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Parametric and Nonparametric Estimators in Fourier Series Semiparametric Regression and Their Characteristics

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Abstract

Consider data pairs $(x_{i1},..., x_{ir}, t_{i1},..., t_{ip}, y_i)$ involving in a semiparametric regression model $y_i = \mu(\underset{\sim}{x}_i', \underset{\sim}{t}_i) + \varepsilon_i$, where $\mu(\underset{\sim}{x}_i', \underset{\sim}{t}_i) = \underset{\sim}{x}_i'\underset{\sim}{\beta} + \sum_{j=1}^{p} g_j(t_{ji}),\ i = 1,\ldots,n;$

$j = 1,..., p$ is the semiparametric regression curve. Response variable y_i is assumed to be proportional to predictor variable $\underset{\sim}{x}_i' = (x_{i1},..., x_{ir})$, but at the same time, its relationship with other predictor variables $\underset{\sim}{t}_i = (t_{i1},..., t_{ip})$ is unidentified. The $\underset{\sim}{x}_i'\underset{\sim}{\beta}$ and $g_j(t_{ji})$ are, parametric and nonparametric components respectively. In this study, the nonparametric component is approximated by Fourier series which is expressed by

$$g_j(t_{ji}) = b_j t_{ji} + \frac{1}{2}a_{0j} + \sum_{k=1}^{K} a_{kj}\cos kt_{ji},\ \ j = 1, 2,..., p.\ \ i = 1, 2,..., n.$$

This report also introduces the mathematical expressions of parametric estimator $\hat{\underset{\sim}{\beta}}_\lambda$, nonparametric estimator $\hat{\underset{\sim}{g}}_\lambda$, estimator for semiparametric regression curve $\hat{\underset{\sim}{\mu}}_\lambda(\underset{\sim}{x}, \underset{\sim}{t})$, and their properties. The estimators are obtained from *Penalized Least Square* (PLS) optimization

$$\underset{\underset{\sim}{\beta}\in\Re^{r+1},\ \underset{\sim}{g}\in C(0,\pi)}{\mathrm{Min}} \left\{ n^{-1}\left\| \underset{\sim}{y} - \mathrm{X}\underset{\sim}{\beta} - \underset{\sim}{g} \right\|^2 + \sum_{j=1}^{p}\lambda_j \int_0^{\pi} \frac{2}{\pi}\left(g_j^{(2)}(t_j)\right)^2 dt_j \right\}.$$

The solution of the PLS approximation produces the estimators $\hat{\underset{\sim}{\beta}}_\lambda = \mathrm{W}(\lambda)\underset{\sim}{\mathrm{y}}$, $\hat{\underset{\sim}{g}}_\lambda = \mathrm{M}(\lambda)\underset{\sim}{\mathrm{y}}$ and $\hat{\underset{\sim}{\mu}}_\lambda(\underset{\sim}{x}, \underset{\sim}{t}) = \mathrm{N}(\lambda)\underset{\sim}{\mathrm{y}}$ for a matrices $\mathrm{W}(\lambda)$, $\mathrm{M}(\lambda)$, and $\mathrm{N}(\lambda)$, that are depending on refined parameter $\underset{\sim}{\lambda} = \left(\lambda_1 \quad ,\ldots, \quad \lambda_p\right)'$. While $\hat{\underset{\sim}{\beta}}_\lambda$, $\hat{\underset{\sim}{g}}_\lambda$, and $\hat{\underset{\sim}{\mu}}_\lambda(\underset{\sim}{x}, \underset{\sim}{t})$ are bias estimators, which are linear with respect to observation $\underset{\sim}{y}$.

Keywords: Fourier series, *Penalized Least Square* (PLS), Semiparametric Regression

1. Introduction

A regression model, in general, describes relationship between response and predictor variables. Suppose that y_i is response variable and t_i is predictor variable, then for n observations, the relationship of the variables can be expressed in a regression model $y_i = g(t_i) + \varepsilon_i,\ i = 1, 2, ..., n,$ where g is a regression function and ε_i is a random *error* having characteristics as identically, independently, and normally distributed with zero mean and variance σ^2. The regression curve g can be estimated by means of three approximations, namely parametric, nonparametric, and semiparametric regressions. The parametric regression model is applied when a strong assumption of the form of the functional relationship between the response and predictor is acquired. In this case, the regression curve estimation is equivalent with the estimation of the parameters within the model (Eubank [1]). The nonparametric model is best applied on the data for which the regression curve is still unrecognized, or there is incomplete knowledge regarding the form of the data (Kayri, et.al. [2]). To estimate the nonparametric regression curve, Amato, et. al. [3] employed Fourier series estimator. The Fourier series estimator was previously developed by Bilodeau [4] to solve the seasonal data.

