Parameter Estimation for Random Amplitude Chirp Signals

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***Abstract*— We consider the problem of estimating the parameters of a chirp signal observed in multiplicative noise, i.e., whose amplitude is randomly time-varying. Two methods for solving this problem are presented. First, an unstructured nonlinear least-squares approach (NLS) is proposed. It is shown that by minimizing the NLS criterion with respect to all samples of the time-varying amplitude, the problem reduces to a two-dimensional (2-D) maximization problem. A theoretical analysis of the NLS estimator is presented, and an expression for its asymptotic variance is derived. It is shown that the NLS estimator has a variance that is very close to the Cramér–Rao bound. The second approach combines the principles behind the high-order ambiguity function (HAF) and the NLS approach. It provides a computationally simpler but suboptimum estimator. A statistical analysis of the HAF-based estimator is also carried out, and closed-form expressions are derived for the asymptotic variance of the HAF estimators based on the data and on the squared data. Numerical examples attest to the validity of the theoretical analyzes and establish a comparison between the two proposed methods.**

***Index Terms*— High-order ambiguity function, multiplicative noise, nonlinear least-squares, random amplitude chirp signals.**

I. INTRODUCTION

THIS PAPER is concerned with the analysis as well as estimation of the parameters of chirp signals with random time-varying amplitude. This kind of signal arises in many applications of signal processing, one of the most important being the radar problem. For instance, consider a radar illuminating a target. Then, the transmitted signal will be affected by two different phenomena. First, it will undergo a phase shift induced by the distance and relative motion between the target and the receiver. Assuming this motion is continuous and differentiable, the phase shift can be adequately modeled as $\phi(t) = a_0 + a_1 t + a_2 t^2$, where the parameters a_1 and a_2 are either related to speed and acceleration or range and speed, depending on what the radar is intended for and on the kind of waveforms transmitted [1, pp. 56–65]. The second phenomenon to be accounted for is amplitude distortion caused either by target fluctuation or scattering of the medium (e.g., fading). In either case, this results in a random time-varying amplitude that can be viewed as an unwanted parameter (hence the terminology multiplicative noise often used in the literature). To summarize, the model to be considered here is given by

$$y(t) = \alpha(t) e^{i(a_0 + a_1 t + a_2 t^2)} + n(t) \qquad t = 0, \cdots, N-1 \quad (1)$$

where $n(t)$ denotes additive noise, and $\alpha(t)$ is the random time-varying amplitude. Although considerable attention has focused on the estimation problem for parts of the model in (1), the literature is scarce on analysis of the complete model (1). More exactly, the two following cases have been addressed thoroughly.

- *Constant amplitude chirp signals* [*i.e.*, $\alpha(t) \equiv A$]: This problem has been dealt with in [2] using rank reduction techniques, in [3] by means of phase unwrapping schemes, and in [4]–[7] using the so-called high-order ambiguity function (HAF). This scheme has become a "standard" tool for analyzing constant amplitude chirp signals since it provides a computationally efficient yet statistically accurate estimator.
- *Exponential signals with time-varying amplitude* (*i.e.*, $a_2 \equiv 0$) have been studied extensively in the recent years. Approaches using high-order statistics [8]–[10], cyclic tools [11], Yule–Walker equations [12], subspace-based methods [13], and nonlinear least-squares estimators [14]–[16] have been proposed and analyzed.

Analysis of signals like (1) can be found in [17] and [18] for the deterministic case (i.e., $\alpha(t)$ deterministic) and [19]–[21] for the random case. In [17], both the amplitude and phase are assumed to be linear combinations of known basis functions and maximum likelihood (ML) estimators are derived and performance compared with the Cramér–Rao bound (CRB). In [18], it is shown that appropriate use of the HAF provides consistent and accurate estimates of the chirp parameters when the amplitude is a deterministic sequence of the form $\rho(t/N)$. In [19], $\alpha(t)$ is assumed to be a stationary Gaussian process whose covariance matrix depends on a finite-dimensional parameter vector and CRB's are derived. Extensions and further results on CRB's and ML estimation can be found in [21]. A broad class of random amplitudes is studied in [20] and cyclostationary solutions are investigated. More precisely, for a chirp signal, use of the cyclic second-order moment is advocated. It should be noted that, in practice, the estimation procedure is equivalent to using the second-order ambiguity function of Peleg and Porat since it amounts to computing a fast Fourier transform of the sequence $y^*(t)y(t+\tau)$.

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In this paper, two approaches are proposed. The first relies on nonlinear least-squares (NLS) estimation of the chirp parameters, following ideas recently published in [15] and [16]. By minimizing the NLS criterion with respect to all samples of the time-varying amplitude, it is shown that the NLS estimator reduces to a 2-D maximization problem over the chirp parameters. Since this approach may be computationally intensive for certain applications, a second approach is proposed that borrows ideas from the HAF and the NLS estimator. More exactly, this method consists of sequentially reducing the order of the polynomial phase using some transformations; this methodology is the essence of the HAF-based estimator. At each step of the method, we are left with the problem of estimating an exponential signal with random time-varying amplitude for which the NLS approach is recommended.

The paper is organized as follows. In Section II, the NLS estimator is derived and a formula for its asymptotic performance is given. A suboptimum but computationally simpler algorithm is presented and analyzed in Section III. Numerical examples are given in Section IV, and our conclusions are drawn in Section V. Technical derivations are deferred to the Appendices.

II. NLS Estimation

To begin with, we recall the model to be used and the hypotheses made. The signal to be dealt with is given by

$$\begin{aligned} y(t) &= \alpha(t)e^{i(a_0+a_1t+a_2t^2)} + n(t) \qquad t=0,\cdots,N-1 \\ &= s(t)+n(t) \end{aligned}$$

where we make the follwing assumptions.

AS1) $\alpha(t)$ is assumed to be a real-valued stationary mixing process (not necessarily Gaussian), whose mean is not assumed to be zero and whose covariance matrix is unknown. We do not make any assumption about the structure of $\alpha(t)$; in particular, it is not assumed to be an ARMA process.

AS2) $n(t)$ is a white complex circular Gaussian process with zero mean and variance σ^2, i.e., $\mathsf{E}\{n(t)n(t+\tau)\} \equiv 0$, $\mathsf{E}\{n^*(t)n(t+\tau)\} = \sigma^2\delta(\tau)$. Additionally, $n(t)$ is assumed to be independent of $\alpha(t)$

Our NLS approach consists of estimating the parameters a_0, a_1, a_2 *as well as* all samples $\{\alpha(t)\}_{t=0,\cdots,N-1}$ of the time-varying amplitude by minimizing the following criterion:

$$J(\boldsymbol{\alpha}, \mathbf{a}) = \frac{1}{N}\sum_{t=0}^{N-1} |y(t) - \alpha(t)e^{i(a_0+a_1t+a_2t^2)}|^2 \qquad (2)$$

where $\boldsymbol{\alpha} = [\alpha(0),\cdots,\alpha(N-1)]^T$, and $\mathbf{a} = [a_0, a_1, a_2]^T$. Note that this is not the "true" NLS estimator since the latter would proceed by minimizing (2) with respect to $\mathbf{a}$ and the parameter vector $\boldsymbol{\lambda}$ on which $\alpha(t)$ would depend. For instance, if $\alpha(t)$ is an autoregressive process, $\alpha(t) = -\Sigma_{k=1}^{p} c_k\alpha(t-k) + v(t)$; then, $\boldsymbol{\lambda} = [c_1,\cdots,c_p]^T$ would denote the vector of autoregressive parameters. The approach we propose tacitly considers that the realization of $\alpha(t)$ is frozen and has to be estimated. However, as will be illustrated below, an estimate of $\{\alpha(t)\}_{t=0,\cdots,N-1}$ is made available [see (5)], which can eventually be used to estimate $\boldsymbol{\lambda}$. The next proposition shows how estimates of $\boldsymbol{\alpha}$ and $\mathbf{a}$ are obtained.

Proposition 1: The vectors $\boldsymbol{\alpha}$ and $\mathbf{a}$ that minimize (2) are given by

$$\hat{a}_1, \hat{a}_2 = \arg\max_{\theta_1,\theta_2} \frac{1}{N}\left|\sum_{t=0}^{N-1} y^2(t)\times e^{-i2(\theta_1t+\theta_2t^2)}\right| \qquad (3)$$

$$\hat{a}_0 = \frac{1}{2}\,\text{angle}\left\{\sum_{t=0}^{N-1} y^2(t)\times e^{-i2(\hat{a}_1t+\hat{a}_2t^2)}\right\} \qquad (4)$$

$$\hat{\alpha}(t) = \text{Re}\,\{y(t)\times e^{-i(\hat{a}_0+\hat{a}_1t+\hat{a}_2t^2)}\}. \qquad (5)$$

Proof: See Appendix A. ∎

Examining (3), it is observed that by minimizing with respect to (wrt) $\boldsymbol{\alpha}$ and not wrt $\boldsymbol{\lambda}$, the problem is reduced to a two-dimensional (2-D) maximization problem, as far as parameters a_1 and a_2 are concerned. Additionally, it should be emphasized that the present approach does not rely on any assumed structure for the amplitude; hence, it has the desirable property of being applicable to a wide class of signals. Before proceeding to the theoretical analysis of the estimator, a few remarks are in order.

Remark 1: It can be seen that the NLS estimates of the phase parameters are decoupled from those of the amplitude parameters, i.e., the amplitude variations are irrelevant to the estimation of the phase parameters (however, they do affect the achievable accuracy; see below). In contrast, the estimates of the amplitude parameters depend on the phase parameters since this estimate essentially involves dephasing.

Remark 2: In the constant amplitude case (i.e., $\alpha(t) \equiv \alpha_0$), the estimate of α_0 would be an average, e.g., $\hat{\alpha}_0 = (1/N)|\Sigma_{t=0}^{N-1}\, y(t)\times e^{-i(\hat{a}_1t+\hat{a}_2t^2)}|$. In the time-varying scenario, each sample of $\alpha(t)$ is estimated [see (5)], which leads to the squaring of the data.

