Abstract

The paper proposes estimators for unknown parameters in some credibility models. Sufficient conditions for asymptotic optimality of empirical credibility estimators in these models are given. Finally we critically discuss the properties of some of the proposed estimators in the case of finite insurance portfolios.

1. Preliminaries

Let m be an unknown random variable. We shall say that an estimator $m^{(1)}$ is a better estimator of m than another estimator $m^{(2)}$ if

$$E(m^{(1)}-m)^2 < E(m^{(2)}-m)^2 ,$$

that is, we use quadratic loss.

Let $x_1, x_2, \ldots, x_n$ be observable random variables. We shall call an estimator $\dot{m}$ of m a linear estimator of m (based on $x_1, \ldots, x_n$) if $\dot{m}$ may be written $\dot{m} = g_0 + \Sigma_{i=1}^n g_i x_i$, where $g_0, g_1, \ldots, g_n$ are non-random numbers. By the credibility estimator of m (based on $x_1, \ldots, x_n$) we shall mean the best linear estimator of m .

2. The Bühlmann-Straub model

2A. The following model was introduced by Bühlmann & Straub (1970).

We consider a ceding insurance company. Let p_j be the direct insurance risk premium and s_j the total reinsurance claims of year j . Then the observed loss ratio of year j is

$$x_j = \frac{s_j}{p_j} .$$

It is assumed that the x_j's are conditionally independent given an unknown random parameter θ, and that for all j

$$E(x_j) = \mu \qquad EV(x_j|\theta) = \frac{\varphi}{p_j} \qquad VE(x_j|\theta) = \lambda$$

with μ, φ, and λ greater than zero.

Then the credibility estimator of $\tilde{x}_{n+1}$ based on $x_1, \ldots, x_n$ is

$$\tilde{x}_{n+1} = \frac{p\lambda}{p\lambda+\varphi} \bar{x}_n + \frac{\varphi}{p\lambda+\varphi} \mu$$

with

$$p = \sum_{j=1}^{n} p_j \qquad \bar{x}_n = \frac{1}{p} \sum_{j=1}^{n} p_j x_j .$$

2B. Assume that we have observed a reinsurance portfolio of N independent ceded portfolios satisfying the conditions of subsection 2A with the same φ, λ, and μ. Portfolio i has been observed for t_i years, and p_{ij} is the direct insurance risk premium and x_{ij} the observed loss ratio from the jth observation year of this portfolio.

Bühlmann and Straub (1970) proposed estimators of φ, λ, and μ based on the x_{ij}'s and p_{ij}'s in the case $t_1 = t_2 = \ldots = t_N$. We are now going to generalize these estimators to the case with general t_i's.

For φ and λ the estimators

$$\varphi^* = \frac{1}{\sum_{i=1}^{N} t_i - N} \sum_{i=1}^{N} \sum_{j=1}^{t_i} p_{ij}(x_{ij} - \bar{x}_i)^2$$

$$\hat{\lambda} = \frac{1}{\sum_{i=1}^{N} p_i \left(1 - \frac{p_i}{\sum_{k=1}^{N} p_k}\right)} \left[\sum_{i=1}^{N} p_i (\bar{x}_i - \bar{\bar{x}})^2 - (N-1)\varphi^*\right]$$

with

$$p_i = \sum_{j=1}^{t_i} p_{ij} \qquad \bar{x}_i = \frac{1}{p_i} \sum_{j=1}^{t_i} p_{ij} x_{ij} \qquad \bar{\bar{x}} = \frac{1}{\sum_{i=1}^{N} p_i} \sum_{i=1}^{N} p_i \bar{x}_i$$

are unbiased. As λ may not be negative, we estimate λ by the adjusted estimator

$$\lambda^* = \max(\hat{\lambda}, 0) .$$

As

$$\hat{\mu} = \frac{\sum_{i=1}^{N} \frac{p_i}{p_i \lambda + \varphi} \bar{x}_i}{\sum_{i=1}^{N} \frac{p_i}{p_i \lambda + \varphi}}$$

is the best linear unbiased estimator of μ (see e.g. Sundt (1978)) based on the x_{ij}'s we propose to estimate μ by

$$\mu^* = \frac{\sum_{i=1}^{N} \frac{p_i}{p_i \lambda^* + \varphi^*} \bar{x}_i}{\sum_{i=1}^{N} \frac{p_i}{p_i \lambda^* + \varphi^*}} . \qquad (1)$$