In many cases, linear or periodical relationship between response and predictor are often found in a number of real-life problems, which in turn, requires the Fourier series semiparametric modelling. The incomplete information of the functional relationship between the response and predictor leads to the inappropriate regression by means of parametric or nonparametric models. Therefore, a semiparametric regression model which combines both the parametric and the nonparametric models is needed (Engle [5,16]).

The semiparametric regression has been widely developed in many forms of the nonparametric component. The semiparametric model with spline function has been applied by (You, at.al. [6], You, J, Chen [7] and Eubank [1]). Speckman [8], Hong [9], You, J, Chen [7] and Manzana [10] employed Kernel function in excuting the semiparametric model. Qu [11,18] and Taylor [12] used Wavelet function. Menawhile, You [13] and Qingguo [14,17] applied local polynomial to estimate the semiparametric regression curve. However, appliying Fourier series function as the nonparametric component in developing the semiparametric model has not been reported yet. Accordingly, this article proposes the estimators for parametric and nonparametric in Fourier series semiparametric regression as well as their characteristics.

2. Semiparametric Regression Model

The mathematical expression of the response and predictor variables in the semiparametric regression model is given by

$$y_i = \underset{\sim}{x}_i' \underset{\sim}{\beta} + \sum_{j=1}^{p} g_j(t_{ji}) + \varepsilon_i, \quad i = 1,\ 2, \ldots,\ n. \tag{1}$$

where ε_i is the random e*rror* which is assumed to be independent and identical with zero mean and variance σ^2. Term $\underset{\sim}{x}_i' \underset{\sim}{\beta}$ with $\underset{\sim}{\beta} = \left(\beta_0, \beta_1, \ldots, \beta_r\right)'$ is the parametric component and $\sum_{j=1}^{p} g_j(t_{ji})$ is the unknown nonparametric component and defined in a continous domain $C(0,\pi)$. The $g_j(t_{ji})$ is expressed by

$$g_j(t_{ji}) = b_j t_{ji} + \frac{1}{2} a_{0j} + \sum_{k=1}^{K} a_{kj} \cos kt_{ji}, \quad i = 1,\ 2, \ldots,\ n, \quad j = 1,\ 2, \ldots,\ p, \tag{2}$$

This expression is recognized as Fourier series function. Fourier series is a trigonometric polynomial function that has the properties of flexibility so that it can adapt effectively to the local nature of the data. Suitable Fourier series is used to describe the curve that shows the sine wave and cosine. The Fourier series estimator, is generally used when a data investigated patterns are not known and there is a recurring tendency (Bilodeau [3] and Tripena [15]).

Eq. (1) can be rewritten in matrix form as follows:

$$\underset{\sim}{y} = \mathrm{X}\underset{\sim}{\beta} + \underset{\sim}{g}(t) + \underset{\sim}{\varepsilon} \tag{3}$$

with $\underset{\sim}{g}(t) = \left(\sum_{j=1}^{p} g_j(t_{j1}), \sum_{j=1}^{p} g_j(t_{j2}), \ldots, \sum_{j=1}^{p} g_j(t_{jn}) \right)'$, $\underset{\sim}{y} = \left(y_1, y_2, \ldots, y_n\right)'$,

$\underset{\sim}{\varepsilon} = \left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\right)'$ and X is matric in which their elements are the predictor of the parametric component.

3. Estimation

The estimation of the parametric component $\underset{\sim}{\beta}$ and the nonparametric component $g_j(t_{ji})$, in (1), is solved using the PLS optimization, is given in the following theorems.