Remark 3: It should be emphasized that the estimates of $\mathbf{a}$ as given by (3) and (4) are equivalent to those that would have been obtained by solving the following minimization problem:

$$\{\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{A}\} = \arg\min_{A,\boldsymbol{\theta}} \frac{1}{N}\sum_{t=0}^{N-1} \cdot |y^2(t) - Ae^{i2(\theta_0+\theta_1t+\theta_2t^2)}|^2. \qquad (6)$$

To see this, let us define $A_1 = Ae^{i2\theta_0}$. Since the criterion in (6) is quadratic in A_1, for any given θ_1 and θ_2, the value of A_1 that minimizes the function in (6) is given by

$$A_1 = \frac{1}{N}\sum_{t=0}^{N-1} y^2(t)\times e^{-i2(\theta_1t+\theta_2t^2)}.$$

Substituting into (6), the estimates of a_1 and a_2 are readily found to be

$$\{\hat{a}_1, \hat{a}_2\} = \arg\max_{\theta_1,\theta_2} \frac{1}{N}\left|\sum_{t=0}^{N-1} y^2(t)e^{-i2(\theta_1t+\theta_2t^2)}\right| \qquad (7)$$

which is exactly (3). Moreover

$$\hat{A}_1 = \frac{1}{N}\sum_{t=0}^{N-1} y^2(t)\times e^{-i2(\hat{a}_1t+\hat{a}_2t^2)}. \qquad (8)$$

Additionally, since $\hat{a}_1, \hat{a}_2$ are consistent estimates of a_1, a_2, it can be inferred that [22]

$$\lim_{N\to\infty} \hat{A}_1 = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \alpha^2(t) \times e^{i2a_0} = r_\alpha(0)e^{i2a_0} \quad (9)$$

where $r_\alpha(0) = \mathsf{E}\{\alpha^2(t)\}$, and $\lim$ is in the mean-square sense. This implies that $\hat{A} \longrightarrow r_\alpha(0)$. Hence, the NLS estimator "views" the signal as

$$y^2(t) = r_\alpha(0)e^{i2\phi(t)} + \Delta(t) \quad (10)$$

where we define $\phi(t) = a_0 + a_1 t + a_2 t^2$. Under the assumptions made on $y(t)$, we have

$$\Delta(t) = [\alpha^2(t) - r_\alpha(0)]e^{i2\phi(t)} + 2\alpha(t)n(t)e^{i2\phi(t)} + n^2(t). \quad (11)$$

Let us examine the mean and covariance sequence of $\Delta(t)$. It is readily verified that, under the assumption that $\alpha(t)$ is a stationary process and $n(t)$ is complex circular white Gaussian noise, $\Delta(t)$ is zero-mean, i.e., $\mathsf{E}\{\Delta(t)\} = 0$. Additionally

$$\begin{aligned}\gamma_\Delta(t;\tau) &= \mathsf{E}\{\Delta^*(t)\Delta(t+\tau)\} \\ &= [m_{4\alpha}(0,\tau,\tau) - r_\alpha^2(0)]e^{i2[\phi(t+\tau)-\phi(t)]} \\ &\quad + 2\sigma^2[2r_\alpha(0) + \sigma^2]\delta(\tau)\end{aligned} \quad (12)$$

where $m_{4\alpha}(0,\tau,\tau) = \mathsf{E}\{\alpha(t)\alpha(t)\alpha(t+\tau)\alpha(t+\tau)\}$. The following facts are worth noting:

- The additive noise $\Delta(t)$ is no longer Gaussian or white.
- In the case of an exponential signal, $\Delta(t)$ is stationary [since $\phi(t+\tau) - \phi(t)$ only depends on τ], whereas, for a chirp signal, the additive noise is nonstationary.

Remark 4: Here, we give some consideration to the implementation of (3) and the associated computational complexity. Since the maximization problem in (3) does not admit an analytical solution, we have to resort to numerical procedures in order to solve this problem. Since the first- and second-order derivatives are available, algorithms that have a quadratic or super linear convergence can be used. The authors' experience is that the criterion in (3) is a rather "smooth" function of a_1 and a_2, and hence, there should not be problems in finding the maximum, provided that a good initial estimate is available. The HAF-based estimator that will be presented in the next section is an excellent candidate for an initial guess. Alternatively, a fast algorithm based on the fast quadratic phase transform [23] can be used to solve (3).

We now analyze the performance of the estimates of a_1 and a_2 as given by (3). The equivalence of (3) and (4) with (6) is used to obtain the following result.

Proposition 2: The large-sample variances of $\hat{a}_1$ and $\hat{a}_2$ in (3) are given by

$$\begin{aligned}\text{var}(\hat{a}_1) &\simeq \frac{96}{N^3}\,\frac{1}{\text{SNR}}\left[1 + \frac{1}{2}\text{SNR}^{-1}\right] \\ \text{var}(\hat{a}_2) &\simeq \frac{90}{N^5}\,\frac{1}{\text{SNR}}\left[1 + \frac{1}{2}\text{SNR}^{-1}\right]\end{aligned} \quad (13)$$

where $\text{SNR} = r_\alpha(0)/\sigma^2$.

Proof: See Appendix B. ∎

We first note that, similar to the constant amplitude case, the variances of $\hat{a}_1$ and $\hat{a}_2$ are of orders $1/N^3$ and $1/N^5$, respectively. We stress that these variance expressions do not assume that $\alpha(t)$ is Gaussian or zero mean. Additionally, it can be observed that although $\alpha(t)$ may be colored, the variance expression (13) involves only the zero-lag term $r_\alpha(0)$. Finally, it is of interest to compare the above expressions with the CRB derived in [19] for the case of Gaussian amplitudes. Although the exact expression for the CRB is available (see [19, (73)]), we will use the *high SNR expression*, which is considerably simpler since it is given by (see [19, (89)] and [21])

$$\begin{aligned}\text{CRB}(\hat{a}_1) &\simeq \frac{96}{N^3}\,\frac{1}{\text{SNR}} \\ \text{CRB}(\hat{a}_2) &\simeq \frac{90}{N^5}\,\frac{1}{\text{SNR}}\end{aligned} \quad (14)$$

Comparing (13) with (14), it is seen that the NLS estimator provides nearly efficient estimates in the Gaussian case.

III. HAF-Based Estimation

Although the NLS estimator achieves the CRB in the Gaussian case, it involves a 2-D maximization problem that could be too intensive for certain applications. In this section, we consider a simpler, yet suboptimum approach with a view to decreasing computational load. It combines the use of the HAF in order to reduce the order of the polynomial phase and that of the NLS approach in order to estimate the frequency of an exponential signal with time-varying amplitude. Before describing the estimation procedure, we make the following observations. Consider first the noiseless case. It is readily verified that

$$s^*(t)s(t+\tau) = \alpha(t)\alpha(t+\tau)e^{ia_1\tau}e^{ia_2\tau^2}e^{i2a_2t\tau} \quad (15)$$

where τ is some positive integer $(\tau > 0)$. Hence, $s_2(t;\tau) = s^*(t)s(t+\tau)$ is an exponential signal with time-varying amplitude $\beta(t;\tau) = \alpha(t)\alpha(t+\tau)$. In the noisy case, we obtain

$$y_2(t;\tau) = y^*(t)y(t+\tau) = s^*(t)s(t+\tau) + n_2(t;\tau) \quad (16)$$

where

$$n_2(t;\tau) = s^*(t)n(t+\tau) + n^*(t)s(t+\tau) + n^*(t)n(t+\tau). \quad (17)$$

Hence, $n_2(t;\tau)$ is a zero-mean (since $\tau > 0$) process with covariance

$$\mathsf{E}\{n_2^*(t;\tau)n_2(t+r;\tau)\} = [2\sigma^2 r_\alpha(0) + \sigma^4]\delta(r). \quad (18)$$

Therefore, $y_2(t;\tau)$ is an exponential signal with random time-varying amplitude $\beta(t;\tau) = \alpha(t)\alpha(t+\tau)$ in complex zero-mean white noise $n_2(t;\tau)$. However, the distributions of $\beta(t;\tau)$ and $n_2(t;\tau)$ are quite complicated to obtain; hence, an optimal (e.g., maximum likelihood) approach appears not to be tractable. Thus, we are naturally led to using a NLS approach that consists of minimizing the following cost function

$$\frac{1}{N}\sum_{t=0}^{N-1} |y_2(t;\tau) - \beta(t;\tau)e^{i(\omega t+\varphi)}|^2$$

with respect to $\{\beta(t;\tau)\}_{t=0,\cdots,N-1}$, φ, and ω. Observe that this estimator is asymptotically efficient in the case of Gaussian amplitude $\beta(t;\tau)$ and additive white complex circular Gaussian noise $n_2(t;\tau)$ [15], [16]. Here, no such claim of optimality can be made since these assumptions are not satisfied. However, the NLS approach should perform well. With these preliminaries, we are now in position to describe the steps involved in the estimation of a_1 and a_2.

Step 1: For a given τ, compute $y_2(t;\tau) = y^*(t)y(t+\tau)$. Then, estimate a_2 as

$$\begin{aligned}\hat{a}_2 &= \frac{1}{2\tau} \arg\min_{\omega} \min_{\boldsymbol{\beta},\varphi} \frac{1}{N} \sum_{t=0}^{N-1} |y_2(t;\tau) - \beta(t;\tau)e^{i(\omega t+\varphi)}|^2 \\ &= \frac{1}{2\tau} \arg\max_{\omega} \frac{1}{N}\left|\sum_{t=0}^{N-1} y_2^2(t;\tau)e^{-i2\omega t}\right|. \end{aligned} \tag{19}$$

Note that $\hat{a}_2$ can be obtained via the fast Fourier transform of $y_2^2(t;\tau)$.

Step 2: Once $\hat{a}_2$ is available, demodulate $y(t)$ to obtain

$$z(t) = y(t) \times e^{-i\hat{a}_2 t^2} \simeq \alpha(t)e^{i(a_0+a_1 t)} + \tilde{n}(t) \tag{20}$$

where $\tilde{n}(t)$ combines the estimation errors in $\hat{a}_2$ and the effect of additive noise. Again, $z(t)$ is an exponential signal with time-varying amplitude, and a_1 is obtained as

$$\begin{aligned}\hat{a}_1 &= \arg\min_{\omega} \min_{\boldsymbol{\alpha},\varphi} \frac{1}{N} \sum_{t=0}^{N-1} |z(t) - \alpha(t)e^{i(\omega t+\varphi)}|^2 \\ &= \arg\max_{\omega} \frac{1}{N}\left|\sum_{t=0}^{N-1} z^2(t)e^{-i2\omega t}\right|. \end{aligned} \tag{21}$$

We note that this HAF-based approach is simpler than the NLS approach. In the next section, we will examine the tradeoffs between statistical accuracy of the NLS estimator and computational simplicity of the HAF estimator. However, as we mentioned in Section II, the NLS estimator needs to be initialized, and the HAF-based estimates can provide good initial values.

