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Parameter Estimation in Multivariate Gamma Distribution

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Abstract Multivariate gamma distribution finds abundant applications in stochastic modelling, hydrology and reliability. Parameter estimation in this distribution is a challenging one as it involves many parameters to be estimated simultaneously. In this paper, the form of multivariate gamma distribution proposed by Mathai and Moschopoulos [9] is considered. This form has nice properties in terms of marginal and conditional densities. A new method of estimation based on optimal search is proposed for estimating the parameters using the marginal distributions and the concepts of maximum likelihood, spacings and least squares. The proposed methodology is easy to implement and is free from calculus. It optimizes the objective function by searching over a wide range of values and determines the estimate of the parameters. The consistency of the estimates is demonstrated in terms of mean, standard deviation and mean square error through simulation studies for different choices of parameters.

Keywords Least squares, Maximum likelihood, Mean square error, Multivariate gamma distribution, Spacing

AMS 2010 subject classifications 62F10, 62E99

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1. Introduction

Three-parameter gamma distribution stands central in the definition of various forms of multivariate gamma distribution. The probability density function of three-parameter gamma distribution with parameters namely, α (shape), β (scale) and μ (location) denoted by Gamma (α, β, μ) is defined as

$$f\left(x;\alpha,\beta,\mu\right)=\frac{1}{\beta^{\alpha}\Gamma\left(\alpha\right)}\left(\mathrm{x}-\mu\right)^{\alpha-1}\exp-\left(\frac{\mathrm{x}-\mu}{\beta}\right),\mathrm{x}>0,\alpha>0,\beta>0,\mu\epsilon\mathbb{R}. \quad (1)$$

For more details on the properties and applications of this distribution, see Johnson et al. [7]. Princy [12] discusses an application of the extended compound gamma model. Multivariate gamma distributions with gamma marginals are common in literature. Several particular cases of these multivariate gamma densities including the bivariate cases have gained prominence over the years. Some examples include bivariate gamma distributions due to Cheriyan [3], Mathai and Moschopoulos [10], McKay [11], Kibble [8], Royen [14], Jensen [6], Sarmanov ([16],[17]). For an elaborate discussion of these distributions along with their applications, generalizations, method of construction and inter-relationships, one may refer to Balakrishnan and Lai [1], Samuel Kotz et al. [15] and Yue et al. [18] and the references cited therein.

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Mathai and Moschopoulos [9] introduced a new form of multivariate gamma distribution using mutually independent three-parameter gamma variates. Let $V_1, V_2, ..., V_k$ be mutually independent random variates where $V_i \sim$ Gamma (α_i, β, μ_i), $i = 1, 2, ..., k$. Let $Z_1 = V_1$, $Z_2 = V_1 + V_2$, ...,$Z_k = V_1 + V_2 + ... + V_k$. The joint distribution of $\mathbf{Z} = (Z_1, Z_2, ..., Z_k)^{'}$ is a k-variate gamma distribution with density function given by

$$f\left(z_1, z_2, ...z_k\right) = \frac{\left(z_1 - \mu_1\right)^{\alpha_1 - 1}\left(z_2 - z_1 - \mu_2\right)^{\alpha_2 - 1} ... \left(z_k - z_{k-1} - \mu_k\right)^{\alpha_k - 1} \exp - \left(\frac{z_k - \sum_{i=1}^{k} \mu_i}{\beta}\right)}{\beta^{\alpha_k^*} \prod_{i=1}^{k} \Gamma\left(\alpha_i\right)} \tag{2}$$

where $\alpha_i > 0, \beta > 0, \mu_i \epsilon \mathbb{R}, z_{i-1} + \mu_i < z_i, i = 2, 3, ..., k, z_k < \infty, \mu_1 < z_1, \alpha_k^* = \alpha_1 + \alpha_2 + ... + \alpha_k$ and zero elsewhere.

The requirement of the common scale parameter β is to ensure that marginals are of the same form. Figure 1 depicts the density function of the multivariate gamma distribution with $k = 2$ for three different choices of parameters.

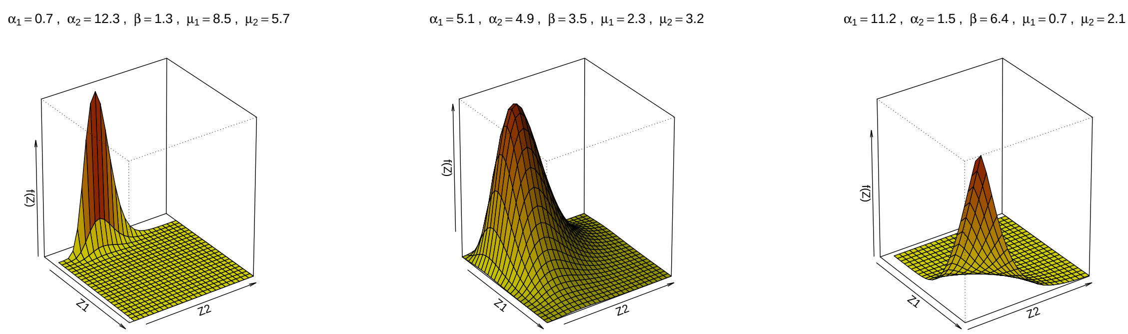

Figure 1. Probability Density Function of Bivariate Gamma Distribution for various choice of parameters

This distribution is especially useful for models in reliability theory and stochastic process. See Mathai and Moschopoulos [9]. In hydrology, it is extensively used in multivariate flood frequency analysis. See Yue [19]. McKay's bivariate gamma distribution is a special case of the form defined in (2) with $\mu_1 = \mu_2 = 0$. Clarke [4] employed McKay's bivariate model for estimating the mean annual streamflow from precipitation data.

Some important properties of k-variate gamma distribution as discussed in Mathai and Moschopoulos [9] are listed below.