Theorem 1.
If the semiparametric regression model is given in Eq. (1) and the regression curve $g_j(t_{ji})$ is approximated by Eq. (2), then the estimator for the parametric $\hat{\underset{\sim}{\beta}}$ and nonparametric $\hat{\underset{\sim}{g}}_\lambda(t)$ will be obtained from PLS optimization

$$\underset{\underset{\sim}{\beta}\in\Re^{r+1},\ \underset{\sim}{g}\in C(0,\pi)}{\mathrm{Min}}\left\{n^{-1}\ \left\|\underset{\sim}{y}-\mathrm{X}\underset{\sim}{\beta}-\underset{\sim}{g}\right\|^2+\sum_{j=1}^{p}\lambda_j\int_0^\pi\frac{2}{\pi}\left(g_j^{(2)}(t_j)\right)^2dt_j\right\}.$$

where $\hat{\underset{\sim}{\beta}}=\left[\mathrm{X'H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X'H}(\lambda)\underset{\sim}{y}$ and $\hat{\underset{\sim}{g}}_\lambda(t)=S(\lambda)\left(\underset{\sim}{y}-\mathrm{X}\left[\mathrm{X'H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X'H}(\lambda)\underset{\sim}{\mathrm{y}}\right)$ and

$$\underset{\sim}{y}=(y_1,y_2,\ldots,y_n)',$$

$$\mathrm{H}(\lambda)=\left(\mathrm{I}-\mathrm{S}(\lambda)\right)'\left(\mathrm{I}-\mathrm{S}(\lambda)\right), \tag{4}$$

$$S(\lambda)=\mathrm{B}\left(\mathrm{B'B}+n\mathrm{D}(\lambda_1,\ldots,\lambda_p)\right)^{-1}\mathrm{B}', \tag{5}$$

$$\mathrm{B}=\left(\underset{\sim}{t}_1\quad \underset{\sim}{1}\quad \cos\underset{\sim}{t}_1\quad \cdots\quad \cos K\underset{\sim}{t}_1\quad \cdots\quad \underset{\sim}{t}_p\quad \cos\underset{\sim}{t}_p\quad \cdots\quad \cos K\underset{\sim}{t}_p\right), \tag{6}$$

$$\underset{\sim}{t}_j=\left(t_{j1}\quad t_{j2}\quad \ldots\quad t_{jn}\right)',\ j=1,2,\ldots,p;\ \underset{\sim}{1}=\left(1\quad 1\quad \ldots\quad 1\right)',$$

$$\cos\underset{\sim}{t}_j=\left(\cos t_{j1}\quad \cos t_{j2}\quad \ldots\quad \cos t_{jn}\right)'$$

$\mathrm{X}=(\underset{\sim}{x}_1',\underset{\sim}{x}_2',\ldots,\underset{\sim}{x}_n')$ where $\underset{\sim}{x}_i'=(1,x_{1i},x_{2i},\ldots,x_{ri})$, $i=1,\ 2,\ldots,\ n$.

Proof:
$\underset{\sim}{\beta}$ and $g_j(t_{ji}),\ j=1,2,\ldots,p$ in the semiparametric model, Eq. (1), can be estimated by taking the optimization of the PLS:

$$\underset{\substack{\underset{\sim}{\beta}\in\Re^{r+1},\ g_j\in C(0,\pi)\\ j=1,2,\ldots,p}}{\mathrm{Min}}\left[n^{-1}\sum_{i=1}^{n}\left(y_i-\underset{\sim}{x}_i'\underset{\sim}{\beta}-\sum_{j=1}^{p}g_j(t_{ji})\right)^2+\sum_{j=1}^{p}\lambda_j\int_0^\pi\frac{2}{\pi}\left(g_j^{(2)}(t_j)\right)^2dt_j\right],$$

Which can be rewritten as

$$\underset{\substack{\underset{\sim}{\beta}\in\Re^{r+1},\ g_j\in C(0,\pi)\\ j=1,2,\ldots,p}}{\mathrm{Min}}\left[R\left(\underset{\sim}{\beta},g_1,\ldots,g_p\right)+\sum_{j=1}^{p}\lambda_jP_j\left(g_j\right)\right],\ \lambda_j>0,\ j=1,2,\ldots,p, \tag{7}$$

where

$$R\left(\underset{\sim}{\beta},g_1,\ldots,g_p\right)=n^{-1}\sum_{i=1}^{n}\left(y_i-\underset{\sim}{x}_i'\underset{\sim}{\beta}-\sum_{j=1}^{p}g_j(t_{ji})\right)^2, \tag{8}$$

and

$$P_j(g_j)=\int_0^{\pi}\frac{2}{\pi}\left(g_j^{(2)}(t_j)\right)^2 dt_j. \tag{9}$$

