PARAMETER ESTIMATION OF CHIRP SIGNALS IN PRESENCE OF STATIONARY NOISE

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Abstract. The problem of parameter estimation of the chirp signals in presence of stationary noise has been addressed. We consider the least squares estimators and it is observed that the least squares estimators are strongly consistent. The asymptotic distributions of the least squares estimators are obtained. The multiple chirp signal model is also considered and we obtain the asymptotic properties of the least squares estimators of the unknown parameters. We perform some small sample simulations to observe how the proposed estimators work for small sample sizes.

1. Introduction

In this paper we consider the estimation procedure of the parameters of the following signal processing model:

$$y(n) = A^0 \cos(\alpha^0 n + \beta^0 n^2) + B^0 \sin(\alpha^0 n + \beta^0 n^2) + X(n); \quad n = 1, \ldots, N. \tag{1}$$

Here $y(n)$ is the real valued signal observed at $n = 1, \ldots, N$. A^0 and B^0 are real-valued amplitudes and α^0 and β^0 are the frequency and frequency rate respectively. So the chirp signal model (1) does not have a constant frequency like the sinusoidal frequency model and the initial frequency changes over time with the rate β. The error random variables $\{X(n)\}$ is a sequence of random variables with mean zero and finite fourth moment. The error random variable $X(n)$ satisfies the following assumption:

Assumption 1. *The error random variable $\{X(n)\}$ can be written in the following form;*

$$X(n) = \sum_{j=-\infty}^{\infty} a(j) e(n-j). \tag{2}$$

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Here $\{e(n)\}$ *is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero and finite fourth moment. The coefficients* $a(j)$*'s satisfy the following condition;*

$$\sum_{j=-\infty}^{\infty} |a(j)| < \infty. \tag{3}$$

The signals as described in (1) are known as the chirp signals in the statistical signal processing literature (Djurić and Kay; 1990). Chirp signals are quite common in various areas of science and engineering, specifically in sonar, radar, communications, etc. Several authors considered the chirp signal model (1) when $X(n)$'s are i.i.d. random variables. See for example, the work of Abatzoglon (1986), Kumaresn and Verma (1987), Djurić and Kay (1990), Gini, Montanari and Verrazzani (2000), Nandi and Kundu (2004) etc. Different approaches of the estimation of chirp parameters in similar kinds of models are found in Giannakis and Zhou (1995), Zhou, Ginnakis and Swami (1996), Shamsunder, Giannakis and Friedlander (1995), Swami (1996) and Zhou and Giannakis (1995). It is well known, that in most of the practical situations, the errors may not be independent. We assume stationarity through assumption 1 to incorporate the dependence structure and make the model more realistic.

Assumption 1 is a standard assumption for a stationary linear process. Any finite dimensional stationary AR, MA or ARMA process can be represented as (2) when the coefficients $a(j)$'s satisfy (3). Thus, the Assumption 1 is true for a large class of stationary random variables.

In this paper, we discuss the problem of parameter estimation of the chirp signal model in presence of stationary noise. We consider the least squares estimators and study their properties, when the errors satisfy assumption 1. It is known, see Kundu (1997), that the simple sum of sinusoidal model does not satisfy the sufficient conditions of Jennrich (1969) or Wu (1981) for the least squares estimators to be consistent. So the chirp signal model as defined in (1) also does not satisfy the sufficient conditions of Jennrich and Wu. Therefore, the results of Wu or Jennrich cannot be applied directly to establish the strong consistency or the asymptotic normality properties of the LSEs. Interestingly, because of the structure of the model, although it does not satisfy the standard sufficient conditions, the strong consistency or the asymptotic normality results can be obtained. It is also observed that the asymptotic variances of the amplitudes, frequency and frequency rate estimators are $O(N^{-1})$, $O(N^{-3})$ and $O(N^{-5})$ respectively. Based on the asymptotic distributions, asymptotic confidence intervals can also be constructed.

The rest of the paper is organized as follows. In section 2, we provide the asymptotic properties of the least squares estimators. Multiple chirp model is discussed in section 3. Some numerical results are presented in section 4 and we conclude the paper in section 5. The proofs of the results of section 2 are provided in the Appendix.

2. Asymptotic Properties of LSEs

Let us use the following notation: $\boldsymbol{\theta} = (A, B, \alpha, \beta)$, $\boldsymbol{\theta}^0 = (A^0, B^0, \alpha^0, \beta^0)$. Then, the least squares estimator (LSE) of $\boldsymbol{\theta}^0$, say $\hat{\boldsymbol{\theta}} = (\hat{A}, \hat{B}, \hat{\alpha}, \hat{\beta})$, can be obtained by minimizing

$$Q(A, B, \alpha, \beta) = Q(\boldsymbol{\theta}) = \sum_{n=1}^{N} \left[y(n) - A\cos(\alpha n + \beta n^2) - B\sin(\alpha n + \beta n^2) \right]^2, \tag{4}$$

with respect to A, B, α and β. In the following, we state the consistency property of $\boldsymbol{\theta}^0$ in theorem 1.

Theorem 1. Let the true parameter vector $\boldsymbol{\theta}^0 = (A^0, B^0, \alpha^0, \beta^0)$ be an interior point of the parameter space $\boldsymbol{\Theta} = (-\infty, \infty) \times (-\infty, \infty) \times (0, \pi) \times (0, \pi)$ and $A^{0^2} + B^{0^2} > 0$. If the error random variables $X(n)$ satisfy assumption 1, then $\hat{\boldsymbol{\theta}}$, the LSE of $\boldsymbol{\theta}^0$, is a strongly consistent estimator of $\boldsymbol{\theta}^0$.