2C. Let us now assume that we have observed a portfolio of N independent insurance policies. Policy i has been observed for t_i insurance years, and x_{ij} is the claim number of the jth observation year of this policy. It is assumed that the x_{ij}'s satisfy the conditions of subsections 2A-B with all $p_{ij} = 1$. In addition we assume that the claim numbers are conditionally Poisson distributed given the underlying θ , that is, we assume $\mu = \varphi$.

Under these conditions we propose to estimate μ and φ

by

$$\mu^{**} = \frac{1}{\sum_{i=1}^{N} t_i} \sum_{i=1}^{N} t_i \bar{x}_i \qquad (2)$$

and λ by

$$\lambda^{**} = \max(\hat{\hat{\lambda}},0)$$

with

$$\hat{\hat{\lambda}} = \frac{1}{\sum_{i=1}^{N} t_i \left(1 - \frac{t_i}{\sum_{k=1}^{N} t_k}\right)} \left[\sum_{i=1}^{N} t_i (\bar{x}_i - \bar{\bar{x}})^2 - (N-1)\mu^{**} \right].$$

3. Generalized Bühlmann-Straub model

3A. Sundt (1980) proposed the following generalization of the Bühlmann-Straub model.

Assume as in subsection 2A that p_j is the direct insurance risk premium and x_j the observed loss ratio of the jth reinsurance year of a ceded insurance portfolio. The x_i's are now assumed conditionally independent given a sequence $\theta = (\theta_1, \theta_2, \ldots)$ of unknown random parameters, and for each i x_i depends on θ only through θ_i. Furthermore, for all i and j we assume that

$$E(x_i) = \mu \qquad EV(x_i|\theta) = \frac{\varphi}{p_i} \qquad C(E(x_i|\theta),E(x_j|\theta)) = \rho^{|i-j|}\lambda$$

with $\varphi,\lambda,\mu > 0$ and $\rho \in [-1,1]$.

Let $\tilde{x}_{n+1}$ be the credibility estimator of $E(x_{n+1}|\theta)$ (and x_{n+1}) based on $x_1,\ldots,x_n$, and

$$\psi_{n+1} = E(E(x_{n+1}|\theta)-\tilde{x}_{n+1})^2$$

the estimation error. Then we have

$$\psi_{n+1} = \rho^2 \frac{\psi_n \varphi}{p_n \psi_n + \varphi} + (1-\rho^2)\lambda \tag{3}$$

$$\tilde{x}_{n+1} = \rho\left(\frac{p_n \psi_n}{p_n \psi_n + \varphi} x_n + \frac{\varphi}{p_n \psi_n + \varphi} \tilde{x}_n\right) + (1-\rho)\mu \tag{4}$$

$$\tilde{x}_1 = \mu \qquad \psi_1 = \lambda . \tag{5}$$

3B. In the present model parameter estimation becomes somewhat more difficult than in the model of Section 2.

Suppose that we have observed for t years a reinsurance portfolio of independent ceded portfolios satisfying the conditions of subsection 3A with the same ρ, φ, λ, and μ. ${}^r_i N$ of these portfolios have been ceded in both year i and year $i+r$, and ${}^r_i x_{kj}$ is the observed loss ratio and ${}^r_i p_{kj}$ the direct premium of the k-th of these portfolios in year j $(k = 1, \ldots, {}^r_i N$; $j = i, i+r)$.