- The marginal distribution of Z_i is three-parameter gamma distribution of the form given in (1), i.e. $Z_i \sim$ Gamma $\left(\alpha_i^*, \beta, \mu_i^*\right), i = 1, 2, ..., k$, where $\alpha_i^* = \alpha_1 + \alpha_2 + ... + \alpha_i$ and $\mu_i^* = \mu_1 + \mu_2 + ... + \mu_i$.
- Z_i and Z_j have positive correlation given by $\rho = \sqrt{\left(\frac{\alpha_i^*}{\alpha_j^*}\right)}$.
- The conditional density of $\left(Z_{i+1} \mid Z_i = z_i\right) \sim$ Gamma $\left(\alpha_{i+1}, \beta, z_i + \mu_{i+1}\right)$.Also, for $j > i$, $\left(\frac{Z_i - \mu_i^*}{Z_j - \mu_j^*}\right) \sim$ Beta$_I$ $\left(\alpha_i^*, \alpha_j^* - \alpha_i^*\right)$ and $\frac{Z_i - \mu_i^*}{Z_j - Z_i + \mu_i^* - \mu_j^*} \sim$ Beta$_{II}$ $\left(\alpha_i^*, \alpha_j^* - \alpha_i^*\right)$, where Beta$_I$ (.) and Beta$_{II}$ (.) denote respectively, beta distribution of first and second kind.

It is interesting to note that, in terms of $V_1, V_2, V_3, ..., V_k$, (2) can be represented as

$$f\left(V_1, V_1 + V_2, ..., V_1 + ... + V_k\right) = \frac{\left(V_1 - \mu_1\right)^{\left(\alpha_1 - 1\right)} .... \left(V_k - \mu_k\right)^{\left(\alpha_k - 1\right)} \exp - \left(\frac{\sum_{i=1}^{k} V_i - \sum_{i=1}^{k} \mu_i}{\beta}\right)}{\beta^{\alpha_k^*} \prod_{i=1}^{k} \Gamma\left(\alpha_i\right)} \tag{3}$$

where, $V_i \sim$ Gamma (α_i, β, μ_i), $i = 1, 2, 3, ..., k$ and $\alpha_i > 0, \beta > 0, \mu_i \epsilon \mathbb{R}$. Owing to the presence of a common scale parameter, this can be expressed as a product of k univariate three-parameter gamma distributions i.e.

$$f\left(V_1, V_1 + V_2, ..., V_1 + ... + V_k\right) = \frac{\left(V_1 - \mu_1\right)^{(\alpha_1 - 1)} \exp - \left(\frac{V_1 - \mu_1}{\beta}\right)}{\beta^{\alpha_1} \Gamma(\alpha_1)} ... \frac{\left(V_k - \mu_k\right)^{(\alpha_k - 1)} \exp - \left(\frac{V_k - \mu_k}{\beta}\right)}{\beta^{\alpha_k} \Gamma(\alpha_k)} \tag{4}$$

The above form serves as an aid to the development of a heuristic methodology to estimate the $(2k + 1)$ unknown parameters of the distribution. In this paper, we propose an approach to parameter estimation for the model in (4) involving the concepts of Maximum Likelihood (ML), Least Squares (LS) and Maximum Product of Spacings (MPS).

The motivation for the present study is due to the fact that parameter estimation has not been attempted for this distribution before. Hence, an attempt in this direction will offer insights into the issues and challenges if any, in parameter estimation. This will provide opportunities to develop new methodologies thereby widening the scope of application of the distribution in other domains of research.

The primary contribution of this paper is to propose a heuristic algorithm for parameter estimation and assess its performance through simulation studies.The proposed approach is easy to implement and is free from calculus. It optimizes the objective function by searching over a wide range of values and determines the estimate of the parameters. Rest of the paper is structured as follows. Section 2 details the methods of estimation in multivariate gamma distribution. Section 3 contains a method for random variate generation from the k-variate gamma distribution and the algorithmic description of the proposed methodology. In Section 4, we present and discuss simulation results for bivariate gamma distribution. The estimates obtained are compared in terms of Average (AVG), Standard Deviation (SD) and Mean Square Error (MSE). Section 5 concludes the paper with discussion.

2. Methods of Estimation

The complex structure of the k-variate gamma distribution requires solving non-linear equations for obtaining estimates. Even typical iterative optimization procedures like Nelder-Mead Algorithm, Genetic Algorithm (GA) often fail to produce results. For instance, Nelder-Mead method results in ”Not a Number” (NaN) when implemented via optim() function in R. Moreover, the accuracy of the estimates obtained through these algorithms decreases when the dimension of parameters increase.

In situations where the marginals follow gamma distribution, method of moments has been extensively used in parameter estimation. See Yue [19]. However, Fisher [5] has reasoned that the classical method of moments in general is inefficient, except when it closely approximates normality and recommends the use of ML estimation. Hence, to estimate the $(2k + 1)$ parameters of k-variate gamma distribution, the methods of ML as well as MPS and LS are considered.

2.1. Maximum Likelihood Estimation

Based on a random sample of size n from k-variate gamma distribution with probability density function defined in (2), the likelihood (L) and log-likelihood $(\log L)$ functions, respectively are given as

$$L = \prod_{i=1}^{n} \frac{\left(z_{1i} - \mu_1\right)^{(\alpha_1 - 1)} ... \left(z_{ki} - z_{(k-1)i} - \mu_k\right)^{(\alpha_k - 1)} \exp - \left(\frac{z_{ki} - (\mu_1 + \mu_2 + ... \mu_k)}{\beta}\right)}{\beta^{\alpha_{k^*}} \prod_{j=1}^{k} \Gamma(\alpha_j)} \tag{5}$$

$$\log L = (\alpha_1 - 1)\sum_{i=1}^{n}\log\left(z_{1i} - \mu_1\right) + ... + (\alpha_k - 1)\sum_{i=1}^{n}\log\left(z_{ki} - z_{(k-1)i} - \mu_k\right) + \frac{\left(n\sum_{j=1}^{k}\mu_j\right)}{\beta}$$
$$-n\left(\log\left(\Gamma\left(\alpha_1\right)\right) + ... + \log\left(\Gamma\left(\alpha_k\right)\right)\right) - \left(\frac{\sum_{i=1}^{n} z_{ki}}{\beta}\right) - n\log\left(\beta\right)\sum_{j=1}^{k}\alpha_j \tag{6}$$

