Parameter Uncertainty in Portfolio Selection: Shrinking the Inverse Covariance Matrix

Apostolos Kourtis[*], George Dotsis[†] and Raphael N. Markellos[‡]

First version: February, 2009
This version: September, 2011

Abstract

The estimation of the inverse covariance matrix plays a crucial role in optimal portfolio choice. We propose a new estimation framework that focuses on enhancing portfolio performance. The framework applies the statistical methodology of shrinkage directly to the inverse covariance matrix using two non-parametric methods. The first minimises the out-of-sample portfolio variance while the second aims to increase out-of-sample risk-adjusted returns. We apply the resulting estimators to compute the minimum variance portfolio weights and obtain a set of new portfolio strategies. These strategies have an intuitive form which allows us to extend our framework to account for short-sale constraints, high transaction costs and singular covariance matrices. A comparative empirical analysis against several strategies from the literature shows that the new strategies generally offer higher risk-adjusted returns and lower levels of risk.

Keywords: Portfolio Optimization, Inverse Covariance Matrix, Estimation Risk, Shrinkage

JEL Classification: C13, C51, C61, G11

[*]Norwich Business School, University of East Anglia, Norwich, Norfolk, NR4 7TJ, UK, a.kourtis@uea.ac.uk

[†]Essex Business School, University of Essex, Colchester, CO4 3SQ, UK, gdotsis@essex.ac.uk

[‡]Norwich Business School, University of East Anglia, Norwich, Norfolk, NR4 7TJ, UK, r.markellos@uea.ac.uk. Tel: +44(0) 1603 59 7395, Fax:+44(0) 1603 59 3343. We are grateful to Michael Wolf and participants of the 28th European Meeting of Statisticians and of the 4th International Conference on Computational and Financial Econometrics for useful comments and suggestions. All remaining errors are our own.

1 Introduction

Under the classical mean-variance framework of Markowitz (1952), optimal portfolio weights are a function of two parameters: the vector of the expected returns on the risky assets and the inverse covariance matrix of these returns. The true values of these parameters are unknown in practice and investors traditionally estimate them using historical data. Several studies over the years have shown that estimation errors lead to poor out-of-sample portfolio performance (see, for example, Jobson and Korkie 1980; Michaud 1989; Best and Grauer 1991). Unsurprisingly, a vast literature has grown around improving portfolio performance using various statistical techniques for the reduction of estimation errors (Brandt 2009 provides a comprehensive review). Despite all the research efforts, the problem of parameter uncertainty remains largely unresolved. Characteristically, DeMiguel et al. (2009a) show that the simple heuristic $1/N$ portfolio outperforms the mean-variance portfolio and most of its extensions. This finding has spawned a new wave of research that seeks to develop portfolio strategies superior to $1/N$ and to reaffirm the practical value of portfolio theory (e.g., DeMiguel et al. 2009b; Tu and Zhou 2011).

The present work deals with parameter uncertainty by developing a new methodology for the estimation of the inverse covariance matrix that aims to improve out-of-sample portfolio performance. We start by studying how the estimation errors in the sample inverse affect out-of-sample portfolio returns. In accordance with recent studies (Jagannathan and Ma 2003; Ledoit and Wolf 2003; DeMiguel et al. 2009b, among others), we constrain our interest to the global minimum variance (GMV) portfolio.[1] On the basis of analytical expressions, we show that in the presence of many assets and/or a small sample, estimation errors in the inverse covariance matrix can substantially increase portfolio risk. Moreover, even though the GMV portfolio generally results in higher risk-adjusted returns than the sample mean-variance portfolio, there is a considerable difference of the risk-adjusted returns between the GMV portfolio and the parameter certainty mean-variance portfolio.

Motivated by our analysis, we apply a popular statistical tool known as "shrinkage" (James and Stein 1961) to the Maximum Likelihood (ML) estimator of the inverse covariance matrix in order to reduce portfolio risk and increase risk-adjusted returns. To the best of our knowledge, this study is the first to directly improve on the sample inverse within the context of portfolio choice. So far, the practice in the literature is to derive an estimator of the covariance matrix and then invert it to compute the portfolio weights. This practice is represented by three popular approaches. The first uses a matrix with more structure as an estimator of the covariance matrix (for instance a matrix implied by a 1-factor or constant correlations model) in order to reduce the number of free parameters (Frost and Savarino 1986; Chan et al. 1999). The second can be interpreted as an extension to the first. It proposes the estimation of the covariance matrix through shrinking the ML covariance matrix to one of the structured matrices proposed by the first approach (Ledoit

[1] This portfolio is appealing since its weights are merely determined by the inverse covariance matrix and not the means. Given that the estimation errors in the means are typically large (e.g., Chopra and Ziemba 1993), the GMV portfolio tends to yield higher out-of-sample performance than the traditional mean-variance portfolio.

and Wolf 2003; 2004a; 2004b). The third uses data of higher frequency, such as daily instead of monthly, to estimate the covariance matrix (e.g., Jagannathan and Ma 2003). All these methods provide superior estimators to the ML inverse under various statistical criteria. However, due to the complexity of matrix inversion, it is difficult to optimise such estimators under investor-oriented objectives, such as portfolio variance or risk-adjusted returns. Our framework resolves these issues, since it produces simple and intuitive expressions of the portfolio weights. In this manner, it enables the derivation of estimators of the inverse covariance matrix that are designed to improve out-of-sample portfolio performance rather than to optimise statistical criteria.

Our framework generates estimators that are a linear combination of the ML estimator of the inverse and a "target" matrix, i.e., a matrix that is less sensitive to estimation errors. In this paper, we use three particular targets: the identity, the inverse covariance matrix generated by the 1-factor model of Sharpe (1963) and a weighted sum of these two matrices. To better interpret our approach, we show that, for each one of these targets, the resulting portfolio is a linear combination of the GMV portfolio and a "target" portfolio. Specifically, the application of the identity as a target matrix gives a target portfolio that coincides with the equal-weighted $1/N$ portfolio investigated by DeMiguel et al. (2009a) while the use of the 1-factor inverse results in a target portfolio that is studied by Chan et al. (1999).[2] Both of these portfolios are known to perform well in the presence of parameter uncertainty, since they have little or no sensitivity to estimation errors. As a result, they can also be efficient in eliminating a part of the estimation risk of the GMV portfolio.[3]

The above interpretation of our framework enables the construction of estimators of the inverse covariance matrix that are optimal under portfolio performance criteria. We specifically propose two non-parametric methods for selecting the linear combination coefficients. The first applies a cross-validation methodology to obtain the combination which minimises out-of-sample portfolio variance. The second exploits the serial dependence of portfolio returns (Campbell et al. 1997) in order to increase risk-adjusted returns. The application of these methods leads to a set of new portfolio strategies. In all cases, the portfolio weights have a simple analytical form that allows us to accommodate for short-sale constraints, high transaction costs and singular covariance matrices.

We evaluate empirically the out-of-sample performance of the new portfolio strategies against that of several benchmark methods from the literature. For this purpose, we employ six commonly used datasets of monthly returns and four performance measures (variance, Sharpe ratio, certainty-equivalent return (CER) and turnover). We can draw four major conclusions from the results. First, the proposed estimators of the inverse covariance matrix offer substantial gains in portfolio performance. In particular, the new portfolio strategies greatly outperform the GMV portfolio constructed using the traditional estimator of the inverse with regards to both risk and risk-adjusted returns. Second, the new strategies outperform the $1/N$ portfolio across most datasets in terms of

[2] Accordingly, for the third target matrix under consideration, the target portfolio is a weighted sum of the target portfolios from the first two cases.

[3] Portfolio strategies that combine two or more funds have received significant attention in recent studies. For instance, Kan and Zhou (2007) propose a linear combination of the mean-variance portfolio, the GMV portfolio and the riskless asset. Tu and Zhou (2011) further augment this combination with the $1/N$ portfolio. For other examples of such strategies, see Jorion (1986) and Garlappi et al. (2007).

variance, Sharpe ratio and CER. This extends the recent contributions of DeMiguel et al. (2009b) and Tu and Zhou (2011) and affirms the practical usefulness of portfolio theory. Third, under most scenarios, the strategies we obtain from the use of both target matrices under consideration outperform the rest of the strategies developed as well as the benchmark strategies considered. This finding indicates that using more than one targets helps to further reduce the effects of estimation errors. Finally, we find that the portfolio strategies that exploit autocorrelation in portfolio returns are associated with a high turnover. Nevertheless, this can be significantly reduced by imposing a short-sale constraint. The corresponding constrained portfolios also lead to higher Sharpe ratios and CERs than the $1/N$ rule and offer a considerably lower turnover than their unconstrained analogues.

The remainder of the paper is organised as follows. Section 2 outlines the mean-variance framework. It also discusses the effects of parameter uncertainty on the performance of the minimum variance portfolio on the basis of analytical expressions. Section 3 develops the framework for the estimation of the inverse covariance matrix and applies it to derive a set of new portfolio strategies. The out-of-sample performance of these strategies is assessed in Section 4. Section 5 concludes this paper whereas the Appendix contains the relevant mathematical derivations.

2 Parameter Uncertainty in Optimal Portfolio Choice

2.1 The mean-variance framework

We study the portfolio choice problem of a mean-variance investor in a market of N risky and a risk-free asset. To fix notation, let r_t and rf_t denote the vector of monthly returns on the risky and the riskless asset at time t, respectively. Then, the excess returns on the risky assets over the risk-free rate are $R_t = r_t - rf_t 1_N$, where 1_N is an N-dimensional vector of ones. Following a common approach in the literature, we assume for now that R_t's are independently and identically ($i.i.d.$) normally distributed with mean μ and covariance matrix Σ.[4] The weights $w \in \mathbb{R}^N$, that the investor should assign to the risky assets, maximise the following quadratic expected utility function:[5]

$$U(w) = \mu_p - \frac{\gamma}{2}\sigma_p^2, \tag{1}$$

where $\mu_p = w'\mu$ and $\sigma_p^2 = w'\Sigma w$ are, respectively, the mean and the variance of the excess return on the portfolio over the risk-free rate and γ is the coefficient of the relative risk aversion of the investor. The maximisation of (1) leads to the optimal mean-variance portfolio:

$$w^{\mathrm{mv}} = \frac{1}{\gamma}\Sigma^{-1}\mu. \tag{2}$$

[4] Tu and Zhou (2004) explore the importance of this assumption and conclude that ignoring tails makes negligible difference in terms of portfolio out-of-sample performance. Nevertheless, this assumption is not required for the derivation of our estimators as we show in the next section.