In this section we compute the asymptotic joint distribution of the least squares estimators of the unknown parameters. We use $Q'(\boldsymbol{\theta})$ and $Q''(\boldsymbol{\theta})$ to denote the 1×4 vector of first derivatives of $Q(\boldsymbol{\theta})$ and the 4×4 second derivative matrix of $Q(\boldsymbol{\theta})$ respectively. Now expanding $Q'(\hat{\boldsymbol{\theta}})$ around the true parameter value $\boldsymbol{\theta}^0$ by Taylor series, we obtain

$$Q'(\hat{\boldsymbol{\theta}}) - Q'(\boldsymbol{\theta}^0) = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) Q''(\bar{\boldsymbol{\theta}}), \tag{5}$$

here $\bar{\boldsymbol{\theta}}$ is a point on the line joining the points $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$. Suppose $\mathbf{D}$ is a 4×4 diagonal matrix as follows;

$$\mathbf{D} = \mathrm{diag}\left\{ N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{3}{2}}, N^{-\frac{5}{2}} \right\}.$$

Since $Q'(\hat{\boldsymbol{\theta}}) = 0$, therefore (5) can be written as

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1} = -\left[\mathbf{Q}'(\boldsymbol{\theta}^0)\mathbf{D}\right] \left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\theta}})\mathbf{D}\right]^{-1}, \tag{6}$$

as $\left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\theta}})\mathbf{D}\right]$ is an invertible matrix *a.e.* for large N. Using theorem 1, it follows that $\hat{\boldsymbol{\theta}}$ converges *a.e.* to $\boldsymbol{\theta}^0$ and since each element of $Q''(\boldsymbol{\theta})$ is a continuous function of $\boldsymbol{\theta}$, therefore,

$$\lim_{N\to\infty}\left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\theta}})\mathbf{D}\right] = \lim_{N\to\infty}\left[\mathbf{D}\mathbf{Q}''(\boldsymbol{\theta}^{\mathbf{0}})\mathbf{D}\right] = 2\boldsymbol{\Sigma}(\boldsymbol{\theta}^0) \quad \text{(say)}.$$

Now let us look at different elements of the matrix $\boldsymbol{\Sigma}(\boldsymbol{\theta}) = (\sigma_{jk}(\boldsymbol{\theta}))$. We will use the following result

$$\lim_{N\to\infty}\frac{1}{N^p}\sum_{n=1}^{N} n^{p-1} = \frac{1}{p} \qquad \text{for } p = 1, 2, \ldots.$$

and the following notation:

$$\lim_{N\to\infty}\frac{1}{N^{p+1}}\sum_{n=1}^{N} n^p \cos^k(\alpha n + \beta n^2) = \delta_k(p,\alpha,\beta), \tag{7}$$

$$\lim_{N\to\infty}\frac{1}{N^{p+1}}\sum_{n=1}^{N} n^p \sin^k(\alpha n + \beta n^2) = \gamma_k(p,\alpha,\beta). \tag{8}$$

Here k takes values 1 and 2. Using these notation for limits, we compute the elements of $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ by routine calculations and are as follows:

The 4×1 random vector $\left[\mathbf{Q}'(\boldsymbol{\theta}^0)\mathbf{D}\right]$ takes the form;

$$\begin{bmatrix} -\frac{2}{\sqrt{N}}\sum_{n=1}^{N} X(n)\cos(\alpha^0 n + \beta^0 n^2) \\ \\ -\frac{2}{\sqrt{N}}\sum_{n=1}^{N} X(n)\sin(\alpha^0 n + \beta^0 n^2) \\ \\ \frac{2}{N^{\frac{3}{2}}}\sum_{n=1}^{N} nX(n)\left[A^0\sin(\alpha^0 n + \beta^0 n^2) - B^0\cos(\alpha^0 n + \beta^0 n^2)\right] \\ \frac{2}{N^{\frac{5}{2}}}\sum_{n=1}^{N} n^2X(n)\left[A^0\sin(\alpha^0 n + \beta^0 n^2) - B^0\cos(\alpha^0 n + \beta^0 n^2)\right] \end{bmatrix}.$$

Now using the central limit theorem of stochastic processes (see Fuller; 1976, page 251), it follows that $\left[\mathbf{Q}'(\boldsymbol{\theta}^0)\mathbf{D}\right]$ tends to a 4-variate normal distribution as given below;

$$\left[\mathbf{Q}'(\boldsymbol{\theta}^0)\mathbf{D}\right] \xrightarrow{d} \mathcal{N}_4(\mathbf{0}, \mathbf{G}(\boldsymbol{\theta}^0)), \tag{9}$$

where the matrix $\mathbf{G}(\boldsymbol{\theta}^0)$ is the asymptotic dispersion matrix of $\left[\mathbf{Q}'(\boldsymbol{\theta}^0)\mathbf{D}\right]$. If we denote $\mathbf{G}(\boldsymbol{\theta}) = ((g_{jk}(\boldsymbol{\theta})))$, then for $k \geq j$, $g_{jk}(\boldsymbol{\theta})$ are as follows:

$$g_{11}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N} E[S_1]^2, \qquad g_{12}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N} E[S_1 S_2], \tag{10}$$

$$g_{13}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N^2} E[S_1 S_3], \qquad g_{14}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N^3} E[S_1 S_4], \tag{11}$$

$$g_{22}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N} E[S_2]^2, \qquad g_{23}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N^2} E[S_2 S_3], \tag{12}$$

$$g_{24}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N^3} E[S_2 S_4], \qquad g_{33}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N^3} E[S_3]^2, \tag{13}$$

$$g_{34}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N^4} E[S_3 S_4], \qquad g_{44}(\boldsymbol{\theta}) = \lim_{N\to\infty} \frac{4}{N^5} E[S_4]^2, \tag{14}$$

where

$$\begin{aligned}
S_1 &= -\sum_{n=1}^{N} X(n)\cos(\alpha n + \beta n^2), \qquad S_2 = -\sum_{n=1}^{N} X(n)\sin(\alpha n + \beta n^2), \\
S_3 &= \sum_{n=1}^{N} nX(n)\left[A\sin(\alpha n + \beta n^2) - B\cos(\alpha n + \beta n^2)\right], \\
S_4 &= \sum_{n=1}^{N} n^2 X(n)\left[A\sin(\alpha n + \beta n^2) - B\cos(\alpha n + \beta n^2)\right].
\end{aligned}$$

For $k < j$, $g_{jk}(\boldsymbol{\theta}) = g_{kj}(\boldsymbol{\theta})$. These limits given in (10) to (14) exist for fixed value of $\boldsymbol{\theta}$ because of (7) and (8). Therefore, from (6) the following theorem follows.