To solve Eq. (7), *Goodness of fit* $R(\underset{\sim}{\beta}, g_1,...,g_p)$, and *Penalty* $P_j(g_j)$ must be derived for all components. The derivation of the *Goodness of fit* $R(\underset{\sim}{\beta}, g_1,...,g_p)$ is given as the following:

Suppose $U_i = y_i - \underset{\sim}{x}_i'\underset{\sim}{\beta}$, then $R(\underset{\sim}{\beta}, g_1,...,g_p)$ in Eq. (8) is given by:

$$R(\underset{\sim}{\beta},\ g_1,...,g_p) = n^{-1}\sum_{i=1}^{n}\left(U_i-\sum_{j=1}^{p}g_j(t_{ji})\right)^2.$$

Since $g_j(t_{ji})$ is a continous function, then $g_j(t_{ji})$ can be approximated by Eq. (2), so that Eq. (8) can be rearranged as

$$R(\underset{\sim}{\beta},\ g_1,...,g_p)=n^{-1}\sum_{i=1}^{n}\left(U_i-\sum_{j=1}^{p}\left(b_jt_{ji}+\frac{1}{2}a_{0j}+\sum_{k=1}^{K}a_{kj}\cos kt_{ji}\right)\right)^2. \tag{10}$$

Consider $\underset{\sim}{U}=\begin{pmatrix}U_1 & U_2 & \cdots & U_n\end{pmatrix}'$ and $\underset{\sim}{\delta}=\begin{pmatrix}\underset{\sim}{w}_1 & \underset{\sim}{w}_2 & \cdots & \underset{\sim}{w}_p\end{pmatrix}'$, where $\underset{\sim}{w}_j=\begin{pmatrix}b_j & \frac{1}{2}a_{0j} & a_{1j} & \cdots & a_{Kj}\end{pmatrix},\ j=1,2,...,p$ and B is given in Eq. (6), the *Goodness of fit* $R(\underset{\sim}{\beta}, g_1,...,g_p)$ in Eq. (8) can be then simplified as follows:

$$R(\underset{\sim}{\beta},\ \underset{\sim}{g}(t))=n^{-1}\left(\underset{\sim}{U}-\mathrm{B}\underset{\sim}{\delta}\right)'\left(\underset{\sim}{U}-\mathrm{B}\underset{\sim}{\delta}\right) \tag{11}$$

Eq. (2) can be used to evaluate the *penalty* $P_j(g_j)$ in Eq. (9)

$$\begin{aligned}P_j(g_j)&=\int_0^{\pi}\frac{2}{\pi}\left(\frac{d^2}{dt_j^2}\left\{b_jt_j+\tfrac{1}{2}a_{0j}+\sum_{k=1}^{K}a_{kj}\cos kt_j\right\}\right)^2 dt_j\\ &=\int_0^{\pi}\frac{2}{\pi}\left(\sum_{k=1}^{K}k^2a_{kj}\cos kt_j\right)^2 dt_j\\ &=\frac{2}{\pi}\int_0^{\pi}\left[\left(\sum_{k=1}^{K}k^2\ a_{kj}\cos kt_j\right)^2+2\sum_{k<m}^{K}\left(k^2\ a_{kj}\cos kt_j\right)\left(m^2a_{mj}\cos mt_j\right)\right]dt_j\end{aligned} \tag{12}$$