[5] The weight given to the riskless asset is then $1 - w'1_N$.

The corresponding expected utility is then

$$U(w^{\mathrm{mv}}) = \frac{\mu'\Sigma^{-1}\mu}{2\gamma} = \frac{\mathrm{SR}^2_{\mathrm{mv}}}{2\gamma}, \tag{3}$$

where $SR^2_{\mathrm{mv}} = \mu'\Sigma^{-1}\mu$ is the squared Sharpe ratio of the optimal portfolio w^{mv}.

The computation of the portfolio weights w^{mv} is not feasible in practice since the values of both μ and Σ are unknown. As a result, the investor needs to estimate these parameters. Generally, a sample of T historical excess returns on the risky assets, i.e., a set of observations $J_T = \{R_1, ..., R_T\}$, is available for estimation. The traditional practice employs the Maximum Likelihood (ML) estimators:

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} R_t \tag{4}$$

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}(R_t - \hat{\mu})(R_t - \hat{\mu})'. \tag{5}$$

and the respective sample mean-variance weights are given by:

$$\hat{w}^{\mathrm{mv}} = \frac{1}{\gamma}\hat{\Sigma}^{-1}\hat{\mu} \tag{6}$$

under the assumption that $T > N$ in order to ensure the non-singularity of the sample covariance matrix. It is well-known that the above simple "plug-in" method introduces a considerable amount of estimation risk in the portfolio choice process and produces severely suboptimal portfolios (for instance, see Michaud 1989; Best and Grauer 1991). This motivates the adoption of alternative portfolio strategies such as the popular global minimum variance portfolio which is the subject of the next subsection.

2.2 The global minimum variance portfolio

The global minimum variance (GMV) portfolio is the minimum risk portfolio with weights that sum to unity. These weights are commonly estimated by:[6]

$$\hat{w}^{\min} := w^{\min}\left(\hat{\Sigma}^{-1}\right) = \frac{\hat{\Sigma}^{-1}1_N}{1_N'\hat{\Sigma}^{-1}1_N}. \tag{7}$$

Clearly, the application of the GMV portfolio does not require the estimation of the "problematic" means. Therefore it is less sensitive to estimation risk that the sample mean-variance portfolio $\hat{w}^{\mathrm{mv}}$. In fact, Jagannathan and Ma (2003) show empirically that when the estimation errors in the means are large, nothing is lost in terms of out-of-sample performance by using the GMV portfolio instead of the mean-variance portfolio. This finding suggests that the GMV port-

[6] It is useful in this paper to express the portfolio weights of the GMV portfolio as a function of the estimator of Σ^{-1} we apply to compute them.

folio is an attractive alternative for the mean-variance investor, even though it is not designed to maximise the objective function (1). To further investigate the validity of this argument, we study the out-of-sample performance of the GMV portfolio on the basis of analytical expressions. We first show that estimation errors in $\hat{\Sigma}^{-1}$ can have significant impact on portfolio performance:

Proposition 1 *If $E(\cdot)$ denotes expectation with respect to the true distribution of J_T, then the expected out-of-sample mean $\tilde{\mu}_{\min}$ and variance $\tilde{\sigma}^2_{\min}$ of the excess return on the sample GMV portfolio are respectively given by*

$$\tilde{\mu}_{\min} := E\left((\hat{w}^{\min})'\hat{\mu}\right) = w^{\min}\mu = \mu_{\min} \tag{8}$$

$$\tilde{\sigma}^2_{\min} := E\left((\hat{w}^{\min})'\Sigma\hat{w}^{\min}\right) = \frac{T-2}{T-N-1}(w^{\min})'\Sigma w^{\min} = \frac{T-2}{T-N-1}\sigma^2_{\min} \tag{9}$$

where $\mu_{\min}$ and $\sigma^2_{\min}$ are respectively the mean and variance of the return on the true GMV portfolio $w^{\min} = \dfrac{\Sigma^{-1}1_{\mathrm{N}}}{1'_N\Sigma^{-1}1_{\mathrm{N}}}$.[7]

Proposition 1 indicates that the expected out-of-sample mean return on the GMV portfolio is unaffected by estimation risk since it is equal to that of its population counterpart. However, the sample portfolio leads to higher out-of-sample variance than its true analogue for $N > 1$. The variance of the sample portfolio increases with the number of assets N and decreases with the sample size T. To better demonstrate the effects of errors in $\hat{\Sigma}^{-1}$ on out-of-sample risk, we apply equation (9) to compute the quantity

$$\frac{\tilde{\sigma}^2_{\min} - \sigma^2_{\min}}{\sigma^2_{\min}} = \frac{N-1}{T-N-1} \tag{10}$$

in percentage terms for four portfolio sizes ($N = 10$, 25, 50 and 100 assets) and six sample lengths ($T = 60$, 120, 240, 480, 960 and 24000 observations). The corresponding results in Table 1 show that the risk of the GMV portfolio raises significantly for large values of the ratio N/T . For instance, when $N = 50$ and $T = 60$, the variance increases by 544.44% when using $\hat{\Sigma}^{-1}$ instead of Σ^{-1}. In such cases, $\hat{\Sigma}^{-1}$ is very inaccurate and an improved estimator of the inverse covariance matrix is particularly valuable.

We further examine the attractiveness of the GMV portfolio to the mean-variance investor. In particular, we compare the out-of-sample performances between the sample GMV and mean-variance portfolios. To this end, we use the opportunity cost (OC) of estimating the optimal weights w^{mv} with a sample-based portfolio $\hat{w}$:

$$\mathrm{OC}(\hat{w}, w^{\mathrm{mv}}) = U(w^{\mathrm{mv}}) - E(U(\hat{w})) \tag{11}$$

[7] The proof is contained in the Appendix. A more comprehensive treatment of the distributional properties of minimum variance portfolios is provided by Ohkrin and Schmid (2006), Kan and Smith (2008) and Basak et al. (2009).

where $E(\cdot)$ is again taken with respect to the true distribution of J_T.[8] The smaller is the OC, the smaller is the difference of the out-of-sample performances between the optimal portfolio and the sample-based portfolio $\hat{w}$. For the sample mean-variance portfolio (6), Kan and Zhou (2007) show that:

$$\mathrm{OC}(\hat{w}^{\mathrm{mv}}, w^{\mathrm{mv}}) = (1-k)\frac{SR_{\mathrm{mv}}^2}{2\gamma} + \frac{NT(T-2)}{2\gamma(T-N-1)(T-N-2)(T-N-4)}, \tag{12}$$

where $k = \dfrac{T}{T-N-2}\left(2 - \dfrac{T(T-2)}{(T-N-1)(T-N-4)}\right)$. Therefore, the performance loss from using $\hat{\mu}$ and $\hat{\Sigma}$ for the computation of the optimal mean-variance weights instead of their true analogues is an increasing function of the market size N and a decreasing function of the number of observations T and of the risk aversion coefficient γ. The next proposition determines the OC of using the weights $\hat{w}^{\mathrm{min}}$ instead of the optimal weights w^{mv}:

Proposition 2 *The opportunity cost of using the GMV portfolio instead of the optimal mean-variance portfolio is:*

$$\mathrm{OC}(\hat{w}^{\mathrm{min}}, w^{mv}) = \frac{SR_{mv}^2}{2\gamma} - \mu_{\mathrm{min}} + \frac{\gamma\,(T-2)}{2(T-N-1)}\sigma_{\mathrm{min}}^2 \tag{13}$$

Similarly to the case of the sample mean-variance portfolio, the opportunity cost $\mathrm{OC}(\hat{w}^{\mathrm{min}}, w^{\mathrm{mv}})$ increases with N and decreases with T.

We apply (13) and (12) to compare the OCs between the sample GMV and mean-variance portfolios, respectively, under several scenarios. Columns 3-6 of Table 2 report the quantity $\mathrm{OC}(\hat{w}^{\mathrm{min}}, w^{\mathrm{mv}})/U\left(w^{\mathrm{mv}}\right)$ in percentage terms for four portfolio sizes (N = 10, 25, 50 and 100 assets), three sample sizes (T = 60, 120 and 240 observations per asset) and two values of SR_{mv} (0.2 and 0.4). For expositional simplicity, we set $\mu = k1_N$ for some constant k in order to account for the quantities $\mu'\Sigma^{-1}1_{\mathrm{N}}$ and $1_{\mathrm{N}}'\Sigma^{-1}1_{\mathrm{N}}$. Then, $\mu'\Sigma^{-1}1_{\mathrm{N}} = SR_{\mathrm{mv}}^2/k$ and $1_N'\Sigma^{-1}1_N = SR_{\mathrm{mv}}^2/k^2$. We consider two values for k, namely 0.05/12 and 0.1/12 which, under the assumption of monthly returns, correspond to expected annual excess returns of 5% and 10%, respectively. The last column of Table 2 expresses the performance losses caused by the use of the sample mean-variance portfolio ($\mathrm{OC}(\hat{w}^{\mathrm{mv}}, w^{\mathrm{mv}})/U\left(w^{\mathrm{mv}}\right)$).[9] We observe that the GMV portfolio greatly outperforms the sample mean-variance portfolio in all but one case. The standardised OC for the GMV portfolio is generally less than 100% for almost every value of N, T and S_{mv} considered. This means that, the GMV portfolio is expected to lead to positive values of the objective function $U\left(\hat{w}^{\mathrm{min}}\right)$ or, in other words, positive risk-adjusted excess returns. In contrast, the OC for the sample mean-variance strategy is significantly higher than 100% which corresponds to highly negative expected risk-adjusted returns.

We also measure the amount of information that the investor should require in order to prefer the mean-variance portfolio over the GMV portfolio. In particular, we compute the number of

[8] OC has also been employed by Brown (1976), Jorion (1986), Kan and Zhou (2007) and DeMiguel et al. (2009a).