The corresponding $(2k+1)$ log-likelihood equations are:

$$\frac{\partial \log L}{\partial \alpha_l} = 0 \Rightarrow (\alpha_l - 1)\sum_{i=1}^{n}\log(z_{li} - z_{(l-1)i} - \mu_l) - n(\log(\beta)) - n\Psi(\alpha_l) = 0 \tag{7}$$

$$\frac{\partial \log L}{\partial \beta} = 0 \Rightarrow \frac{\left((\sum_{i=1}^{n} z_{ki}) - (n\sum_{j=1}^{k}\mu_j)\right)}{\beta^2} - \frac{\left(n\sum_{j=1}^{k}\alpha_j\right)}{\beta} = 0 \tag{8}$$

$$\frac{\partial \log L}{\partial \mu_l} = 0 \Rightarrow \left(\frac{n}{\beta}\right) - (\alpha_l - 1)\sum_{i=1}^{n}\left(\frac{1}{z_{li} - z_{(l-1)i} - \mu_l}\right) = 0 \tag{9}$$

where $z_{0i} = 0, l = 1, 2, 3, ..., k, i = 1, 2, ..., n$ and $\Psi(\alpha_l) = \frac{\partial \log \Gamma(\alpha_l)}{\partial \alpha_l}$.

The non-linear equations defined in (7)-(9) have no closed form solutions. As pointed out earlier, numerical optimization algorithms like Nelder-Mead, GA often fail to provide solutions owing to the large number of parameters. It is found that these methods fail to produce numerical estimates even when $k = 2$. Also, Fisher's scoring method cannot be used since the regularity conditions are not met. It is interesting to note that the likelihood function defined in (5) can be expressed as a product of k univariate three-parameter gamma likelihoods using (4). As a consequence, the unknown parameters of the k-variate gamma distribution are estimated using its marginal distributions. In this perspective, we consider n random samples from a three-parameter gamma distribution with density defined in (1). The corresponding likelihood L and log-likelihood functions $\log L$ are

$$L = \left(\frac{1}{\beta^{\alpha}\Gamma\left(\alpha\right)}\right)^{n}\prod_{i=1}^{n}(x_i - \mu)^{\alpha-1}\exp -\left(\sum_{i=1}^{n}\frac{(x_i - \mu)}{\beta}\right) \tag{10}$$

$$\log L = (\alpha - 1)\sum_{i=1}^{n}\log(x_i - \mu) - \sum_{i=1}^{n}\left(\frac{x_i - \mu}{\beta}\right) - n\alpha\log\beta - n\log\Gamma\alpha \tag{11}$$

In order to obtain estimates for the unknown parameters, the proposed methodology resorts to direct maximization of the function given in (10).

2.2. Maximum Product of Spacings Estimation

In order to estimate the $(2k+1)$ parameters of the k-variate gamma distribution, we use the form defined in (4) that involves k univariate three-parameter gamma distributions. Let $X_1, X_2, ..., X_n$ be n random samples from a three-parameter gamma distribution with density defined in (1). Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ represent the corresponding ordered observations where $X_{(1)} < X_{(2)} < ... < X_{(n)}$. Also, let $X_{(0)} = -\infty$ and $X_{(n+1)} = +\infty$. Spacings (Cheng and Amin [2], Ranneby [13]) are defined as the "gaps" between the values of the distribution function at adjacent observed ordered points. Define $D_i(\theta) = F(x_{(i)};\theta) - F(x_{(i-1)};\theta), \theta = (\alpha, \beta, \mu)$. i.e.

$$D_i(\theta) = \int_{x_{(i-1)}}^{x_{(i)}}\frac{(t-\mu)^{(\alpha-1)}}{\beta^{\alpha}\Gamma(\alpha)}\exp -\left(\frac{t-\mu}{\beta}\right)dt, i = 1, 2, 3, ..., n+1 \tag{12}$$

with $D_1(\theta) = F\left(x_{(1)};\theta\right)$ and $D_{n+1}(\theta) = 1 - F\left(x_{(n)};\theta\right)$.
The maximum product of spacings estimator for θ is the one which maximizes the logarithm of the geometric mean of sample spacings i.e. $\hat{\theta} = \arg\max_{\theta\epsilon\Theta} S_n(\theta)$, where

$$S_n(\theta) = \log \sqrt[n+1]{D_1 D_2 ... D_{n+1}} = \log(F(x_{(1)};\theta)) + \sum_{i=2}^{n} \log(D_i(\theta)) + \log(1 - F(x_{(n)};\theta)) \tag{13}$$

The direct maximization of the function defined in (13) would in turn, lead to estimates for the unknown parameters.

2.3. Least Squares Estimation

Based on the fact that the k-variate gamma distribution defined in (2) can be expressed as a product of k univariate gamma distributions as given in (4), we consider n samples from each of the marginal densities. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ represent the corresponding ordered observations. LS estimation involves estimating the parameters by minimizing the function $L_n(\theta)$ defined as

$$L_n(\theta) = \sum_{i=1}^{n} \left[(F(x_{(i)});\theta) - \left(\frac{i}{n+1}\right)^2 \right], \theta = (\alpha, \beta, \mu) \tag{14}$$

where $F\left(x_{(i)};\theta\right), i = 1, 2, ..., n$ represents the distribution function evaluated at the observed value of $X_{(i)}$. The minimization of the function defined in (14) produces the required estimates for the unknown parameters.

3. Proposed Methodology

In this section, we present a method for generating observations from k-variate gamma distribution of the form given in (2). Following this, the algorithm for determining the estimates of $(2k+1)$ unknown parameters of k-variate gamma distribution using ML, MPS and LS methods is described.