Theorem 2. Under the same assumptions as in Theorem 1,

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1} \xrightarrow{d} \mathcal{N}_4\left[\mathbf{0},\ \frac{1}{4}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}^0)\mathbf{G}(\boldsymbol{\theta}^0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}^0)\right]. \tag{15}$$

Remark 1. When $X(n)$'s are *i.i.d.* random variables, then the covariance matrix takes the simplified form

$$\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}^0)\mathbf{G}(\boldsymbol{\theta}^0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}^0) = \sigma^2\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}^0).$$

Remark 2. Although we could not prove it theoretically, but it is observed by extensive numerical computations that the right hand side limits of (7) and (8) for $k = 1, 2$ do not depend on α. So we assume that these quantities are independent of their second argument and we write them as

$$\delta_k(p;\beta) = \delta_k(p,\alpha,\beta), \qquad \gamma_k(p;\beta) = \gamma_k(p,\alpha,\beta).$$

Let us denote

$$c_c = \sum_{k=-\infty}^{\infty} a(k)\cos(\alpha^0 k + \beta^0 k^2), \qquad c_s = \sum_{k=-\infty}^{\infty} a(k)\sin(\alpha^0 k + \beta^0 k^2).$$

c_c and c_s are functions of α^0 and β^0, but we do not make it explicit here to keep the notation simple.

Now according to the above assumption, δ's and γ's are independent of α and based on it, we can explicitly compute the elements of $\mathbf{G}(\boldsymbol{\theta})$ matrix for a given $\boldsymbol{\theta}$. For different entries of the matrix $\mathbf{G}(\boldsymbol{\theta})$ in terms of δ's and γ's, one can see at http://www.isid.ac.in/~statmath/eprints/ (isid/ms/2005/08) or it can be available from the authors on request.

Thus, obtaining the explicit expression of different entries of the variance-covariance matrix $\frac{1}{4}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}^0)\mathbf{G}(\boldsymbol{\theta}^0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}^0)$ of $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)\mathbf{D}^{-1}$ is possible by inverting the matrix $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ at $\boldsymbol{\theta}^0$. But they are not provided here due to the complex (notational) structure of matrices $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ and $\mathbf{G}(\boldsymbol{\theta})$. If the true value of β is zero (i.e. frequency does not change over time) and if this information is used in the model, then the model (1) is nothing but the usual sinusoidal model. In that case, the asymptotic distribution can be obtained in compact form and the amplitude is asymptotically independent of the frequency. This has not been observed in case of the chirp signal model.

3. Multiple Chirp Signal

In this section, we introduce the multiple chirp signal model in stationary noise. The complex-valued single chirp model was generalized as superimposed chirp model by Saha and Kay (2002). The following model is a similar generalization of model (1). We assume that the observed data $y(n)$ have the following representation.

$$y(n) = \sum_{k=1}^{p} \left[A_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2) + B_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2)\right] + X(n); \quad n = 1, \ldots, N. \tag{16}$$

Similarly as the single chirp model, the parameters $\alpha_k^0, \beta_k^0 \in (0, \pi)$ are the frequency and frequency rate respectively. A_k^0's and B_k^0's are real-valued amplitudes. Again our aim is to estimate the parameters and study their properties. We assume that the number of components, p is known and $X(n)$'s satisfy assumption 1. Estimation of p is an important problem and will be addressed elsewhere. Now let us define, $\boldsymbol{\theta}_k = (A_k, B_k, \alpha_k, \beta_k)$ and $\boldsymbol{\nu} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_p)$ be the parameter vector. The least squares estimators of the parameters are obtained by minimizing the objective function, say $R(\boldsymbol{\nu})$ (defined similarly as $Q(\boldsymbol{\theta})$; see eq. (4), sec. 2). Let $\hat{\boldsymbol{\nu}}$ and $\boldsymbol{\nu}^0$ denote the least squares estimator and the true value of $\boldsymbol{\nu}$. The consistency of $\hat{\boldsymbol{\nu}}$ follows similarly as the consistency of $\hat{\boldsymbol{\theta}}$, considering the parameter vector as $\boldsymbol{\nu}$. We will state the asymptotic distribution of $\hat{\boldsymbol{\nu}}$ here. The proof involves routine calculations and use of the multiple Taylor series expansion and the central limit theorem for stochastic processes. For the asymptotic distribution of $\hat{\boldsymbol{\nu}}$, we introduce the following notation; $\boldsymbol{\psi}_k^N = (\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k^0)\mathbf{D}^{-1} = \Big(N^{1/2}(\hat{A}_k - A_k^0), N^{1/2}(\hat{B}_k - B_k^0), N^{3/2}(\hat{\alpha}_k - \alpha_k^0), N^{5/2}(\hat{\beta}_k - \beta_k^0)\Big)$, moreover c_c^k and c_s^k are obtained from c_c and c_s by replacing α^0 and β^0 by α_k^0 and β_k^0 respectively. Let us denote $\beta_j + \beta_k = \beta_{jk}^+$, $\beta_j - \beta_k = \beta_{jk}^-$, $d_1 = c_c^1 c_c^2 + c_s^1 c_s^2$, $d_2 = c_c^1 c_s^2 + c_s^1 c_c^2$, $d_3 = c_c^1 c_c^2 - c_s^1 c_s^2$ and $d_4 = c_c^1 c_s^2 - c_s^1 c_c^2$. Then the asymptotic distribution of $(\boldsymbol{\psi}_1^N, \ldots, \boldsymbol{\psi}_p^N)$ is as follows.

$$(\boldsymbol{\psi}_1^N, \ldots, \boldsymbol{\psi}_p^N) \xrightarrow{d} \mathcal{N}_{4p}\left(\mathbf{0}, 2\sigma^2 \Lambda^{-1}(\boldsymbol{\nu}^0)\mathbf{H}(\boldsymbol{\nu}^0)\Lambda^{-1}(\boldsymbol{\nu}^0)\right), \tag{17}$$

$$\Lambda(\boldsymbol{\nu}) = \begin{pmatrix} \mathbf{\Lambda}_{11} & \mathbf{\Lambda}_{12} & \cdots & \mathbf{\Lambda}_{1p} \\ \mathbf{\Lambda}_{21} & \mathbf{\Lambda}_{22} & \cdots & \mathbf{\Lambda}_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{\Lambda}_{p1} & \mathbf{\Lambda}_{p2} & \cdots & \mathbf{\Lambda}_{pp} \end{pmatrix}, \qquad \mathbf{H}(\boldsymbol{\nu}) = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \cdots & \mathbf{H}_{1p} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \cdots & \mathbf{H}_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{H}_{p1} & \mathbf{H}_{p2} & \cdots & \mathbf{H}_{pp} \end{pmatrix}. \tag{18}$$

