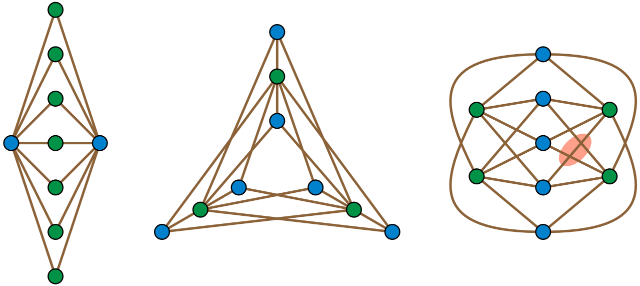
Parameterized Complexity of 1-Planarity

Michael J. Bannister, Sergio Cabello, and **David Eppstein**

Algorithms and Data Structures Symposium (WADS 2013)
London, Ontario, August 2013

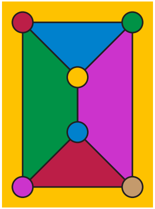
What is 1-planarity?

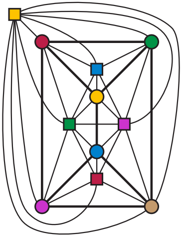
A graph is 1-planar if it can be drawn in the plane (vertices as points, edges as curves disjoint from non-incident vertices) so that each edge is crossed at most once (in one point, by one edge)



E.g. $K_{2,7}$ is planar, $K_{3,6}$ is 1-planar, and $K_{4,5}$ is not 1-planar
[Czap and Hudák 2012]

History and properties





Original application of 1-planarity: simultaneously coloring vertices and faces of planar maps [Ringel 1965]

1-planar graphs have:

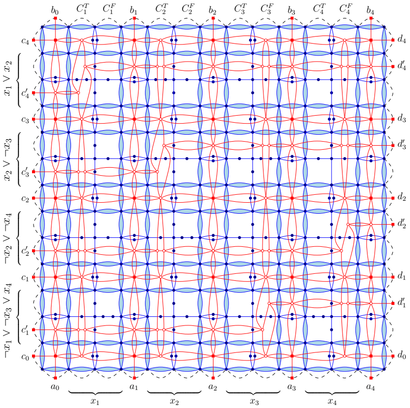
- At most $4n - 8$ edges [Schumacher 1986]
- At most $n - 2$ crossings [Czap and Hudák 2013]
- Chromatic number ≤ 6 [Borodin 1984]
- Sparse *shallow minors* [Nešetřil and Ossona de Mendez 2012]

Computational complexity of 1-planarity

NP-complete …
[Grigoriev and Bodlaender 2007; Korzhik and Mohar 2013]

even for planar + one edge
[Cabello and Mohar 2012]

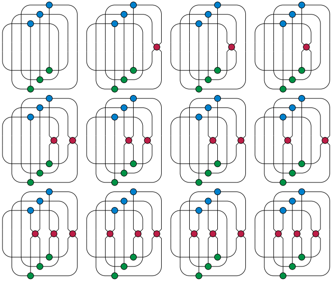
But that shouldn't stop us from seeking exponential or parameterized algorithms for instances of moderate size



Reduction from Cabello and Mohar [2012]

A naive exponential-time algorithm

1. Check that #edges $\leq 4n - 8$
2. For each pairing of edges
   - Replace each pair by $K_{1,4}$
   - Check if result is planar
   - If so, return success
3. If loop terminated normally, return failure



Time dominated by #pairings (*telephone numbers*)

$\approx m^{m/2 - o(m)}$ [Chowla et al. 1951]

E.g. the 9 edges of $K_{3,3}$ have 2620 pairings

Graphs with 18 edges have approximately a billion pairings

Parameterized complexity

NP-hard $\Rightarrow$ we expect time to be (at least) exponential

But exponential in what?

Maybe something smaller than instance size

Goals:

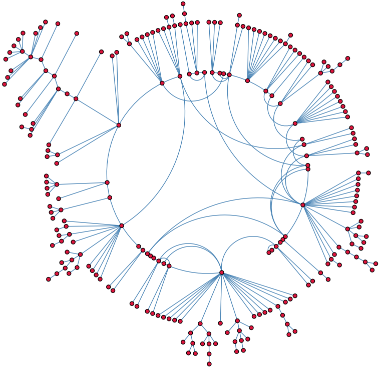
- Find a parameter p defined from inputs that is often small
- Find an algorithm with time $O(f(p)n^c)$
- f must be *computable* and c must be independent of p