3.1. Random Variate Generation

The following steps are used to generate n observations from k-variate gamma distribution.
Step 1:
Generate n independent samples each from $V_1, V_2, V_3, ..., V_k$ where $V_i \sim$ Gamma (α_i, β, μ_i), $i = 1, ..., k$.
Step 2:
Define $Z_1 = V_1$, $Z_2 = V_1 + V_2$, ...,$Z_k = V_1 + V_2 + ... + V_k$ and $\mathbf{Z} = (Z_1, Z_2, ..., Z_k)'$.
Thus $\mathbf{Z} = (Z_1, Z_2, ..., Z_k)'$ contains n samples from k-variate gamma distribution.

3.2. Split-Join Algorithm

Consider a random sample of size n from k-variate gamma distribution with $(2k+1)$ unknown parameters. In order to estimate the parameters, we propose an algorithm called Split-Join algorithm. This involves 'Splitting' the k-variate gamma distribution into k univariate three-parameter gamma distributions followed by estimation of each of its parameters and then 'Joining' the resulting estimates thereby giving estimates for the $(2k+1)$ parameters.

Given the k-variate gamma variable Z, the algorithm requires splitting of Z to k univariate three-parameter gamma variables V_i where $V_i = Z_i - Z_{i-1}, i = 2, 3, ..., k$ with $V_1 = Z_1$. Let $\theta_i = (\alpha_i, \beta, \mu_i)$ denote the parameters to be estimated in V_i. Specify the boundaries of parameter search space by defining a lower bound (low_{θ_i}) and upper bound (up_{θ_i}) for each parameter in θ_i. Let (seq_{α_i}),(seq_{β}), and (seq_{μ_i}) represent the sequence of values generated between (low_{θ_i}) and (up_{θ_i}) by an incrementing factor I. The algorithm is given below.

Algorithm 1 Split-Join Methodology

for i in 1 and k **do**
  Sort V_i
  Initialize I, (seq_{α_i}),(seq_{β}), and (seq_{μ_i})
  for p in (seq_{α_i}) **do**
    for q in (seq_{β}) **do**
      for r in (seq_{μ_i}) **do**
        for s in 1 and n **do**
          Compute $obj()$
  End Loops
  $optim(obj) \leftarrow$ optimum of $obj()$
  $\hat{\theta} \leftarrow$ (p,q,r) corresponding to $optim(obj)$
End Loop
Combine all $\hat{\theta}_i, i = 1, 2, ..., k$ to obtain required estimates.

The objective function $obj()$ in the above algorithm correspond to any of the equations (10),(13) and (14) and $optim(obj)$ represents the optimum value of the objective function. Thus, an implementation of the above algorithm will result in estimates $\hat{\theta}_i$ for θ_i by ML, MPS, or LS method. Combining $\hat{\theta}_i, i = 1, 2, ..., k$ gives the required estimates for k-variate gamma distribution. It is important to note that the algorithm results in k estimates for the common scale parameter β . To get a single estimate for β , one may take the arithmetic mean of the k estimates or any other meaningful function, thereof. Alternatively, one may fix the estimate of β obtained from the first run of the algorithm for the successive runs. However, this approach is meaningful only when ML method is employed due to the fact that the likelihood function of the k-variate gamma distribution can be expressed as product of the likelihoods of marginal distribution.

4. Simulation results

The proposed methodology is implemented in R 3.1.0 for the case of bivariate gamma distribution. Datasets of sizes $n = 20, 50$ and 100 are simulated for MC runs$(m) = 100, 500$ times each with the choice of parameter values as shown in Table (I) using the random variate generation technique given in Section 3.1.

Table I. Choice of parameter values for simulating samples from bivariate gamma distribution

Case	α_1	α_2	β	μ_1	μ_2
1	0.70	12.30	1.30	8.50	5.70
2	5.10	4.90	3.50	2.30	3.20
3	11.20	1.50	6.40	0.70	2.10

The parameters are chosen so that the correlations of observations typically fall under three categories namely low, moderate and high. The correlations under the above three cases are 0.232, 0.714 and 0.939 respectively. The mean value of the estimates (AVG), its standard deviation (SD) and mean square error (MSE) obtained by ML, MPS and LS methods based on $m = 100, 500$ runs of the algorithm for each of the above cases are presented in Tables (II) to (X).

Note that LSE-S, MLE-S and MPS-S refer to LS, ML and MPS methods under Split-Join Methodology. MLE-M refers to ML method wherein the estimate of β obtained from the first run of the Split-Join algorithm is used throughout.

Table II. AVG, SD and MSE of estimates for n=20, Case 1

Measure	Estimates of	LSE-S	MLE-S	MPS-S	MLE-M	LSE-S	MLE-S	MPS-S	MLE-M
		m=100				m=500			
AVG	α_1	0.8520	0.7700	0.8290	0.7700	0.8374	0.8000	0.8820	0.8000
	α_2	12.2600	12.5210	12.5730	12.9670	12.4462	12.6478	12.8132	12.9384
	β	1.3505	1.2590	1.1875	1.2685	1.3149	1.2193	1.1402	1.2114
	μ_1	8.4560	8.4990	8.4770	8.4990	8.4702	8.5004	8.4750	8.5004
	μ_2	5.6290	6.0770	5.6540	6.3530	5.8346	6.2966	5.8302	6.3382
SD	α_1	0.4019	0.2342	0.3591	0.2342	0.3700	0.2367	0.3729	0.2367
	α_2	0.8764	1.0499	1.0935	1.3134	0.9906	1.1171	1.1052	1.3062
	β	0.3061	0.2213	0.2433	0.4349	0.2979	0.2274	0.2402	0.4352
	μ_1	0.0903	0.0100	0.0423	0.0100	0.0918	0.0127	0.0477	0.0127
	μ_2	1.0145	1.0858	0.9799	1.1962	1.0537	1.1100	1.0438	1.2064
MSE	α_1	0.1830	0.0592	0.1443	0.0592	0.1555	0.0659	0.1719	0.0659
	α_2	0.7620	1.1401	1.2583	2.1527	1.0008	1.3665	1.4824	2.1104
	β	0.0953	0.0502	0.0713	0.1882	0.0888	0.0581	0.0831	0.1969
	μ_1	0.0100	0.0001	0.0023	0.0001	0.0093	0.0002	0.0029	0.0002
	μ_2	1.0239	1.3093	0.9528	1.8431	1.1263	1.5855	1.1043	1.8598