Remark 5: It can be readily verified that the estimate $\hat{a}_2$ in (19) implicitly relies on a fourth-order transformation of the data since

$$\begin{aligned}\hat{a}_2 = &\frac{1}{2\tau} \arg\max_{\omega} \frac{1}{N} \\ &\cdot\left|\sum_{t=0}^{N-1} y^*(t)y^*(t)y(t+\tau)y(t+\tau)e^{-i2\omega t}\right|. \end{aligned} \tag{22}$$

Note that such a transformation has also been proposed in [24] for the detection of signals in white multiplicative noise. However, this is to be contrasted with [20], where a second-order transformation is used. Indeed, it is generally admitted that the "classical" HAF estimator (i.e., the estimator derived for constant amplitude chirps) could handle the case of time-varying amplitudes provided that the process $\alpha(t)$ is lowpass and has a second-order HAF (i.e., power spectral density) maximum at frequency zero [6, p. 396]. Hence, we should wonder if it is worth resorting to a higher order transformation. To clarify this point, first note that [20] estimates a_2 as

$$\hat{a}_2^{(1)} = \frac{1}{2\tau} \arg\max_{\omega} \frac{1}{N}\left|\sum_{t=0}^{N-1} y^*(t)y(t+\tau)e^{-i\omega t}\right|. \tag{23}$$

To motivate this latter approach, note that

$$\begin{aligned} m_{2y}(t;\tau) &= \mathsf{E}\{y^*(t)y(t+\tau)\} = m_{1y_2}(t) \\ &= \mathsf{E}\{\alpha(t)\alpha(t+\tau)\}e^{ia_1\tau}e^{ia_2\tau^2}e^{i2a_2 t\tau}. \\ &= r_\alpha(\tau)e^{ia_1\tau}e^{ia_2\tau^2}e^{i2a_2 t\tau} \end{aligned}$$

Hence, the cyclic mean of $y_2(t;\tau)$ [or, equivalently, the cyclic second-order moment of $y(t)$], which is the generalized Fourier series expansion of $m_{2y}(t;\tau) = m_{1y_2}(t)$, will peak at $\omega = 2a_2\tau$. This is because the process $\beta(t;\tau) = \alpha(t)\alpha(t+\tau)$ is not zero mean. Additionally, for large N, we have from [22]

$$\begin{aligned} &\frac{1}{N}\sum_{t=0}^{N-1} y^*(t)y(t+\tau)e^{-i\omega t} \\ &\quad\longrightarrow \frac{1}{N}\sum_{t=0}^{N-1} \mathsf{E}\{y^*(t)y(t+\tau)\}e^{-i\omega t} \\ &\quad\longrightarrow r_\alpha(\tau)e^{ia_1\tau}e^{ia_2\tau^2}\delta(\omega - 2a_2\tau) \end{aligned}$$

and hence, (23) is a consistent estimate of a_2. Thus, it should be sufficient to use a second-order transformation. However, this statement should be revisited in light of the following observations. In [16], it is shown that even if $\beta(t;\tau)$ is not zero mean, the estimate (23) based on the cyclic mean of $y_2(t;\tau)$ does not necessarily outperform the estimate (19) based on the cyclic variance of $y_2(t;\tau)$. Briefly stated, the relative performance of the two estimates depends on the respective values of the "coherent" signal-to-noise ratio (SNR) $R_1 = \mathsf{E}^2\{\beta(t;\tau)\}/\text{var}\{n_2(t;\tau)\}$ and the "noncoherent" SNR $R_2 = \text{var}\{\beta(t;\tau)\}/\text{var}\{n_2(t;\tau)\}$. Additionally, it was shown that for white Gaussian additive noise, if $R_2 > 0.5$, the estimator based on the cyclic variance outperforms the estimator based on the cyclic mean. In the present case, it is readily verified that $R_2 = (SNR/2)(1+r_\alpha^2(\tau)/r_\alpha^2(0)/1+0.5SNR^{-1})$, and hence, R_2 is generally greater than 0.5. Although the conclusions of [16] cannot be directly transposed to the present case since $n_2(t;\tau)$ is not Gaussian and independent of $\beta(t;\tau)$, they clearly indicate that superiority of (23) over (19) is not immediate. A more theoretically sound response on this point will be given in Proposition 3. Finally, we note that the NLS approach does not make any distinction between the zero-mean and the nonzero-mean cases; it leads naturally to the estimate (19). Additionally, the computational increase compared with using the classical HAF amounts to N multiplications in order to compute the sequence $y^2(t)$.

Remark 6: It should be pointed out that the present approach, in its implementation, is equivalent to the classical HAF estimator with $y(t)$ replaced by $y^2(t)$ in the estimation procedure. Therefore, it tacitly considers that the square of the data is a constant-amplitude chirp signal. A similar remark has also been made for the NLS estimator.

Since the HAF-based scheme sequentially estimate a_2, a_1, a_0, its performance will highly depend on the variance of the a_2 estimate. Therefore, we concentrate on this parameter and now derive its asymptotic variance.

Proposition 3: Assuming that $N-\tau \gg 1$, the large sample variance of the HAF estimate of a_2 (see (19)) is given by

$$\text{var}(\hat{a}_2) \simeq \frac{3}{(N-\tau)^3} \frac{D_2(\tau)}{4\tau^2 m_{4\alpha}^2(0,\tau,\tau)} \tag{24}$$

with

$$\begin{aligned} D_2(\tau) =& 4\sigma^2 m_{6\alpha}(0,0,0,\tau,\tau) + 2\sigma^4 m_{4\alpha}(0,0,0) \\ &+ 8\sigma^4 m_{4\alpha}(0,\tau,\tau) + 8\sigma^6 r_\alpha(0) + 2\sigma^8 \\ &- [4\sigma^2 m_{6\alpha}(0,\tau,\tau,2\tau,2\tau) + 2\sigma^4 m_{4\alpha}(0,2\tau,2\tau)] \\ &\times \frac{(N-2\tau)(N^2-4\tau N+\tau^2)}{(N-\tau)^3} \mathbf{1}(N-2\tau) \end{aligned} \tag{25}$$

where $\mathbf{1}(\cdot)$ is the unit step function, and $m_{k\alpha}(\cdot)$ denotes the kth-order moment of $\alpha(t)$, i.e., $m_{k\alpha}(\tau_1,\tau_2,\cdots,\tau_{k-1}) = \mathsf{E}\{\alpha(t)\alpha(t+\tau_1)\cdots\alpha(t+\tau_{k-1})\}$.

Proof: See Appendix C. ∎

Observe that the variance of the HAF-based estimator depends on τ and the fourth- and sixth-order moments of $\alpha(t)$. Hence, derivation of an optimal τ solely as a function of N, as in the constant amplitude case, appears not to be directly feasible. However, the form of (24) suggests than an optimal τ should be close to $0.5N$. For $\tau = N/2$ and in the high SNR case, onee could readily show that [assuming $\alpha(t)$ is Gaussian]

$$\text{var}(\hat{a}_2)_{\tau=N/2} \simeq \frac{288}{N^5 \text{SNR}}$$

which is approximately 3.2 times the corresponding CRB. The variance at $\tau = N/2$ depends on $\alpha(t)$ only through its power $r_\alpha(0)$. Therefore, although the performance of the estimator depends on the spectral characteristics of $\alpha(t)$, the variance (24) should not be too sensitive to it. Finally, we stress that in contrast with the constant-amplitude chirp case, the HAF estimator does not provide a nearly efficient estimator.

Remark 7: A similar analysis can be carried out for the "conventional" HAF estimator, i.e., that based on $y(t)$. We omit the derivations since it follows along the lines of Appendix C. It can be proved that the variance of a_2's estimate is given by

$$\text{var}(\hat{a}_2^{\text{HAF}(y)}) \simeq \frac{3}{(N-\tau)^3} \frac{D_1(\tau)}{\tau^2 r_\alpha^2(\tau)} \tag{26}$$

with

$$\begin{aligned} D_1(\tau) =& \sigma^2 r_\alpha(0) + \frac{1}{2}\sigma^4 - \sigma^2 r_\alpha(2\tau) \\ &\cdot \frac{(N-2\tau)(N^2-4\tau N+\tau^2)}{(N-\tau)^3} \mathbf{1}(N-2\tau). \end{aligned}$$

Numerical evaluation of (24) and (26) clearly indicates that the variance of the classical HAF estimator is (very) superior to the variance of the estimator proposed here. This confirms the ideas of Remark 5. The numerical examples of the next section will also corroborate this fact.

In the case of a chirp signal with *constant amplitude* A, it can be verified that (26) coincides with the expression established by Peleg and Porat (see, e.g., [4, (31)]). Additionally, in the constant amplitude case and assuming terms in $\text{SNR}^{-2}, \text{SNR}^{-3}, \cdots$ are negligible, it is straightforward to verify that

$$\begin{aligned} \text{var}(\hat{a}_2^{\text{HAF}(y^2)}) &\simeq \frac{5}{(N-\tau)^3 \text{SNR}} \times \frac{1}{\tau^2} \\ \text{var}(\hat{a}_2^{\text{HAF}(y)}) &\simeq \frac{3}{(N-\tau)^3 \text{SNR}} \times \frac{1}{\tau^2}. \end{aligned} \tag{27}$$

Hence, in the constant amplitude case, the HAF estimator based on $y(t)$ should generally be preferred to the HAF estimator based on $y^2(t)$.

IV. Numerical Examples

The aim of this section is threefold. First, we study the performance of the HAF-based scheme and the validity of the theoretical analysis. Accordingly, the influence of τ on the performance of the estimator will be emphasized. Next, we compare the empirical performance of the NLS estimator with the CRB and verify the validity of the theoretical formulas for the asymptotic variances. Finally, we compare the performances of the suboptimum HAF-based scheme with that of the NLS estimator. Additionally, we provide a comparison with the "classical" HAF estimator based on $y(t)$ (which does not take into account the time variation of the amplitude but is expected to perform well under certain conditions). Note that the method of [20] is essentially equivalent to the HAF scheme based on $y(t)$, and only the results of the latter will be reported. In all the simulations, the time-varying amplitude $\alpha(t)$ is generated as a zero-mean $AR(p)$ process, and the additive noise is complex circular white Gaussian with variance σ^2. The SNR is defined as $\text{SNR} = r_\alpha(0)/\sigma^2$. In all simulations, the chirp parameters are $a_0 = 2\pi \times 0.1$, $a_1 = 2\pi \times 0.18$, and $a_2 = 2\pi \times 3 \times 10^{-4}$. The cases of an $AR(1)$ process with parameter ρ and of an $AR(2)$ process with poles at $\rho e^{\pm 2\pi i \times f}$ will be considered. Five hundred Monte Carlo trials were run to estimate the mean square errors of the estimates.