The sub-matrices $\mathbf{\Lambda}_{jk}$ and $\mathbf{H}_{jk}$ are square matrices of order four and $\mathbf{\Lambda}_{jk} \equiv \mathbf{\Lambda}_{jk}(\boldsymbol{\theta}_j, \boldsymbol{\theta}_k)$, $\mathbf{H}_{jk} \equiv \mathbf{H}_{jk}(\boldsymbol{\theta}_j, \boldsymbol{\theta}_k)$. $\mathbf{\Lambda}_{jj}$ and $\mathbf{H}_{jj}$ can be obtained from $\mathbf{\Sigma}(\boldsymbol{\theta})$ and $\mathbf{G}(\boldsymbol{\theta})$ by putting $\boldsymbol{\theta} = \boldsymbol{\theta}_j$. As in case of $\mathbf{G}(\boldsymbol{\theta})$, the entries of the off-diagonal sub-matrices $\mathbf{\Lambda}_{jk} = ((\lambda_{rs}))$ and $\mathbf{H}_{jk} = ((h_{rs}))$ are available at http://www.isid.ac.in/~statmath/eprints/ (isid/ms/2005/08). The elements of the matrices $\mathbf{\Lambda}_{jk}$ and $\mathbf{H}_{jk}$ are non-zero. So the parameters corresponding to different components, $\boldsymbol{\psi}_j^N$ and $\boldsymbol{\psi}_k^N$ for $j \neq k$, are not asymptotically independent. If the frequencies do not change over time, i.e. the frequency rates β's vanish, the model (16) is equivalent to the multiple frequency model. Then the off-diagonal matrices in $\mathbf{H}$ and $\mathbf{\Lambda}$ are zero matrices and the estimators of the unknown parameters in different components are independent. This is due to the reason that $\delta_1(p, \alpha, 0) = 0 = \gamma_1(p, \alpha, 0)$ for all $p \geq 0$ and $\alpha \in (0, \pi)$.

4. Numerical Experiments

In this section, we present the results of the numerical experiments based on simulations. For this purpose, we consider a single chirp model with $A = 2.93$, $B = 1.91$, $\alpha = 2.5$ and $\beta = .10$. We use sample size $N = 50$ and $N = 100$. Though, $\alpha, \beta \in (0, \pi)$, we have considered the true value of β, much less than the initial frequency α, as β, being the frequency rate is comparatively small in general. We consider different stationary processes as the error random variables for our simulations. The errors are generated from (a) $X(t) =$

$\rho e(t+1) + e(t)$, (b) $X(t) = \rho_1 e(t-1) + \rho_2 e(t-2) + e(t)$ and (c) $X(t) = \rho X(t-1) + e(t)$. The random variables $\{e(t)\}$ are distributed as $\mathcal{N}(0, \sigma^2)$. The processes (a), (b) and (c) are stationary $MA(1)$, $MA(2)$ and $AR(1)$ processes. Here $MA(q)$ and $AR(p)$ are usual notation for the moving average process of order q and the autoregressive process of order p respectively. For simulations, $\rho = .5$, $\rho_1 = .5$ and $\rho_2 = -.4$ have been used. We consider different values of σ^2 and accordingly the variances of $X(t)$ are different depending on the model of the error process and their true parameter values. We generate the data using (1) and the parameters as mentioned above. The LSEs of the parameters are obtained by minimizing the residual sum of squares. The starting estimates of the frequency and the frequency rate are obtained by maximizing the following periodogram like function;

$$I(\omega_1, \omega_2) = \frac{1}{N} \left| \sum_{t=1}^{N} y(t) e^{-i(\omega_1 t + \omega_2 t^2)} \right|^2$$

over a fine two-dimensional (2-d) grid of $(0,\pi) \times (0,\pi)$. The linear parameters, A and B are expressible in terms of α and β. So the minimization of $Q(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ involves a 2-d search. Once the non-linear parameters, α and β are estimated, A and B are estimated using the linear regression technique. We replicate this procedure of data generation and the estimation for parameters 1000 times resulting 1000 estimated values of each parameter. Then we calculate the average estimate (AVEEST), the bias (BIAS) and the mean squared error (MSE) of each parameter. We summarize results in tables 1-2 when the errors are of type (c). In table 1, results with $N = 50$ are reported and results with $N = 100$ are in table 2. We did not report the results with errors of types (a) and (b) for limitation of space. These results are available at http://www.isid.ac.in/~statmath/eprints/ (isid/ms/2005/08) or it can be obtained from the authors on request. In section 2, we have obtained the asymptotic distributions of the LSEs of the unknown parameters of a single chirp signal model under quite general assumptions. So, it is possible to obtain the confidence intervals of the unknown parameters for fixed finite length data using Theorem 2. But due to the complexity involved in the distribution, it is extremely complicated to implement it in practice. Also in numerical experiments, it has been observed that the convergence of sequences, δ's as well γ's highly depends on the parameters and in many cases we need a very large value to stabilize the convergence. For this reason, we have used the percentile bootstrap method for interval estimation of the different parameters as a simple alternative method as suggested by Nandi, Iyer and Kundu (2002). In each replication of our experiment, we generate 1000 bootstrap resamples using the estimated parameters and then the bootstrap confidence intervals using the bootstrap quantiles at 95% nominal level. So we have 1000 intervals for each parameter from the replicated experiment. Then we estimate the 95% bootstrap coverage probability

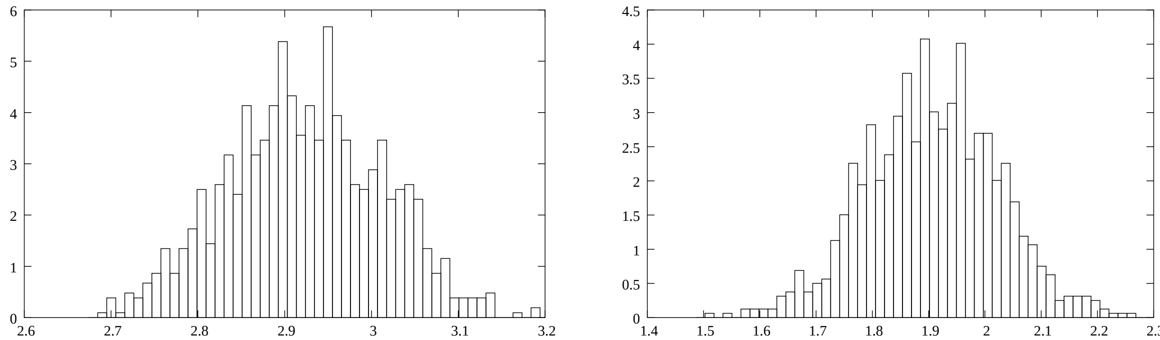

Figure 1. Plot of the histograms of LSEs of A (left plot) and B (right plot).

