\left\|\vec{w}_3^{n+1}\right\|_0^2\\
&\qquad + \frac{2C}{\varepsilon_2}\triangle t\Big[\left\|w_2^{n+1}\right\|_{0,\Gamma}^2 + \left\|w_4^{n+1}\right\|_{0,\Gamma}^2 + 2\left\|\vec{\rho}_1^{\,n}\right\|_{0,\Gamma}^2 + 2\|\rho^n\|_{0,\Gamma}^2\\
&\qquad + \left\|\rho^{n+1}\right\|_{0,\Gamma}^2 + \left\|\vec{\rho}_1^{\,n+1}\right\|_{0,\Gamma}^2\Big] + 2C\triangle t\left[\frac{1}{\varepsilon_3\varepsilon_4}\left\|\vec{\theta}_1^n\right\|_0^2 + \frac{1}{\varepsilon_3\varepsilon_4}\|\theta^n\|_0^2\right]\\
&\qquad + 2C\triangle t\left[\frac{\varepsilon_4}{\varepsilon_3}\left\|\vec{\theta}_1^n\right\|_1^2 + \frac{\varepsilon_4}{\varepsilon_3}\|\theta^n\|_1^2\right].
\end{aligned} \tag{6.57}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Let $\varepsilon = \frac{2C\varepsilon_4}{\varepsilon_3}$. Then,

$$\begin{aligned}
&\|\theta^{n+1}\|_0^2 + \left\|\vec{\theta}_1^{n+1}\right\|_0^2 + \varepsilon \triangle t \|\theta^{n+1}\|_1^2 + \varepsilon \triangle t \left\|\vec{\theta}_1^{n+1}\right\|_1^2 \\
&\le (1 + C \triangle t) \left[\|\theta^n\|_0^2 + \left\|\vec{\theta}_1^n\right\|_0^2 + \varepsilon \triangle t \left\|\vec{\theta}_1^n\right\|_1^2 + \varepsilon \triangle t \|\theta^n\|_1^2 \right] \\
&\quad + C \triangle t \Big[\|w_1^{n+1}\|_0^2 + \|w_2^{n+1}\|_{0,\Gamma}^2 + \|\vec{w}_3^{n+1}\|_0^2 + \|w_4^{n+1}\|_{0,\Gamma}^2 \\
&\quad + \|\vec{\rho}_1^{\,n}\|_{0,\Gamma}^2 + \|\rho^n\|_{0,\Gamma}^2 + \|\rho^{n+1}\|_{0,\Gamma}^2 + \|\vec{\rho}_1^{\,n+1}\|_{0,\Gamma}^2 \Big].
\end{aligned}$$

Straightforward manipulation leads to

$$\begin{aligned}
&\|\theta^{n+1}\|_0^2 + \left\|\vec{\theta}_1^{n+1}\right\|_0^2 + \varepsilon \triangle t \|\theta^{n+1}\|_1^2 + \varepsilon \triangle t \left\|\vec{\theta}_1^{n+1}\right\|_1^2 \\
&\le (1 + C \triangle t)^{n+1} (\|\theta^0\|_0^2 + \left\|\vec{\theta}_1^{\,0}\right\|_0^2 + \varepsilon \triangle t \left\|\vec{\theta}_1^{\,0}\right\|_1^2 + \varepsilon \triangle t \|\theta^0\|_1^2) \\
&\quad + C \triangle t \sum_{j=0}^{n} (1 + C \triangle t)^j \Big[\left\|w_1^{j+1}\right\|_0^2 + \left\|w_2^{j+1}\right\|_{0,\Gamma}^2 + \left\|\vec{w}_3^{j+1}\right\|_0^2 + \left\|w_4^{j+1}\right\|_{0,\Gamma}^2 \\
&\quad + \left\|\vec{\rho}_1^{\,j}\right\|_{0,\Gamma}^2 + \left\|\vec{\rho}_1^{\,j+1}\right\|_{0,\Gamma}^2 + \|\rho^j\|_{0,\Gamma}^2 + \|\rho^{j+1}\|_{0,\Gamma}^2 \Big] \\
&\le C e^{CT} \Big[\|\theta^0\|_0^2 + \left\|\vec{\theta}_1^{\,0}\right\|_0^2 + \varepsilon \triangle t \left\|\vec{\theta}_1^{\,0}\right\|_1^2 + \varepsilon \triangle t \|\theta^0\|_1^2 \\
&\quad + \triangle t \sum_{j=0}^{n} \Big(\left\|w_1^{j+1}\right\|_0^2 + \left\|w_2^{j+1}\right\|_{0,\Gamma}^2 + \left\|\vec{w}_3^{j+1}\right\|_0^2 + \left\|w_4^{j+1}\right\|_{0,\Gamma}^2 \\
&\quad + \left\|\vec{\rho}_1^{\,j}\right\|_{0,\Gamma}^2 + \left\|\vec{\rho}_1^{\,j+1}\right\|_{0,\Gamma}^2 + \|\rho^j\|_{0,\Gamma}^2 + \|\rho^{j+1}\|_{0,\Gamma}^2 \Big]. \qquad (6.58)
\end{aligned}$$

Then by (6.4)–(6.10), we complete the proof of (6.25). □

Remark 6.5. The constant C in Theorems 6.2 and 6.4 is independent of the parameter α in the BJSJ interface condition but depends on ν and $\|\mathbb{K}\|$ and therefore, it depends on (see Remark 2.1) on ν and $\|\Pi\|$. It can be shown that this constant grows with decreasing ν or $\|\Pi\|$.

7. The second parallel non-iterative domain decomposition method

We now discuss a multi-step method to improve the accuracy in time descretization. Assume the spatial error is of $O(h^3)$. Then we may apply a three-step backward differentiation method that results in $O(h^3 + \triangle t^3)$ errors. The method is complicated by the need to use one- and two-step methods to set up the three-step method in such a way that accuracy is not lost. Again, as in Section 5, we use a uniform partition of the interval $[0, T]$, although, again, this is not essential to the algorithm, applying the algorithm to non-uniform partitions now requires changes in the coefficients of the two- and three-step methods to avoid loss of accuracy. The algorithm proceeds as follows.

Algorithm 2 – Second non-iterative domain decomposition method. Set $\widehat{\phi}_h^0 = P_h \phi_0$ and $\widehat{\vec{u}}_h^{\,0} = \mathbb{P}_h \vec{u}_0$. Then,
1. set

$$\xi_{Dh}^0 = \widehat{\vec{u}}_h^{\,0} \cdot \vec{n}_S + g \widehat{\phi}_h^0 \qquad \text{and} \qquad \xi_{Sh}^0 = \widehat{\vec{u}}_h^{\,0} \cdot \vec{n}_S - g \widehat{\phi}_h^0 + gz \qquad \text{on } \Gamma$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

and independently solve

$$(\frac{\widetilde{\phi}_h^1 - \widehat{\phi}_h^0}{\triangle t}, \psi_h)_{\Omega_D} + a_D(\widetilde{\phi}_h^1, \psi_h) + \langle g\widetilde{\phi}_h^1, \psi_h\rangle = (f_D^1, \psi_h)_{\Omega_D} + \langle \xi_{Dh}^0, \psi_h\rangle \quad \forall\, \psi_h \in X_{Dh}$$

for $\widetilde{\phi}_h^1$ and

$$(\frac{\widetilde{\vec{u}}_h^1 - \widehat{\vec{u}}_h^0}{\triangle t}, \vec{v}_h)_{\Omega_S} + a_S(\widetilde{\vec{u}}_h^1, \vec{v}_h) + b_S(\vec{v}_h, \widetilde{p}_h^1) + \langle \widetilde{\vec{u}}_h^1 \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\Pi)}}\langle P_\tau \widetilde{\vec{u}}_h^1, P_\tau \vec{v}_h\rangle = (\vec{f}_S^{\,1}, \vec{v}_h)_{\Omega_S} + \langle \widetilde{\xi}_{Sh}^0, \vec{v}_h \cdot \vec{n}_S\rangle \quad \forall\, \vec{v}_h \in X_{Sh}$$

$$b_S(\widetilde{\vec{u}}_h^1, q_h) = 0 \quad \forall\, q_h \in Q_{Sh}$$

for $\widetilde{\vec{u}}_h^1$ and $\widetilde{p}_h^1$;