A. HAF-Based Scheme: Influence of τ

In this subsection, we study the influence of τ on the performance of the HAF-based estimators. We will refer to the "classical" HAF-estimator that uses $y(t)$ as the HAF(y) estimator, whereas the new HAF-based scheme proposed here will be denoted HAF(y^2) in the sequel. Figs. 1 and 2 display the theoretical (dotted lines) and empirical variances ("+") of the HAF(y^2) estimator as a function of τ in the case of an $AR(1)$ and $AR(2)$ process, respectively.

It can be observed that the variance begins to decrease when τ is increased. Then, a nearly constant variance is obtained for $\tau \in [0.2N, 0.5N]$. When τ becomes large, the variance tends to increase. As was expected, the optimal τ is around $0.4N - 0.5N$. However, we can choose τ in the range $[0.2N, 0.5N]$ without penalizing the performance of estimation too much. This is an interesting feature of the method. Finally, it is observed that the theoretical analysis predicts fairly well the simulation results, provided that τ is not too large (note that the analysis assumes $N - \tau \gg 1$, which is no longer the case when $\tau \approx 0.7N$ and $N = 256$).

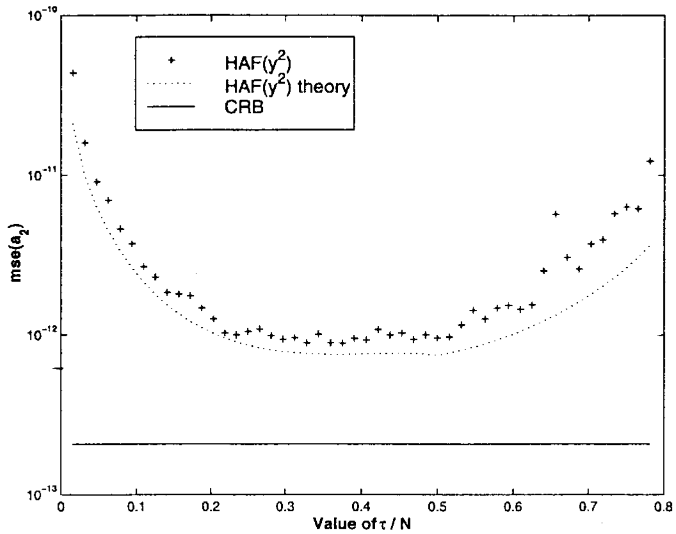

Fig. 1. Influence of τ on the performance of the HAF(y^2) estimator in the $AR(1)$ case. Dotted lines: theoretical variance. "+": empirical variance. solid lines: CRB. $\rho = 0.95$, $N = 256$, and $SNR = 10$ dB.

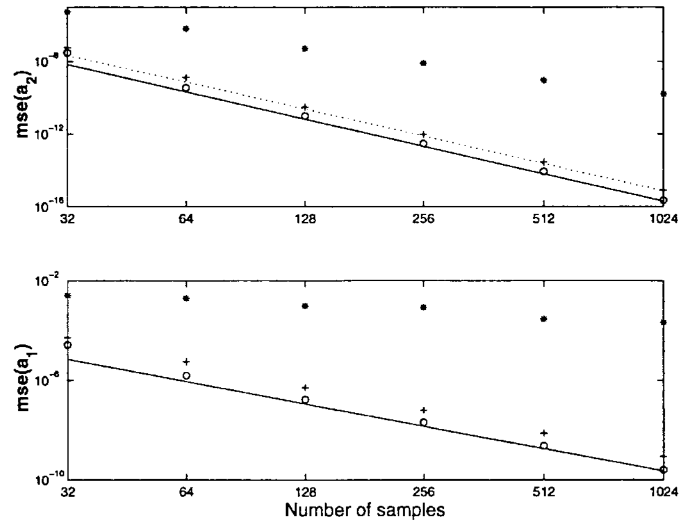

Fig. 3. CRB (solid lines) and mean square errors of $\hat{a}_1$ and $\hat{a}_2$ versus number of samples in the $AR(1)$ case. "*": HAF(y). Dotted lines: HAF(y^2) theory. "+": HAF(y^2). "o": NLS. $\rho = 0.95$ and $SNR = 10$ dB.

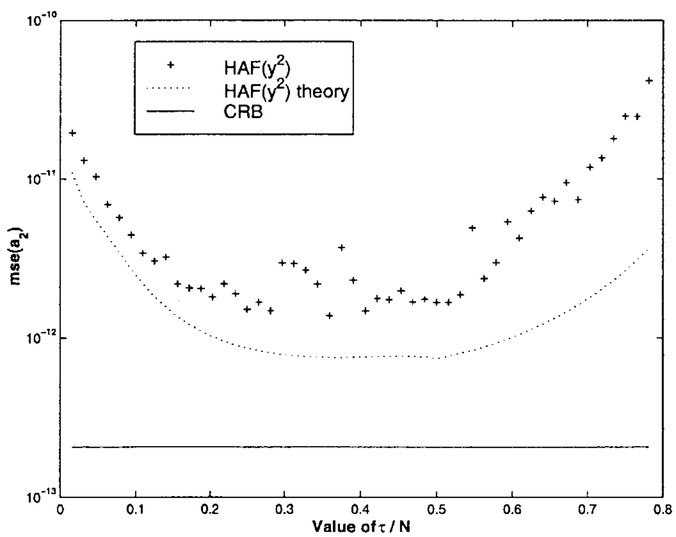

Fig. 2. Influence of τ on the performance of the HAF(y^2) estimator in the $AR(2)$ case. Dotted lines: theoretical variance. "+": empirical variance. Solid lines: CRB. $\rho = 0.95$, $f = 0.01$, $N = 256$, and $SNR = 10$ dB.

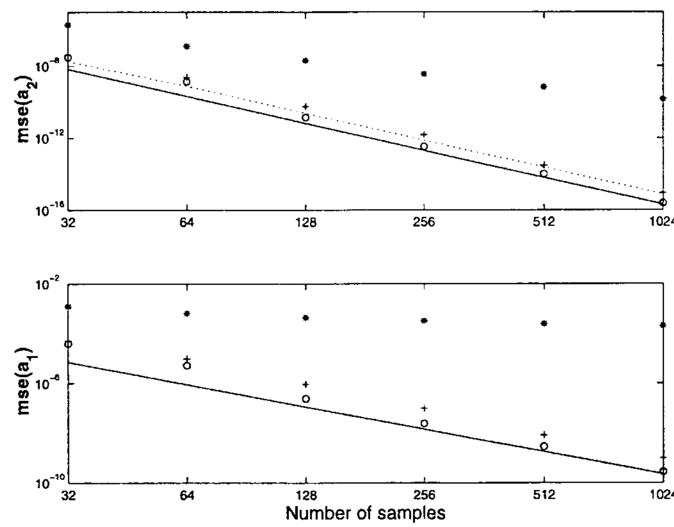

Fig. 4. CRB (solid lines) and mean square errors of $\hat{a}_1$ and $\hat{a}_2$ versus number of samples in the $AR(2)$ case. "*": HAF(y). Dotted lines: HAF(y^2) theory. "+": HAF(y^2). "o": NLS. $\rho = 0.95$, $f = 0.01$, and $SNR = 10$ dB.

B. Comparison Between the NLS and HAF Estimators

We now compare the performance of the NLS estimator with that of the HAF estimators. In what follows, the amplitude is either an $AR(2)$ process with poles at $0.95e^{\pm 2i\pi \times 0.01}$ or an $AR(1)$ process whose pole modulus is $\rho = 0.95$. In all simulations, τ is chosen as $\tau = 0.4N$. Figs. 3–6 display the influences of N and SNR, respectively, on the performance of the estimators. Since the theoretical variance of the NLS estimator is almost indistinguishable from the CRB's, only the latter of these are plotted.

The following points are worth noting:

- The NLS estimator is seen to come close to the CRB, provided that N and SNR are sufficiently large (typically $N \geq 256$ and $SNR \geq 10$ dB). This validates the theoretical analysis.
- Using $y^2(t)$ in lieu of $y(t)$ in the HAF procedure considerably improves the estimation performance. As a matter of fact, the HAF(y^2) estimator outperforms the classical HAF estimator, whose performance is quite poor. Indeed, in the case of zero-mean amplitude, the classical HAF does not provide a consistent estimate: a fact also noted in [16]. The method of [20] offers a slight improvement at least for the estimation of a_1 [note that it provides the same estimate of a_2 as the HAF estimator that uses $y(t)$].
- The HAF(y^2) scheme performs comparably with the NLS estimator for small N or low SNR. In contrast, the NLS estimator performs better for large N or high SNR. The ratio between the variance of the HAF(y^2) estimator and the variance of the NLS estimator is about 3.2 for large N, as predicted by the theory. Hence, the gain in computation

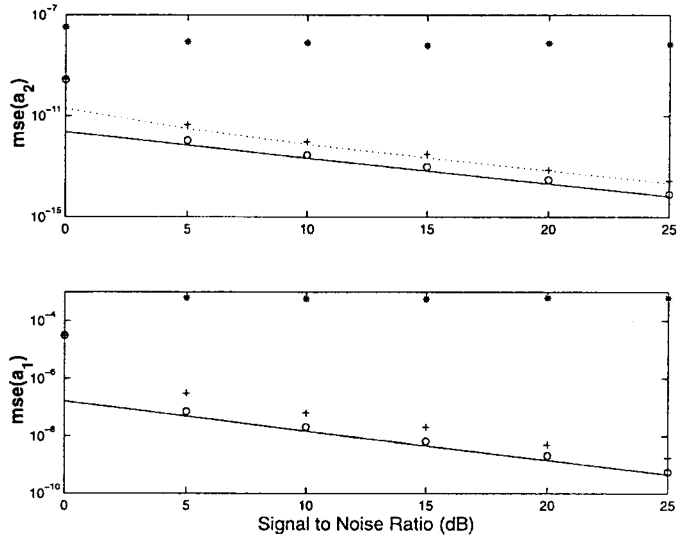

Fig. 5. CRB (solid lines) and mean square errors of $\hat{a}_1$ and $\hat{a}_2$ versus SNR in the $AR(1)$ case. "$*$": HAF(y). Dotted lines: HAF(y^2) theory. "$+$": HAF(y^2). "$\circ$": NLS. $\rho = 0.95$ and $N = 256$.