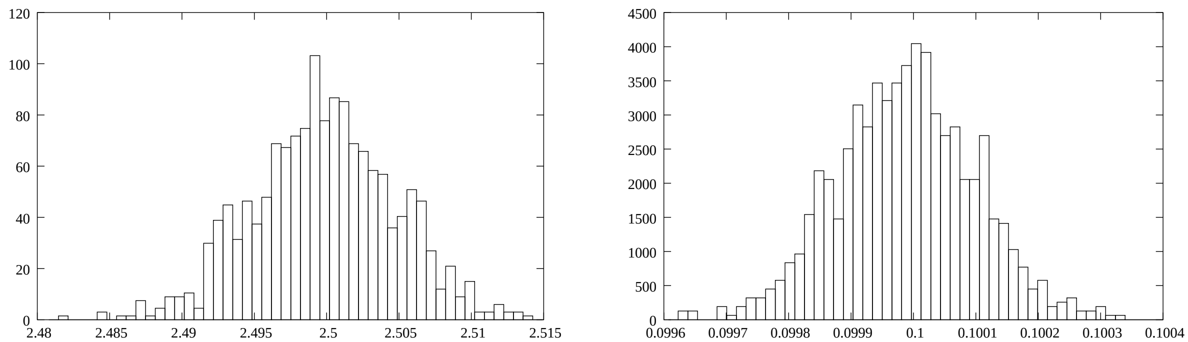

Figure 2. Plot of the histograms of LSEs of α (left plot) and β (right plot).

by calculating the proportion of covering the true parameter value by the interval in each replication. We report them as B-COVP in Tables 1-2. We also report the average length of the bootstrap confidence interval as B-AVEL. So in each table, we report the average estimate, its bias and mean squared error and the 95% bootstrap coverage probability and the average length. We have seen in simulations, that the maximizer of the periodogram like function defined above over a fine grid provides reasonably good initial estimates of the non-linear parameters, α and β in most of the cases.

In the above discussed experiments, we have collected the LSEs of all the parameters estimated in all replications in case of $N = 50$ with type (c) error and $\sigma^2 = .1$. The type (c) error, being an AR(1) process, has the variance $\frac{\sigma^2}{1-\rho^2} = .13$, of its stationary distribution. To understand their sample distributions, we plot the histograms of the LSEs of A and B in Fig. 1 and histograms of the LSEs of α and β in Fig. 2. We wanted to see how the fitted signal looks like, so we have generated a realization using the type of error and σ^2, same as above. The fitted signal is plotted in Fig. 3 along with the original one.

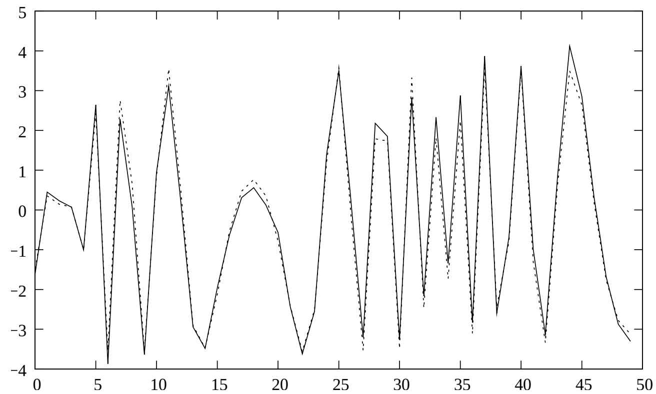

FIGURE 3. Plot of original signal (solid line) and estimated signals (dotted line).

TABLE 1. Average estimates, biases, MSEs, coverage probabilities and average lengths using bootstrap technique when errors are of type (c) and sample size $N = 50$

		Parameters			
σ^2		A	B	α	β
0.1	AVEEST	2.92884326	1.91418457	2.50015926	.0999963731
	BIAS	-1.15680695e-3	4.18460369e-3	1.59263611e-4	-3.62843275e-6
	MSE	8.06691125e-3	1.39705129e-2	2.45984756e-5	1.25075923e-8
	B-COVP	.976	.973	.897	.848
	B-AVEL	.435647398	.585830092	1.85300075e-2	3.61990707e-4
0.5	AVEEST	2.92720795	1.91918254	2.50034285	.0999920592
	BIAS	-2.79211998e-3	9.18257236e-3	3.42845917e-4	-7.94231892e-6
	MSE	4.05863188e-2	6.96346015e-2	1.23846767e-4	6.29799075e-8
	B-COVP	.977	.971	.895	.849
	B-AVEL	.965029001	1.29620469	4.16616127e-2	8.12350831e-4
1.0	AVEEST	2.92532682	1.92366946	2.50049949	0.0999882892
	BIAS	-4.67324257e-3	1.36694908e-2	4.99486923e-4	-1.17123127e-5
	MSE	8.10585618e-2	.138886198	2.47357559e-4	1.25821842e-7
	B-COVP	.977	.972	.894	.852
	B-AVEL	1.36005151	1.81864798	5.9380278e-2	1.15502137e-3

Now we summarize the findings of the experiments discussed above. We observe that the average estimates are quite good which is reflected in the fact that the biases are quite small in absolute value. The MSEs are reasonably small and we observe that they are in decreasing

Table 2. Average estimates, biases, MSEs, coverage probabilities and average lengths using bootstrap technique when errors are of type (c) and sample size $N = 100$