2. set

$$\widetilde{\xi}_{Dh}^1 = \widetilde{\vec{u}}_h^1 \cdot \vec{n}_S + g\widetilde{\phi}_h^1 \qquad \text{and} \qquad \widetilde{\xi}_{Sh}^1 = \widetilde{\vec{u}}_h^1 \cdot \vec{n}_S - g\widetilde{\phi}_h^1 + gz \qquad \text{on } \Gamma$$

and independently solve

$$(\frac{\widehat{\phi}_h^1 - \widehat{\phi}_h^0}{\triangle t}, \psi_h)_{\Omega_D} + a_D(\frac{\widehat{\phi}_h^1 + \widehat{\phi}_h^0}{2}, \psi_h) + \langle g\frac{\widehat{\phi}_h^1 + \widehat{\phi}_h^0}{2}, \psi_h\rangle = (\frac{f_D^1 + f_D^0}{2}, \psi_h)_{\Omega_D} + \langle \frac{\widetilde{\xi}_{Dh}^1 + \xi_{Dh}^0}{2}, \psi_h\rangle \quad \forall\, \psi_h \in X_{Dh}$$

for $\widehat{\phi}_h^1$ and

$$(\frac{\widehat{\vec{u}}_h^1 - \widehat{\vec{u}}_h^0}{\triangle t}, \vec{v}_h)_{\Omega_S} + a_S(\frac{\widehat{\vec{u}}_h^1 + \widehat{\vec{u}}_h^0}{2}, \vec{v}_h) + b_S(\vec{v}_h, \frac{\widehat{p}_h^1 + \widehat{p}_h^0}{2}) + \langle \frac{\widehat{\vec{u}}_h^1 + \widehat{\vec{u}}_h^0}{2} \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\Pi)}}\langle P_\tau \frac{\widehat{\vec{u}}_h^1 + \widehat{\vec{u}}_h^0}{2}, P_\tau \vec{v}_h\rangle = (\frac{\vec{f}_S^{\,1} + \vec{f}_S^{\,0}}{2}, \vec{v}_h)_{\Omega_S} + \langle \frac{\widetilde{\xi}_{Sh}^1 + \xi_{Sh}^0}{2}, \vec{v}_h \cdot \vec{n}_S\rangle \quad \forall\, \vec{v}_h \in X_{Sh}$$

$$b_S(\frac{\widehat{\vec{u}}_h^1 + \widehat{\vec{u}}_h^0}{2}, q_h) = 0 \quad \forall\, q_h \in Q_{Sh}$$

for $\widehat{\vec{u}}_h^1$ and $\widehat{p}_h^1$;

3. set

$$\xi_{Dh}^1 = \widehat{\vec{u}}_h^1 \cdot \vec{n}_S + g\widehat{\phi}_h^1 \qquad \text{and} \qquad \xi_{Sh}^1 = \widehat{\vec{u}}_h^1 \cdot \vec{n}_S - g\widehat{\phi}_h^1 + gz \qquad \text{on } \Gamma$$

and

$$\widetilde{\xi}_{Dh}^2 = 2\xi_{Dh}^1 - \xi_{Dh}^0 \qquad \text{and} \qquad \widetilde{\xi}_{Sh}^2 = 2\xi_{Sh}^1 - \xi_{Sh}^0 \qquad \text{on } \Gamma$$

and independently solve

$$(\frac{3\widehat{\phi}_h^2 - 4\widehat{\phi}_h^1 + \widehat{\phi}_h^0}{2 \triangle t}, \psi_h)_{\Omega_D} + a_D(\widehat{\phi}_h^2, \psi_h) + \langle g\widehat{\phi}_h^2, \psi_h\rangle = (f_D^2, \psi_h)_{\Omega_D} + \langle \widetilde{\xi}_{Dh}^2, \psi_h\rangle \quad \forall\, \psi_h \in X_{Dh}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

for $\widehat{\phi}_h^2$

$$(\frac{3\widehat{\vec{u}}_h^2 - 4\widehat{\vec{u}}_h^1 + \widehat{\vec{u}}_h^0}{2\triangle t}, \vec{v}_h)_{\Omega_S} + a_S(\widehat{\vec{u}}_h^2, \vec{v}_h) + b_S(\vec{v}_h, \widehat{p}_h^2)$$
$$+\langle \widehat{\vec{u}}_h^2 \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}}\langle P_\tau(\widehat{\vec{u}}_h^2, P_\tau \vec{v}_h\rangle$$
$$= (\vec{f}_S^2, \vec{v}_h)_{\Omega_S} + \langle \widehat{\xi}_{Sh}^2, \vec{v}_h \cdot \vec{n}_S\rangle \quad \forall\, \vec{v}_h \in X_{Sh}$$
$$b_S(\widehat{\vec{u}}_h^2, q_h) = 0 \quad \forall\, q_h \in Q_{Sh}$$

for $\widehat{\vec{u}}_h^2$ and $\widehat{p}_h^2$;

4. for $n = 2, \ldots, N-1$, set

$$\xi_{Dh}^n = \widehat{\vec{u}}_h^n \cdot \vec{n}_S + g\widehat{\phi}_h^n \qquad \text{and} \qquad \xi_{Sh}^n = \widehat{\vec{u}}_h^n \cdot \vec{n}_S - g\widehat{\phi}_h^n + gz \qquad \text{on } \Gamma$$

and then independently solve

$$(\frac{11\widehat{\phi}_h^{n+1} - 18\widehat{\phi}_h^n + 9\phi_h^{n-1} - 2\widehat{\phi}_h^{n-2}}{6\triangle t}, \psi_h)_{\Omega_D} + a_D(\widehat{\phi}_h^{n+1}, \psi_h) + \langle g\widehat{\phi}_h^{n+1}, \psi_h\rangle$$
$$= (f_D^{n+1}, \psi_h)_{\Omega_D} + \langle \xi_{Dh}^n, \psi_h\rangle \quad \forall\, \psi_h \in X_{Dh}$$