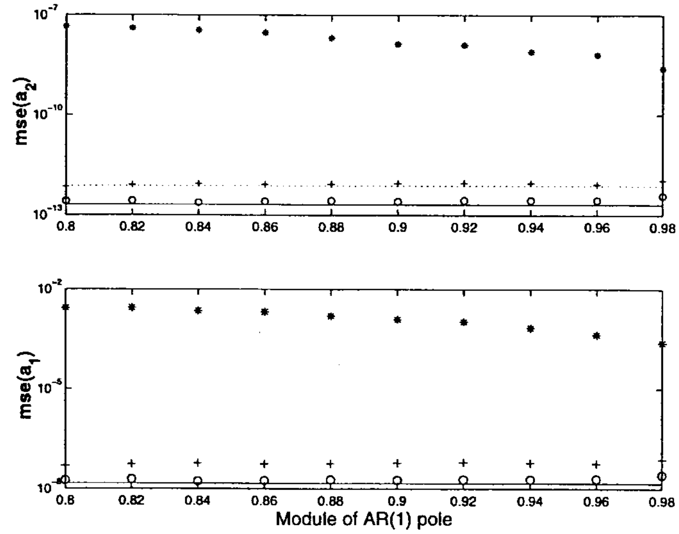

Fig. 7. CRB (solid lines) and mean square errors of $\hat{a}_1$ and $\hat{a}_2$ versus module of AR(1) pole. "$*$": HAF(y). Dotted lines: HAF(y^2) theory, "$+$": HAF(y^2). "$\circ$": NLS. $N = 256$ and $SNR = 10$ dB.

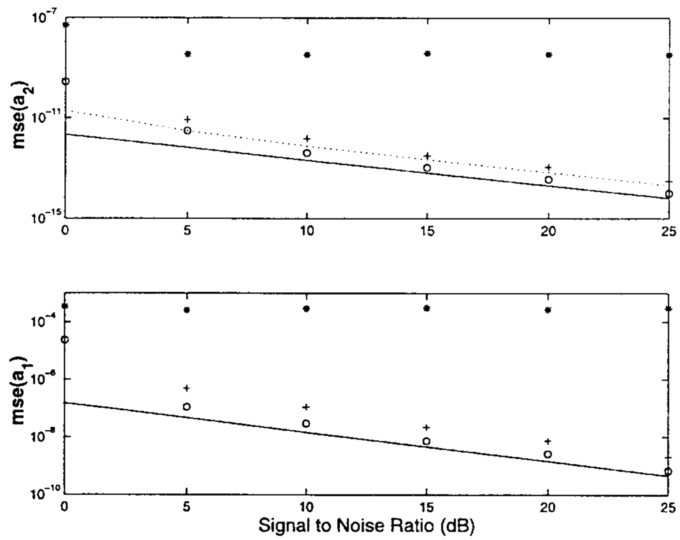

Fig. 6. CRB (solid lines) and mean square errors of $\hat{a}_1$ and $\hat{a}_2$ versus SNR in the $AR(2)$ case. "$*$": HAF(y). Dotted lines: HAF(y^2) theory. "$+$": HAF(y^2). "$\circ$": NLS. $\rho = 0.95$, $f = 0.01$, and $N = 256$.

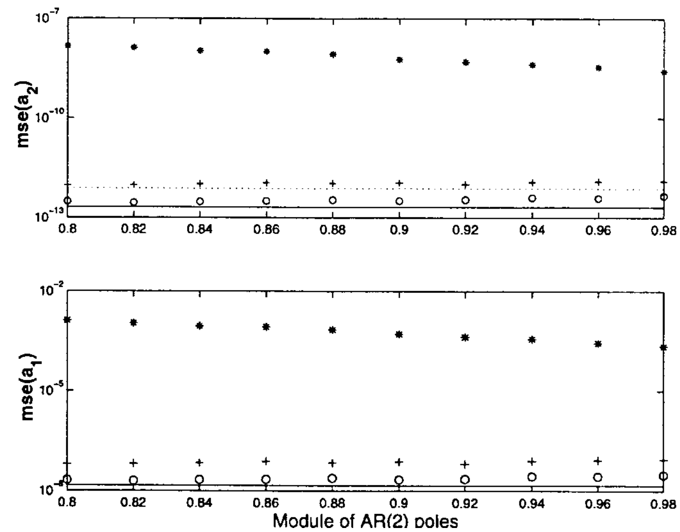

Fig. 8. CRB (solid lines) and mean square errors of $\hat{a}_1$ and $\hat{a}_2$ versus module of AR(2) poles. "$*$": HAF(y). Dotted lines: HAF(y^2) theory. "$+$": HAF(y^2). "$\circ$": NLS. $f = 0.01$, $N = 256$, and $SNR = 10$ dB.

of the HAF(y^2) scheme is counterbalanced by some loss of accuracy.

- The HAF(y^2) estimator (and in certain respect the NLS estimator) exhibits the threshold effect in SNR, which is inherent to nonlinear transformations and has already been reported in other studies on the same kind of algorithms.

Next, we study the influence of the bandwidth of the time-varying amplitude on the performance of the estimators. To this end, Monte Carlo simulations were run for the $AR(1)$ and $AR(2)$ cases by varying ρ (the modulus of the AR poles) and f [frequency of the $AR(2)$ poles]. The results are shown in Figs. 7–9. As can be seen, the performance remains stable wrt variations of the amplitude bandwidth and corroborates the "hierarchy" between the estimators established in the previous simulations.

V. Conclusions

We addressed the problem of estimating the parameters of chirp signals with randomly time-varying amplitude. Two methods were proposed: First, an unstructured nonlinear least-squares approach was presented and analyzed from a theoretical point of view. It was shown that the NLS estimator achieves the CRB for large N. Since the NLS estimator requires a 2-D search for a maximum, an alternative and simpler approach was proposed. It utilizes the HAF scheme in order to reduce polynomial order along with the NLS approach to estimate the remaining component, which is a complex exponential signal with time-varying amplitude. Statistical analysis was carried out showing that this estimator has a variance only 3.2 times greater than the CRB when the amplitude is a Gaussian process. Closed-form expressions were derived for the large sample variances of the NLS

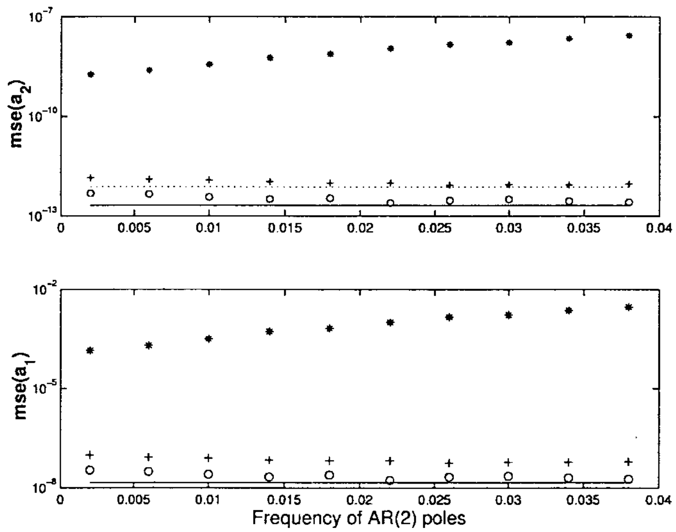

Fig. 9. CRB (solid lines) and mean square errors of $\hat{a}_1$ and $\hat{a}_2$ versus frequency of AR(2) poles. "$*$": HAF(y). Dotted lines: HAF(y^2) theory. "+": HAF(y^2). "o": NLS. $\rho = 0.95$, $N = 256$, and $SNR = 10$ dB.

estimator and the HAF estimators based on the data and squared data. Simulation results were presented that attested to the validity of the theoretical analysis. The NLS estimator was shown to provide slightly better performance than the HAF-based estimator. Additionally, these two estimators were shown to outperform the classical HAF estimator, which was previously proposed to solve this problem.

Appendix A
Derivation of the NLS Estimator

Let

$$y(t) = \alpha(t)e^{i\phi(t)} + n(t) \qquad t = 0, \cdots, N-1 \tag{28}$$

where $\alpha(t)$ is a real-valued stationary process, and $\phi(t) = \Sigma_{n=0}^{M} a_n t^n$. We will focus on the case $M = 2$, which corresponds to a chirp signal but the results holds for any M. The NLS estimates of $\{\alpha(t)\}_{t=0,\cdots,N-1}$ and $\mathbf{a} = [a_0, \cdots, a_M]^T$ are obtained as the minimizing arguments of the following criterion:

$$\{\hat{\boldsymbol{\alpha}}, \hat{\mathbf{a}}\} = \arg \min_{\mathbf{x}, \boldsymbol{\theta}} \frac{1}{N} \sum_{t=0}^{N-1} |y(t) - x(t)e^{i\psi(t)}|^2 \tag{29}$$

where $\psi(t) = \Sigma_{n=0}^{M} \theta_n t^n$. Let $\mathbf{y} = [y(0), \cdots, y(N-1)]^T$, $\mathbf{x} = [x(0), \cdots, x(N-1)]^T$, and $\mathbf{H} = \mathrm{diag}(e^{i\psi(0)}, \cdots, e^{i\psi(N-1)})$ so that the criterion in (29) can be written as

$$J(\mathbf{x}, \boldsymbol{\theta}) = \frac{1}{N} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2. \tag{30}$$

Differentiating with respect to $\mathbf{x}$, we obtain

$$\frac{\partial J}{\partial \mathbf{x}} = \frac{1}{N}(-\mathbf{H}\mathbf{y}^* - \mathbf{H}^H\mathbf{y} + 2\mathbf{x}). \tag{31}$$

Hence, for any given value of $\boldsymbol{\theta}$, the vector $\mathbf{x}$ that minimizes (30) is given by

$$\mathbf{x} = \tfrac{1}{2}\{\mathbf{H}\mathbf{y}^* + \mathbf{H}^H\mathbf{y}\}. \tag{32}$$