Concequently, Eq. (12) can be reexpressed as

$$P_j(g_j)=\sum_{k=1}^{K}k^4a_{kj}^2 \tag{13}$$

By taking into consideration the Eq. (13), the second term of the PLS in Eq. (7) becomes

$$\sum_{j=1}^{p}\lambda_j P_j(g_j)=\sum_{j=1}^{p}\lambda_j\left(\sum_{k=1}^{K}k^4a_{kj}^2\right)$$
$$=\lambda_1\left(\sum_{k=1}^{K}k^4a_{k1}^2\right)+\lambda_2\left(\sum_{k=1}^{K}k^4a_{k2}^2\right)+\cdots+\lambda_p\left(\sum_{k=1}^{K}k^4a_{kp}^2\right)$$
$$=\lambda_1\left(1^4a_{11}^2+2^4a_{21}^2+3^4a_{31}^2+\cdots+K^4a_{K1}^2\right)+$$
$$+\lambda_2\left(1^4a_{12}^2+2^4a_{22}^2+3^4a_{32}^2+\cdots+K^4a_{K2}^2\right)+\cdots$$
$$+\lambda_p\left(1^4a_{1p}^2+2^4a_{2p}^2+3^4a_{3p}^2+\cdots+K^4a_{Kp}^2\right) \tag{14}$$

Given

$$\underset{\sim}{\delta}=\begin{bmatrix}b_1 & \tfrac{1}{2}a_{01} & a_{11} & \cdots & a_{K1} & b_2 & \tfrac{1}{2}a_{01} & a_{12} & \cdots & a_{K2} & \cdots & b_p & \tfrac{1}{2}a_{0p} & a_{1p} & \cdots & a_{Kp}\end{bmatrix}'$$
$$\mathrm{D}(\lambda_1,\ldots,\lambda_p)=diag\left(\mathrm{D}_1(\lambda_1)\quad \mathrm{D}_2(\lambda_2)\quad\cdots\quad \mathrm{D}_p(\lambda_p)\right),$$
$$\mathrm{D}_j(\lambda_j)=\left(0\quad 0\quad \lambda_j1^4\quad \lambda_j2^4\quad\cdots\quad\lambda_jK^4\right),\ \ j=1,2,\ldots,p$$

Eq. (14) can be rearranged as

$$\sum_{j=1}^{p}\lambda_j P_j(g_j)=\underset{\sim}{\delta}'\ \mathrm{D}\ \underset{\sim}{\delta} \tag{15}$$

Combining *Goodness of fit* (11) dan *Penalty* (15), the optimaization of the (7) is given by:

$$\underset{\underset{\sim}{\delta}\in\Re^{(K+2)p}}{Min}\left\{n^{-1}\left(\underset{\sim}{U}-\mathrm{B}\underset{\sim}{\delta}\right)'\left(\underset{\sim}{U}-\mathrm{B}\underset{\sim}{\delta}\right)+\underset{\sim}{\delta}'D(\lambda_1,\ldots,\lambda_p)\underset{\sim}{\delta}\right\}$$
$$=\underset{\underset{\sim}{\delta}\in\Re^{(K+2)p}}{Min}\left\{n^{-1}\underset{\sim}{U}'\underset{\sim}{U}-2n^{-1}\underset{\sim}{\delta}'\mathrm{B}'\underset{\sim}{U}+\underset{\sim}{\delta}'\left(n^{-1}\mathrm{B}'\mathrm{B}+\mathrm{D}(\lambda_1,\ldots,\lambda_p)\right)\underset{\sim}{\delta}\right\}$$
$$=\underset{\underset{\sim}{\delta}\in\Re^{(K+2)p}}{Min}\left\{\mathrm{Q}(\underset{\sim}{\delta})\right\}.$$

Taking the partial derivative of $Q(\underset{\sim}{\delta})$ with respect to $\underset{\sim}{\delta}$ and taking its zero value, we obtain:

$$\frac{\partial Q(\underset{\sim}{\delta})}{\partial\underset{\sim}{\delta}}=\frac{\partial}{\partial\underset{\sim}{\delta}}\left(n^{-1}\underset{\sim}{U}'\underset{\sim}{U}-2n^{-1}\underset{\sim}{\delta}'\mathrm{B}'\underset{\sim}{U}+\underset{\sim}{\delta}'\left(n^{-1}\mathrm{B}'\mathrm{B}+\mathrm{D}(\lambda_1,\ldots,\lambda_p)\right)\underset{\sim}{\delta}\right)$$
$$0=-2n^{-1}\mathrm{B}'\underset{\sim}{U}+2\left(n^{-1}\mathrm{B}'\mathrm{B}+\mathrm{D}(\lambda_1,\ldots,\lambda_p)\right)\underset{\sim}{\delta}$$
$$-2\left(n^{-1}\mathrm{B}'\mathrm{B}+\mathrm{D}(\lambda_1,\ldots,\lambda_p)\right)\underset{\sim}{\delta}=-2n^{-1}\mathrm{B}'\underset{\sim}{U}$$
$$\hat{\underset{\sim}{\delta}}=\left(n^{-1}\mathrm{B}'\mathrm{B}+\mathrm{D}(\lambda_1,\ldots,\lambda_p)\right)^{-1}n^{-1}\mathrm{B}'\underset{\sim}{U}$$