for $\widehat{\phi}_h^{n+1}$ and

$$(\frac{11\widehat{\vec{u}}_h^{n+1} - 18\widehat{\vec{u}}_h^n + 9\widehat{\vec{u}}_h^{n-1} - 2\widehat{\vec{u}}_h^{n-2}}{6\triangle t}, \vec{v}_h)_{\Omega_S} + a_S(\widehat{\vec{u}}_h^{n+1}, \vec{v}_h)$$
$$+b_S(\vec{v}_h, \widehat{p}_h^{n+1}) + \langle \widehat{\vec{u}}_h^{n+1} \cdot \vec{n}_S, \vec{v}_h \cdot \vec{n}_S\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}}\langle P_\tau(\widehat{\vec{u}}_h^{n+1}, P_\tau \vec{v}_h\rangle$$
$$= (\vec{f}_S^{n+1}, \vec{v}_h)_{\Omega_S} + \langle \xi_S^{n+1}, \vec{v}_h \cdot \vec{n}_S\rangle \quad \forall\, \vec{v}_h \in X_{Sh}$$
$$b_S(\widehat{\vec{u}}_h^{n+1}, q_h) = 0 \quad \forall\, q_h \in Q_{Sh}$$

for $\widehat{\vec{u}}_h^{n+1}$ and $\widehat{p}_h^{n+1}$.

The three-step backward differentiation method requires information from the previous three time levels so that it cannot be applied until one has determined an approximate solution for the first three time levels, i.e., for $n = 0, 1, 2$. Of course, for $n = 0$, that approximate solution is determined from the initial conditions. Steps 1 and 2 implement a backward-Euler predictor/Crank-Nicholson corrector method to determine the approximate solution for $n = 1$ whereas Step 3 implements a two-step backward differentiation method to determine the approximate solution for $n = 2$. After that, in Step 4, the three-step backward differentiation is implemented. In all steps, the coupling functions ξ_{Dh} and ξ_{Sh} are treated in an explicit manner so that, in all steps, the discretized Darcy and Stokes systems need to be solved only once and can be solved in parallel.

The global error of three-step backward differentiation method, the local error of two-step backward differentiation method, and the local error of Crank-Nicolson scheme are all of $O(\triangle t^3)$ [6, 62, 96]. We also use piecewise quadratic polynomial spatial approximations for ϕ and $\vec{u}$. Thus, one has reason to expect that the global accuracy of the approximations to ϕ and $\vec{u}$ obtained using the second parallel non-iterative domain decomposition method are of $O(h^3 + \triangle t^3)$; this is numerically

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

verified in Section 8. Note that to obtain this optimal accuracy, the approximations of the coupling functions ξ_D and ξ_S were also carefully defined in Algorithm 2.

At each time step, the corresponding linear algebraic systems appearing in Algorithms 1 and 2 are the same size. The two methods require information from the previous steps 1 and 3, respectively, so that Algorithm 2 requires greater storage and incurs greater costs for matrix assembly and for start up. However, because of the much higher temporal accuracy of Algorithm 2, these extra costs are more than made up by the fact that, for the same accuracy, one can take much larger time steps. Of course, because both methods are non-iterative domain decomposition methods, in both cases only single uncoupled Stokes and Darcy problems need be solved at each time step.

8. Numerical examples

We use a manufactured solution to illustrate the accuracy and stability of the two non-iterative domain decomposition algorithms introduced in Sections 5 and 7.

Let $\Omega = [0,\pi]\times[-1,1]$ with $\Omega_D = [0,\pi]\times[0,1]$ and $\Omega_S = [0,\pi]\times[-1,0]$. Set $T=1$, $\alpha=1$, $\nu=1$, $g=1$, $z=0$, and $\mathbb{K}=\mathbb{I}$, where $\mathbb{I}$ denotes the identity tensor. We subdivide Ω_D and Ω_S into rectangles of height $h=1/M$ and width πh, where M denotes a positive integer, and then subdivide each rectangle into two triangles by drawing a diagonal. Clearly, faces of the triangles in Ω_D and Ω_S are aligned with and match at the interface $\Gamma = \{y=0, 0\le x\le\pi\}$. For both non-iterative domain decomposition methods, we effect spatial discretization using the Taylor-Hood element pair for the Stokes equation, i.e., continuous piecewise linear and quadratic finite element spaces for the approximation of p_S and $\vec{u}_S$, respectively, and for the Darcy equation, continuous piecewise quadratic finite element spaces for the approximation of ϕ_D.

The initial condition data, boundary condition data, and source terms are chosen to correspond to the exact solution[3]

$$\begin{cases} \phi_D = (e^y - e^{-y})\sin(x)e^t, \\ \vec{u}_S = [\frac{1}{\pi^2}\sin(2\pi y)\cos(x)e^t,\ (-2+\frac{1}{\pi^2}\sin^2(\pi y))\sin(x)e^t]^T, \\ p_S = 0, \end{cases}$$

All numerical results given below are for $t=T=1$, i.e., for $n=N$.

Errors are measured using high-accuracy quadrature rules to approximate $L^2(\Omega_D)$, $[L^2(\Omega_S)]^2$, and $L^2(\Omega_S)$ norms and $H^1(\Omega_D)$ and $[H^1(\Omega_S)]^2$ semi-norms. We also abbreviate notation so that $\vec{u}$ represents $\vec{u}_S|_{t=1}$ and $\vec{u}_h$ represents $\widehat{\vec{u}}_h^N$ and similarly for the other variables.

8.1. Results for the first non-iterative domain decomposition method.

We first choose $\triangle t = 8h^3$. Table 1 provides errors for different choices of h for the first non-iterative domain decomposition algorithm. Using linear regression, the errors in Table 1 satisfy

$$\begin{aligned} &\|\vec{u}_h - \vec{u}\|_0 \approx 1.980\ h^{3.017}, \qquad |\vec{u}_h - \vec{u}|_1 \approx 9.644\ h^{1.957}, \\ &\|p_h - p\|_0 \approx 20.736\ h^{3.155}, \\ &\|\phi_h - \phi\|_0 \approx 16.457\ h^{2.946}, \qquad |\phi_h - \phi|_1 \approx 15.239\ h^{2.375}. \end{aligned}$$

[3]We also ran examples with exact solutions such that $p_S \neq 0$; the results obtained are entirely similar to those reported on here, except that the pressure approximations now conform more fully with expectations; see the discussion below.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

The Taylor-Hood element pair is used for the Stokes part and quadratic elements are used for the hydraulic head. Hence, we see the $O(h^3 + \triangle t) = O(h^3)$ convergence rates with respect to L^2 norms for $\vec{u}$ and ϕ. The one exception is that the convergence rate for the pressure approximation is better than expected; seemingly, this occurs because the exact solution for the pressure vanishes; in other tests with $p_S \neq 0$, we obtained the expected second-order rate of convergence.