Substituting the last equation in (30), we need to minimize

$$\begin{aligned}\tilde{J}(\boldsymbol{\theta}) &= \frac{1}{N}\left\|\mathbf{y} - \frac{1}{2}\mathbf{H}^2\mathbf{y}^* - \frac{1}{2}\mathbf{y}\right\|^2 \\ &= \frac{1}{4N}\|\mathbf{y} - \mathbf{H}^2\mathbf{y}^*\|^2 \\ &= \frac{1}{2N}\mathbf{y}^H\mathbf{y} - \frac{1}{2N}\,\mathrm{Re}\,\{\mathbf{y}^T\mathbf{H}^{2*}\mathbf{y}\}\end{aligned} \tag{33}$$

or, equivalently, to maximize

$$\begin{aligned}&\frac{1}{N}\,\mathrm{Re}\,\{\mathbf{y}^T\mathbf{H}^{2*}\mathbf{y}\} \\ &\quad= \frac{1}{N}\mathrm{Re}\left\{\sum_{t=0}^{N-1} y^2(t)e^{-i2\psi(t)}\right\} \\ &\quad= \frac{1}{N}\mathrm{Re}\left\{\sum_{t=0}^{N-1} y^2(t)e^{-2i\Sigma_{n=1}^{M}\theta_n t^n} \times e^{-2i\theta_0}\right\}\end{aligned} \tag{34}$$

where $\mathrm{Re}\{x\}$ denotes the real part of x. For any complex number $x = \rho e^{i\theta}$, the maximum value of $\mathrm{Re}\{xe^{-i\varphi}\}$ is ρ and is obtained for $\varphi = \theta$. Hence, the parameters $\{a_1, \cdots, a_M\}$ and a_0 are given by

$$\begin{aligned}\{\hat{a}_1, \cdots, \hat{a}_M\} &= \arg \max_{\theta_1, \cdots, \theta_M} \frac{1}{N} \\ &\qquad \cdot \left|\sum_{t=0}^{N-1} y^2(t)\exp\left\{-2i\left(\sum_{n=1}^{M}\theta_n t^n\right)\right\}\right| \\ \hat{a}_0 &= \frac{1}{2}\ \mathrm{angle} \\ &\qquad \cdot \left\{\sum_{t=0}^{N-1} y^2(t)\exp\left\{-2i\left(\sum_{n=1}^{M}\hat{a}_n t^n\right)\right\}\right\}\end{aligned} \tag{35}$$

which concludes the derivation.

Appendix B
Proof of (13)

In this Appendix, we derive the large-sample variances of the NLS estimates of a_1 and a_2. First, we show that $\hat{a}_1$ and $\hat{a}_2$ in (3) are consistent. Recall that the NLS estimate of $\overline{\mathbf{a}} \triangleq [a_1\ a_2]^T$ is obtained as

$$\hat{\overline{\mathbf{a}}} = \arg \max_{\overline{\theta}} |f_N(\overline{\boldsymbol{\theta}})|$$

with $\overline{\boldsymbol{\theta}} \triangleq [\theta_1\ \theta_2]^T$ and where

$$f_N(\overline{\boldsymbol{\theta}}) = \frac{1}{N} \sum_{t=0}^{N-1} y^2(t)e^{-i2(\theta_1 t + \theta_2 t^2)}.$$

To prove consistency, we need to show that $f_\infty(\overline{\boldsymbol{\theta}}) = \lim_{N\to\infty} |f_N(\overline{\boldsymbol{\theta}})|$ achieves a global maximum at $\overline{\mathbf{a}}$ and that this maximum is unique. Using the results of Dandawaté and Giannakis [22] and under the assumptions made on $\alpha(t)$ and

$n(t)$, we have

$$\begin{aligned}\lim_{N\to\infty} f_N(\bar{\boldsymbol{\theta}}) &= \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} y^2(t)e^{-i2(\theta_1 t+\theta_2 t^2)} \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} \mathsf{E}\{y^2(t)\}e^{-i2(\theta_1 t+\theta_2 t^2)} \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} r_\alpha(0)e^{i2(a_1 t+a_2 t^2)} \\ &\quad \cdot e^{-i2(\theta_1 t+\theta_2 t^2)} \\ &= r_\alpha(0)\lim_{N\to\infty} g_N(\bar{\boldsymbol{\theta}}) \end{aligned} \tag{36}$$

with

$$\begin{aligned} g_N(\bar{\boldsymbol{\theta}}) &= \langle s_N(t;\bar{\boldsymbol{\theta}}), s_N(t;\bar{\mathbf{a}})\rangle \\ s_N(t;\bar{\boldsymbol{\theta}}) &= \frac{1}{\sqrt{N}}e^{i2(\theta_1 t+\theta_2 t^2)} \end{aligned}$$

and the scalar product is defined as $\langle f(t), g(t)\rangle = \Sigma_{t=0}^{N-1} f^*(t)g(t)$. By the Cauchy–Schwartz inequality, we have

$$|g_N(\bar{\boldsymbol{\theta}})|^2 \le 1 = |g_N(\bar{\mathbf{a}})|^2$$

where equality holds if and only if

$$\begin{aligned}|g_N(\bar{\boldsymbol{\theta}})| = |g_N(\bar{\mathbf{a}})| &\Leftrightarrow s_N(t;\bar{\boldsymbol{\theta}}) = Ks_N(t;\bar{\mathbf{a}}) \\ &\Leftrightarrow \bar{\boldsymbol{\theta}} = \bar{\mathbf{a}}. \end{aligned}$$

Using (36) along with the fact that $g_N(\bar{\boldsymbol{\theta}})$ has a unique global maximum at $\bar{\mathbf{a}}$, it follows, by a continuity argument, that $f_\infty(\bar{\boldsymbol{\theta}})$ achieves its unique global maximum at $\bar{\mathbf{a}}$, which proves consistency of $\hat{a}_1$ and $\hat{a}_2$. Next, we establish an expression for their asymptotic variances. As was pointed out in Remark 3, the estimate of $\mathbf{a}$ in (3) and (4) is equivalent to the estimate of $\mathbf{a}$ in (6), i.e.,

$$\begin{aligned}\{\hat{\mathbf{a}}, \hat{A}\} &= \arg\min_{\tilde{\mathbf{a}},\tilde{A}} J(\tilde{\mathbf{a}},\tilde{A}) = \frac{1}{N}\sum_{t=0}^{N-1} |y^2(t) - \tilde{A}e^{i2\tilde{\phi}(t)}|^2 \\ &= \arg\min_{\tilde{\mathbf{a}},\tilde{A}} \frac{1}{N}\|\mathbf{y}\odot\mathbf{y} - \tilde{A}\tilde{\mathbf{e}}\|^2 \end{aligned} \tag{37}$$

where $\tilde{\phi}(t) = \Sigma_{n=0}^{2}\tilde{a}_n t^n$, $\mathbf{y} = [y(0),\cdots,y(N-1)]^T$, $\tilde{\mathbf{e}} = [1, e^{i2\tilde{\phi}(1)},\cdots,e^{i2\tilde{\phi}(N-1)}]^T$, and $\odot$ denotes the element-wise (i.e., Hadamard) product. We focus on (37) in order to derive the asymptotic performance of the NLS estimate of $\mathbf{a}$. We assume that N is large so that we can make use of a standard Taylor series expansion to obtain the asymptotic covariance of the NLS estimates. Toward this objective, we first approximate $\tilde{A}e^{i2(\tilde{a}_0+\tilde{a}_1 t+\tilde{a}_2 t^2)}$ in (37) by its first-order Taylor expansion to obtain

$$\begin{aligned} y^2(t) - \tilde{A}e^{i2\tilde{\phi}(t)} &\simeq y^2(t) - r_\alpha(0)e^{i2\phi(t)} \\ &\quad - (\tilde{A} - r_\alpha(0))e^{i2\phi(t)} \\ &\quad - 2ir_\alpha(0)(\tilde{a}_0 - a_0)e^{i2\phi(t)} \\ &\quad - 2itr_\alpha(0)(\tilde{a}_1 - a_1)e^{i2\phi(t)} \\ &\quad - 2it^2 r_\alpha(0)(\tilde{a}_2 - a_2)e^{i2\phi(t)} \end{aligned} \tag{38}$$

where $\phi(t) = a_0 + a_1 t + a_2 t^2$. Noting that $y^2(t) - r_\alpha(0)e^{i2\phi(t)} = \Delta(t)$ [cf. (10)], we can write

$$\begin{aligned}\mathbf{y}\odot\mathbf{y} - \tilde{A}\tilde{\mathbf{e}} &\simeq \boldsymbol{\Delta} - (\tilde{A} - r_\alpha(0))\mathbf{e} \\ &\quad - 2ir_\alpha(0)\sum_{k=0}^{2}(\tilde{a}_k - a_k)[\mathbf{t}^k\odot\mathbf{e}] \end{aligned} \tag{39}$$

with $\mathbf{e} = [1, e^{i2\phi(1)},\cdots,e^{i2\phi(N-1)}]^T$ and where we define $\mathbf{t}^0 = [1,1,\cdots,1]^T$ and $\mathbf{t}^m = \mathbf{t}^{m-1}\odot[0,1,\cdots,N-1]^T$ for $m \ge 1$. Differentiating the squared norm of the expression on the right-hand side of the previous equation with respect to $\tilde{a}_m$, $m = 0,\cdots,2$, and equating it to 0 at the estimated values, we readily obtain

$$\mathrm{Im}\{(\mathbf{t}^m\odot\mathbf{e})^H\boldsymbol{\Delta}\} - 2r_\alpha(0)\sum_{k=0}^{2}(\hat{a}_k - a_k)(\mathbf{t}^m)^T\mathbf{t}^k = 0 \tag{40}$$

where Im$\{x\}$ denotes the imaginary part of a complex variable x. We normalize the above equation by $N^{m+1/2}$ to get

$$\begin{aligned}\frac{1}{2\sqrt{N}}&\sum_{t=0}^{N-1}\mathrm{Im}\left\{\left(\frac{t}{N}\right)^m\Delta(t)e^{-2i\phi(t)}\right\} \\ &\simeq N^{1/2}(\hat{a}_0 - a_0)\frac{r_\alpha(0)}{N}\sum_{t=0}^{N-1}\left(\frac{t}{N}\right)^m \\ &\quad + N^{3/2}(\hat{a}_1 - a_1)\frac{r_\alpha(0)}{N}\sum_{t=0}^{N-1}\left(\frac{t}{N}\right)^{m+1} \\ &\quad + N^{5/2}(\hat{a}_2 - a_2)\frac{r_\alpha(0)}{N}\sum_{t=0}^{N-1}\left(\frac{t}{N}\right)^{m+2} \end{aligned} \tag{41}$$

for $m = 0,1,2$. Using [25]

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\left(\frac{t}{N}\right)^k = \frac{1}{k+1}, \qquad k = 0,1,2,\cdots \tag{42}$$

along with (41), we obtain the asymptotic expression

$$\mathbf{K}_N(\hat{\mathbf{a}} - \mathbf{a}) \simeq r_\alpha^{-1}(0)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varepsilon} \tag{43}$$