Let

$${}^r_i \bar{x}_j = \sum_{k=1}^{{}^r_i N} {}^r_i a_{kj} \, {}^r_i x_{kj} ,$$

where the non-random weights ${}^r_i a_{kj}$ satisfy $\sum_{k=1}^{{}^r_i N} {}^r_i a_{kj} = 1$. Then some tedious calculus give

$$E(({}^r_i x_{ki} - {}^r_i \bar{x}_i)({}^r_i x_{k,i+r} - {}^r_i \bar{x}_{i+r})) = \begin{cases} c_{ki}\varphi + {}^o_i c_k \lambda & r = 0 \\ {}^r_i c_k \rho^r \lambda & r = 1,2,\ldots,t-1 \end{cases}$$

with

$$c_{ki} = \frac{1 - 2\,{}^o_i a_{ki}}{{}^o_i p_{ki}} + \sum_{j=1}^{{}^o_i N} \frac{{}^o_i a_{ji}^2}{{}^o_i p_{ji}}$$

$$ {}^r_i c_k = 1 - {}^r_i a_{ki} - {}^r_i a_{k,i+r} + \sum_{j=1}^{{}^r_i N} {}^r_i a_{ji}\, {}^r_i a_{j,i+r} \,. $$

Let

$$ e_r = \sum_{i=1}^{t-r} \sum_{k=1}^{{}^r_i N} {}^r_i b_k ({}^r_i x_{ki} - {}^r_i \bar{x}_i)({}^r_i x_{k,i+r} - {}^r_i \bar{x}_{i+r}) \qquad k = 0,1,\ldots,t-1 \,, $$

where the constants ${}^r_i b_k$ are chosen so as to satisfy

$$ \sum_{i=1}^{t-r} \sum_{k=1}^{{}^r_i N} {}^r_i b_k \, {}^r_i c_k = 1 \,. $$

Then we have

$$ E(e_r) = \begin{cases} c\varphi + \lambda & r = 0 \\ \rho^r \lambda & r = 1,2,\ldots,t-1 \end{cases} $$

with

$$ c = \sum_{i=1}^{t} \sum_{k=1}^{{}^o_i N} {}^o_i b_k \, c_{ki} \,, $$

and for $\rho, \lambda,$ and φ the estimators

$$ \hat{\rho} = \frac{\sum_{r=2}^{t-1} d_r e_r}{\sum_{r=2}^{t-1} d_r e_{r-1}} \qquad (6) $$

$$ \hat{\lambda} = \frac{e_1}{\hat{\rho}} $$

$$ \hat{\varphi} = \frac{e_0 - \hat{\lambda}}{c} $$

where the d_r's are constants, seem reasonable. (If the denominator of (6) is equal to zero, we put $\hat{\rho} = 0$.) As the absolute value of ρ may not exceed 1 , and φ and λ may not be negative, we propose the adjusted estimators

$$\rho^* = \begin{cases} -1 & \hat{\rho} < -1 \\ \hat{\rho} & |\hat{\rho}| \leq 1 \\ 1 & \hat{\rho} > 1 \end{cases}$$

$$\lambda^* = \begin{cases} 0 & \frac{e_1}{\rho^*} < 0 \\ \frac{e_1}{\rho^*} & 0 \leq \frac{e_1}{\rho^*} \leq e_0 \\ e_0 & \frac{e_1}{\rho^*} > e_0 \end{cases} \tag{7}$$

$$\varphi^* = \frac{e_0 - \lambda^*}{c} .$$

From (7) we see that λ^* (and thereby φ^*) is not defined if $\rho^* = 0$. In that case we put $\lambda^* = 0$ and $\varphi^* = \frac{e_0}{c}$.

In the case $t = 3$ (6) reduces to

$$\hat{\rho} = \frac{e_2}{e_1}$$

As $E(e_2) = \rho^2\lambda \geq 0$, we propose to estimate ρ by 0 if e_2 is negative.