TABLE 1. Errors of the first non-iterative domain decomposition algorithm for $\triangle t = 8h^3$.

h	$\|\vec{u}_h - \vec{u}\|_0$	$\vert\vec{u}_h - \vec{u}\vert_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\vert\phi_h - \phi\vert_1$
1/4	3.018×10^{-2}	6.284×10^{-1}	2.689×10^{-1}	2.657×10^{-1}	6.240×10^{-1}
1/8	3.737×10^{-3}	1.671×10^{-1}	2.886×10^{-2}	3.741×10^{-2}	1.042×10^{-1}
1/12	1.099×10^{-3}	7.561×10^{-2}	7.998×10^{-3}	1.121×10^{-2}	3.830×10^{-2}
1/16	4.608×10^{-4}	4.286×10^{-2}	3.238×10^{-3}	4.743×10^{-3}	1.966×10^{-2}
1/20	2.349×10^{-4}	2.754×10^{-2}	1.615×10^{-3}	2.430×10^{-3}	1.198×10^{-2}
1/24	1.356×10^{-4}	1.917×10^{-2}	9.179×10^{-4}	1.407×10^{-3}	8.081×10^{-3}
1/28	8.518×10^{-5}	1.411×10^{-2}	5.708×10^{-4}	8.862×10^{-4}	5.830×10^{-3}
1/32	5.698×10^{-5}	1.081×10^{-2}	3.788×10^{-4}	5.937×10^{-4}	4.410×10^{-3}

Next, we choose $\triangle t = h$. The first non-iterative domain decomposition algorithm is still stable; error information is given in Table 2. Again, all L^2 norm errors are consistent with the theoretical results of Section 6. For example, for $\vec{u}$ and ϕ we see the expected $O(h^3 + \triangle t) = O(h)$ behavior.

TABLE 2. Errors of the first non-iterative domain decomposition algorithm for $\triangle t = h$.

h	$\|\vec{u}_h - \vec{u}\|_0$	$\vert\vec{u}_h - \vec{u}\vert_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\vert\phi_h - \phi\vert_1$
1/4	3.341×10^{-2}	6.301×10^{-1}	5.697×10^{-1}	4.630×10^{-1}	1.022×10^{0}
1/8	7.148×10^{-3}	1.683×10^{-1}	2.079×10^{-1}	2.643×10^{-1}	5.711×10^{-1}
1/16	2.475×10^{-3}	4.373×10^{-2}	8.549×10^{-2}	1.416×10^{-1}	3.048×10^{-1}
1/32	1.035×10^{-3}	1.146×10^{-2}	3.848×10^{-2}	7.333×10^{-2}	1.577×10^{-1}
1/64	4.687×10^{-4}	3.227×10^{-3}	1.826×10^{-2}	3.731×10^{-2}	8.025×10^{-2}

To further illustrate the unconditional stability of the first non-iterative domain decomposition method, we also considered the choice $\triangle t = \sqrt{h} > h$. The numerical results did indeed show that the method remained stable and the accuracy obtained was what was expected, given the estimates obtained in Section 6. For the sake of brevity, we do not to report details about the numerical results for this case.

8.2. **Results for the second non-iterative domain decomposition method.** We choose $\triangle t = h$. Table 3 provides errors for different choices of h for the second non-iterative domain decomposition algorithm. Using linear regression, the errors in Table 3 satisfy

$$\begin{aligned} &\|\vec{u}_h - \vec{u}\|_0 \approx 1.9480\ h^{3.0128}, && |\vec{u}_h - \vec{u}|_1 \approx 9.8117\ h^{1.9663}, \\ &\|p_h - p\|_0 \approx 13.8987\ h^{3.1332}, && \\ &\|\phi_h - \phi\|_0 \approx 2.8731\ h^{3.0012}, && |\phi_h - \phi|_1 \approx 4.4010\ h^{2.0050}. \end{aligned}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

TABLE 3. Errors of the second non-iterative domain decomposition algorithm for $\triangle t = h$.

h	$\|\vec{u}_h - \vec{u}\|_0$	$\|\vec{u}_h - \vec{u}\|_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$
1/4	2.980×10^{-2}	6.285×10^{-1}	1.739×10^{-1}	5.115×10^{-2}	2.760×10^{-1}
1/8	3.726×10^{-3}	1.671×10^{-1}	2.185×10^{-2}	4.865×10^{-3}	6.734×10^{-2}
1/16	4.589×10^{-4}	4.286×10^{-2}	2.341×10^{-3}	6.620×10^{-4}	1.687×10^{-2}
1/32	5.673×10^{-5}	1.081×10^{-2}	2.603×10^{-4}	8.747×10^{-5}	4.225×10^{-3}
1/64	7.051×10^{-6}	2.711×10^{-3}	3.065×10^{-5}	1.159×10^{-5}	1.058×10^{-3}

These rates of convergence are entirely consistent with the expectations in Section 7; in particular, we see the $O(h^3 + \triangle t^3) = O(h^3)$ convergence rates with respect to L^2 norms for $\vec{u}$ and ϕ.

9. Conclusions

In this paper, we present two parallel non-iterative domain decomposition methods for the approximate solution of the time-dependent Stokes-Darcy system with the Beavers-Joseph-Saffman-Jones interface condition. From the numerical experiments, it can be seen that both methods are unconditionally stable and optimally convergent with respect to the finite element spaces and the temporal discretizations used. The stability and convergence of the first method are proven and illustrated through numerical experiments whereas, for the second method, only numerical experiments are provided. The analysis of the second method, studying more general configurations of the domain and the interface, error estimation for the interface approximation and for non-matching meshes on the interface are of interest for future work; see [6, 11, 13, 29, 63] and the references therein for related topics.

Acknowledgments

This work was supported in part by the CMG program of the National Science Foundation under grant numbers DMS-0620035 (for MG, XH, and XW) and DMS-0914554 (for YC) and the University of Missouri Research Board (for XH). We also thank the referees for helpful comments.

References

[1] M. Amara, D. Capatina, and L. Lizaik, *Coupling of Darcy-Forchheimer and compressible Navier-Stokes equations with heat transfer*, SIAM J. Sci. Comput. **31** (2008/09), no. 2, 1470–1499, DOI 10.1137/070709517. MR2486839 (2010g:65141)

[2] Todd Arbogast and Dana S. Brunson, *A computational method for approximating a Darcy-Stokes system governing a vuggy porous medium*, Comput. Geosci. **11** (2007), no. 3, 207–218, DOI 10.1007/s10596-007-9043-0. MR2344200 (2009b:76155)

[3] Ivo Babuška and Gabriel N. Gatica, *A residual-based a posteriori error estimator for the Stokes-Darcy coupled problem*, SIAM J. Numer. Anal. **48** (2010), no. 2, 498–523, DOI 10.1137/080727646. MR2646106 (2011e:65254)

[4] Lori Badea, Marco Discacciati, and Alfio Quarteroni, *Numerical analysis of the Navier-Stokes/Darcy coupling*, Numer. Math. **115** (2010), no. 2, 195–227, DOI 10.1007/s00211-009-0279-6. MR2606960 (2011f:76158)

[5] Santiago Badia and Ramon Codina, *Unified stabilized finite element formulations for the Stokes and the Darcy problems*, SIAM J. Numer. Anal. **47** (2009), no. 3, 1971–2000, DOI 10.1137/08072632X. MR2519591 (2010h:65216)

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[6] Garth A. Baker, Vassilios A. Dougalis, and Ohannes A. Karakashian, *On a higher order accurate fully discrete Galerkin approximation to the Navier-Stokes equations*, Math. Comp. **39** (1982), no. 160, 339–375, DOI 10.2307/2007319. MR669634 (84h:65096)

[7] Christine Bernardi, Frédéric Hecht, and Fatma Zohra Nouri, *A new finite-element discretization of the Stokes problem coupled with the Darcy equations*, IMA J. Numer. Anal. **30** (2010), no. 1, 61–93, DOI 10.1093/imanum/drn054. MR2580547 (2011a:65388)