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \tag{44}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{1}{2\sqrt{N}}\sum_{t=0}^{N-1}\mathrm{Im}\{\Delta(t)e^{-i2(a_0+a_1 t+a_2 t^2)}\} \\ \frac{1}{2\sqrt{N}}\sum_{t=0}^{N-1}\left(\frac{t}{N}\right)\mathrm{Im}\{\Delta(t)e^{-i2(a_0+a_1 t+a_2 t^2)}\} \\ \frac{1}{2\sqrt{N}}\sum_{t=0}^{N-1}\left(\frac{t}{N}\right)^2\mathrm{Im}\{\Delta(t)e^{-i2(a_0+a_1 t+a_2 t^2)}\} \end{bmatrix} \tag{45}$$

and $\mathbf{K}_N = \mathrm{diag}(N^{1/2}, N^{3/2}, N^{5/2})$. To derive the asymptotic performance of the NLS estimate $\hat{\mathbf{a}}$, we need to compute the asymptotic covariance matrix of the random vector $\boldsymbol{\varepsilon}$

$$\mathbf{R}_\varepsilon = \lim_{N\to\infty}\mathsf{E}\{\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\}. \tag{46}$$

Observing that [see (11)]

$$\mathrm{Im}\,\{\Delta(t)e^{-i2\phi(t)}\} = \mathrm{Im}\,\{2\alpha(t)n(t) + n^2(t)e^{-i2\phi(t)}\} \triangleq \mathrm{Im}\,\{\tilde{n}(t)\}$$

and noting that $\mathrm{Im}[a]\mathrm{Im}[b] = \frac{1}{2}\{\mathrm{Re}[ab^*] - \mathrm{Re}[ab]\}$, the (k,ℓ)th element of R_ε is given by

$$\begin{aligned}\mathbf{R}_\varepsilon(k,\ell) &= \lim_{N\to\infty} \sum_{t,s=0}^{N-1} \left(\frac{t}{N}\right)^k \left(\frac{s}{N}\right)^\ell \cdot \mathsf{E}\,\{\mathrm{Im}[\tilde{n}(t)]\mathrm{Im}[\tilde{n}(s)]\}\\ &= \lim_{N\to\infty} \frac{1}{8N} \sum_{t,s=0}^{N-1} \left(\frac{t}{N}\right)^k \left(\frac{s}{N}\right)^\ell \mathrm{Re}\{E\{\tilde{n}(t)\tilde{n}^*(s)\} - E\{\tilde{n}(t)\tilde{n}(s)\}\}. \quad (47)\end{aligned}$$

Furthermore

$$\begin{aligned}\mathsf{E}\,\{\tilde{n}(t)\tilde{n}^*(s)\} &= 4r_\alpha(0)\sigma^2\delta(t-s) + 2\sigma^4\delta(t-s)\\ \mathsf{E}\,\{\tilde{n}(t)\tilde{n}(s)\} &= 0 \quad (48)\end{aligned}$$

where we use the fact that $\mathsf{E}\{n^2(t)n^{*2}(s)\} = 2\sigma^4\delta(t-s)$, and $\mathsf{E}\{n^2(t)n^2(s)\} = 0$ since $n(t)$ is a white complex circular Gaussian noise. Reporting (48) in (47), it follows that

$$\begin{aligned}\mathbf{R}_\varepsilon(k,\ell) &= \lim_{N\to\infty} \frac{1}{8N} \sum_{t=0}^{N-1} \left(\frac{t}{N}\right)^{k+\ell} \{4r_\alpha(0)\sigma^2 + 2\sigma^4\}\\ &= \frac{1}{2}[r_\alpha(0)\sigma^2 + 0.5\sigma^4] \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \left(\frac{t}{N}\right)^{k+\ell}\\ &= \frac{1}{2(k+\ell+1)}[r_\alpha(0)\sigma^2 + 0.5\sigma^4]. \quad (49)\end{aligned}$$

It should be stressed that it is not required that $\alpha(t)$ be a Gaussian process to obtain the previous equation. The entire matrix $\mathbf{R}_\varepsilon$ is then found to be

$$\begin{aligned}\mathbf{R}_\varepsilon &= \tfrac{1}{2}[r_\alpha(0)\sigma^2 + 0.5\sigma^4] \begin{bmatrix} 1 & 1/2 & 1/3\\ 1/2 & 1/3 & 1/4\\ 1/3 & 1/4 & 1/5 \end{bmatrix}\\ &= \tfrac{1}{2}[r_\alpha(0)\sigma^2 + 0.5\sigma^4]\boldsymbol{\Lambda}. \quad (50)\end{aligned}$$

Finally, the asymptotic covariance of the NLS estimate of $\mathbf{a}$ is

$$\begin{aligned}&\mathbf{K}_N\,\mathsf{E}\,\{(\hat{\mathbf{a}}-\mathbf{a})(\hat{\mathbf{a}}-\mathbf{a})^T\}\mathbf{K}_N\\ &\quad\simeq r_\alpha^{-2}(0)\boldsymbol{\Lambda}^{-1}\mathbf{R}_\varepsilon\boldsymbol{\Lambda}^{-1}\\ &\quad= \tfrac{1}{2}r_\alpha^{-2}(0)[r_\alpha(0)\sigma^2 + 0.5\sigma^4]\boldsymbol{\Lambda}^{-1}. \quad (51)\end{aligned}$$

The asymptotic variances of a_1 and a_2 are thus given by

$$\mathrm{var}(\hat{a}_1) \simeq \frac{96}{N^3}\,\frac{r_\alpha(0)\sigma^2 + 0.5\sigma^4}{r_\alpha^2(0)} \quad (52)$$

$$\mathrm{var}(\hat{a}_2) \simeq \frac{90}{N^5}\,\frac{r_\alpha(0)\sigma^2 + 0.5\sigma^4}{r_\alpha^2(0)} \quad (53)$$

which are equivalent to the expressions in Proposition 2, where $\mathrm{SNR} = r_\alpha(0)/\sigma^2$.

Remark 8: Note that this result extends a similar result that was obtained in [15] and [16] for the exponential case (i.e., $a_2 \equiv 0$). Interestingly enough, although the present derivation and the approach of [15] are conceptually different (the orders in which derivations and Taylor series expansion are done are reversed), we get the same type of formula.

APPENDIX C
ANALYSIS OF THE HAF-BASED ESTIMATORS

In this Appendix, we derive the asymptotic variances of the two HAF-based estimators, i.e., the estimator based on $y(t)$ [referred to as HAF(y) in the sequel] and the estimator based on $y^2(t)$ [referred to as HAF(y^2)]. In order to analyze their performance, first note that in both cases, the first step in obtaining the HAF estimate of a_2 consists of solving the following minimization problem:

$$\{\hat{A},\hat{\phi},\hat{\omega}\} = \arg\min_{A,\phi,\omega} \frac{1}{N} \sum_{t=0}^{N-1-\tau} |z(t;\tau) - Ae^{i(\omega t+\phi)}|^2 \quad (54)$$

with

$$z(t;\tau) = \begin{cases} y_2(t;\tau) = y^*(t)y(t+\tau), & \text{for HAF}(y)\\ y_2^2(t;\tau) = y^{*2}(t)y^2(t+\tau), & \text{for HAF}(y^2). \end{cases}$$

In a second step, an estimate of a_2 is obtained as $\hat{a}_2 = \hat{\omega}/(2\tau)$ for HAF(y) and $\hat{a}_2 = \hat{\omega}/(4\tau)$ for HAF(y^2). Next, using the definition of $y_2(t;\tau)$, it is readily verified that

$$\begin{aligned}\mathsf{E}\,\{y_2(t;\tau)\} &= \mathsf{E}\,\{y^*(t)y(t+\tau)\} = \mathsf{E}\,\{s^*(t)s(t+\tau)\}\\ &= m_{2\alpha}(\tau)e^{ia_1\tau}e^{ia_2\tau^2}e^{2ia_2t\tau} \quad (55)\\ \mathsf{E}\,\{y_2^2(t;\tau)\} &= \mathsf{E}\,\{y^*(t)y^*(t)y(t+\tau)y(t+\tau)\}\\ &= \mathsf{E}\,\{s^*(t)s^*(t)s(t+\tau)s(t+\tau)\}\\ &= m_{4\alpha}(0,\tau,\tau)e^{2ia_1\tau}e^{2ia_2\tau^2}e^{4ia_2t\tau} \quad (56)\end{aligned}$$

where we use the notation $m_{n\alpha}(\cdot)$ to denote the nth-order moment of $\alpha(t)$ at appropriate lags. Assuming that $N-\tau \gg 1$ (e.g. the "effective" number of points is large), it can be inferred that $\hat{\boldsymbol{\theta}} = [\hat{A},\hat{\phi},\hat{\omega}]^T$ as given by (54) will be a consistent estimate of $\boldsymbol{\theta}_0 = [A_0,\phi_0,\,\omega_0]^T$, where

$$\boldsymbol{\theta}_0 = \begin{cases} [m_{2\alpha}(\tau) \quad a_1\tau + a_2\tau^2 \quad 2a_2\tau]^T\\ \qquad \text{for HAF}(y)\\ [m_{4\alpha}(0,\tau,\tau) \quad 2(a_1\tau + a_2\tau^2) \quad 4a_2\tau]^T\\ \qquad \text{for HAF}(y^2). \end{cases}$$

Similar to the analysis of the NLS estimator, we make use of a Taylor series expansion to approximate the objective function in (54) as

$$\begin{aligned}&z(t;\tau) - Ae^{i(\omega t+\phi)}\\ &\quad\simeq z(t;\tau) - A_0e^{i(\omega_0t+\phi_0)}\\ &\qquad - e^{i(\omega_0t+\phi_0)}(A-A_0) - iA_0e^{i(\omega_0t+\phi_0)}(\phi-\phi_0)\\ &\qquad - itA_0e^{i(\omega_0t+\phi_0)}(\omega-\omega_0). \quad (57)\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}f(A,\phi,\omega) &= \frac{1}{N} \sum_{t=0}^{N-1-\tau} |z(t;\tau) - Ae^{i(\omega t+\phi)}|^2\\ &\simeq \frac{1}{N}\|\boldsymbol{\delta} - (A-A_0)\mathbf{e} - iA_0(\phi-\phi_0)\\ &\qquad \cdot\mathbf{e} - iA_0(\omega-\omega_0)(\mathbf{t}^1 \odot \mathbf{e})\|^2 \quad (58)\end{aligned}$$

with $\boldsymbol{\delta} = [\delta(0;\tau),\cdots,\delta(N'-1;\tau)]^T$, $\mathbf{e} = [e^{i\phi_0}, e^{i(\omega_0+\phi_0)}, \cdots, e^{i((N'-1)\omega_0+\phi_0)}]^T$, $\delta(t;\tau) = z(t;\tau) - A_0e^{i(\omega_0t+\phi_0)}$, and $N' = N-\tau$. Differentiating the previous equation with respect to ω and ϕ and setting the derivative to zero, we get

$$\mathrm{Im}\{(\mathbf{t}^1 \odot \mathbf{e})^H \boldsymbol{\delta}\} \simeq A_0(\hat{\phi}-\phi_0)(\mathbf{t}^1)^T\mathbf{t}^0 + A_0(\hat{\omega}-\omega_0)(\mathbf{t}^1)^T\mathbf{t}^1 \tag{59}$$

$$\mathrm{Im}\{\mathbf{e}^H \boldsymbol{\delta}\} \simeq N' A_0(\hat{\phi}-\phi_0) + A_0(\hat{\omega}-\omega_0)(\mathbf{t}^1)^T\mathbf{t}^0. \tag{60}$$