If possible, then the problem is **fixed-parameter tractable**

Cyclomatic number

Remove a spanning tree, count remaining edges $\Rightarrow m - n + 1$

Often $\ll n$ for social networks (if closing cycles is rare) and utility networks (redundant links are expensive)



HIV transmission network
[Potterat et al. 2002]

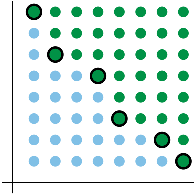
$n = 243$ cyclomatic$\# = 15$
[Bannister et al. 2013]

A hint of fixed-parameter tractability

For any fixed bound k on cyclomatic number,
all properties preserved when degree ≤ 2 vertices are suppressed
(e.g. non-1-planarity) can be tested in linear time

Proof idea:

- Delete degree-1 vertices
- Partition into paths of degree-2 vertices
- Find $O(k)$-tuple of path lengths
- Check vs $O(1)$ minimal forbidden tuples



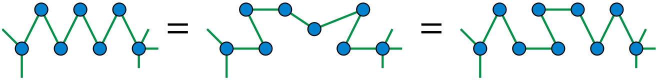
Every set of $O(1)$-tuples of positive integers has $O(1)$ minimal tuples [Dickson 1913]

But don't know how to find minimal tuples or construct drawing

Not FPT because dependence on k isn't explicit and computable

Kernelization

Suppose sufficiently long paths of degree-2 vertices – longer than some bound $\ell(k)$ – are indistinguishable with respect to 1-planarity



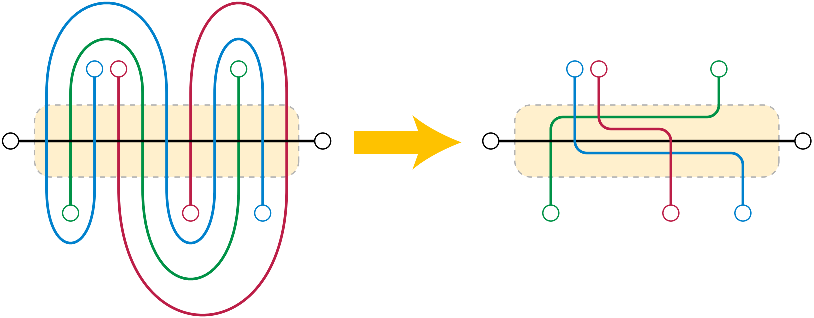
Leads to a simple algorithm:

- Delete degree-1 vertices
- Compress paths longer than $\ell(k)$ to length exactly $\ell(k)$, giving a kernel of size $O(k \cdot \ell(k))$
- Apply the naive algorithm to the resulting kernel
- Uncompress paths and restore deleted vertices, updating drawing to incorporate restored vertices

FPT: Running time $O(n + \text{naive}(\text{kernel size}))$

Rewiring

Suppose that path p is crossed by t other paths, each $\geq t$ times



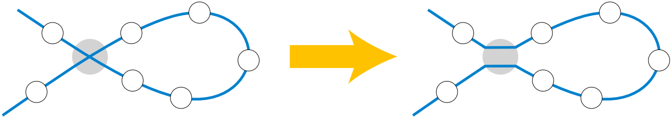
Then can reconnect near p, remove parts of paths elsewhere so:

- Each other path crosses p at most once
- Crossings on other paths do not increase

How long is a long path?

In a crossing-minimal 1-planar drawing, with q degree-two paths:

- No path crosses itself



- No path has $2(q-1)!$ or more crossings
  ...else we have a rewirable sequence of crossings

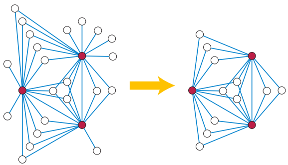
Path length longer than #crossings does not change 1-planarity

$q \leq 3k - 3 \quad \Rightarrow \quad \ell(k) \leq 2(3k-4)! - 1 \quad \Rightarrow \quad$ FPT

FPT algorithms for other parameters

- k-almost-tree number: max cyclomatic number of biconnected components
- Vertex cover number: min size of a vertex set that touches all edges

  "the *Drosophila* of fixed-parameter algorithmics" [Guo et al. 2005]
- Tree-depth: min depth of a tree such that every edge connects ancestor-descendant