[9] Note that this quantity does not depend on γ.

observations T needed for the estimation of the mean-variance weights (6) in order for the latter to result in a lower OC than the GMV portfolio. According to the results in Table 3, T ranges from 108 in the case of 10 assets to 3602 in the case of 100 assets. For monthly observations, the latter value of T corresponds to an unrealistically long historical dataset which exceeds 300 years. We conclude that the GMV portfolio dominates the mean-variance portfolio in practical situations.

The results in Tables 2 and 3 illustrate the appeal of the GMV portfolio to the mean-variance investor. Nevertheless, there still exists a considerable difference of the performance between the GMV and its parameter certainty analogue (in terms of out-of-sample variance) or the optimal mean-variance strategy (as measured by the opportunity cost). The next section proposes a framework for the estimation of the inverse covariance matrix that attempts to reduce these differences.

3 Estimation of the Inverse Covariance Matrix

3.1 Estimation framework

We develop a shrinkage approach for the estimation of the inverse covariance matrix. James and Stein (1961) first proposed shrinkage as a mean to reduce the error of an estimator by optimally exploiting the trade-off between bias and variance. Shrinkage has been applied to portfolio selection problems in estimating excess returns (e.g., Jobson and Korkie 1981; Jorion 1986) and covariances (Ledoit and Wolf 2003; 2004a; 2004b). The method of Ledoit and Wolf, in particular, is considered one of the most efficient methods in estimating the covariance matrix.[10] It involves the estimation of the covariance matrix by a convex combination of its sample counterpart $\hat{\Sigma}$ and a shrinkage target Λ:

$$S_{\mathrm{LW}} = (1-f)\hat{\Sigma} + f\Lambda \tag{14}$$

The set of typical shrinkage targets includes the identity matrix (Ledoit and Wolf 2004a), the constant correlations matrix (Ledoit and Wolf 2004b) and a covariance matrix that corresponds to a 1-factor model (Ledoit and Wolf 2003). Ledoit and Wolf (2003; 2004b) estimate the optimal shrinkage parameter f under a statistical loss function and then invert the resulting estimator $\hat{S}_{\mathrm{LW}}$ to compute the GMV weights ($w^{\min}\left(\hat{S}_{\mathrm{LW}}^{-1}\right)$). They find that the latter portfolio outperforms the standard GMV portfolio $\hat{w}^{\min}$ in terms of out-of-sample variance and Sharpe ratio.

Despite its apparent efficiency in reducing the estimation error in the ML covariance matrix, the method of Ledoit and Wolf is difficult to be optimised with respect to portfolio performance measures due to the complexity of the inversion of sum of matrices. To overcome this difficulty, we suggest the estimation of the inverse covariance matrix in a different manner. Instead of shrinking the covariance matrix and then inverting it, we propose to apply shrinkage directly to the inverse covariance matrix. Specifically, we propose the estimation of the inverse with a weighted sum of

[10] In comparison with the other commonly used methodologies for estimating the covariance matrix, the method of Ledoit and Wolf is a dominating extension of the approach which imposes more structure to the covariance matrix (e.g., Frost and Savarino 1986; Chan et al. 1999) while it provides results similar to estimation using more frequent historical data (Jagannathan and Ma 2003).

the ML estimator and a target matrix $\hat{\Omega}$

$$S_{inv} = c_1\hat{\Sigma}^{-1} + c_2\hat{\Omega}, \tag{15}$$

where $\hat{\Omega}$ is an N x N symmetric and positive definite matrix. Note that S_{inv} retains the basic properties of $\hat{\Sigma}^{-1}$ such as symmetry, positive definiteness and invertibility for positive values of $c_1,\ c_2$.

Although this paper is the first to apply the class of estimators (15) to portfolio choice, the multivariate statistics literature has studied a special case of this class (Efron and Morris 1976, Haff 1977, 1979, Dey 1987 and Kubokawa 2005, among others). These studies suggest the estimation of Σ^{-1} by shrinking $\hat{\Sigma}^{-1}$ to $\hat{\Sigma}$ or to the identity matrix. The selection of the parameters c_1 and c_2 is performed on the basis of statistical loss functions under the assumption that the covariance matrix follows a Wishart distribution. Our interest here lies in a different direction since our ultimate aim is to improve portfolio performance. Moreover, our framework has two important advantages over previous efforts. First, as shown below, we do not need to assume any specific distribution to derive the parameters c_1 and c_2. Second, our framework can easily accommodate any $N \times N$ matrix as a target. Here, we consider three different targets:[11]

1. The identity matrix I which is a common choice for the estimation of random matrices due to its constancy and simplicity.
2. The inverse covariance matrix resulting from the 1-factor model of Sharpe (1963) where the factor is the market ($\hat{F}$).
3. A linear combination of I and $\hat{F}$. Then,

$$S_{inv} = c_1\hat{\Sigma}^{-1} + c_2 I + c_3\hat{F} \tag{16}$$

We further provide some interpretation on how the use of (15)-(16) can improve out-of-sample performance. If we apply (15) to estimate the GMV weights, we obtain the following portfolio:

$$w^{\min}(S_{inv}) = \frac{c_1\hat{\Sigma}^{-1}1_N + c_2\hat{\Omega}1_N}{c_1 1_N'\hat{\Sigma}^{-1}1_N + c_2 1_N'\hat{\Omega}1_N} \tag{17}$$

We rewrite the above equation in a simpler form as

$$w^{\min}(S_{inv}) = \hat{d}_0\hat{w}^{\min} + (1-\hat{d}_0)w^{\min}(\hat{\Omega}) \tag{18}$$

where $\hat{d}_0 = \dfrac{c_1 1_N'\hat{\Sigma}^{-1}1_N}{c_1 1_N'\hat{\Sigma}^{-1}1_N + c_2 1_N'\hat{\Omega}1_N}$. Therefore, the application of the inverse covariance matrix estimator (15) results in a "two-fund" portfolio, i.e., a convex combination of the standard GMV

[11] More sophisticated targets are likely to produce superior estimators. We leave such extensions for future research since our focus here is on the introduction of the concept of shrinking the inverse covariance matrix within the context of portfolio selection.

portfolio $\hat{w}_{\min}$ and the minimum variance portfolio constructed using $\hat{\Omega}$ instead of Σ^{-1}. This result provides further intuition on how our approach deals with parameter uncertainty. Specifically, the "target portfolio" $w^{\min}(\hat{\Omega})$ helps in eliminating a part of the estimation risk of the GMV portfolio and in improving its out-of-sample performance. This is because, for the target matrices under consideration, $w^{\min}(\hat{\Omega})$ is less subject to estimation errors than $\hat{w}^{\min}$. When the identity is used as a target, then the target portfolio is $w^{\min}(I) = 1_N/N$ representing the $1/N$ strategy. This equal-weighted portfolio appears to outperform several sophisticated sample-based portfolios, since it does not require the estimation of any parameter (see DeMiguel et al. 2009a). When we extract the target matrix from the market model of Sharpe (1963), the corresponding target portfolio $w^{\min}(\hat{F})$ belongs to the class of portfolios studied by Chan et al. (1999). These authors show that when estimation errors are significant, portfolios resulting from factor models result in considerably higher performance than the GMV portfolio. Finally, when S_{inv} is given by (16), the target portfolio $w^{\min}(\hat{\Omega})$ is a linear combination of $w^{\min}(I)$ and $w^{\min}(\hat{F})$. In this case, we end up with the following three-fund strategy:

$$w^{\min}(S_{inv}) = \hat{d}_1\hat{w}^{\min} + \hat{d}_2\hat{w}^{\min}(I) + (1 - \hat{d}_1 - \hat{d}_2)\hat{w}^{\min}(\hat{F}). \tag{19}$$

where $\hat{d}_1 = \dfrac{c_1 1_N'\hat{\Sigma}^{-1}1_N}{c_1 1_N'\hat{\Sigma}^{-1}1_N + Nc_2 + c_3 1_N'\hat{F}1_N}$ and $\hat{d}_2 = \dfrac{Nc_2}{c_1 1_N'\hat{\Sigma}^{-1}1_N + Nc_2 + c_3 1_N'\hat{F}1_N}$.

Besides providing intuition, the expressions (18)-(19) modify the problem of selecting the combination coefficients c_i in two useful ways. First, they reduce the number of coefficients by one and we now only have to determine the value(s) of $\hat{d}_i$ instead of c_i. Second, they enable the choice of $\hat{d}_i$ on the basis of portfolio performance criteria. We next present two methods to compute $\hat{d}_i$; the first aims to minimise portfolio risk while the second attempts to increase portfolio return.

3.2 Selection of the combination coefficients

3.2.1 Minimisation of the out-of-sample variance

Our first approach in determining the combination coefficients in (18)-(19) aims at reducing the out-of-sample variance of the GMV portfolio. For this purpose, we adopt the non-parametric statistical method of *cross-validation*.[12] This method allows us to exploit the information contained in the sample without the need of any specific distributional assumption for the excess returns. As a result, it can be easily used for any target matrix avoiding hard expectation computations that could be required by a parametric method. We apply cross-validation as follows. For each historical return R_t known to the investor, we first compute the portfolio weights $\hat{w}_t{}^{\min}$ and $w_t{}^{\min}(\hat{\Omega})$ ($\hat{\Omega} = I, \hat{F}$) using the remaining returns in the sample ($J_T - \{R_t\}$). We then apply these weights to extract the corresponding excess returns $R_t^{\min} = R_t'\hat{w}_t{}^{\min}$ and $R_t^{\hat{\Omega}} = R_t'w_t{}^{\min}(\hat{\Omega})$. The outcome of this procedure is a time-series of out-of-sample excess returns for each combination strategy. We finally compute the parameter(s) $\hat{d}_i^{var}$ that minimise(s) the out-of-sample variance of the returns $R_t'w_t^{\min}(S_{inv})$:

[12] Cross-validation has also been used in portfolio choice applications by Brandt (1999) and DeMiguel et al. (2009b).

$$\hat{d}_i^{var} = \arg\min_{\hat{d}_i} var\left(\left(w_t^{\min}\left(S_{inv}\right)' R_t\right)_{(t=1,\dots,T)}\right), \tag{20}$$