		Parameters			
σ^2		A	B	α	β
0.1	AVEEST	2.92976546	1.91153288	2.50007677	0.0999990776
	BIAS	-2.34603882e-4	1.53291225e-3	7.67707825e-5	-9.23871994e-7
	MSE	5.03146602e-3	1.00822281e-2	2.46111904e-6	2.25312574e-10
	B-COVP	.946	.955	.962	.960
	B-AVEL	.309637368	.458883166	7.0124059e-3	6.7486304e-5
0.5	AVEEST	2.92800331	1.91039991	2.50014496	0.0999983251
	BIAS	-1.9967556e-3	3.99947166e-4	1.44958496e-4	-1.67638063e-6
	MSE	2.51741707e-2	5.0748501e-2	1.23751706e-5	1.13128373e-9
	B-COVP	.946	.957	.962	.959
	B-AVEL	.686493933	1.01766443	1.56924874e-2	1.51613291e-4
1.0	AVEEST	2.92434359	1.90953135	2.50021315	0.0999974459
	BIAS	-5.65648079e-3	-4.68611717e-4	2.1314621e-4	-2.55554914e-6
	MSE	5.05070463e-2	.101672225	2.4809211e-5	2.26738139e-9
	B-COVP	.943	.957	.965	.961
	B-AVEL	.967028975	1.42990315	2.22291183e-2	2.15437249e-4

order of the linear parameters, α and β. The similar findings are also seen in case of the average bootstrap confidence lengths i.e. the average lengths decrease with $(A, B), \alpha, \beta$. The asymptotic distribution (Theorem 2) also suggests accordingly as the rates of convergence are $N^{-1/2}$, $N^{-3/2}$, $N^{-5/2}$ respectively. This has been reflected in the bootstrap intervals to some extent. Also considering all the cases reported here, we can say that the order of the MSEs, approximately match with the order given in the asymptotic distribution of the LSEs and that we expect in case of finite samples of moderate size. For each type of error, the average lengths of intervals, biases and MSEs increase as the error variance increases for all the parameters. Also as we increase the sample size, these values decrease and that has to be according to the Theorem 1 which says that LSEs are strongly consistent. By comparing the different types of errors and σ^2, we see that with $N = 50$, the coverage probabilities do not attain the nominal level mainly for the frequency rate β except type (b) error. However, in case of $N = 100$, the bootstrap coverage probabilities are quite close to the nominal level. In some cases, mainly for the linear parameters, the bootstrap method overestimate the coverage probabilities. We understand that using the given sample size in calculating the limiting quantities, δ's and γ's, may cause the overestimation.

We have plotted the histograms of the LSEs in Figs 1 and 2. It is clear from the plots that the LSEs are distributed symmetrically around the true value for all parameters. Though, we have reported the MSEs, the histograms gives a very good idea of the variability of the estimates. In Fig. 3, the fitted signal have been plotted with the observed one for a particular case. We see that the fitted one match reasonably well with the observed one. So, from this discussion, we see that the performance of the LSEs and the bootstrap method in obtaining the confidence intervals are quite good and can be used in practice.

5. Conclusions

In this paper, we study the problem of estimation of the parameters of the real single chirp signal model as well as the multiple chirp signal model in stationary noise. It is a generalization of the multiple frequency model similar to the way the complex-valued chirp signal is a generalization of the exponential model. We propose the least squares estimators to estimate the unknown parameters and study their asymptotic properties. As the joint asymptotic distribution of the LSEs of the unknown parameters is quite complicated for practical implementation purposes, we have used a parametric bootstrap method for interval estimation. We observe that the results are quite satisfactory and can be used in practice. In simulations study, the initial estimates of the frequency and the frequency rate are obtained by maximizing a periodogram like function. It will be interesting to explore the properties of the estimators obtained by maximizing the periodogram like function defined in section 4. Also generalization of some of the existing iterative and non-iterative methods for the frequency model to the chirp signal model is another problem which needs to be addressed as well as the estimation of the number of chirp components for the multiple chirp model.

Appendix

In this appendix, we first state Lemmas 1 and 2 and then state and prove the lemmas A-1 to A-6. Then these lemmas are used to prove lemma 2.

Lemma 1: Let us denote

$$S_{C,M} = \left\{\boldsymbol{\theta}; \boldsymbol{\theta} = (A_R, A_I, \alpha, \beta), |\boldsymbol{\theta} - \boldsymbol{\theta}^0| \geq 4C, |A_R| \leq M, |A_I| \leq M\right\}.$$

If for any $C > 0$ and for some $M < \infty$,

$$\liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in S_{C,M}} \frac{1}{N}\left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right] > 0 \quad a.s.$$

then $\hat{\boldsymbol{\theta}}$ is a strongly consistent estimator of $\boldsymbol{\theta}^0$.

Proof of Lemma 1: The proof can be obtained by contradiction along the same line as the lemma 1 of Wu (1981).

Lemma 2: As $N \to \infty$,

$$\sup_{\alpha,\beta}\left|\frac{1}{N}\sum_{n=1}^{N} X(n)e^{i(\alpha n+\beta n^2)}\right| \to 0 \quad a.s.$$

In the following, we first state and prove the lemmas A-1 to A-6 and then these lemmas are used to prove lemma 2.

Lemma A-1: Let $\{e(n)\}$ be a sequence of i.i.d. random variables with mean zero and finite fourth moment, then

$$E\left|\sum_{n=1}^{N-2} e(n)e(n+1)^2 e(n+2)\right| = O(N^{\frac{1}{2}}), \tag{19}$$

$$E\left|\sum_{n=1}^{N-k-1} e(n)e(n+1)e(n+k)e(n+k+1)\right| = O(N^{\frac{1}{2}}), \tag{20}$$

for $k = 2, 3, \ldots, N-2$.