Finally we have to take care if both $\rho^* = \pm 1$ and $\varphi^* = 0$, because replacing ρ and φ by these values in (3) would make the denominator $p_n\psi_n + \varphi$ equal to zero. If $\rho^* = -1$ and $\varphi^* = 0$, we propose to put $\rho^* = \lambda^* = 0$ and $\varphi^* = e_0/c$. If $\rho^* = 1$, we propose to estimate φ and λ by the estimators φ^* and λ^* of subsection 2B.

It seems to be complicated to find constants ${}_i^r a_{kj}$, ${}_i^r b_k$, and d_r that are optimal according to quadratic loss. One might perhaps proceed along the lines of De Vylder (1977), but even if we found optimal constants, they would probably be too complicated for practical applications. We propose to choose

$$ {}^{r}_{i}a_{kj} = \frac{{}^{r}_{i}p_{kj}}{{}^{r}_{i}p_{j}} \qquad {}^{r}_{i}b_{k} = \frac{1}{g_r}\sqrt{{}^{r}_{i}p_{ki}\;{}^{r}_{i}p_{k,i+r}} \qquad d_r = \sum_{i=1}^{t-r} {}^{r}_{i}N $$

with

$$ {}^{r}_{i}p_{j} = \sum_{k=1}^{{}^{r}_{i}N} {}^{r}_{i}p_{kj} \qquad g_r = \sum_{i=1}^{t-r} \sum_{k=1}^{{}^{r}_{i}N} \sqrt{{}^{r}_{i}p_{ki}\;{}^{r}_{i}p_{k,i+r}}\; c_{ki} \; . $$

For the estimation of μ we propose to use μ^* defined by (1) with φ^* and λ^* being the estimators developed in the present subsection. This estimator should be reasonable if ρ does not differ too much from 1 . An alternative estimator is μ^{**} defined by (2). This is the best linear unbiased estimator based on the observed ${}^{r}_{i}x_{kj}$'s if $\rho = 0$.

3C. Let us now assume that a portfolio of independent insurance policies has been observed for t years. ${}^{r}_{i}N$ of the policies have been observed in both year i and i+r , and ${}^{r}_{i}x_{kj}$ is the observed claim amount of the k-th of these policies in year j ($k = 1,\dots,{}^{r}_{i}N$; $j = i,i+r$). It is assumed that the ${}^{r}_{i}x_{kj}$'s satisfy the conditions of subsections 3A-B with all ${}^{r}_{i}p_{kj} = 1$.

In this special case of the model the choice of constants ${}^{r}_{i}a_{kj}$, ${}^{r}_{i}b_{k}$, and d_r in the previous subsection seems more obvious, and we get

$$ {}^{r}_{i}\bar{x}_{j} = \frac{1}{{}^{r}_{i}N} \sum_{k=1}^{{}^{r}_{i}N} {}^{r}_{i}x_{kj} $$

$$ e_r = \frac{1}{\sum_{i=1}^{t-r} {}^{r}_{i}N - t + r} \sum_{i=1}^{t-r} \sum_{k=1}^{{}^{r}_{i}N} ({}^{r}_{i}x_{ki} - {}^{r}_{i}\bar{x}_{i})({}^{r}_{i}x_{k,i+r} - {}^{r}_{i}\bar{x}_{i+r}) $$

$$ \hat{\rho} = \frac{\sum_{r=2}^{t-1} \left(\sum_{i=1}^{t-r} {}^{r}_{i}N\right) e_r}{\sum_{r=2}^{t-1} \left(\sum_{i=1}^{t-r} {}^{r}_{i}N\right) e_{r-1}} \; . $$