Solving for ω

$$\begin{aligned}\frac{A_0}{12}N'^{3/2}(\hat{\omega}-\omega_0) &\simeq \frac{1}{N'^{3/2}}\,\mathrm{Im}\{(\mathbf{t}^1\odot\mathbf{e})^H\boldsymbol{\delta}\} - \frac{1}{2\sqrt{N'}}\,\mathrm{Im}\{\mathbf{e}^H\boldsymbol{\delta}\}\\ &= \frac{1}{\sqrt{N'}}\,\mathrm{Im}\left\{\sum_{t=0}^{N'-1}\frac{t}{N'}\delta(t;\tau)e^{-i(\omega_0 t+\phi_0)}\right\} - \frac{1}{2\sqrt{N'}}\,\mathrm{Im}\left\{\sum_{t=0}^{N'-1}\delta(t;\tau)e^{-i(\omega_0 t+\phi_0)}\right\}\\ &= \frac{1}{\sqrt{N'}}\,\mathrm{Im}\left\{\sum_{t=0}^{N'-1}\frac{t-N'/2}{N'}z(t;\tau)e^{-i(\omega_0 t+\phi_0)}\right\}.\end{aligned} \tag{61}$$

In the sequel, we focus on the analysis of HAF(y^2) since it constitutes the main novelty of this paper. Analysis of HAF(y) could be carried out along the same lines, and only the results will be stated. Using the definition of $z(t;\tau)$ for HAF(y^2) and recalling that $\omega_0 t+\phi_0 = 2[\phi(t+\tau)-\phi(t)]$, we can write

$$\begin{aligned}z(t;\tau)e^{-i(\omega_0 t+\phi_0)} &= \alpha^2(t)\alpha^2(t+\tau) + \alpha^2(t)n^2(t+\tau)e^{-i2\phi(t+\tau)}\\ &+ \alpha^2(t+\tau)n^{2*}(t)e^{i2\phi(t)}\\ &+ n^{2*}(t)n^2(t+\tau)e^{-i(\omega_0 t+\phi_0)}\\ &+ 2\alpha^2(t)\alpha(t+\tau)n(t+\tau)e^{-i\phi(t+\tau)}\\ &+ 2\alpha(t)\alpha^2(t+\tau)n^*(t)e^{i\phi(t)}\\ &+ 4\alpha(t)\alpha(t+\tau)n^*(t)n(t+\tau)e^{-i\phi(t+\tau)}e^{i\phi(t)}\\ &+ 2\alpha(t)n^2(t+\tau)n^*(t)e^{i\phi(t)}e^{-i2\phi(t+\tau)}\\ &+ 2\alpha(t+\tau)n^{2*}(t)n(t+\tau)e^{i2\phi(t)}e^{-i\phi(t+\tau)}.\end{aligned}$$

Let $\xi(t;\tau) = \Sigma_{k=1}^{8}\, T_k(t;\tau)$, where the $T_k(t;\tau)$ correspond to the last eight terms of the previous equation. Then, we have

$$\begin{aligned}\frac{A_0^2}{144}N'^3\,\mathrm{var}(\hat{\omega}-\omega_0) &\simeq \frac{1}{4N'}\sum_{t,s=0}^{N'-1}\left(\frac{t-N'/2}{N'}\right)\left(\frac{s-N'/2}{N'}\right)\\ &\times\ \mathsf{E}\,\{\xi(t;\tau)\xi^*(s;\tau)+\xi(s;\tau)\xi^*(t;\tau)\\ &-\xi(t;\tau)\xi(s;\tau)-\xi^*(t;\tau)\xi^*(s;\tau)\}.\end{aligned} \tag{62}$$

Using the fact that $n(t)$ is a white circular noise, the only nonzero terms in $\mathsf{E}\,\{\xi(t;\tau)\xi(s;\tau)\}$ are

$$\begin{aligned}&\mathsf{E}\,\{T_1(s;\tau)T_2(t;\tau)\}\\ &\quad=\gamma_{4n}\ \mathsf{E}\,\{\alpha^2(s)\alpha^2(s+2\tau)\}\delta(t,s+\tau)\\ &\mathsf{E}\,\{T_4(s;\tau)T_5(t;\tau)\}\\ &\quad=4\gamma_{2n}\ \mathsf{E}\,\{\alpha^2(s)\alpha^2(s+\tau)\alpha^2(s+2\tau)\}\delta(t,s+\tau).\end{aligned} \tag{63}$$

The nonzero terms of $\mathsf{E}\,\{\xi(t;\tau)\xi^*(s;\tau)\}$ are given by

$$\begin{aligned}\mathsf{E}\,\{T_1(t;\tau)T_1^*(s;\tau)\} &= \gamma_{4n}\ \mathsf{E}\,\{\alpha^4(t)\}\delta(t,s)\\ \mathsf{E}\,\{T_2(t;\tau)T_2^*(s;\tau)\} &= \gamma_{4n}\ \mathsf{E}\,\{\alpha^4(t)\}\delta(t,s)\\ \mathsf{E}\,\{T_3(t;\tau)T_3^*(s;\tau)\} &= \gamma_{4n}^2\delta(t,s)\\ \mathsf{E}\,\{T_4(t;\tau)T_4^*(s;\tau)\} &= 4\gamma_{2n}\ \mathsf{E}\,\{\alpha^4(t)\alpha^2(t+\tau)\}\delta(t,s)\\ \mathsf{E}\,\{T_5(t;\tau)T_5^*(s;\tau)\} &= 4\gamma_{2n}\ \mathsf{E}\,\{\alpha^2(t)\alpha^4(t+\tau)\}\delta(t,s)\\ \mathsf{E}\,\{T_6(t;\tau)T_6^*(s;\tau)\} &= 16\gamma_{2n}^2\ \mathsf{E}\,\{\alpha^2(t)\alpha^2(t+\tau)\}\delta(t,s)\\ \mathsf{E}\,\{T_7(t;\tau)T_7^*(s;\tau)\} &= 4\gamma_{2n}\gamma_{4n}m_{2\alpha}(0)\delta(t,s)\\ \mathsf{E}\,\{T_8(t;\tau)T_8^*(s;\tau)\} &= 4\gamma_{2n}\gamma_{4n}m_{2\alpha}(0)\delta(t,s)\end{aligned} \tag{64}$$

where we used the notation $\gamma_{kn} = \mathsf{E}\{|n(t)|^k\}$. Reporting the previous equation in (62), it ensues that

$$\mathrm{var}(\hat{\omega}-\omega_0) \simeq \frac{12}{(N-\tau)^3}\ \frac{D_2(\tau)}{m_{4\alpha}^2(0,\tau,\tau)} \tag{65}$$

where

$$\begin{aligned}D_2(\tau) &= 4\gamma_{2n}m_{6\alpha}(0,0,0,\tau,\tau) + \gamma_{4n}m_{4\alpha}(0,0,0)\\ &+ 8\gamma_{2n}^2 m_{4\alpha}(0,\tau,\tau) + 4\gamma_{2n}\gamma_{4n}m_{2\alpha}(0) + \frac{1}{2}\gamma_{4n}^2\\ &- [\gamma_{4n}m_{4\alpha}(0,2\tau,2\tau) + 4\gamma_{2n}m_{6\alpha}(0,\tau,\tau,2\tau,2\tau)]\\ &\cdot \frac{(N-2\tau)(N^2-4\tau N+\tau^2)}{(N-\tau)^3}\mathbf{1}(N-2\tau)\end{aligned} \tag{66}$$

and $\mathbf{1}(\cdot)$ denotes the unit step function. Finally, the asymptotic variance of $\hat{a}_2$ is given by

$$\mathrm{var}(\hat{a}_2^{(y^2)}) \simeq \frac{3}{(N-\tau)^3}\ \frac{D_2(\tau)}{4\tau^2 m_{4\alpha}^2(0,\tau,\tau)}. \tag{67}$$

In the case of white Gaussian noise $n(t)$ and assuming $\alpha(t)$ is a zero-mean Gaussian process, we have $\gamma_{2n} = \sigma^2$, $\gamma_{4n} = 2\sigma^4$, $m_{4\alpha}(0,0,0) = 3r_\alpha^2(0)$ and expressions for the higher order moments are given by [26]

$$\begin{aligned}m_{4\alpha}(0,\tau,\tau) &= r_\alpha^2(0) + 2r_\alpha^2(\tau)\\ m_{6\alpha}(0,0,0,\tau,\tau) &= 3r_\alpha^3(0) + 12r_\alpha(0)r_\alpha^2(\tau)\\ m_{6\alpha}(0,\tau,\tau,2\tau,2\tau) &= r_\alpha^3(0) + 2r_\alpha(0)r_\alpha^2(\tau)\\ &+ 4r_\alpha^2(\tau)[r_\alpha(0)+2r_\alpha(2\tau)].\end{aligned}$$

Hence, $D_2(\tau)$ simplifies to

$$\begin{aligned}D_2(\tau) &= 4\sigma^2[3r_\alpha^3(0) + 12r_\alpha(0)r_\alpha^2(\tau)]\\ &+ 2\sigma^4[7r_\alpha^2(0) + 8r_\alpha^2(\tau)]\\ &+ 86\sigma^6 r_\alpha(0) + 2\sigma^8\\ &- [2\sigma^4[r_\alpha^2(0) + 2r_\alpha^2(2\tau)]\\ &+ 4\sigma^2 m_{6\alpha}(0,0,\tau,\tau,2\tau,2\tau)]\\ &\cdot \frac{(N-2\tau)(N^2-4\tau N+\tau^2)}{(N-\tau)^3}\mathbf{1}(N-2\tau).\end{aligned}$$

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