Table III. AVG, SD and MSE of estimates for n=50, Case 1

Measure	Estimates of	LSE-S	MLE-S	MPS-S	MLE-M	LSE-S	MLE-S	MPS-S	MLE-M
		m=100				m=500			
AVG	α_1	0.7750	0.7380	0.7240	0.7380	0.7798	0.7350	0.7262	0.7350
	α_2	12.3720	12.4450	12.6940	12.9010	12.4256	12.5162	12.7386	12.8884
	β	1.3095	1.2765	1.2490	1.2680	1.3066	1.2760	1.2472	1.2906
	μ_1	8.4800	8.5000	8.4970	8.5000	8.4822	8.4998	8.4962	8.4998
	μ_2	5.7230	5.8360	5.4770	6.1710	5.7916	5.9658	5.5948	6.1492
SD	α_1	0.2528	0.1339	0.1700	0.1339	0.2706	0.1269	0.1748	0.1269
	α_2	0.8757	0.9786	1.0531	1.2910	0.8982	1.0409	1.0637	1.2708
	β	0.2377	0.1857	0.1848	0.3375	0.2366	0.1771	0.1778	0.3210
	μ_1	0.0449	0.0000	0.0171	0.0000	0.0539	0.0045	0.0202	0.0045
	μ_2	0.8870	0.9324	0.7282	1.0936	0.9409	0.9561	0.8145	1.0915
MSE	α_1	0.0689	0.0192	0.0292	0.0192	0.0794	0.0173	0.0312	0.0173
	α_2	0.7644	0.9691	1.2532	2.0113	0.8208	1.1281	1.3217	1.9578
	β	0.0560	0.0347	0.0364	0.1138	0.0559	0.0319	0.0343	0.1029
	μ_1	0.0024	0.0000	0.0003	0.0000	0.0032	0.0000	0.0004	0.0000
	μ_2	0.7795	0.8792	0.5747	1.4059	0.8919	0.9829	0.6731	1.3907

Table IV. AVG, SD and MSE of estimates for n=100, Case 1

Measure	Estimates of	LSE-S	MLE-S	MPS-S	MLE-M	LSE-S	MLE-S	MPS-S	MLE-M
		m=100				m=500			
AVG	α_1	0.7550	0.7270	0.7240	0.7270	0.7428	0.7254	0.7124	0.7254
	α_2	12.5910	12.4940	12.9150	12.7820	12.3788	12.4720	12.6710	12.7810
	β	1.2800	1.2610	1.2350	1.2540	1.2977	1.2731	1.2579	1.2690
	μ_1	8.4890	8.5000	8.4990	8.5000	8.4914	8.5000	8.4990	8.5000
	μ_2	5.7530	5.9200	5.5110	6.0440	5.7494	5.7816	5.5350	5.8716
SD	α_1	0.2007	0.1072	0.1415	0.1072	0.1861	0.1008	0.1166	0.1008
	α_2	1.0030	1.0152	0.9743	1.2555	0.8456	0.9764	1.0175	1.2390
	β	0.1777	0.1323	0.1313	0.2280	0.1710	0.1308	0.1318	0.2285
	μ_1	0.0345	0.0000	0.0100	0.0000	0.0314	0.0063	0.0100	0.0063
	μ_2	0.8132	0.8255	0.6456	0.9874	0.8001	0.7966	0.6803	0.9147
MSE	α_1	0.0429	0.0121	0.0204	0.0121	0.0364	0.0108	0.0137	0.0108
	α_2	1.0807	1.0580	1.3179	1.7928	0.7198	0.9811	1.1709	1.7634
	β	0.0317	0.0189	0.0213	0.0536	0.0292	0.0178	0.0191	0.0531
	μ_1	0.0013	0.0000	0.0001	0.0000	0.0011	0.0000	0.0001	0.0000
	μ_2	0.6575	0.7230	0.4483	1.0836	0.6413	0.6400	0.4891	0.8645

Table V. AVG, SD and MSE of estimates for n=20, Case 2

Measure	Estimates of	LSE-S	MLE-S	MPS-S	MLE-M	LSE-S	MLE-S	MPS-S	MLE-M
		m=100				m=500			
AVG	α_1	5.0560	5.3260	5.4330	5.3260	4.9768	5.1660	5.2870	5.1660
	α_2	4.9360	4.9460	5.2150	5.0910	5.0306	5.0650	5.3146	4.9332
	β	3.6815	3.4410	3.1300	3.3360	3.6693	3.4346	3.1030	3.4530
	μ_1	2.2380	2.7110	2.1000	2.7110	2.1076	2.6900	1.9888	2.6900
	μ_2	2.8190	3.5330	2.5920	3.5360	2.7714	3.5312	2.5582	3.6200
SD	α_1	0.8710	0.8608	0.7764	0.8608	0.8866	0.9272	0.8669	0.9272
	α_2	0.9367	0.9644	0.8557	0.8580	0.8755	0.9008	0.7832	0.8896
	β	0.5280	0.5248	0.6324	0.6791	0.5291	0.5004	0.6531	0.7078
	μ_1	1.0441	0.9162	1.0673	0.9162	1.0287	0.9174	1.0383	0.9174
	μ_2	1.2939	1.0846	1.3937	1.0385	1.3537	1.0627	1.3702	0.9378
MSE	α_1	0.7530	0.7846	0.7077	0.7846	0.7997	0.8624	0.7850	0.8624
	α_2	0.8700	0.9228	0.8241	0.7653	0.7820	0.8371	0.7840	0.7909
	β	0.3089	0.2762	0.5309	0.4834	0.3081	0.2542	0.5814	0.5021
	μ_1	1.0830	0.9999	1.1678	0.9999	1.0931	0.9920	1.1728	0.9920
	μ_2	1.8025	1.2755	2.2926	1.1806	2.0124	1.2368	2.2857	1.0541