3D. We see that to estimate ρ, λ, and φ by the procedure outlined above we need data from at least three years. If the ${}_{i}^{r}x_{kj}$'s are claim numbers instead of claim amounts, and we may assume that they are conditionally Poisson distributed given the underlying random parameters, we may manage with two years as we then have $\varphi = \mu$. We propose to estimate μ and φ by μ^{**} defined by (2), λ by

$$\lambda^{**} = \max(e_0 - \mu^{**}, 0) ,$$

and ρ by

$$\rho^{**} = \begin{cases} -1 & \hat{\hat{\rho}} < -1 \\ \hat{\hat{\rho}} & |\hat{\hat{\rho}}| \leq 1 \\ 1 & \hat{\hat{\rho}} > 1 \end{cases}$$

with

$$\hat{\hat{\rho}} = \frac{\sum_{r=1}^{t-1} \left(\sum_{i=1}^{t-r} {}_{i}^{r}N \right) e_r}{\sum_{r=1}^{t-1} \left(\sum_{i=1}^{t-r} {}_{i}^{r}N \right) e_{r-1} - \left(\sum_{i=1}^{t-1} {}_{i}N \right) \mu^{**}} . \tag{8}$$

(If the denominator in (8) is equal to zero, we put $\hat{\hat{\rho}} = 0$.)

In the case $t = 2$ (8) reduces to

$$\hat{\hat{\rho}} = \frac{e_1}{e_0 - \mu^{**}} .$$

As $E(e_0 - \mu^{**}) = \lambda \geq 0$, we propose to estimate ρ by 0 if $e_0 < \mu^{**}$.

4. Asymptotic optimality

4A. We assume the model of subsection 3A. Let $({}_N\mu, {}_N\rho, {}_N\varphi, {}_N\lambda)$ be a set of estimators of $(\mu, \rho, \varphi, \lambda)$ based on observations from N independent ceded portfolios. It is assumed that for all N

$$\begin{aligned} &{}_N\varphi, {}_N\lambda, {}_N\mu \geq 0 \; ; \; {}_N\rho \in [-1,1] \; ; \\ &({}_N\varphi + 1 - {}_N\rho^2)({}_N\varphi + {}_N\lambda) > 0 \; ; \end{aligned} \tag{9}$$

and that

$$({}_N\mu, {}_N\rho, {}_N\varphi, {}_N\lambda) \xrightarrow{P} (\mu, \rho, \varphi, \lambda) \tag{10}$$

as N approaches infinity.

Let ${}_N\tilde{x}_{n+1}$ be the estimator we obtain as replacement of $\tilde{x}_{n+1}$ when replacing $(\mu, \rho, \varphi, \lambda)$ by $({}_N\mu, {}_N\rho, {}_N\varphi, {}_N\lambda)$ in the recursion (3) - (5). Then ${}_N\tilde{x}_{n+1}$ is an empirical credibility estimator in Norberg's (1980) sense. We shall now discuss in Norberg's (1980) sense asymptotic optimality of this estimator.

Let

$$\tilde{x}_{n+1} = \alpha_{no} + \Sigma_{i=1}^{n} \alpha_{ni} x_i$$

and

$${}_N\tilde{x}_{n+1} = {}_N\alpha_{no} + \Sigma_{i=1}^{n} {}_N\alpha_{ni} x_i$$

define the credibility coefficients α_{ni} and the empirical credibility coefficients ${}_N\alpha_{ni}$. Then we have the following lemma.

<u>Lemma 1</u>. If

$$\lim_{N\to\infty} E({}_N\mu - \mu)^2 = 0 \; , \tag{11}$$

then

$$\lim_{N\to\infty} E({}_N\alpha_{no}-\alpha_{no})^2 = 0 .$$

<u>Proof</u>. From (3) - (5) it is clear that we have

$$\alpha_{no} = \eta_n \mu ,$$

where η_n is defined by

$$\eta_n = \rho \frac{\varphi}{p_n\psi_n+\varphi} \eta_{n-1} + 1-\rho \qquad \eta_0 = 1 .$$