[8] Christine Bernardi, Frédéric Hecht, and Olivier Pironneau, *Coupling Darcy and Stokes equations for porous media with cracks*, M2AN Math. Model. Numer. Anal. **39** (2005), no. 1, 7–35, DOI 10.1051/m2an:2005007. MR2136198 (2006a:76107)

[9] Christine Bernardi, Tomás Chacón Rebollo, Frédéric Hecht, and Zoubida Mghazli, *Mortar finite element discretization of a model coupling Darcy and Stokes equations*, M2AN Math. Model. Numer. Anal. **42** (2008), no. 3, 375–410, DOI 10.1051/m2an:2008009. MR2423791 (2009d:65162)

[10] Yassine Boubendir and Svetlana Tlupova, *Stokes-Darcy boundary integral solutions using preconditioners*, J. Comput. Phys. **228** (2009), no. 23, 8627–8641, DOI 10.1016/j.jcp.2009.08.014. MR2558769 (2011d:65335)

[11] James H. Bramble and J. Thomas King, *A finite element method for interface problems in domains with smooth boundaries and interfaces*, Adv. Comput. Math. **6** (1996), no. 2, 109–138 (1997), DOI 10.1007/BF02127700. MR1431789 (98e:65094)

[12] J. H. Bramble, J. E. Pasciak, and A. H. Schatz, *The construction of preconditioners for elliptic problems by substructuring. I*, Math. Comp. **47** (1986), no. 175, 103–134, DOI 10.2307/2008084. MR842125 (87m:65174)

[13] Susanne C. Brenner and L. Ridgway Scott, *The mathematical theory of finite element methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008. MR2373954 (2008m:65001)

[14] Erik Burman and Peter Hansbo, *Stabilized Crouzeix-Raviart element for the Darcy-Stokes problem*, Numer. Methods Partial Differential Equations **21** (2005), no. 5, 986–997, DOI 10.1002/num.20076. MR2154230 (2006i:65190)

[15] Erik Burman and Peter Hansbo, *A unified stabilized method for Stokes' and Darcy's equations*, J. Comput. Appl. Math. **198** (2007), no. 1, 35–51, DOI 10.1016/j.cam.2005.11.022. MR2250387 (2007i:65076)

[16] Mingchao Cai, Mo Mu, and Jinchao Xu, *Numerical solution to a mixed Navier-Stokes/Darcy model by the two-grid approach*, SIAM J. Numer. Anal. **47** (2009), no. 5, 3325–3338, DOI 10.1137/080721868. MR2551196 (2010j:65226)

[17] Mingchao Cai, Mo Mu, and Jinchao Xu, *Preconditioning techniques for a mixed Stokes/Darcy model in porous media applications*, J. Comput. Appl. Math. **233** (2009), no. 2, 346–355, DOI 10.1016/j.cam.2009.07.029. MR2568530 (2010j:65046)

[18] Xiao-Chuan Cai, *Multiplicative Schwarz methods for parabolic problems*, Iterative methods in numerical linear algebra (Copper Mountain Resort, CO, 1992). SIAM J. Sci. Comput. **15** (1994), no. 3, 587–603, DOI 10.1137/0915039. MR1273154 (95c:65178)

[19] Yanzhao Cao, Max Gunzburger, Xiaoming He, and Xiaoming Wang, *Robin-Robin domain decomposition methods for the steady-state Stokes-Darcy system with the Beavers-Joseph interface condition*, Numer. Math. **117** (2011), no. 4, 601–629, DOI 10.1007/s00211-011-0361-8. MR2776912 (2012g:65253)

[20] Y. Cao, M. Gunzburger, X. Hu, F. Hua, X. Wang, and W. Zhao, *Finite element approximations for Stokes-Darcy flow with Beavers-Joseph interface conditions*, SIAM. J. Numer. Anal. **47** (2010), no. 6, 4239–4256. MR2585186 (2011b:65220)

[21] Yanzhao Cao, Max Gunzburger, Fei Hua, and Xiaoming Wang, *Coupled Stokes-Darcy model with Beavers-Joseph interface boundary condition*, Commun. Math. Sci. **8** (2010), no. 1, 1–25. MR2655899 (2011c:35438)

[22] A. Çeşmelioğlu and B. Rivière, *Analysis of time-dependent Navier-Stokes flow coupled with Darcy flow*, J. Numer. Math. **16** (2008), no. 4, 249–280, DOI 10.1515/JNUM.2008.012. MR2493168 (2009m:35375)

[23] Ayçıl Çeşmelioğlu and Béatrice Rivière, *Primal discontinuous Galerkin methods for time-dependent coupled surface and subsurface flow*, J. Sci. Comput. **40** (2009), no. 1-3, 115–140, DOI 10.1007/s10915-009-9274-4. MR2511730 (2010f:65185)

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[24] P. Chatzipantelidis, R. D. Lazarov, V. Thomée, and L. B. Wahlbin, *Parabolic finite element equations in nonconvex polygonal domains*, BIT **46** (2006), no. suppl., S113–S143, DOI 10.1007/s10543-006-0087-7. MR2283311 (2007m:65084)

[25] Nan Chen, Max Gunzburger, and Xiaoming Wang, *Asymptotic analysis of the differences between the Stokes-Darcy system with different interface conditions and the Stokes-Brinkman system*, J. Math. Anal. Appl. **368** (2010), no. 2, 658–676, DOI 10.1016/j.jmaa.2010.02.022. MR2643831 (2011c:35395)

[26] Wenbin Chen, Puying Chen, Max Gunzburger, and Ningning Yan, *Superconvergence analysis of FEMs for the Stokes-Darcy system*, Math. Methods Appl. Sci. **33** (2010), no. 13, 1605–1617, DOI 10.1002/mma.1279. MR2680670 (2011h:76072)

[27] Wenbin Chen, Max Gunzburger, Fei Hua, and Xiaoming Wang, *A parallel Robin-Robin domain decomposition method for the Stokes-Darcy system*, SIAM J. Numer. Anal. **49** (2011), no. 3, 1064–1084, DOI 10.1137/080740556. MR2812558 (2012h:65294)

[28] Yumei Chen, Feiteng Huang, and Xiaoping Xie, $H(\mathrm{div})$ *conforming finite element methods for the coupled Stokes and Darcy problem*, J. Comput. Appl. Math. **235** (2011), no. 15, 4337–4349, DOI 10.1016/j.cam.2011.03.023. MR2802009 (2012e:65264)

[29] Zhiming Chen and Jun Zou, *Finite element methods and their convergence for elliptic and parabolic interface problems*, Numer. Math. **79** (1998), no. 2, 175–202, DOI 10.1007/s002110050336. MR1622502 (99d:65313)

[30] Prince Chidyagwai and Béatrice Rivière, *On the solution of the coupled Navier-Stokes and Darcy equations*, Comput. Methods Appl. Mech. Engrg. **198** (2009), no. 47-48, 3806–3820, DOI 10.1016/j.cma.2009.08.012. MR2557499 (2010j:76038)

[31] Philippe G. Ciarlet, *The finite element method for elliptic problems*, North-Holland Publishing Co., Amsterdam, 1978. Studies in Mathematics and its Applications, Vol. 4. MR0520174 (58 #25001)