Table VI. AVG, SD and MSE of estimates for n=50, Case 2

Measure	Estimates of	LSE-S	MLE-S	MPS-S	MLE-M	LSE-S	MLE-S	MPS-S	MLE-M
		m=100				m=500			
AVG	α_1	5.0470	5.1240	5.2490	5.1240	4.9904	5.0904	5.2988	5.0904
	α_2	5.1440	5.1160	5.4050	4.9000	4.9994	5.0340	5.3730	4.8986
	β	3.5420	3.4570	3.3435	3.5090	3.6303	3.4959	3.3609	3.5274
	μ_1	2.2520	2.5230	2.0400	2.5260	2.2804	2.6076	1.9850	2.6076
	μ_2	2.9370	3.3180	2.5760	3.3960	2.9044	3.3318	2.4972	3.4464
SD	α_1	0.8236	0.8390	0.7817	0.8326	0.8144	0.8348	0.7903	0.8337
	α_2	0.7701	0.8336	0.6752	0.7602	0.8057	0.8254	0.7090	0.8297
	β	0.4599	0.4542	0.3835	0.5719	0.4366	0.4212	0.3705	0.5910
	μ_1	0.9724	0.9492	1.0371	0.9509	0.9793	0.8978	0.9790	0.8982
	μ_2	1.2151	1.0251	1.2598	0.9879	1.2578	1.0754	1.2359	0.9396
MSE	α_1	0.6743	0.6974	0.6271	0.6875	0.6739	0.6955	0.6629	0.6937
	α_2	0.6466	0.7346	0.7063	0.5722	0.6577	0.6979	0.7254	0.6871
	β	0.2112	0.2061	0.1701	0.3239	0.2072	0.1770	0.1563	0.3493
	μ_1	0.9384	0.9417	1.1324	0.9462	0.9575	0.8990	1.0558	0.8998
	μ_2	1.5309	1.0542	1.9606	1.0046	1.6662	1.1715	2.0184	0.9418

Table VII. AVG, SD and MSE of estimates for n=100, Case 2

Measure	Estimates of	LSE-S	MLE-S	MPS-S	MLE-M	LSE-S	MLE-S	MPS-S	MLE-M
		m=100				m=500			
AVG	α_1	4.9750	5.1210	5.4200	5.1210	5.0388	5.1374	5.4406	5.1374
	α_2	4.8650	4.9240	5.2610	4.9270	4.8604	4.9368	5.3030	4.9536
	β	3.6555	3.5255	3.3845	3.5290	3.6432	3.5188	3.3751	3.5118
	μ_1	2.4100	2.5180	1.8970	2.5180	2.3400	2.4918	1.8560	2.4918
	μ_2	3.0150	3.2770	2.5990	3.3050	2.9922	3.2530	2.5026	3.2682
SD	α_1	0.7179	0.7400	0.5805	0.7400	0.7473	0.7449	0.6080	0.7449
	α_2	0.7792	0.7412	0.7128	0.7774	0.7723	0.7580	0.7091	0.7542
	β	0.3941	0.3010	0.2655	0.4728	0.3956	0.3154	0.2803	0.4770
	μ_1	0.9275	0.9217	0.9214	0.9217	0.9294	0.9405	0.9177	0.9405
	μ_2	1.1732	1.1471	1.1962	0.9354	1.1853	1.1271	1.1827	0.9413
MSE	α_1	0.5259	0.5425	0.4360	0.5425	0.5611	0.5552	0.4849	0.5552
	α_2	0.6023	0.5444	0.6333	0.5991	0.5968	0.5747	0.6642	0.5706
	β	0.1779	0.0903	0.0831	0.2221	0.1767	0.0996	0.0940	0.2272
	μ_1	0.8638	0.8886	1.0029	0.8886	0.8636	0.9195	1.0376	0.9195
	μ_2	1.3969	1.3085	1.7779	0.8773	1.4453	1.2707	1.8824	0.8889

Table VIII. AVG, SD and MSE of estimates for n=20, Case 3

Measure	Estimates of	LSE-S	MLE-S	MPS-S	MLE-M	LSE-S	MLE-S	MPS-S	MLE-M
		m=100				m=500			
AVG	α_1	10.9940	11.3220	11.1060	11.3220	10.9492	11.2916	11.1238	11.2916
	α_2	1.5100	1.3210	1.5240	1.3380	1.5796	1.3804	1.6010	1.3778
	β	6.5950	6.4505	6.1210	6.3910	6.5415	6.4165	6.0530	6.3986
	μ_1	0.7610	0.9520	0.7160	0.9520	0.7174	1.0474	0.7410	1.0474
	μ_2	1.7080	2.6270	1.9090	2.5830	1.6980	2.6164	1.8418	2.5816
SD	α_1	0.9458	0.9430	0.9898	0.9430	0.9303	0.9458	0.9920	0.9458
	α_2	0.3863	0.3836	0.4461	0.3345	0.4043	0.3994	0.4464	0.3421
	β	0.5806	0.6394	0.6522	0.7040	0.5921	0.6222	0.6056	0.6891
	μ_1	0.7136	0.7885	0.7580	0.7885	0.7128	0.7652	0.7573	0.7652
	μ_2	1.1244	0.5698	0.9540	0.6032	1.1197	0.5597	0.9468	0.5842
MSE	α_1	0.9280	0.8952	0.9788	0.8952	0.9266	0.9012	0.9879	0.9012
	α_2	0.1478	0.1777	0.1976	0.1370	0.1694	0.1735	0.2091	0.1317
	β	0.3718	0.4073	0.4990	0.4907	0.3699	0.3867	0.4864	0.4739
	μ_1	0.5079	0.6790	0.5690	0.6790	0.5074	0.7051	0.5741	0.7051
	μ_2	1.4054	0.5991	0.9375	0.5935	1.4129	0.5794	0.9614	0.5726