When $w_t^{\min}\left(S_{inv}\right)$ is expressed as in (18), it is simple to show that

$$\hat{d}_0^{var} = \frac{var\left(R_t^{\hat{\Omega}}\right) - cov\left(R_t^{\min}, R_t^{\hat{\Omega}}\right)}{var\left(R_t^{\min}\right) - 2cov\left(R_t^{\min}, R_t^{\hat{\Omega}}\right) + var\left(R_t^{\hat{\Omega}}\right)} \tag{21}$$

To obtain the quantities $\hat{d}_1^{var}$, $\hat{d}_2^{var}$ for the three-fund rule (19), we first introduce notation for the covariance matrix $V = (v_{ij})$ of the out-of-sample excess returns of the three combination strategies:

$$V = \begin{pmatrix} var\left(R_t^{\min}\right) & cov\left(R_t^{\min}, R_t^{I}\right) & cov\left(R_t^{\min}, R_t^{\hat{F}}\right) \\ cov\left(R_t^{\min}, R_t^{I}\right) & var\left(R_t^{I}\right) & cov\left(R_t^{I}, R_t^{\hat{F}}\right) \\ cov\left(R_t^{\min}, R_t^{\hat{F}}\right) & cov\left(R_t^{I}, R_t^{\hat{F}}\right) & var\left(R_t^{\hat{F}}\right) \end{pmatrix} \tag{22}$$

Then, the solution of (20) is given by

$$\begin{pmatrix} \hat{d}_1^{var} \\ \hat{d}_2^{var} \end{pmatrix} = \begin{pmatrix} v_{11} + v_{33} - 2v_{13} & v_{12} - v_{13} - v_{23} + v_{33} \\ v_{12} - v_{13} - v_{23} + v_{33} & v_{22} + v_{33} - 2v_{23} \end{pmatrix}^{-1} \begin{pmatrix} v_{33} - v_{13} \\ v_{33} - v_{23} \end{pmatrix} \tag{23}$$

The substitution of $\hat{d}_i^{var}$ in the corresponding portfolio weights in (18) and (19) gives three new "variance-based" portfolio strategies. We denote them ICVAR$_I$, ICVAR$_F$ and ICVAR$_{IF}$ in accordance to the specific target that each one employs for the estimation of the inverse covariance matrix.

3.2.2 Maximisation of the last period's return

In section 2, we show that the GMV strategy is appealing to the mean-variance investor, since it generally leads to higher risk-adjusted returns than the sample mean-variance portfolio. To further enhance the out-of-sample returns of the GMV strategy, we provide an alternative method to select the parameters $\hat{d}_i$. This method is motivated by the strong positive autocorrelation of portfolio returns that has been reported in the literature (Campbell et al. 1997).[13] In particular, the combination coefficients are chosen to maximise the last period's return on the portfolio $w^{\min}(S_{inv})$ or, equivalently, they solve the following optimisation problem:

$$\hat{d}_i^{ret} = \arg\max_{\hat{d}_i} w^{\min}(S_{inv})' R_T, \tag{24}$$

subject to the constraint

$$0 \leq \hat{d}_i \leq 1, \tag{25}$$

[13] In a similar fashion, DeMiguel et al. (2009b) exploit the autocorrelation of portfolio returns to calibrate their norm-constrained portfolios.

that ensures that $\hat{d}_i$ do not acquire any extreme value. The optimization problem (24)-(25) also has a simple solution. In the case that S_{inv} is given by (15), the solution can be expressed by

$$\hat{d}_0^{ret} = \delta\left((\hat{w}^{\min})' R_T - w^{\min}(\hat{\Omega})' R_T\right), \tag{26}$$

where $\delta(x) = 1$ if x is positive and $\delta(x) = 0$ otherwise. In essence, the investor adopts either $\hat{w}^{\min}$ or $w^{\min}(\hat{\Omega})$ depending on which portfolio yielded higher return in the last period. Similarly, for the three-fund strategy (19), the investor chooses among $\hat{w}^{\min}$, $w^{\min}(I)$ and $w^{\min}(\hat{F})$:

$$\hat{d}_1^{ret} = \delta\big((\hat{w}^{\min})' R_T - w^{\min}(I)' R_T\big)\delta\Big(\hat{w}^{\min})' R_T - w^{\min}(\hat{F})' R_T\Big) \tag{27}$$

$$\hat{d}_2^{ret} = \delta\big(w^{\min}(I)' R_T - (\hat{w}^{\min})' R_T\big)\delta\Big(w^{\min}(I)' R_T - w^{\min}(\hat{F})' R_T\Big). \tag{28}$$

In the remainder of the paper, we use the abbreviations ICRET$_I$, ICRET$_F$ and ICRET$_{IF}$ to denote the "return-based" portfolio strategies that result from the corresponding target matrices and the application of $\hat{d}_i^{ret}$ to (18)-(19).

3.3 Practical issues

Despite its generality, the portfolio choice framework developed in this work has still some limitations that may impose difficulties in its practical implementation. Namely, we have not yet accounted for the case that the number of assets exceeds the sample size and for the possibility of short-sale constraints and transaction costs. With respect to the first limitation, the ML estimator of the covariance matrix $\hat{\Sigma}$ is not invertible when $N > T$. In this case, we suggest the application of the Moore-Penrose inverse of $\hat{\Sigma}$ instead of $\hat{\Sigma}^{-1}$ in (15) and (16) :

$$W_{MP} = HL^{-1}H' \tag{29}$$

where H is a matrix of eigenvectors of $\hat{\Sigma}$, $L = diag\{l_1, l_2, ..., l_N\}$ is a diagonal matrix and $l_1 > l_2 > ... > l_N$ are the eigenvalues of $\hat{\Sigma}$.[14] W_{MP} is well-defined for all positive values of N and T. Hence, it enables the computation of the weights for the six portfolio strategies developed earlier when $N > T$.

Further, the new portfolios may involve negative positions in some assets. In the presence of short-sale constraints, they cannot be applied in their current form. Additionally, the effects of parameter uncertainty on portfolio performance can be more hazardous when transaction costs are high. This is because parameter uncertainty magnifies the volatility of the portfolio weights across time and, consequently, increases portfolio turnover. A high portfolio turnover corresponds to a significant decrease in the out-of-sample performance after accounting for transaction costs. To address these issues, we provide two alternative strategies that result from the constraint-based inverse covariance matrix studied in Jagannathan and Ma (2003). These authors show that

[14] Kubokawa and Srivastava (2008) also propose a combination of W_{MP} and I for the estimation of the singular inverse Wishart matrix.

imposing a nonnegativity constraint to the minimum variance portfolio is equivalent to applying:

$$\hat{\Sigma}_{cons} = \hat{\Sigma} - \left(\zeta 1_N' + 1_N \zeta'\right) \tag{30}$$

to compute the GMV portfolio weights (7), where ζ is the N-dimensional vector of the Lagrange multipliers for the nonnegativity constraint at the solution of the short-sale constrained minimum variance problem. If we use $\hat{\Sigma}_{cons}^{-1}$ in the place of $\hat{\Sigma}^{-1}$ in (15) and apply the identity as a target while constraining $\hat{d}_0$ to lie in [0,1], then the corresponding portfolio weights are a convex combination of the short-sale constrained minimum variance weights and $1/N$. Since the weights of both portfolios are nonnegative, so is their combination. Moreover, the turnover of the short-sale constrained minimum variance portfolio as well as that of the $1/N$ rule are known to be relatively low. In accordance, we expect a similar feature for the combination.[15] To obtain the coefficient $\hat{d}_0$ in this case, we again apply the two selection criteria discussed earlier. We denote the resulting portfolios with "ICVAR$_I$ (cons)" and "ICRET$_I$ (cons)".

4 Performance Evaluation

4.1 Benchmarks, data and methodology

We carry out a comparative performance analysis of the eight portfolio strategies derived in the previous section against several strategies from the literature. The set of benchmark strategies naturally includes the sample mean-variance (MEAN) portfolio (6), the sample global minimum variance (GMV) portfolio (7) and the $1/N$ strategy. We also consider five extensions of the GMV portfolio. The first two are based on the shrinkage estimation of the covariance matrix developed in Ledoit and Wolf (2003) and in Ledoit and Wolf (2004a) respectively denoted with LW (1f) and LW (id). We consider these specific portfolios, since they are known to produce lower out-of-sample variance than most other strategies in the literature. The third extension we include in our study is the short-sale constrained minimum variance portfolio (MV (cons)). This portfolio is reported to outperform the unconstrained GMV portfolio for large values of N (Jagannathan and Ma 2003). The remaining two benchmark strategies we consider are the 2-norm constrained GMV portfolio (MV (2-norm)) that exploits autocorrelation in portfolio returns (DeMiguel et al. 2009b) and the "four-fund" rule of Tu and Zhou (2011) that combines the MEAN, GMV, $1/N$ strategies and the risk-free asset (TZ4).[16] DeMiguel et al. (2009b) and Tu and Zhou (2011) respectively find that these two strategies dominate $1/N$ as well as most sample-based portfolios from earlier works,. Hence, we exclude from this study several other strategies that have been proposed in the literature. Table 5 lists the competing strategies.

[15] For a further reduction in the turnover, one can also restrict $\hat{d}_0$ to take values close to 0 or to 1.

[16] Among the different norm-constrained portfolios developed by DeMiguel et al. (2009b) we just consider the return-based 2-norm constrained portfolio for two reasons. First, the variance-based portfolios appear to perform worse than the Ledoit-Wolf portfolios in terms of out-of-sample variance. Second, the 2-norm return-based portfolio dominates the 1-norm and A-norm portfolios with regards to the Sharpe ratio measure. Moreover, it performs similarly to the partial minimum variance portfolio while it is easier to compute.