[32] Ming Cui and Ningning Yan, *A posteriori error estimate for the Stokes-Darcy system*, Math. Methods Appl. Sci. **34** (2011), no. 9, 1050–1064, DOI 10.1002/mma.1422. MR2829467 (2012e:76124)

[33] Mark C. Curran, *An iterative finite-element collocation method for parabolic problems using domain decomposition*, Domain decomposition methods in science and engineering (Como, 1992), Contemp. Math., vol. 157, Amer. Math. Soc., Providence, RI, 1994, pp. 245–253, DOI 10.1090/conm/157/01424. MR1262624

[34] Carlo D'Angelo and Paolo Zunino, *Robust numerical approximation of coupled Stokes' and Darcy's flows applied to vascular hemodynamics and biochemical transport*, ESAIM Math. Model. Numer. Anal. **45** (2011), no. 3, 447–476, DOI 10.1051/m2an/2010062. MR2804646 (2012d:76077)

[35] Daoud S. Daoud and Bruce A. Wade, *A two-stage multi-splitting method for non-overlapping domain decomposition for parabolic equations*, Domain decomposition methods in sciences and engineering (Chiba, 1999), DDM.org, Augsburg, 2001, pp. 101–108 (electronic). MR1827527

[36] Clint Dawson, *Analysis of discontinuous finite element methods for ground water/surface water coupling*, SIAM J. Numer. Anal. **44** (2006), no. 4, 1375–1404, DOI 10.1137/050639405. MR2257109 (2007g:76129)

[37] Clint N. Dawson, Qiang Du, and Todd F. Dupont, *A finite difference domain decomposition algorithm for numerical solution of the heat equation*, Math. Comp. **57** (1991), no. 195, 63–71, DOI 10.2307/2938663. MR1079011 (91m:65254)

[38] Clint N. Dawson and Todd F. Dupont, *Explicit/implicit conservative Galerkin domain decomposition procedures for parabolic problems*, Math. Comp. **58** (1992), no. 197, 21–34, DOI 10.2307/2153018. MR1106964 (92h:65183)

[39] Clint N. Dawson and Todd F. Dupont, *Explicit/implicit, conservative domain decomposition procedures for parabolic problems based on block-centered finite differences*, SIAM J. Numer. Anal. **31** (1994), no. 4, 1045–1061, DOI 10.1137/0731055. MR1286216 (95c:65148)

[40] M. Discacciati, *Domain decomposition methods for the coupling of surface and groundwater flows*, Ph.D. thesis, Ecole Polytechnique Fédérale de Lausanne, Switzerland, 2004.

[41] Marco Discacciati, *Iterative methods for Stokes/Darcy coupling*, Domain decomposition methods in science and engineering, Lect. Notes Comput. Sci. Eng., vol. 40, Springer, Berlin, 2005, pp. 563–570, DOI 10.1007/3-540-26825-1_59. MR2236665

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[42] Marco Discacciati, Edie Miglio, and Alfio Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows*, 19th Dundee Biennial Conference on Numerical Analysis (2001). Appl. Numer. Math. **43** (2002), no. 1-2, 57–74, DOI 10.1016/S0168-9274(02)00125-3. MR1936102 (2003h:76087)

[43] M. Discacciati and A. Quarteroni, *Analysis of a domain decomposition method for the coupling of Stokes and Darcy equations*, Numerical mathematics and advanced applications, Springer Italia, Milan, 2003, pp. 3–20. MR2360703 (2008i:65288)

[44] Marco Discacciati and Alfio Quarteroni, *Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations*, Comput. Vis. Sci. **6** (2004), no. 2-3, 93–103, DOI 10.1007/s00791-003-0113-0. MR2061270 (2005e:65142)

[45] Marco Discacciati, Alfio Quarteroni, and Alberto Valli, *Robin-Robin domain decomposition methods for the Stokes-Darcy coupling*, SIAM J. Numer. Anal. **45** (2007), no. 3, 1246–1268 (electronic), DOI 10.1137/06065091X. MR2318811 (2008e:65390)

[46] Jim Douglas Jr. and Todd Dupont, *Galerkin methods for parabolic equations*, SIAM J. Numer. Anal. **7** (1970), 575–626. MR0277126 (43 #2863)

[47] Maksymilian Dryja, *Substructuring methods for parabolic problems*, Partial Differential Equations (Moscow, 1990), SIAM, Philadelphia, PA, 1991, pp. 264–271. MR1106468 (92a:65326)

[48] Maksymilian Dryja and Xuemin Tu, *A domain decomposition discretization of parabolic problems*, Numer. Math. **107** (2007), no. 4, 625–640, DOI 10.1007/s00211-007-0103-0. MR2342646 (2008i:65205)

[49] V. J. Ervin, E. W. Jenkins, and S. Sun, *Coupled generalized nonlinear Stokes flow with flow through a porous medium*, SIAM J. Numer. Anal. **47** (2009), no. 2, 929–952, DOI 10.1137/070708354. MR2485439 (2010b:65254)

[50] V. J. Ervin, E. W. Jenkins, and S. Sun, *Coupling nonlinear Stokes and Darcy flow using mortar finite elements*, Appl. Numer. Math. **61** (2011), no. 11, 1198–1222, DOI 10.1016/j.apnum.2011.08.002. MR2842139 (2012j:65398)

[51] Min-fu Feng, Rui-sheng Qi, Rui Zhu, and Bing-tao Ju, *Stabilized Crouzeix-Raviart element for the coupled Stokes and Darcy problem*, Appl. Math. Mech. (English Ed.) **31** (2010), no. 3, 393–404, DOI 10.1007/s10483-010-0312-z. MR2655474 (2011f:65255)

[52] Juan Galvis and Marcus Sarkis, *Balancing domain decomposition methods for mortar coupling Stokes-Darcy systems*, Domain decomposition methods in science and engineering XVI, Lect. Notes Comput. Sci. Eng., vol. 55, Springer, Berlin, 2007, pp. 373–380, DOI 10.1007/978-3-540-34469-8_46. MR2334125

[53] Juan Galvis and Marcus Sarkis, *Non-matching mortar discretization analysis for the coupling Stokes-Darcy equations*, Electron. Trans. Numer. Anal. **26** (2007), 350–384. MR2391227 (2009a:76120)

[54] Juan Galvis and Marcus Sarkis, *FETI and BDD preconditioners for Stokes-Mortar-Darcy systems*, Commun. Appl. Math. Comput. Sci. **5** (2010), 1–30, DOI 10.2140/camcos.2010.5.1. MR2600819 (2011a:35411)

[55] Martin J. Gander, Laurence Halpern, and Frederic Nataf, *Optimized Schwarz methods*, Domain decomposition methods in sciences and engineering (Chiba, 1999), DDM.org, Augsburg, 2001, pp. 15–27 (electronic). MR1827519

[56] Gabriel N. Gatica, Salim Meddahi, and Ricardo Oyarzúa, *A conforming mixed finite-element method for the coupling of fluid flow with porous media flow*, IMA J. Numer. Anal. **29** (2009), no. 1, 86–108, DOI 10.1093/imanum/drm049. MR2470941 (2010b:76118)