Proof of Lemma A-1: We prove (19) and then (20) follows similarly. Note that

$$E\left|\sum_{n=1}^{N-2} e(n)e(n+1)^2 e(n+2)\right| \le \left[E\left(\sum_{n=1}^{N-2} e(n)e(n+1)^2 e(n+2)\right)^2\right]^{\frac{1}{2}} = O(N^{\frac{1}{2}}).$$

Lemma A-2: For an arbitrary integer m,

$$E\sup_{\theta}\left|\sum_{n=1}^{N} e(n)e(n+k)e^{im\theta n}\right| = O(N^{\frac{3}{4}}).$$

PROOF OF LEMMA A-2:

$$E\sup_{\theta}\left|\sum_{n=1}^{N}e(n)e(n+k)e^{im\theta n}\right| \le \left[E\sup_{\theta}\left|\sum_{n=1}^{N}e(n)e(n+k)e^{im\theta n}\right|^2\right]^{\frac{1}{2}}$$
$$=\left[E\sup_{\theta}\left(\sum_{n=1}^{N}e(n)e(n+k)e^{im\theta n}\right)\left(\sum_{n=1}^{N}e(n)e(n+k)e^{-im\theta n}\right)\right]^{\frac{1}{2}}$$
$$\le\left[E\sum_{n=1}^{N}e(n)^2e(n+k)^2+2E\left|\sum_{n=1}^{N-1}e(n)e(n+1)e(n+k)e(n+k+1)\right|+\dots\right.$$
$$\left.+2E\left|e(1)e(1+k)e(N)e(N+k)\right|\right]^{\frac{1}{2}}=O(N+N.N^{\frac{1}{2}})^{\frac{1}{2}}\text{ (using Lemma A-1) }=O(N^{\frac{3}{4}}).$$

LEMMA A-3:

$$E\sup_{\alpha,\beta}\left|\sum_{n=1}^{N}e(n)e^{i(\alpha n+\beta n^2)}\right|^2=O(N^{\frac{7}{4}}).$$

PROOF OF LEMMA A-3:

$$E\sup_{\alpha,\beta}\left|\sum_{n=1}^{N}e(n)e^{i(\alpha n+\beta n^2)}\right|^2=E\sup_{\alpha,\beta}\left[\sum_{n=1}^{N}e(n)e^{i(\alpha n+\beta n^2)}\right]\left[\sum_{n=1}^{N}e(n)e^{-i(\alpha n+\beta n^2)}\right]$$
$$\le O(N+NN^{\frac{3}{4}})\text{ (using Lemma A-2) }=O(N^{\frac{7}{4}}).$$

LEMMA A-4:

$$E\sup_{\alpha,\beta}\left|\frac{1}{N}\sum_{n=1}^{N}e(n)e^{i(\alpha n+\beta n^2)}\right|\le O(N^{-\frac{1}{8}}).$$

PROOF OF LEMMA A-4:

$$E\sup_{\alpha\beta}\left|\frac{1}{N}\sum_{n=1}^{N}e(n)e^{i(\alpha n+\beta n^2)}\right|\le\left[E\sup_{\alpha,\beta}\left|\frac{1}{N}\sum_{n=1}^{N}e(n)e^{i(\alpha n+\beta n^2)}\right|^2\right]^{\frac{1}{2}}=O(N^{-\frac{1}{8}})\text{ (using Lemma A-3).}$$

LEMMA A-5:

$$E\sup_{\alpha\beta}\left|\frac{1}{N}\sum_{n=1}^{N}X(n)e^{i(\alpha n+\beta n^2)}\right|\le O(N^{-\frac{1}{8}}).$$

Proof of Lemma A-5:

$$E \sup_{\alpha,\beta} \left| \frac{1}{N} \sum_{n=1}^{N} X(n) e^{i(\alpha n + \beta n^2)} \right| = E \sup_{\alpha,\beta} \left| \frac{1}{N} \sum_{n=1}^{N} \sum_{k=-\infty}^{\infty} a(k) e(n-k) e^{i(\alpha n + \beta n^2)} \right|$$
$$\leq \sum_{k=-\infty}^{\infty} |a(k)| \left[E \sup_{\alpha,\beta} \frac{1}{N} \left| \sum_{n=1}^{N} e(n-k) e^{i(\alpha n + \beta n^2)} \right| \right].$$

Note that $E \sup_{\alpha,\beta} \frac{1}{N} \left| \sum_{n=1}^{N} e(n-k) e^{i(\alpha n + \beta n^2)} \right|$ is independent of k and therefore the result follows using Lemma A-4.

Lemma A-6:

$$\sup_{\alpha,\beta} \left| \frac{1}{N} \sum_{n=1}^{N} X(n) e^{i(\alpha n + \beta n^2)} \right| \longrightarrow 0, \qquad a.s.$$

Proof of Lemma A-6:

Consider the sequence N^9, then using Lemma A-5 we obtain

$$E \sup_{\alpha,\beta} \frac{1}{N^9} \left| \sum_{n=1}^{N^9} X(n) e^{i(\alpha n + \beta n^2)} \right| \leq O(N^{-\frac{9}{8}}).$$

Therefore, using Borel Cantelli lemma it follows that

$$\sup_{\alpha,\beta} \frac{1}{N^9} \left| \sum_{n=1}^{N^9} X(n) e^{i(\alpha n + \beta n^2)} \right| \longrightarrow 0, \qquad a.s.$$

Now consider for J, such that $N^9 < J \leq (N+1)^9$, then

$$\sup_{\alpha,\beta} \sup_{N^9 < J \leq (N+1)^9} \left| \frac{1}{N^9} \sum_{n=1}^{N^9} X(n) e^{i(\alpha n + \beta n^2)} - \frac{1}{J} \sum_{n=1}^{J} X(n) e^{i(\alpha n + \beta n^2)} \right|$$
$$= \sup_{\alpha,\beta} \sup_{N^9 < J \leq (N+1)^9} \left| \frac{1}{N^9} \sum_{n=1}^{N^9} X(n) e^{i(\alpha n + \beta n^2)} - \frac{1}{N^9} \sum_{n=1}^{J} X(n) e^{i(\alpha n + \beta n^2)} + \right.$$
$$\left. \frac{1}{N^9} \sum_{n=1}^{J} X(n) e^{i(\alpha n + \beta n^2)} - \frac{1}{J} \sum_{n=1}^{J} X(n) e^{i(\alpha n + \beta n^2)} \right|$$
$$\leq \frac{1}{N^9} \sum_{n=N^9+1}^{(N+1)^9} |X(n)| + \sum_{n=1}^{(N+1)^9} |X(n)| \left(\frac{1}{N^9} - \frac{1}{(N+1)^9} \right).$$

Note that the mean squared error of the first term is of the order $O\left(\frac{1}{N^{18}} \times ((N+1)^9 - N^9)^2\right) = O(N^{-2})$. Similarly, the mean squared error of the second term is of the order

$O\left(N^{18} \times \left(\frac{(N+1)^9 - N^9}{N^{18}}\right)^2\right) = O(N^{-2})$. Therefore, both terms converge to zero almost surely and that proves the lemma.