Table IX. AVG, SD and MSE of estimates for n=50, Case 3

Measure	Estimates of	LSE-S	MLE-S	MPS-S	MLE-M	LSE-S	MLE-S	MPS-S	MLE-M
		m=100				m=500			
AVG	α_1	11.0230	11.2680	11.1920	11.2680	11.0318	11.2372	11.1820	11.2372
	α_2	1.5760	1.4300	1.6390	1.4450	1.5878	1.4354	1.6460	1.4512
	β	6.4935	6.4525	6.1370	6.3930	6.4880	6.4588	6.1390	6.4014
	μ_1	0.8910	1.1170	0.7840	1.1170	0.8856	1.1414	0.8056	1.1414
	μ_2	1.8460	2.3990	1.8830	2.3760	1.8128	2.3884	1.8644	2.3620
SD	α_1	0.9162	0.9005	0.9143	0.9005	0.9212	0.9135	0.9291	0.9135
	α_2	0.3699	0.2684	0.3318	0.2181	0.3791	0.2711	0.3361	0.2245
	β	0.5445	0.5258	0.5819	0.6354	0.5481	0.5123	0.5684	0.6350
	μ_1	0.7007	0.7317	0.7562	0.7317	0.6981	0.7221	0.7576	0.7221
	μ_2	0.9944	0.4428	0.6030	0.4048	0.9885	0.4376	0.5994	0.4017
MSE	α_1	0.8623	0.8074	0.8276	0.8074	0.8753	0.8341	0.8618	0.8341
	α_2	0.1412	0.0762	0.1283	0.0501	0.1511	0.0775	0.1341	0.0527
	β	0.3022	0.2764	0.4044	0.3997	0.3075	0.2654	0.3906	0.4024
	μ_1	0.5225	0.7039	0.5732	0.7039	0.5208	0.7153	0.5840	0.7153
	μ_2	1.0434	0.2835	0.4071	0.2384	1.0577	0.2743	0.4140	0.2297

Table X. AVG, SD and MSE of estimates for n=100, Case 3

Measure	Estimates of	LSE-S	MLE-S	MPS-S	MLE-M	LSE-S	MLE-S	MPS-S	MLE-M
		m=100				m=500			
	α_1	11.0860	11.3070	11.2720	11.3070	11.0506	11.2142	11.1898	11.2142
	α_2	1.6060	1.4590	1.6070	1.4880	1.6086	1.4546	1.6100	1.4770
	β	6.4155	6.4345	6.2165	6.3480	6.4138	6.4704	6.2269	6.4030
	μ_1	0.9450	1.0570	0.7190	1.0570	0.8936	1.0218	0.6792	1.0218
	μ_2	1.7990	2.2670	1.9390	2.2210	1.8270	2.2716	1.9414	2.2406
SD	α_1	0.9360	0.8299	0.8411	0.8299	0.9368	0.8521	0.8556	0.8521
	α_2	0.3120	0.2216	0.2555	0.1849	0.3157	0.2161	0.2578	0.1772
	β	0.6253	0.5664	0.5965	0.5799	0.5887	0.5274	0.5479	0.5813
	μ_1	0.6700	0.7350	0.7243	0.7350	0.6640	0.7366	0.7162	0.7366
	μ_2	0.7593	0.2288	0.3051	0.2358	0.7404	0.2426	0.3235	0.2473
MSE	α_1	0.8804	0.6933	0.7056	0.6933	0.8981	0.7248	0.7306	0.7248
	α_1	0.1076	0.0503	0.0761	0.0340	0.1113	0.0487	0.0784	0.0319
	β	0.3873	0.3188	0.3859	0.3356	0.3461	0.2825	0.3295	0.3372
	μ_1	0.5045	0.6623	0.5197	0.6623	0.4775	0.6451	0.5123	0.6451
	μ_2	0.6613	0.0797	0.1181	0.0697	0.6217	0.0882	0.1296	0.0808

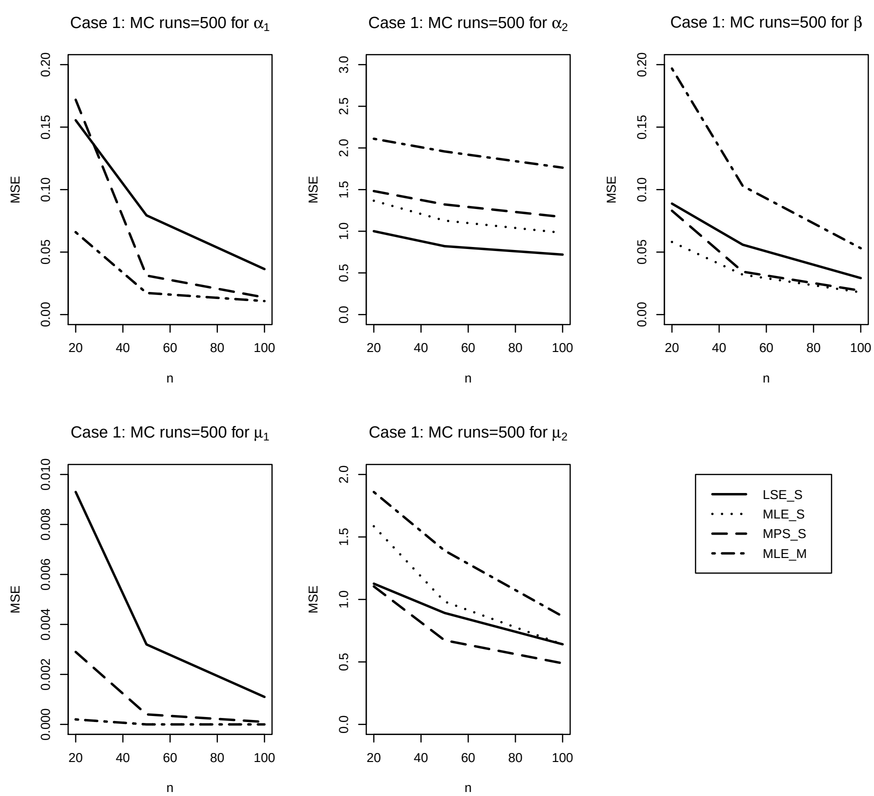

Figure 2. Line plot of MSEs corresponding to Case 1

In Case 1, which is characterised by low correlation, the simulation results in Tables (II) to (IV) reveal that the average value of estimates (AVG) obtained by LSE-S is closer to the true parameter values among the different methods proposed. However, LSE-S does not produce consistent estimates for all the parameters as is evident from Figure 2.