We have

$$|\eta_n| \leq |\eta_{n-1}| + 2 ,$$

and induction gives

$$|\eta_n| \leq 2n+1 . \tag{12}$$

Likewise we have

$${}_N\alpha_{no} = {}_N\eta_n \; {}_N\mu$$

with ${}_N\eta_n$ satisfying

$$|{}_N\eta_n| \leq 2n+1 . \tag{13}$$

We now get

$$E({}_N\alpha_{no}-\alpha_{no})^2 = E({}_N\eta_n \; {}_N\mu-\eta_n\mu)^2 =$$

$$E({}_N\eta_n({}_N\mu-\mu) + \mu({}_N\eta_n-\eta_n))^2 \leq$$

$$E(4\{[{}_N\eta_n({}_N\mu-\mu)\;^2 + \;\mu({}_N\eta_n-\eta_n)]^2\}) \leq$$

$$4(2n+1)^2 E({}_N\mu-\mu)^2 + 4\mu^2 E({}_N\eta_n-\eta_n)^2 .$$

Because of (11) it now only remains to show that

$$\lim_{N\to\infty} E({}_N\eta_n - \eta_n)^2 = 0 . \qquad (14)$$

By (12) and (13)

$$({}_N\eta_n - \eta_n)^2 \leq 4(2n+1)^2 ,$$

that is, the sequence $\{({}_N\eta_n - \eta_n)^2\}$ is dominated.

As in addition $({}_N\eta_n - \eta_n)^2 \xrightarrow{P} 0$, (14) follows from Lemma 5.1 in Norberg (1980), and Lemma 1 is thereby proved.

Q.E.D.

Theorem 1. If (11) is satisfied, then ${}_N\tilde{x}_{n+1}$ is asymptotically optimal for all n .

Proof. From (3) - (5) it is clear that

$$\alpha_{ni} = \rho \frac{\varphi}{p_n\psi_n+\varphi} \alpha_{n-1,i} \qquad \begin{array}{l} i = 1,\ldots,n-1 \\ n = 2,3,\ldots \end{array}$$

$$\alpha_{nn} = \rho \frac{p_n \psi_n}{p_n\psi_n+\varphi} \qquad n = 1,2,\ldots .$$

From this we easily see that

$$|\alpha_{ni}| \leq 1 \qquad \begin{array}{l} i = 1,2,\ldots,n \\ n = 1,2,\ldots \end{array} .$$

Now all the conditions of Theorem 5.5 in Norberg (1980) is satisfied, and Theorem 1 follows by that theorem.

Q.E.D.

4B. Now consider the situation of subsection 3C and assume that we have observed N independent and identical policies for the same t years ($t \geq 3$). We let $({}_N\mu, {}_N\rho, {}_N\varphi, {}_N\lambda)$ be the adjusted estimators $(\mu^*, \rho^*, \varphi^*, \lambda^*)$ of subsection 3B. These estimators satisfy (9) and (11), and if we add the assumption that $\rho \neq 0$, then (10) is satisfied. Then the estimators ${}_N\tilde{x}_{n+1}$ (with $p_n \equiv 1$) are asymptotically optimal by Theorem 1.

The asymptotic optimality is still valied if we add the Poisson assumption of subsection 3D and let $({}_N\mu, {}_N\rho, {}_N\varphi, {}_N\lambda)$ be the estimators $(\mu^{**}, \rho^{**}, \varphi^{**}, \lambda^{**})$.

5. Numerical example. Critical comments.

5A. A homogeneous portfolio of $N = 2697$ automobile liability insurance policies was observed for $t = 3$ years, and it was assumed that both claim amounts (adjusted for inflation) and claim numbers satisfied the conditions of subsection 3C.

For the claim numbers we found

$e_0 = 0.02605$ $e_1 = 0.001743$ $e_2 = 0.0001364$

$\mu^* = 0.02546$ $\rho^* = 0.07825$ $\lambda^* = 0.02228$ $\varphi^* = 0.003778$.

The estimate of ρ definitely does not seem convincing. In classical credibility theory it is customary to assume $\rho = 1$. Hence one should for the present portfolio expect a value of ρ not differing very much from 1. And the difference between 1 and 0.07825 is definitely great.