[57] Gabriel N. Gatica, Ricardo Oyarzúa, and Francisco-Javier Sayas, *A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes-Darcy coupled problem*, Comput. Methods Appl. Mech. Engrg. **200** (2011), no. 21-22, 1877–1891, DOI 10.1016/j.cma.2011.02.009. MR2787543 (2012a:65328)

[58] Gabriel N. Gatica, Ricardo Oyarzúa, and Francisco-Javier Sayas, *Convergence of a family of Galerkin discretizations for the Stokes-Darcy coupled problem*, Numer. Methods Partial Differential Equations **27** (2011), no. 3, 721–748, DOI 10.1002/num.20548. MR2809968 (2012g:65260)

[59] Vivette Girault and Pierre-Arnaud Raviart, *Finite element methods for Navier-Stokes equations*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986. Theory and algorithms. MR851383 (88b:65129)

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[60] Vivette Girault and Béatrice Rivière, *DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition*, SIAM J. Numer. Anal. **47** (2009), no. 3, 2052–2089, DOI 10.1137/070686081. MR2519594 (2010f:76084)

[61] James K. Guest and Jean H. Prévost, *Topology optimization of creeping fluid flows using a Darcy-Stokes finite element*, Internat. J. Numer. Methods Engrg. **66** (2006), no. 3, 461–484, DOI 10.1002/nme.1560. MR2222192 (2006k:76050)

[62] Max D. Gunzburger, *On the stability of Galerkin methods for initial-boundary value problems for hyperbolic systems*, Math. Comp. **31** (1977), no. 139, 661–675. MR0436624 (55 #9567)

[63] Max D. Gunzburger, *Finite element methods for viscous incompressible flows*, Computer Science and Scientific Computing, Academic Press Inc., Boston, MA, 1989. A guide to theory, practice, and algorithms. MR1017032 (91d:76053)

[64] Ronald H. W. Hoppe, Paulo Porta, and Yuri Vassilevski, *Computational issues related to iterative coupling of subsurface and channel flows*, Calcolo **44** (2007), no. 1, 1–20, DOI 10.1007/s10092-007-0126-z. MR2301278 (2008a:76088)

[65] Fei (Neil) Hua, *Modeling, analysis and simulation of the Stokes-Darcy system with Beavers-Joseph interface condition*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–The Florida State University. MR2714080

[66] Willi Jäger and Andro Mikelić, *On the interface boundary condition of Beavers, Joseph, and Saffman*, SIAM J. Appl. Math. **60** (2000), no. 4, 1111–1127 (electronic), DOI 10.1137/S003613999833678X. MR1760028 (2001e:76122)

[67] Bin Jiang, *A parallel domain decomposition method for coupling of surface and groundwater flows*, Comput. Methods Appl. Mech. Engrg. **198** (2009), no. 9-12, 947–957, DOI 10.1016/j.cma.2008.11.001. MR2498862 (2010c:76042)

[68] I. Jones, *Low Reynolds number flow past a porous spherical shell*, Proc. Camb. Phil. Soc. **73** (1973), 231–238.

[69] Younbae Jun and Tsun-Zee Mai, *ADI method—domain decomposition*, Appl. Numer. Math. **56** (2006), no. 8, 1092–1107, DOI 10.1016/j.apnum.2005.09.008. MR2234842 (2007h:65097)

[70] Younbae Jun and Tsun-Zee Mai, *IPIC domain decomposition algorithm for parabolic problems*, Appl. Math. Comput. **177** (2006), no. 1, 352–364, DOI 10.1016/j.amc.2005.11.017. MR2234525

[71] G. Kanschat and B. Rivière, *A strongly conservative finite element method for the coupling of Stokes and Darcy flow*, J. Comput. Phys. **229** (2010), no. 17, 5933–5943, DOI 10.1016/j.jcp.2010.04.021. MR2657851 (2011h:76081)

[72] Trygve Karper, Kent-Andre Mardal, and Ragnar Winther, *Unified finite element discretizations of coupled Darcy-Stokes flow*, Numer. Methods Partial Differential Equations **25** (2009), no. 2, 311–326, DOI 10.1002/num.20349. MR2483769 (2010a:65240)

[73] Sondes Khabthani, Lassaad Elasmi, and François Feuillebois, *Perturbation solution of the coupled Stokes-Darcy problem*, Discrete Contin. Dyn. Syst. Ser. B **15** (2011), no. 4, 971–990, DOI 10.3934/dcdsb.2011.15.971. MR2786363 (2012e:76130)

[74] Yu. A. Kuznetsov, *Overlapping domain decomposition methods for parabolic problems*, Domain decomposition methods in science and engineering (Como, 1992), Contemp. Math., vol. 157, Amer. Math. Soc., Providence, RI, 1994, pp. 63–69, DOI 10.1090/conm/157/01406. MR1262606

[75] William Layton, Hoang Tran, and Xin Xiong, *Long time stability of four methods for splitting the evolutionary Stokes-Darcy problem into Stokes and Darcy subproblems*, J. Comput. Appl. Math. **236** (2012), no. 13, 3198–3217, DOI 10.1016/j.cam.2012.02.019. MR2912685

[76] William J. Layton, Friedhelm Schieweck, and Ivan Yotov, *Coupling fluid flow with porous media flow*, SIAM J. Numer. Anal. **40** (2002), no. 6, 2195–2218 (2003), DOI 10.1137/S0036142901392766. MR1974181 (2004c:76048)

[77] Jingzhi Li, Jens Markus Melenk, Barbara Wohlmuth, and Jun Zou, *Optimal a priori estimates for higher order finite elements for elliptic interface problems*, Appl. Numer. Math. **60** (2010), no. 1-2, 19–37, DOI 10.1016/j.apnum.2009.08.005. MR2566075 (2011a:65370)

[78] S. Li, H. Zheng, and W. Layton, *A decoupling method with different sub-domain time steps for the nonstationary Stokes-Darcy model*, Numer. Methods Partial Differential Equations **29** (2013), doi: 10.1002/num.21720. MR3022899

[79] P.-L. Lions, *On the Schwarz alternating method. I*, Partial Differential Equations (Paris, 1987), SIAM, Philadelphia, PA, 1988, pp. 1–42. MR972510 (90a:65248)

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[80] P.-L. Lions, *On the Schwarz alternating method. III. A variant for nonoverlapping subdomains*, Partial Differential Equations (Houston, TX, 1989), SIAM, Philadelphia, PA, 1990, pp. 202–223. MR1064345 (91g:65226)

[81] Kent Andre Mardal, Xue-Cheng Tai, and Ragnar Winther, *A robust finite element method for Darcy-Stokes flow*, SIAM J. Numer. Anal. **40** (2002), no. 5, 1605–1631, DOI 10.1137/S0036142901383910. MR1950614 (2003m:76110)

[82] Arif Masud, *A stabilized mixed finite element method for Darcy-Stokes flow*, Internat. J. Numer. Methods Fluids **54** (2007), no. 6-8, 665–681, DOI 10.1002/fld.1508. MR2333004 (2008b:76120)

[83] Mo Mu and Jinchao Xu, *A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow*, SIAM J. Numer. Anal. **45** (2007), no. 5, 1801–1813, DOI 10.1137/050637820. MR2346360 (2008i:65264)