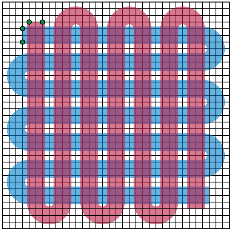


Kernelization for vertex cover

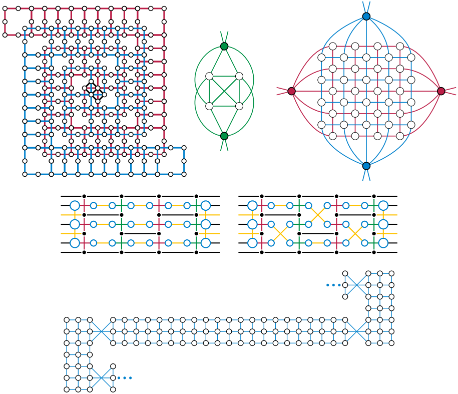
For vertex cover and tree-depth, *existence* of a finite set of forbidden subgraphs follows from known results [Nešetřil and Ossona de Mendez 2012]; difficulty is making dependence *explicit*

Negative results

NP-hard for graphs of bounded treewidth, pathwidth, or bandwidth



Reduction from satisfiability with three parts: substrate (black), variables (blue), and clauses (red)



Some of the gadgets

Conclusions

Results:

- First algorithmic investigation of 1-planarity
- Semi-practical exact exponential algorithm (18-20 edges)
- Impractical but explicit FPT algorithms
- Hardness results for other natural parameters

For future research:

- Make usable by reducing dependence on parameter
- Parameterize by feedback vertex set number?
  Would unify vertex cover and cyclomatic number
- Use similar kernelization for cyclomatic number / almost-trees in other graph drawing problems [Bannister et al. 2013]

References, I

Michael J. Bannister, David Eppstein, and Joseph A. Simons. Fixed parameter tractability of crossing minimization of almost-trees. In *Graph Drawing*, 2013. To appear.

O. V. Borodin. Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs. *Metody Diskret. Analiz.*, 41:12–26, 108, 1984.

Sergio Cabello and Bojan Mohar. Adding one edge to planar graphs makes crossing number and 1-planarity hard. Electronic preprint arxiv:1203.5944, 2012.

S. Chowla, I. N. Herstein, and W. K. Moore. On recursions connected with symmetric groups. I. *Canad. J. Math.*, 3:328–334, 1951. doi: 10.4153/CJM-1951-038-3.

Július Czap and Dávid Hudák. 1-planarity of complete multipartite graphs. *Disc. Appl. Math.*, 160(4-5):505–512, 2012. doi: 10.1016/j.dam.2011.11.014.

References, II

Július Czap and Dávid Hudák. On drawings and decompositions of 1-planar graphs. *Elect. J. Combin.*, 20(2):P54, 2013. URL http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i2p54.

L. E. Dickson. Finiteness of the odd perfect and primitive abundant numbers with *n* distinct prime factors. *Amer. J. Math.*, 35(4): 413–422, 1913. doi: 10.2307/2370405.

Alexander Grigoriev and Hans L. Bodlaender. Algorithms for graphs embeddable with few crossings per edge. *Algorithmica*, 49(1):1–11, 2007. doi: 10.1007/s00453-007-0010-x.

Jiong Guo, Rolf Niedermeier, and Sebastian Wernicke. Parameterized complexity of generalized vertex cover problems. In Frank Dehne, Alejandro López-Ortiz, and Jörg-Rüdiger Sack, editors, *9th International Workshop, WADS 2005, Waterloo, Canada, August 15-17, 2005, Proceedings*, volume 3608 of *Lecture Notes in Computer Science*, pages 36–48. Springer, 2005. doi: 10.1007/11534273_5.

References, III

Vladimir P. Korzhik and Bojan Mohar. Minimal Obstructions for 1-Immersions and Hardness of 1-Planarity Testing. *J. Graph Th.*, 72 (1):30–71, 2013. doi: 10.1002/jgt.21630.

Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity: Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and Combinatorics*. Springer, 2012. doi: 10.1007/978-3-642-27875-4.

J. J. Potterat, L. Phillips-Plummer, S. Q. Muth, R. B. Rothenberg, D. E. Woodhouse, T. S. Maldonado-Long, H. P. Zimmerman, and J. B. Muth. Risk network structure in the early epidemic phase of HIV transmission in Colorado Springs. *Sexually transmitted infections*, 78 Suppl 1:i159–63, April 2002. doi: 10.1136/sti.78.suppl_1.i159.

Gerhard Ringel. Ein Sechsfarbenproblem auf der Kugel. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 29: 107–117, 1965. doi: 10.1007/BF02996313.

H. Schumacher. Zur Struktur 1-planarer Graphen. *Mathematische Nachrichten*, 125:291–300, 1986.