Following the literature, we evaluate the performance of the competing portfolio strategies in 6 datasets of monthly excess returns (see Table 6 for a summary). Our selection of data allows us to account for various numbers of assets and different asset characteristics. Moreover, apart from the dataset of stocks returns, the remaining datasets are a common choice in portfolio choice studies. We assess out-of-sample portfolio performance by employing the rolling window method of DeMiguel et al. (2009a). In particular, for each month $t > T$, we estimate the portfolio weights for each strategy s using the returns for the months t-T,...,t-1. We then calculate the corresponding portfolio returns R_t^s and obtain a time-series of excess returns for each portfolio strategy. We let $\hat{\mu}_s$ and $\hat{\sigma}_s$ denote the sample mean and standard deviation of the time-series and compute the following out-of-sample performance metrics:

$$\text{Variance: } \hat{\sigma}_s^2 \tag{31}$$

$$\text{Sharpe ratio: } \widehat{SR}_s = \frac{\hat{\mu}_s}{\hat{\sigma}_s} \tag{32}$$

$$\text{Certainty-equivalent return: } CER_s = \hat{\mu}_s - \frac{\gamma}{2}\hat{\sigma}_s^2 \tag{33}$$

$$\text{Turnover: } \hat{\tau} = \frac{1}{M-T-1}\sum_{t=T+1}^{M-1} \left|\left|w_{t+1}^s - w_{t+}^s\right|\right|_1 , \tag{34}$$

where M is the total number of observations in the dataset and w_{t+}^s and w_{t+1}^s are respectively the portfolio weights for strategy s at time t before and after rebalancing. We derive the above measures assuming a sample size of 60 which corresponds to 5 years of monthly returns.[17] We also assume a risk aversion parameter of $\gamma = 3$ for the computation of the weights of the mean-variance portfolio and the portfolio of Tu and Zhou (2011) as well as for the computation of the CERs. We note that in the case that $N = 100$ all unconstrained portfolio strategies are implemented using W_{MP} instead of $\hat{\Sigma}$.

It is also necessary to consider the statistical significance of the difference of the variances, the Sharpe ratios and the CERs between two specific strategies. To test the null hypothesis "$\mathrm{H}_0 : \hat{\sigma}_s^2 - \hat{\sigma}_p^2 = 0$" or "$H_0 : CER_s - CER_p = 0$" for two strategies s and p, we employ the circular studentized bootstrap of Politis and Romano (1994) assuming an expected block size of 5. We then use the resulting two-sided confidence interval to compute the p-value following the recommendation of Ledoit and Wolf (2008). We also test the hypothesis "$\mathrm{H}_0 : \widehat{SR}_s - \widehat{SR}_p = 0$" by employing the studentised circular block bootstrap methodology of Ledoit and Wolf (2008) to obtain the respective p-value.[18]

[17] This sample length allows us to assess the new strategies in a competing situation where estimation errors are typically large. It also enables us to study portfolio performance in the case that $N > T$ using the dataset of 100 stocks. Nevertheless, we have also accounted for $T = 120$, but these results lead to comparable conclusions and, for conservation of space, we exclude them from the paper.

[18] We are grateful to Michael Wolf for making available the Matlab code that performs this test at his website (http://www.econ.uzh.ch/faculty/wolf/).

4.2 Out-of-sample variance

Table 6 reports in percentage terms the monthly out-of-sample variance of the excess returns for each portfolio strategy considered. We also report in brackets the p-value for the difference between this variance and that of the GMV portfolio. We first observe that GMV leads to lower variance than MEAN, $1/N$ and TZ4. It appears that a large increase in the risk aversion parameter is required for the mean-variance and the Tu and Zhou (2011) portfolios to reach the risk levels of the GMV portfolio. However, in line with the results in Table 1, the variance of the latter portfolio significantly increases with the number of assets. For instance, it attains a value of 0.157% in the dataset of 6 size and book-to-market portfolios while it increases to 0.183% in the set of the 25 size and book-to-market portfolios.

The three new strategies ICVAR$_I$, ICVAR$_F$ and ICVAR$_{IF}$ calibrated using the minimum variance criterion outperform the GMV portfolio in all cases. In most of them, the difference between the variances is statistically significant. Among the new strategies, the lowest variance is obtained when both the identity and the 1-factor inverse covariance matrix are applied as targets. For example, in the 30 industry portfolios dataset, the variance of GMV portfolio is equal to 0.219% while the variance for the ICVAR$_I$, ICVAR$_F$ and ICVAR$_{IF}$ strategies are 0.173%, 0.142% and 0.137%, respectively. In comparison to the strategies that use the covariance matrix estimators of Ledoit and Wolf, the latter lead to lower variance than ICVAR$_{IF}$ in half of the datasets while ICVAR$_{IF}$ dominates in the remaining.

The strategies ICRET$_I$, ICRET$_F$, ICRET$_{IF}$ and MV (2-norm) also lead to lower risk levels than the GMV portfolio in four out of six datasets, even though they aim to improve out-of-sample returns. Finally, the constrained portfolio ICVAR$_I$ (cons) appears to attain similar risk levels to the original MV (cons) portfolio. We deduce that no significant variance reduction can be achieved by shrinking the constrained inverse covariance matrix.

4.3 Out-of-sample Sharpe ratios and certainty-equivalent returns

We now turn our focus on risk-adjusted returns. In Table 7, we present the out-of-sample Sharpe ratio for the 16 portfolio strategies under study. We observe that the Sharpe ratio for the GMV portfolio is lower than that of the strategies ICVAR$_I$, ICVAR$_F$ and ICVAR$_{IF}$ in most datasets. The new strategies that exploit the positive autocorrelation in portfolio returns (ICRET$_I$, ICRET$_F$ and ICRET$_{IF}$) generally outperform GMV, LW (id), LW (1f) and all the variance-based portfolios developed in this work. More importantly, the new return-based strategies result in considerably higher Sharpe ratios than the $1/N$ rule in five out of six datasets with the difference being statistical significant. $1/N$ only wins in the international indices dataset, but its superiority is not statistical significant. This result shows that portfolio optimisation can be useful in practice extending the contributions of Demiguel et al. (2009b) and Tu and Zhou (2011). We further observe that ICRET$_{IF}$ results in the highest Sharpe ratios among the theory-based portfolios in the majority of the datasets considered. For instance, it attains a Sharpe ratio of almost 29% in the 25 size and book-to-market portfolios while TZ4, MEAN and MV (2-norm) follow with a Sharpe ratio of around 25%, 24.5% and 21%, respectively. The only exception occurs for the dataset of stocks, where ICRET_I yields the highest Sharpe ratio.

We can draw at least two more conclusions from Table 7. First, in the presence of short-sale constraints, the constrained portfolio ICRET_I (cons) can be applied instead of MV (cons). The former portfolio leads to considerably higher Sharpe ratios than $1/N$. Second, when the number of assets exceeds the size of the sample as in the last dataset, the generalised inverse W_{MP} appears to be an efficient replacement of $\hat{\Sigma}^{-1}$ in our framework. As we can see in the last column, the highest Sharpe ratios are attained for ICRET_I and ICRET_{IF}.

The insights from the above discussion are also valid when we employ the CER measure instead of SR to measure risk-adjusted returns. This is evident in Table 8 which reports, for $\gamma = 3$, the CERs for each strategy. Nevertheless, we include these results in the paper for two reasons. First, the CER metric is an out-of-sample analogue to the objective function (1). Clearly, the portfolio strategies developed adapt well on this objective and are appealing to the mean-variance investor. Second, the four-fund strategy of Tu and Zhou (2011) is constructed to maximise CER. Yet, this strategy generally underperforms the portfolios ICVAR_I, ICVAR_F and ICVAR_{IF} due to the magnitude of the estimation errors in the means.

4.4 Portfolio turnover

Table 9 contains the turnover generated by each strategy under consideration. Naturally, the $1/N$ portfolio has the lowest turnover. We observe that, besides outperforming the GMV portfolio in terms of risk and risk-adjusted returns, the portfolios ICVAR_I, ICVAR_F and ICVAR_{IF} also offer lower levels of turnover. We further find that the high level of Sharpe ratio for the return-based strategies developed in this work comes at the cost of high turnover. This is comparable to that generated by the norm-constrained portfolio and the four-fund strategy of Tu and Zhou (2011). However, when turnover is an issue, the portfolios ICVAR_I (cons) and ICRET_I (cons) can be applied instead. The constrained strategies offer significantly lower turnover than their unconstrained counterparts. At the same time, they result in higher out-of-sample Sharpe ratios and certainty-equivalent returns than both the $1/N$ strategy and the short-sale constrained minimum variance portfolio MV (cons).

5 Conclusions

This paper focused on the estimation of the inverse covariance matrix in the context of optimal portfolio choice. We derived analytical expressions which shed light on the circumstances under which estimation errors in the inverse lead to a substantial deterioration in portfolio performance. Based on our findings, we proposed a new framework for the estimation of the inverse that aims to improve this performance. Our framework shrinks the sample inverse covariance matrix to a target matrix which is less subject to estimation errors. We accounted for three different targets and derived our shrinkage estimators using two non-parametric methods. The application of our estimators to the computation of the minimum variance weights led to a set of new portfolio strategies. We investigated the performance of these strategies in several datasets of monthly returns in terms of four metrics.

The estimation of the inverse covariance through the framework we developed offers several advantages. First, the estimators are investor-oriented, i.e., they are optimised under portfolio performance criteria rather statistical metrics. Second, the portfolio weights have an intuitive form and are simple to compute. Third, our framework can easily accommodate a small sample size or short-sale constraints. Fourth, the resulting portfolios perform very well out-of-sample. They result in relatively low levels of variance and outperform several competing strategies in terms of risk-adjusted returns.