[84] Mo Mu and Xiaohong Zhu, *Decoupled schemes for a non-stationary mixed Stokes-Darcy model*, Math. Comp. **79** (2010), no. 270, 707–731, DOI 10.1090/S0025-5718-09-02302-3. MR2600540 (2011c:65219)

[85] Steffen Münzenmaier and Gerhard Starke, *First-order system least squares for coupled Stokes-Darcy flow*, SIAM J. Numer. Anal. **49** (2011), no. 1, 387–404, DOI 10.1137/100805108. MR2783231 (2012e:65282)

[86] V. Nassehi and J. Petera, *A new least-squares finite element model for combined Navier-Stokes and Darcy flows in geometrically complicated domains with solid and porous boundaries*, Internat. J. Numer. Methods Engrg. **37** (1994), no. 9, 1609–1620, DOI 10.1002/nme.1620370912. MR1274560 (95a:76061)

[87] Weihong Peng, Guohua Cao, Dongzhengzhu, and Shuncai Li, *Darcy-Stokes equations with finite difference and natural boundary element coupling method*, CMES Comput. Model. Eng. Sci. **75** (2011), no. 3-4, 173–188. MR2867757

[88] Peter Popov, Yalchin Efendiev, and Guan Qin, *Multiscale modeling and simulations of flows in naturally fractured Karst reservoirs*, Commun. Comput. Phys. **6** (2009), no. 1, 162–184, DOI 10.4208/cicp.2009.v6.p162. MR2537310

[89] Lizhen Qin and Xuejun Xu, *Optimized Schwarz methods with Robin transmission conditions for parabolic problems*, SIAM J. Sci. Comput. **31** (2008), no. 1, 608–623, DOI 10.1137/070682149. MR2460791 (2009j:65234)

[90] Alfio Quarteroni and Alberto Valli, *Domain decomposition methods for partial differential equations*, Numerical Mathematics and Scientific Computation, The Clarendon Press Oxford University Press, New York, 1999. Oxford Science Publications. MR1857663 (2002i:65002)

[91] Béatrice Rivière, *Analysis of a discontinuous finite element method for the coupled Stokes and Darcy problems*, J. Sci. Comput. **22/23** (2005), 479–500, DOI 10.1007/s10915-004-4147-3. MR2142206 (2006b:65175)

[92] Béatrice Rivière and Ivan Yotov, *Locally conservative coupling of Stokes and Darcy flows*, SIAM J. Numer. Anal. **42** (2005), no. 5, 1959–1977, DOI 10.1137/S0036142903427640. MR2139232 (2006a:76035)

[93] Hongxing Rui and Ran Zhang, *A unified stabilized mixed finite element method for coupling Stokes and Darcy flows*, Comput. Methods Appl. Mech. Engrg. **198** (2009), no. 33-36, 2692–2699, DOI 10.1016/j.cma.2009.03.011. MR2532369 (2010g:76045)

[94] P. Saffman, *On the boundary condition at the interface of a porous medium*, Stud. in Appl. Math. **1** (1971), 77–84.

[95] Xue-Cheng Tai and Ragnar Winther, *A discrete de Rham complex with enhanced smoothness*, Calcolo **43** (2006), no. 4, 287–306, DOI 10.1007/s10092-006-0124-6. MR2283095 (2007j:76050)

[96] Vidar Thomée, *Galerkin finite element methods for parabolic problems*, 2nd ed., Springer Series in Computational Mathematics, vol. 25, Springer-Verlag, Berlin, 2006. MR2249024 (2007b:65003)

[97] Svetlana Tlupova and Ricardo Cortez, *Boundary integral solutions of coupled Stokes and Darcy flows*, J. Comput. Phys. **228** (2009), no. 1, 158–179, DOI 10.1016/j.jcp.2008.09.011. MR2464074 (2009j:76168)

[98] J. M. Urquiza, D. N'Dri, A. Garon, and M. C. Delfour, *Coupling Stokes and Darcy equations*, Appl. Numer. Math. **58** (2008), no. 5, 525–538, DOI 10.1016/j.apnum.2006.12.006. MR2407730 (2009a:76053)

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[99] Mary Fanett Wheeler, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal. **10** (1973), 723–759. MR0351124 (50 #3613)

[100] Xiaoping Xie, Jinchao Xu, and Guangri Xue, *Uniformly-stable finite element methods for Darcy-Stokes-Brinkman models*, J. Comput. Math. **26** (2008), no. 3, 437–455. MR2421892 (2009g:76087)

[101] Jinchao Xu and Jun Zou, *Some nonoverlapping domain decomposition methods*, SIAM Rev. **40** (1998), no. 4, 857–914, DOI 10.1137/S0036144596306800. MR1659681 (99m:65241)

[102] Xuejun Xu and Shangyou Zhang, *A new divergence-free interpolation operator with applications to the Darcy-Stokes-Brinkman equations*, SIAM J. Sci. Comput. **32** (2010), no. 2, 855–874, DOI 10.1137/090751049. MR2609343 (2011g:76108)

[103] Shiquan Zhang, Xiaoping Xie, and Yumei Chen, *Low order nonconforming rectangular finite element methods for Darcy-Stokes problems*, J. Comput. Math. **27** (2009), no. 2-3, 400–424. MR2495068 (2010d:65339)

[104] Zheming Zheng, Bernd Simeon, and Linda Petzold, *A stabilized explicit Lagrange multiplier based domain decomposition method for parabolic problems*, J. Comput. Phys. **227** (2008), no. 10, 5272–5285, DOI 10.1016/j.jcp.2008.01.057. MR2414854 (2010a:65180)

[105] Liyong Zhu, Guangwei Yuan, and Qiang Du, *An explicit-implicit predictor-corrector domain decomposition method for time dependent multi-dimensional convection diffusion equations*, Numer. Math. Theory Methods Appl. **2** (2009), no. 3, 301–325, DOI 10.4208/nmtma.2009.m8016. MR2605862 (2011g:65188)

[106] Liyong Zhu, Guangwei Yuan, and Qiang Du, *An efficient explicit/implicit domain decomposition method for convection-diffusion equations*, Numer. Methods Partial Differential Equations **26** (2010), no. 4, 852–873, DOI 10.1002/num.20461. MR2642324 (2011d:65268)

[107] Yu Zhuang, *An alternating explicit-implicit domain decomposition method for the parallel solution of parabolic equations*, J. Comput. Appl. Math. **206** (2007), no. 1, 549–566, DOI 10.1016/j.cam.2006.08.024. MR2337462 (2008i:65197)

[108] Yu Zhuang and Xian-He Sun, *Stabilized explicit-implicit domain decomposition methods for the numerical solution of parabolic equations*, SIAM J. Sci. Comput. **24** (2002), no. 1, 335–358 (electronic), DOI 10.1137/S1064827501384755. MR1924428 (2003g:65112)

Department of Mathematics and Statistics, Auburn University, Auburn, Alabama 36830

E-mail address: yzc0009@auburn.edu

Department of Scientific Computing, Florida State University, Tallahassee, Florida 32306

E-mail address: gunzburg@fsu.edu

Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, Missouri 65409

E-mail address: hex@mst.edu

Department of Mathematics, Florida State University, Tallahassee, Florida 32306

E-mail address: wxm@math.fsu.edu

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use