For the inflation-adjusted claim amounts the case is even worse:

$e_0 = 401198.77$ $e_1 = 1830.42$ $e_2 = -2328.17$

$\hat{\rho} = -1.2719$

$\mu^* = 70.4338$ $\rho^* = \lambda^* = 0$ $\varphi^* = 401198.77$.

5B. To get some indication of the quality of the estimates from the observed portfolio, we shall study a bit more closely the quantity e_2 in the case of claim numbers.

Let x_{ij} be the claim number of policy i in year j. As an approximation it is assumed that these claim numbers may take only the values 1 and 0. (In the observed portfolio very few policies had more than one claim a year.) The probability distribution of (x_{i1}, x_{i3}) is then given by

$$\Pr(x_{i1}=1) = \Pr(x_{i2}=1) = \mu$$

$$\nu = \Pr(x_{i1}=x_{i3}=1) = C(x_{i1},x_{i3}) + \mu^2 .$$

We are going to estimate the variance of e_2, and hence we need an expression for $E(e_2^2)$. We have

$$e_2^2 = \frac{1}{(N-1)^2}\Big[(\sum_{i=1}^{N} x_{i1}x_{i3})^2 - \frac{2}{N}(\sum_{i=1}^{N} x_{i1})(\sum_{i=1}^{N} x_{i3})(\sum_{i=1}^{N} x_{i1}x_{i3}) + \frac{1}{N^2}(\sum_{i=1}^{N} x_{i1})^2(\sum_{i=1}^{N} x_{i3})^2\Big] . \tag{15}$$

As

$$E(\sum_{i=1}^{N} x_{i1}x_{i3})^2 = V(\sum_{i=1}^{N} x_{i1}x_{i3}) + (E(\sum_{i=1}^{N} x_{i1}x_{i3}))^2 ,$$

we easily obtain

$$E(\sum_{i=1}^{N} x_{i1}x_{i3})^2 = N\nu(1-\nu) + N^2\nu^2 . \tag{16}$$

Some trivial, but tedious, combinatorical reasoning gives

$$E((\sum_{i=1}^{N} x_{i1})(\sum_{i=1}^{N} x_{i3})(\sum_{i=1}^{N} x_{i1}x_{i3})) = N\nu + 2N(N-1)\nu\mu + N(N-1)(N-2)\nu\mu^2 + N(N-1)\nu^2 \tag{17}$$

$$E((\sum_{i=1}^{N} x_{i1})^2(\sum_{i=1}^{N} x_{i3})^2) = N\nu + 4N(N-1)\nu\mu + 4N(N-1)(N-2)\nu\mu^2 + \tag{18}$$

$$2N(N-1)\nu^2 + N(N-1)(N-2)(N-3)\mu^4 + 2N(N-1)(N-2)\mu^3 + N(N-1)\mu^2 .$$

We estimate ν by $\nu^* = e_1 + \mu^{*2} = 0.0007847$. Using the estimates μ^* and ν^* in (16) - (18) gives the estimates

$$E^*(\sum_{i=1}^{N} x_{i1} x_{i3})^2 = 6.5930$$

$$E^*((\sum_{i=1}^{N} x_{i1})(\sum_{i=1}^{N} x_{i3})(\sum_{i=1}^{N} x_{i1} x_{i3})) = 10264.28936$$

$$E^*((\sum_{i=1}^{N} X_{i1})^2(\sum_{i=1}^{N} x_{i3})^2) = 22874842.40.$$

By these estimates and (15) we obtain the estimate

$$E^*(e_2^2) = 2.9253 \cdot 10^{-7}$$

for $E(e_2^2)$, and we estimate $V(e_2)$ by

$$V^*(e_2) = E^*(e_2^2) - e_2^2 = 2.7391 \cdot 10^{-7}$$

and find a standard deviation 0.0005234 for e_2.