A Appendix: Mathematical Derivations

Proof of Proposition 1. Define the vector of m weights of the sample global minimum variance (GMV) portfolio by:

$$\hat{w}_m^{\min} = \left(e_1 \hat{w}^{\min}, ..., e_m \hat{w}^{\min}\right), \tag{A.1}$$

where $\{e_i\}_{i=1,...m}$ is the standard base of $\mathbb{R}^m$ and $1 \leq m < N$. Ohkrin and Schmid (2006) show that $\hat{w}_m^{\min}$ follows an m-variate elliptical distribution with $T - N + 1$ degrees of freedom and parameters $w_m^{\min} = \left(e_1 w^{\min}, ..., e_m w^{\min}\right)$ and $\frac{1}{T-N+1} \frac{Q^m}{1_N' \Sigma^{-1} 1_N}$ where $Q^m = \{e_i' Q e_j\}_{i,j=1,...m}$ and $Q = \Sigma^{-1} - \frac{\Sigma^{-1} 1_N 1_N' \Sigma^{-1}}{1_N' \Sigma^{-1} 1_N}$. This implies that the mean and covariance matrix of the sample GMV weights are given by:

$$E\left(\hat{w}^{\min}\right) = w^{\min} \text{ and } COV\left(\hat{w}^{\min}\right) = \frac{1}{T-N+1} \frac{Q}{1_N' \Sigma^{-1} 1_N} \tag{A.2}$$

It follows that the expected out-of-sample return on the GMV portfolio is simply:

$$E\left(\left(\hat{w}^{\min}\right)' \mu\right) = \left(w^{\min}\right)' \mu \tag{A.4}$$

We proceed with the calculation of the expected out-of-sample variance of the excess return on the GMV portfolio. Let $\hat{w}_i = e_i' \hat{w}^{\min}$ for $i = 1, ..., N$ and $\Sigma = \{\sigma_{ij}\}_{i,j=1,...,N}$. Then

$$E\left(\left(\hat{w}^{\min}\right)' \Sigma \hat{w}^{\min}\right) = \sum_{i=1}^{N} \sum_{j=1}^{N} E\left(\hat{w}_i \hat{w}_j\right) \sigma_{ij} = tr\left(E\left(\hat{W}\right) \Sigma\right), \tag{A.5}$$

where $\hat{W} = \{\hat{w}_i \hat{w}_j\}_{i,j=1,...,N}$. The expectation of $\hat{W}$ can be derived using (A.2) as:

$$E\left(\hat{W}\right) = \{E\left(\hat{w}_i \hat{w}_j\right)\}_{i,j=1,...,N} = \left\{e_i' COV\left(\hat{w}^{\min}\right) e_j + E\left(\hat{w}_i\right) E\left(\hat{w}_j\right)\right\}_{i,j=1,...,N} \overset{(A.2)}{\Leftrightarrow}$$

$$E\left(\hat{W}\right) = \frac{1}{T-N+1} \frac{Q}{1_N' \Sigma^{-1} 1_N} + w^{\min} \left(w^{\min}\right)' \tag{A.6}$$

Applying (A.6) to (A.5) gives

$$E(\left(\hat{w}^{\min}\right)' \Sigma \hat{w}^{\min}) = tr\left(\frac{1}{T-N-1} \frac{Q\Sigma}{1_N' \Sigma^{-1} 1_N} + w^{\min} \left(w^{\min}\right)' \Sigma\right)$$

$$= tr\left(\frac{1}{T-N-1} \frac{1_N' \Sigma^{-1} 1_N I - \Sigma^{-1} 1_N 1_N'}{\left(1_N' \Sigma^{-1} 1_N\right)^2} + \frac{\Sigma^{-1} 1_N 1_N'}{\left(1_N' \Sigma^{-1} 1_N\right)^2}\right)$$

Simplifying the above expression, results in

$$E(\left(\hat{w}^{\min}\right)' \Sigma \hat{w}^{\min}) = \frac{T-2}{T-N-1} \frac{1}{1_N' \Sigma^{-1} 1_N}, \tag{A.7}$$

since $tr\left(\Sigma^{-1}1_N 1'\right) = 1'\Sigma^{-1}1_N$. This completes the proof. ■

References

Basak, G. K., R. Jagannathan, T. Ma. 2009. Jackknife estimator for tracking error variance of optimal portfolios. *Management Science* **55** 990-1002.

Best, M. J., R. R. Grauer. 1991. On the sensitivity of mean-variance-efficient portfolios to changes in asset means: Some analytical and computational results. *Review of Financial Studies* **4** 315-342.

Brandt, M. W. 1999. Estimating portfolio and consumption choice: A conditional Euler equations approach. *Journal of Finance* **54** 1609-1645.

Brandt, M. W. 2009. Portfolio choice problems, in: Aït-Sahalia, Y., Hansen, L.P. (Eds.), *Handbook of Financial Econometrics: Tools and techniques*. Elsevier, pp. 269-336.

Brown, S. J. 1976. Optimal portfolio choice under uncertainty: A Bayesian approach. Ph.D. dissertation, University of Chicago.

Campbell, J. Y., A. W. Lo, A. G. MacKinlay. 1997. *The Econometrics Of Financial Markets*. Princeton University Press, Princeton NJ.

Chan, L. K. C., J. Karceski, J. Lakonishok. 1999. On portfolio optimization: Forecasting covariances and choosing the risk model. *Review of Financial Studies* **12** 937-974.

Chopra, V., W. Ziemba. 1993. The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice. *Journal of Portfolio Management* **19** 6-12.

DeMiguel, V., L. Garlappi, R. Uppal. 2009a. Optimal versus naive diversification: How inefficient is the $1/N$ portfolio strategy? *Review of Financial Studies* **22** 1915-1953.

DeMiguel, V., L. Garlappi, F. J. Nogales, R. Uppal. 2009b. A generalized approach to portfolio optimization: Improving performance by constraining portfolio norms. *Management Science* **55** 798-812.

Dey, D. K. 1987. Improved estimation of a multinormal precision matrix. *Statistics and Probability Letters* **6** 125-128.

Efron, B., C. Morris. 1976. Multivariate empirical Bayes and estimation of covariance matrices. *The Annals of Statistics* **4** 22-32.

Frost, P. A., J. E. Savarino. 1986. An empirical Bayes approach to efficient portfolio selection. *Journal of Financial and Quantitative Analysis* **21** 293-305.

Garlappi, L., R. Uppal, T. Wang. 2007. Portfolio selection with parameter and model uncertainty: a multi-prior approach. *Review of Financial Studies* **20** 41-81.

Haff, L. R. 1977. Minimax estimators for a multinormal precision matrix. *Journal of Multivariate Analysis* **7** 374-385.

Haff, L. R. 1979. Estimation of the inverse covariance matrix: Random mixtures of the inverse Wishart matrix and the identity. *The Annals of Statistics* **7** 1264-1276.

Jagannathan, R., T. Ma. 2003. Risk reduction in large portfolios: Why imposing the wrong constraints helps. *Journal of Finance* **58** 1651-1684.

James, W., C. Stein. 1961. Estimation with quadratic loss. In *Proceedings of the 4th CityBerkeley Symposium on Mathematical Statistics and Probability*. University of California Press, pp. 361-379.

Jobson, J. D., B. Korkie. 1980. Estimation for Markowitz efficient portfolios. *Journal of the American Statistical Association* **75** 544-554.

Jobson, J. D., B. Korkie. 1981. Putting Markowitz theory to work. *Journal of Portfolio Management* **7** 70-74.

Jorion, P. 1986. Bayes-Stein estimation for portfolio analysis. *Journal of Financial and Quantitative Analysis* **21** 279-292.

Kan, R., D. R. Smith. 2008. The distribution of the sample minimum-variance frontier. *Management Science* **54** 1364-1380.

Kan, R., G. Zhou. 2007. Optimal portfolio choice with parameter uncertainty. *Journal of Financial and Quantitative Analysis* **42** 621-656.

Kubokawa, T. 2005. A revisit to estimation of the precision matrix of the Wishart distribution. *Journal of Statistical Research* **39** 91-114.

Kubokawa, T., M. S. Srivastava. 2008. Estimation of the precision matrix of a singular Wishart distribution and its application in high-dimensional data. *Journal of Multivariate Analysis* **99** 1906-1928.

Ledoit, O., M. Wolf. 2003. Improved estimation of the covariance matrix of stock returns with an application to portfolio selection. *Journal of Empirical Finance* **10** 603-621.

Ledoit, O., M. Wolf. 2004a. Honey, I shrunk the sample covariance matrix: Problems in mean-variance optimization. *Journal of Portfolio Management* **30** 110-119.

Ledoit, O., M. Wolf. 2004b. A well-conditioned estimator for large-dimensional covariance matrices. *Journal of Multivariate Analysis*. **88** 365-411.

Ledoit, O., M. Wolf. 2008. Robust performance hypothesis testing with the Sharpe ratio. *Journal of Empirical Finance* **15** 850-859.

Markowitz, H. M. 1952. Portfolio selection. *Journal of Finance* **7** 77-91.

Michaud, R. O. 1989. The Markowitz optimization enigma: Is optimized optimal? *Financial Analysts Journal* **45** 31-42.

Okhrin, Y., W. Schmid. 2006. Distributional properties of portfolio weights. *Journal of Econometrics* **134** 235-256.

Politis, D. N., J. P. Romano. 1994. The stationary bootstrap. *Journal of the American Statistical Association* **89** 1303-1313.

Sharpe, W. F. 1963. A simplified model for portfolio analysis. *Management Science* **9** 277-293.

Tu, J., G. Zhou. 2004. Data-generating process uncertainty: what difference does it make in portfolio decisions? *Journal of Financial Economics* **72** 385-421.

Tu, J., G. Zhou. 2011. Markowitz meets Talmud: A combination of sophisticated and naive diversification strategies. *Journal of Financial Economics* **99** 204-215.

Table 1: Increase in the out-of-sample variance due to estimation errors

This table presents the increase in the out-of-sample variance of the global minimum variance portfolio when using the ML estimator of the inverse covariance matrix ($\hat{\Sigma}^{-1}$) instead of its population counterpart. In particular, it reports the quantity $\dfrac{\tilde{\sigma}^2_{\min} - \sigma^2_{\min}}{\sigma^2_{\min}} = \dfrac{N-1}{T-N-1}$ in percentage terms for different values of the sample length (T) and of the number of available risky assets (N), where $\tilde{\sigma}^2_{\min}$ and $\sigma^2_{\min}$ are the expected out-of-sample variances of the sample and true portfolios, respectively.