With an increase in correlation (Case 2), it is noted that the proposed methods provide similar estimates for the parameters as seen in Tables (V) to (VII). But, from Figure 3, it is evident that MLE-M outperforms others in terms of MSE .

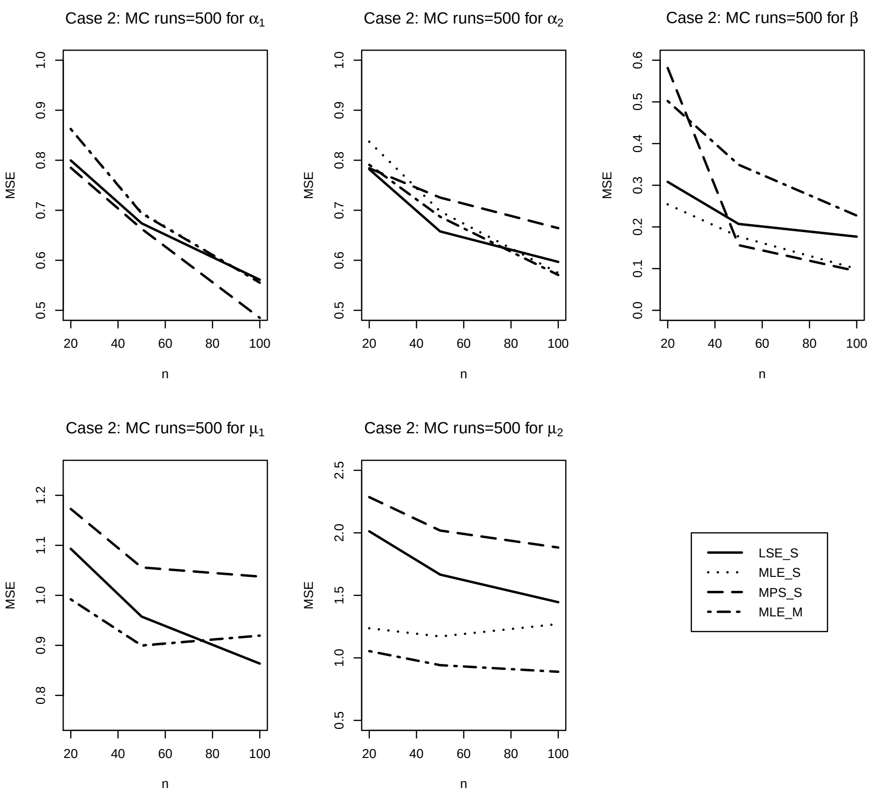

Figure 3. Line plot of MSEs corresponding to Case 2

Considering Case 3, which is marked by high correlation, it can be seen from Tables (VIII) to (X) that all the proposed methods perform equally well in estimating the shape and scale parameters. However, the methods fail to provide estimates closer to the true value for the location parameters. In general, it is observed that the performance of the proposed methods improve with an increase in sample size and number of simulations.

5. Discussion

This paper introduces a heuristic approach for parameter estimation in k-variate gamma distribution introduced by Mathai and Moschopoulos [9]. The proposed methodology makes use of the marginal distributions instead of directly using the complex structure of the k-variate gamma distribution. This is due to the fact that the probability density function of k-variate gamma distribution defined in (2) can be expressed as product of three-parameter gamma marginals as given in (4). Thus, the complexity associated with estimating all the $(2k+1)$ parameters simultaneously is reduced as only the parameters of the marginals need to be estimated independently. Since the density in (2) has a common scale parameter β, the $3k$ estimates obtained for β have to be meaningfully combined to provide a single estimate. It is observed from simulation studies that the arithmetic mean of the $3k$ estimates obtained for β provides a single estimate that is close to the true parameter value. Alternatively, one can estimate the scale parameter from any of the marginal distributions and fix this value for estimating the parameters of the remaining marginal distributions.

It is to be noted that the proposed methodology directly optimizes the objective function (e.g. likelihood function) of the corresponding univariate marginal distributions through an extensive search in the three-parameter space rather than solving non-linear derivatives. This offers better chances of attaining global optima which is achieved by increasing the width of the interval in which the search is carried out. The steps involved in the proposed methodology are presented in Split-Join Algorithm explained in Section 3.2. It is observed that the precision of the estimates increases by reducing the factor of incrementation. Moreover, the algorithm places no restrictions on the parameter space. However, the algorithm takes more time to arrive at the estimates owing to the fact that it searches the parameter space corresponding to each marginal distribution separately.

Simulation studies performed to obtain estimates for the parameters of bivariate gamma distribution through the proposed methodology involving ML, MPS and LS methods suggest that the resulting values are closer to the parameters.

For implementing the algorithm on real-life datasets that follow k-variate gamma distribution, the following considerations will serve as an aid. A test procedure for testing for common scale parameter of the marginal distributions can be employed before implementing the algorithm. For fixing the lower and upper bounds of the parameters in the search process, one may initially start with a random set of points in the three-dimensional search space and evaluate the objective function corresponding to each of these points. An interval can be defined around the point that optimizes the objective function and the algorithm can then be implemented by setting the endpoints of the interval as the lower and upper bounds.

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