Proof of Theorem 1: In this proof, we denote $\hat{\boldsymbol{\theta}}$ by $\hat{\boldsymbol{\theta}}_N = (\hat{A}_N, \hat{B}_N, \hat{\alpha}_N, \hat{\beta}_N)$ to emphasize that $\hat{\boldsymbol{\theta}}$ depends on the sample size. If $\hat{\boldsymbol{\theta}}_N$ is not consistent for $\boldsymbol{\theta}^0$, then there exits a subsequence $\{N_k\}$ of $\{N\}$ such that $\hat{\boldsymbol{\theta}}_{N_k}$ does not converge to $\boldsymbol{\theta}^0$. Then either:

Case I: $|\hat{A}_{N_k}| + |\hat{B}_{N_k}|$ is not bounded. So $|\hat{A}_{N_k}| + |\hat{B}_{N_k}| \to \infty$ *i.e.* at least one of the $|\hat{A}_{N_k}|$ or $|\hat{B}_{N_k}|$ tends to ∞. This implies $\frac{1}{N_k} Q(\hat{\boldsymbol{\theta}}_{N_k}) \to \infty$. Since

$$\lim \frac{1}{N_k} Q(\boldsymbol{\theta}^0) < \infty,$$

hence,

$$\frac{1}{N_k}\left[Q(\hat{\boldsymbol{\theta}}_{N_k}) - Q(\boldsymbol{\theta}^0)\right] \to \infty.$$

But as $\hat{\boldsymbol{\theta}}_{N_k}$ is the LSE of $\boldsymbol{\theta}^0$, therefore,

$$Q(\hat{\boldsymbol{\theta}}_{N_k}) - Q(\boldsymbol{\theta}^0) < 0,$$

which leads to a contradiction. So $\hat{\boldsymbol{\theta}}_N$ is a strongly consistent estimator of $\boldsymbol{\theta}^0$.

Case II: $|\hat{A}_{N_k}| + |\hat{B}_{N_k}|$ is bounded, that means there exists a set $S_{C,M}$ (as defined in Lemma 1) such that $\hat{\boldsymbol{\theta}}_{N_k} \in S_{C,M}$, for some $C > 0$ and for an $0 < M < \infty$. Now let us write

$$\frac{1}{N}\left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right] = f_1(\boldsymbol{\theta}) + f_2(\boldsymbol{\theta}),$$

where

$$\begin{aligned} f_1(\boldsymbol{\theta}) &= \frac{1}{N}\sum_{n=1}^{N}\left[A^0\cos(\alpha^0 n + \beta^0 n^2) - A\cos(\alpha n + \beta n^2)\right. \\ &\qquad \left. + B^0 \sin(\alpha^0 n + \beta^0 n^2) - B\sin(\alpha n + \beta n^2)\right]^2, \end{aligned}$$

$$\begin{aligned} f_2(\boldsymbol{\theta}) &= \frac{2}{N}\sum_{n=1}^{N} X(n)\left[A^0\cos(\alpha^0 n + \beta^0 n^2) - A\cos(\alpha n + \beta n^2)\right. \\ &\qquad \left. + B^0 \sin(\alpha^0 n + \beta^0 n^2) - B\sin(\alpha n + \beta n^2)\right]. \end{aligned}$$

Using lemma 2, it follows that

$$\lim_{N\to\infty} \sup_{\boldsymbol{\theta}\in S_{C,M}} f_2(\boldsymbol{\theta}) = 0 \quad a.s. \tag{21}$$

Now consider the following sets;

$$\begin{aligned}
S_{C,M,1} &= \left\{\boldsymbol{\theta} : \boldsymbol{\theta} = (A,B,\alpha,\beta), |A - A^0| \geq C, |A| \leq M, |B| \leq M\right\}, \\
S_{C,M,2} &= \left\{\boldsymbol{\theta} : \boldsymbol{\theta} = (A,B,\alpha,\beta), |B - B^0| \geq C, |A| \leq M, |B| \leq M\right\}, \\
S_{C,M,3} &= \left\{\boldsymbol{\theta} : \boldsymbol{\theta} = (A,B,\alpha,\beta), |\alpha - \alpha^0| \geq C, |A| \leq M, |B| \leq M\right\}, \\
S_{C,M,4} &= \left\{\boldsymbol{\theta} : \boldsymbol{\theta} = (A,B,\alpha,\beta), |\beta - \beta^0| \geq C, |A| \leq M, |B| \leq M\right\}.
\end{aligned}$$

Note that

$$S_{C,M} \subset S_{C,M,1} \cup S_{C,M,2} \cup S_{C,M,3} \cup S_{C,M,4} = S \quad \text{(say)}.$$

Therefore,

$$\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M}} \frac{1}{N}\left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right] \geq \underline{\lim} \inf_{\boldsymbol{\theta} \in S} \frac{1}{N}\left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right]. \tag{22}$$

First we show that

$$\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M,j}} \frac{1}{N}\left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right] > 0 \qquad a.s., \tag{23}$$

for $j = 1, \ldots, 4$ and then because of (22), it implies

$$\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M}} \frac{1}{N}\left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right] > 0 \qquad a.s.$$

Therefore, due to lemma 1, theorem 1 is proved, provided we can show (23). First consider $j = 1$ to prove (23). Using (21), it follows that

$$\begin{aligned}
&\underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M,1}} \frac{1}{N}\left[Q(\boldsymbol{\theta}) - Q(\boldsymbol{\theta}^0)\right] = \underline{\lim} \inf_{\boldsymbol{\theta} \in S_{C,M,1}} f_1(\boldsymbol{\theta}) \\
&= \underline{\lim} \inf_{|A-A^0| \geq C} \frac{1}{N} \sum_{n=1}^{N} \Big[A^0 \cos(\alpha^0 n + \beta^0 n^2) - A\cos(\alpha n + \beta n^2) + \\
&\qquad\qquad B^0 \sin(\alpha^0 n + \beta^0 n^2) - B \sin(\alpha n + \beta n^2)\Big]^2 \\
&= \lim_{N\to\infty} \inf_{|A-A^0| \geq C} \frac{1}{N} \sum_{n=1}^{N} \cos^2(\alpha^0 n + \beta^0 n^2)(A - A^0)^2 \\
&\geq C^2 \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \cos^2(\alpha^0 n + \beta^0 n^2) > 0.
\end{aligned}$$

For other j also, it can be shown along the same line and that proves theorem 1.

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