	N			
T	10	25	50	100
60	18.37	70.59	544.44	-
120	8.26	25.53	71.01	521.05
240	3.93	11.22	25.93	71.22
480	1.92	5.29	11.42	26.12
960	0.95	2.57	5.39	11.53
24000	0.04	0.10	0.20	0.41

Table 2: Opportunity costs

This table presents the expected out-of-sample performance losses from using the global minimum variance portfolio or the sample mean-variance portfolio instead of the optimal mean-variance portfolio. It reports the percentage normalized opportunity cost $100 \cdot \mathrm{OC}(\hat{w}^{\min}, w^{\mathrm{mv}})/U(w^{\mathrm{mv}})$ for different sample lengths (T) and numbers of assets (N), where $\hat{w}^{\min}$ and w^{mv} are respectively the weights of the sample global minimum variance and the parameter certainty mean-variance portfolios. The opportunity cost is computed using equation (13) in the text. The table accounts for two values for the relative risk aversion coefficient ($\gamma = 1, 3$), for the Sharpe ratio of the optimal mean variance portfolio ($SR_{\mathrm{mv}} = 0.2, 0.4$) and for the mean excess returns ($\mu = 0.05\ 1_N, 0.1\ 1_N$), respectively. The last column includes the normalised opportunity cost $100 \cdot \mathrm{OC}(\hat{w}^{\mathrm{mv}}, w^{\mathrm{mv}})/U(w^{\mathrm{mv}})$ for the sample mean-variance portfolio $\hat{w}^{\mathrm{mv}}$ computed using the relevant formula in Kan and Zhou (2007) (see equation (12) in the text).

Panel A: $SR_{\mathrm{mv}} = 0.2$

N	T	Annual μ= 5% $\gamma = 1$	Annual μ= 5% $\gamma = 3$	Annual μ= 10% $\gamma = 1$	Annual μ= 10% $\gamma = 3$	$\frac{100 \cdot \mathrm{OC}(\hat{w}^{\mathrm{mv}}, w^{\mathrm{mv}})}{U(w^{\mathrm{mv}})}$
10	60	95.11	80.45	87.26	49.06	847.12
	120	95.09	80.34	87.02	48.07	297.64
	240	95.07	80.29	86.91	47.65	126.67
25	60	95.25	81.02	88.54	54.16	6589.91
	120	95.13	80.53	87.44	49.76	1168.01
	240	95.09	80.37	87.09	48.36	388.37
50	60	96.54	86.16	100.11	100.43	1053877.78
	120	95.26	81.02	88.55	54.20	6011.49
	240	95.13	80.53	87.45	49.80	1132.46
100	120	96.48	85.91	99.54	98.15	676749.12
	240	95.26	81.02	88.56	54.22	5751.55

Panel B: $SR_{\mathrm{mv}} = 0.4$

N	T	Annual μ= 5% $\gamma = 1$	Annual μ= 5% $\gamma = 3$	Annual μ= 10% $\gamma = 1$	Annual μ= 10% $\gamma = 3$	$\frac{100 \cdot \mathrm{OC}(\hat{w}^{\mathrm{mv}}, w^{\mathrm{mv}})}{U(w^{\mathrm{mv}})}$
10	60	98.72	94.87	96.27	85.10	244.02
	120	98.72	94.87	96.26	85.04	84.87
	240	98.72	94.86	96.25	85.01	35.91
25	60	98.73	94.91	96.35	85.42	1899.98
	120	98.72	94.88	96.29	85.14	333.65
	240	98.72	94.87	96.26	85.05	109.98
50	60	98.81	95.23	97.08	88.31	298669.44
	120	98.73	94.91	96.35	85.42	1724.70
	240	98.72	94.88	96.29	85.14	322.20
100	120	98.80	95.21	97.04	88.17	191551.75
	240	98.73	94.91	96.36	85.42	1646.14

Table 3: Number of observations required for the estimation of the mean-variance portfolio to outperform the global minimum variance portfolio

This table reports the number of historical observations required for the estimation of the sample mean-variance portfolio in order to outperform the global minimum variance portfolio. This value is calculated by comparing the analytical expressions of the opportunity cost for each portfolio choice method (see equations (12) and (13) in the text). The table considers four numbers of assets (N). It also accounts for two values for the relative risk aversion coefficient ($\gamma = 1, 3$), for the Sharpe ratio of the optimal mean variance portfolio ($SR_{\mathrm{mv}} = 0.2, 0.4$) and for the mean excess returns ($\mu = 0.05\ 1_N, 0.1\ 1_N$), respectively.

	$SR_{\mathrm{mv}} = 0.2$				$SR_{\mathrm{mv}} = 0.4$			
N	Annual μ= 5%		Annual μ= 10%		Annual μ= 5%		Annual μ= 10%	
	$\gamma = 1$	$\gamma = 3$	$\gamma = 1$	$\gamma = 3$	$\gamma = 1$	$\gamma = 3$	$\gamma = 1$	$\gamma = 3$
10	309	323	335	366	108	109	110	113
25	764	797	828	906	260	263	265	272
50	1521	1587	1650	1804	514	519	524	538
100	3035	3167	3293	3602	1021	1031	1041	1069

Table 4: List of portfolio strategies

This table reports the portfolio strategies considered in the empirical analysis. Panel A lists strategies from the literature while panel B outlines the strategies developed in the paper. For the variance-based portfolios, the estimator of the inverse covariance matrix is constructed to minimise out-of-sample portfolio variance using a cross-validation technique. For the return-based strategies, the estimator of the inverse covariance matrix is constructed to maximise the last month's portfolio return.

Abbreviation	Description
	Panel A: Strategies from the literature
MEAN	Sample mean-variance portfolio
GMV	Sample global minimum variance portfolio
1/N	Equal-weighted portfolio
LW (id)	Minimum variance portfolio that results from shrinking the covariance matrix to the identity (Ledoit and Wolf 2004a)
LW(1f)	Minimum variance portfolio that results from shrinking the covariance matrix to the covariance matrix from an 1-factor model. (Ledoit and Wolf 2003)
MV (cons)	Short-sale constrained minimum variance portfolio
MV (2-norm)	2-norm constrained minimum variance portfolio that exploits the positive autocorrelation of portfolio returns (DeMiguel et al. 2009b)
TZ4	The linear combination of MEAN, GMV, $1/N$ portfolios and the risk-free asset proposed in Tu and Zhou (2011).
	Panel B: Strategies developed in the paper
	Variance-based portfolios
ICVAR$_I$	Minimum variance portfolio that results from shrinking the ML estimator of the inverse covariance matrix to the identity, under the minimum variance criterion.
ICVAR$_F$	Minimum variance portfolio that results from shrinking the ML estimator of the inverse covariance matrix to the inverse covariance matrix from a 1-factor model, under the minimum variance criterion.
ICVAR$_{IF}$	Minimum variance portfolio that results from shrinking the ML estimator of the inverse covariance matrix to a linear combination the identity and the inverse covariance matrix from a 1-factor model, under the minimum variance criterion.
	Return-based portfolios
ICRET$_I$	Minimum variance portfolio that results from shrinking the ML estimator of the inverse covariance matrix to the identity, under the return-based criterion.
ICRET$_F$	Minimum variance portfolio that results from shrinking the ML estimator of the inverse covariance matrix to the inverse covariance matrix from a 1-factor model, under the return-based criterion.
ICRET$_{IF}$	Minimum variance portfolio that results from shrinking the ML estimator of the inverse covariance matrix to a linear combination of the identity and the inverse covariance matrix from a 1-factor model, under the return-based criterion.
	Short-sale constrained portfolios
ICVAR$_I$ (cons)	Minimum variance portfolio that results from shrinking the short-sale constrained estimator of the inverse covariance matrix to the identity, under the minimum variance criterion.
ICRET$_I$ (cons)	Minimum variance portfolio that results from shrinking the short-sale constrained estimator of the inverse covariance matrix to the identity, under the return-based criterion.

Table 5: List of datasets

This table contains the datasets of monthly excess returns that we employ in the empirical analysis. For the computation of excess returns, the 30-days T-bill is employed. For each dataset we also report its abbreviation and the time period spanned. The source of the first 5 datasets is Kenneth French's website (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html). Data for the S&P 500 stocks was obtained by Reuters.

Abbreviation	Time Period	Description
6FF	07/1963-12/2008	6 industry portfolios
10Ind	07/1963-12/2008	10 size and book-to-market portfolios
16Int	01/1977-12/2008	16 international indices
25FF	07/1963-12/2008	25 size and book-to-market portfolios
30Ind	07/1963-12/2008	30 Industry portfolios
100S&P	01/1981-09/2009	100 randomly selected S&P 500 stocks

Table 6: Variances

This table reports in percentage terms the monthly out-of-sample variance of the excess portfolio returns for each strategy of Table 4 and for each dataset of Table 5. It also includes in brackets the p-value that the variance of each strategy differs from that of the GMV portfolio. The p-values are computed using the stationary bootstrap approach of Politis and Romano (1994) and the recommendation of Ledoit and Wolf (2008) with an expected block size of 5.

	6FF	10Ind	16Int	25FF	30Ind	100S&P
Panel A: Strategies from the literature						
Mean	6.7527	5.7518	13.5067	68.5813	87.8512	143.5053
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
GMV	0.1567	0.1374	0.2009	0.1833	0.2188	0.2493
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
1/*N*	0.2430	0.1862	0.1948	0.2649	0.2317	0.2178
	(0.00)	(0.00)	(0.52)	(0.00)	(0.31)	(0.08)
LW (id)	0.1584	0.1269	0.1672	0.1378	0.1296	0.1361
	(0.56)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
LW (1f)	0.1559	0.1295	0.1670	0.1471	0.1285	0.1349
	(0.45)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
MV (cons)	0.1901	0.1366	0.1653	0.1852	0.1391	0.1276
	(0.00)	(0.83)	(0.00)	(0.79)	(0.00)	(0.00)
MV (2-norm)	0.1703	0.1473	0.1940	0.1754	0.1601	0.2178
	(0.00)	(0.01)	(0.36)	(0.15)	(0.00)	(0.07)
TZ4	1.4857	0.3071	0.4582	1.1577	0.3059	0.5786
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Panel B: Strategies developed in the paper						
Variance-based portfolios						
ICVAR$_I$	0.1563	0.1364	0.1743	0.1660	0.1725	0.1774
	(0.74)	(0.63)	(0.00)	(0.00)	(0.00)	(0.00)
ICVAR$_F$	0.1554	0.1323	0.1693	0.1650	0.1422	0.1561
	(0.19)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)
ICVAR$_{IF}$	0.1551	0.1321	0.1618	0.1515	0.1369	0.1295
	(0.29)	(0.04)	(0.00)	(0.00)	(0.00)	(0.00)
Return-based portfolios						
ICRET$_I$	0.1734	0.1496	0.1954	0.1927	0.1934	0.2094
	(0.00)	(0.00)	(0.42)	(0.06)	(0.00)	(0.00)
ICRET$_F$	0.1630	0.1352	0.1795	0.2008	0.1758	0.2012
	(0.09)	(0.43)	(0.00)	(0.02)	(0.00)	(0.00)
ICRET$_{IF}$	0.1695	0.1435	0.1814	0.2006	0.1648	0.1784
	(0.01)	(0.17)	(0.02)	(0.03)	(0.00)	(0.00)
Short-sale constrained portfolios						
ICVAR$_I$ **(cons)**	0.1903	0.1388	0.1646	0.1855	0.1414	0.1278
	(0.00)	(0.00)	(0.00)	(0.00)	(0.03)	(0.00)
ICRET$_I$ **(cons)**	0.2027	0.1513	0.1813	0.2129	0.1793	0.1480
	(0.00)	(0.01)	(0.03)	(0.00)	(0.00)	(0.00)