By using normal approximation to the distribution of e_2 and letting g be the 0.95-fractile of the normal distribution we find

$$[e_2 - g\sqrt{V^*(e_2)}, e_2 + g\sqrt{V^*(e_2)}] = [-0.0007244, 0.0009973]$$

as a 90% confidence interval for $C(x_{i1}, x_{i3})$ based on e_2.

5C. The investigation in the previous subsection obviously indicates that our estimation procedure does not work satisfactorily.

To see how the standard deviation of e_2 behaves when N is increased, $\sqrt{E(e_2)}$ is given in Table 1 for some values of N. For ν and μ we use the estimated values ν^* and μ^*. We see that by increasing N to 10000 we get the standard deviation reduced to about the half of the value in the observed portfolio.

In Table 2 values of $E(e_2)$ and $\sqrt{V(e_2)}$ are given for different values of μ and ν for $N = 2697$.

5D. It would of course be desirable to compute $V(e_2)$ also for $t > 3$, but this seems to give very awkward expressions. Furthermore, what we are primarily interested in, is not the specific e_k's, but ρ^*, λ^*, and φ^*. However, to compute expectations, variances, and confidence regions for these quantities seems to be immensely difficult, even if we make approximations as assuming that the policies may not have more than one claim a year. It seems that simulations would be the only realistic way to obtain such quantities.

5E. As a conclusion it seems that to use the proposed estimators in individual automobile insurance one ought to have more observations than in the given example. The picture may be different when estimating loss ratios in reinsurance, where the observed loss ratios usually differ from zero.

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N	$\sqrt{V(e_2)}$
2	$2.6021 \cdot 10^{-2}$
50	$3.8758 \cdot 10^{-3}$
100	$2.7290 \cdot 10^{-3}$
500	$1.2164 \cdot 10^{-3}$
1000	$8.5973 \cdot 10^{-4}$
4000	$4.2973 \cdot 10^{-4}$
7000	$3.2483 \cdot 10^{-4}$
10000	$2.7177 \cdot 10^{-4}$
20000	$1.9216 \cdot 10^{-4}$
50000	$1.2153 \cdot 10^{-4}$
100000	$8.5937 \cdot 10^{-5}$

Table 1

μ	ν	$E(e_2)$	$\sqrt{V(e_2)}$
0.01	μ^2	0	$1.9067\ 10^{-4}$
	0.0025	0.0024	$9.4279\ 10^{-4}$
	0.005	0.0049	$1.3313\ 10^{-3}$
	0.0075	0.0074	$1.6283\ 10^{-3}$
	0.01	0.0099	$1.8776\ 10^{-3}$
0.02	μ^2	0	$3.7748\ 10^{-4}$
	0.005	0.0046	$1.3063\ 10^{-3}$
	0.01	0.0096	$1.8409\ 10^{-3}$
	0.015	0.0146	$2.2478\ 10^{-3}$
	0.02	0.0196	$2.5880\ 10^{-3}$
0.04	μ^2	0	$7.3956\ 10^{-4}$
	0.01	0.0084	$1.7768\ 10^{-3}$
	0.02	0.0184	$2.4892\ 10^{-3}$
	0.03	0.0284	$3.0267\ 10^{-3}$
	0.04	0.0384	$3.4715\ 10^{-3}$
0.08	μ^2	0	$1.4175\ 10^{-3}$
	0.02	0.0136	$2.3450\ 10^{-3}$
	0.04	0.0336	$3.2220\ 10^{-3}$
	0.06	0.0536	$3.8688\ 10^{-3}$
	0.08	0.0736	$4.3883\ 10^{-3}$
0.12	μ^2	0	$2.0338\ 10^{-3}$
	0.03	0.0156	$2.7179\ 10^{-3}$
	0.06	0.0456	$3.6237\ 10^{-3}$
	0.09	0.0756	$4.2672\ 10^{-3}$
	0.12	0.1056	$4.7559\ 10^{-3}$

Table 2