Table 7: Sharpe ratios

This table reports in percentage terms the monthly out-of-sample Sharpe ratios for each portfolio strategy of Table 4 and for each dataset of Table 5. It also includes in brackets the p-value that the Sharpe ratio of each strategy differs from that of the $1/N$ portfolio. The p-values are computed using the studentized circular bootstrap of Ledoit and Wolf (2008) with an expected block size of 5.

	6FF	10Ind	16Int	25FF	30Ind	100S&P
			Panel A: Strategies from the literature			
Mean	22.9419	2.2160	8.4165	24.5826	4.9673	-9.0485
	(0.02)	(0.21)	(0.10)	(0.04)	(0.53)	(0.02)
GMV	20.3735	13.1767	13.8147	20.2081	6.2139	13.8470
	(0.01)	(0.39)	(0.12)	(0.04)	(0.60)	(0.97)
$1/N$	10.1801	9.5254	20.2858	10.3339	8.7357	13.5740
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
LW (id)	15.4522	13.3689	18.2761	18.8106	11.2836	6.4553
	(0.07)	(0.21)	(0.47)	(0.03)	(0.54)	(0.26)
LW (1f)	19.2836	13.8102	15.1082	20.2838	11.7729	8.9997
	(0.01)	(0.28)	(0.18)	(0.03)	(0.54)	(0.43)
MV (cons)	11.1360	11.1357	18.7276	10.8669	10.2177	10.8830
	(0.60)	(0.54)	(0.58)	(0.81)	(0.61)	(0.47)
MV (2-norm)	23.1322	18.3923	19.0307	21.5498	18.1341	13.5759
	(0.00)	(0.00)	(0.53)	(0.00)	(0.01)	(0.23)
TZ4	20.2322	6.0897	13.3063	25.0833	6.7939	0.4040
	(0.04)	(0.34)	(0.13)	(0.02)	(0.66)	(0.06)
			Panel B: Strategies developed in the paper			
			Variance-based portfolios			
ICVAR$_I$	19.4444	13.2175	18.2894	17.8754	9.4415	11.2871
	(0.01)	(0.25)	(0.44)	(0.03)	(0.81)	(0.58)
ICVAR$_F$	19.6185	14.3242	15.1362	18.8909	11.2191	6.5407
	(0.02)	(0.25)	(0.23)	(0.10)	(0.64)	(0.40)
ICVAR$_{IF}$	18.1596	14.5096	18.1618	17.2551	12.1511	7.6968
	(0.02)	(0.11)	(0.48)	(0.07)	(0.40)	(0.28)
			Return-based portfolios			
ICRET$_I$	22.7646	17.9636	17.1046	21.5872	14.0824	19.0613
	(0.00)	(0.01)	(0.23)	(0.01)	(0.18)	(0.29)
ICRET$_F$	17.1273	14.9796	13.7175	23.6955	12.3222	9.3374
	(0.10)	(0.21)	(0.10)	(0.02)	(0.51)	(0.57)
ICRET$_{IF}$	23.2405	19.2813	18.6364	28.9572	21.1849	14.2316
	(0.00)	(0.00)	(0.57)	(0.00)	(0.01)	(0.90)
			Short-sale constrained portfolios			
ICVAR$_I$ (cons)	11.0855	11.3612	19.9041	11.0011	10.2228	10.8880
	(0.59)	(0.45)	(0.89)	(0.76)	(0.61)	(0.51)
ICRET$_I$ (cons)	14.0257	13.6725	19.8520	14.8786	15.0144	13.3534
	(0.00)	(0.03)	(0.79)	(0.00)	(0.00)	(0.92)

Table 8: Certainty-equivalent returns

This table reports in percentage terms the monthly certainty-equivalent return for each portfolio strategy of Table 4 in each dataset of Table 5. It also includes in brackets the p-value that the variance of each strategy differs from that of the $1/N$ portfolio. The p-values are computed using the stationary bootstrap approach of Politis and Romano (1994) and the recommendation of Ledoit and Wolf (2008) with an expected block size of 5. The relative risk aversion coefficient is equal to 3.

	6FF	10Ind	16Int	25FF	30Ind	100S&P
Panel A: Strategies from the literature						
Mean	-4.1674	-8.0962	-17.1669	-82.5141	-127.1210	-226.0975
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
GMV	0.5715	0.2823	0.3179	0.5902	-0.0375	0.3174
	(0.00)	(0.09)	(0.01)	(0.00)	(0.39)	(0.96)
$1/N$	0.1373	0.1317	0.6031	0.1345	0.0729	0.3068
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
LW (id)	0.3774	0.2859	0.4965	0.4916	0.2118	0.0329
	(0.00)	(0.02)	(0.12)	(0.00)	(0.15)	(0.07)
LW (1f)	0.5276	0.3027	0.3669	0.5573	0.2293	0.1282
	(0.00)	(0.04)	(0.01)	(0.00)	(0.15)	(0.23)
MV (cons)	0.2004	0.2067	0.5135	0.1899	0.1724	0.1974
	(0.20)	(0.21)	(0.16)	(0.37)	(0.21)	(0.29)
MV (2-norm)	0.6992	0.4850	0.5472	0.6394	0.4855	0.3069
	(0.00)	(0.00)	(0.32)	(0.00)	(0.00)	(0.04)
TZ4	0.2375	-0.1231	0.2134	0.9623	-0.0831	-0.8371
	(0.79)	(0.02)	(0.07)	(0.01)	(0.17)	(0.00)
Panel B: Strategies developed in the paper						
Variance-based portfolios						
ICVAR$_I$	0.5343	0.2836	0.5021	0.4793	0.1334	0.2093
	(0.00)	(0.02)	(0.10)	(0.00)	(0.46)	(0.33)
ICVAR$_F$	0.5402	0.3226	0.3688	0.5198	0.2098	0.0243
	(0.00)	(0.03)	(0.02)	(0.00)	(0.25)	(0.11)
ICVAR$_{IF}$	0.4825	0.3292	0.4879	0.4443	0.2442	0.0827
	(0.00)	(0.01)	(0.10)	(0.00)	(0.06)	(0.07)
Return-based portfolios						
ICRET$_I$	0.6878	0.4704	0.4630	0.6586	0.3292	0.5581
	(0.00)	(0.00)	(0.05)	(0.00)	(0.01)	(0.03)
ICRET$_F$	0.4470	0.3480	0.3119	0.7606	0.2530	0.1170
	(0.01)	(0.01)	(0.00)	(0.00)	(0.16)	(0.30)
ICRET$_{IF}$	0.7025	0.5152	0.5216	0.9961	0.6128	0.3335
	(0.00)	(0.00)	(0.28)	(0.00)	(0.00)	(0.85)
Short-sale constrained portfolios						
ICVAR$_I$ (cons)	0.1981	0.2151	0.5606	0.1956	0.1723	0.1975
	(0.21)	(0.15)	(0.44)	(0.36)	(0.20)	(0.28)
ICRET$_I$ (cons)	0.3274	0.3049	0.5734	0.3672	0.3668	0.2917
	(0.00)	(0.00)	(0.47)	(0.00)	(0.00)	(0.85)

Table 9: Portfolio turnovers

This table reports the monthly turnover for each portfolio strategy of Table 4 and for each dataset of Table 5.

	6FF	10Ind	16Int	25FF	30Ind	100S&P
			Panel A: Strategies from the literature			
Mean	16.9883	7.7554	15.8747	336.7108	341.1199	568.6349
GMV	0.4082	0.3376	0.4345	1.8737	1.3702	1.9252
1/*N*	0.0163	0.0233	0.0329	0.0182	0.0289	0.0553
LW (id)	0.1346	0.1798	0.1942	0.4829	0.4442	0.3922
LW (1f)	0.3544	0.2568	0.2599	0.9287	0.4687	0.4389
MV (cons)	0.0828	0.0979	0.1231	0.1306	0.1268	0.2422
MV (2-norm)	1.7464	1.0726	1.2014	4.1989	2.4231	0.0553
TZ4	3.7526	0.6827	0.7270	5.0123	1.5983	2.7856
			Panel B: Strategies developed in the paper			
			Variance-based portfolios			
ICVAR$_I$	0.3480	0.2653	0.2669	1.2797	0.8550	1.0008
ICVAR$_F$	0.3950	0.2725	0.2879	1.3371	0.6503	0.8002
ICVAR$_{IF}$	0.3452	0.2435	0.2463	0.9270	0.5342	0.5548
			Return-based portfolios			
ICRET$_I$	2.0196	1.2040	1.3920	5.0754	3.1508	2.4612
ICRET$_F$	1.9160	0.9370	1.0899	4.5742	2.8014	2.8219
ICRET$_{IF}$	2.1959	1.2867	1.4260	4.4276	2.7769	2.1472
			Short-sale constrained portfolios			
ICVAR$_I$ **(cons)**	0.0883	0.0955	0.1227	0.1352	0.1323	0.2412
ICRET$_I$ **(cons)**	0.5995	0.6293	0.6703	0.7826	0.7513	0.9242