$$=\left(\mathrm{B}'\mathrm{B}+n\mathrm{D}(\lambda_1,\dots,\lambda_p)\right)^{-1}\mathrm{B}'\underset{\sim}{U}$$

where

$$\underset{\sim}{\hat{\delta}}=\left[\hat{b}_1 \quad \tfrac{1}{2}\hat{a}_{01} \quad \hat{a}_{11} \quad \cdots \quad \hat{a}_{K1} \quad \hat{b}_2 \quad \tfrac{1}{2}\hat{a}_{01} \quad \hat{a}_{12} \quad \cdots \quad \hat{a}_{K2} \quad \cdots \quad \hat{b}_p \quad \tfrac{1}{2}\hat{a}_{0p} \quad \hat{a}_{1p} \quad \cdots \quad \hat{a}_{Kp}\right]'.$$

From Eq. (11), it is found that $\underset{\sim}{g}(t)=\mathrm{B}\underset{\sim}{\delta}.$ Hence

$$\begin{aligned}\underset{\sim}{\hat{g}}_{\lambda}(t)&=\mathrm{B}\underset{\sim}{\hat{\delta}}(\lambda).\\ &=\mathrm{B}\left(\mathrm{B}'\mathrm{B}+n\mathrm{D}(\lambda_1,\dots,\lambda_p)\right)^{-1}\mathrm{B}'\underset{\sim}{U}\\ &=\mathrm{S}(\lambda)\underset{\sim}{U}\\ &=S(\lambda)\left(\underset{\sim}{\mathrm{y}}-\mathrm{X}\underset{\sim}{\hat{\beta}}\right).\end{aligned} \tag{16}$$

Where $\mathrm{S}(\lambda)=\mathrm{B}\left(\mathrm{B}'\mathrm{B}+n\mathrm{D}(\lambda_1,\dots,\lambda_p)\right)\mathrm{B}'.$

Furthermore, the estimation of the parameter $\underset{\sim}{\beta}$ is evaluated from least square method by minimizing the error squared as follows

$$\begin{aligned}Q(\underset{\sim}{\beta})&=\sum_{i=1}^{n}\varepsilon_i^2=\sum_{i=1}^{n}\left(y_i-\underset{\sim}{x}_i'\underset{\sim}{\beta}-\sum_{j=1}^{p}\hat{g}_j(t_{ji})\right)^2\\ &=\left[y_i-\underset{\sim}{x}_i'\underset{\sim}{\beta}-\underset{\sim}{\hat{g}}_{\lambda}(t)\right]'\left[y_i-\underset{\sim}{x}_i'\underset{\sim}{\beta}-\underset{\sim}{\hat{g}}_{\lambda}(t)\right]\\ &=\left[\underset{\sim}{y}-\mathrm{X}\underset{\sim}{\beta}-\left(\mathrm{S}(\lambda)(\underset{\sim}{y}-\mathrm{X}\underset{\sim}{\beta})\right)\right]'\left[\underset{\sim}{y}-\mathrm{X}\underset{\sim}{\beta}-\left(\mathrm{S}(\lambda)(\underset{\sim}{y}-\mathrm{X}\underset{\sim}{\beta})\right)\right]\end{aligned}$$

Deriving $Q(\underset{\sim}{\beta})$ with respect to $\underset{\sim}{\beta}$ and taking its zero, we get

$$\begin{aligned}\frac{\partial Q(\underset{\sim}{\beta})}{\partial\underset{\sim}{\beta}}=\frac{\partial}{\partial\underset{\sim}{\beta}}&\Big(\underset{\sim}{y}'\left(\mathrm{I}-S(\lambda)\right)'\left(\mathrm{I}-S(\lambda)\right)\underset{\sim}{y}-2\underset{\sim}{\beta}'\mathrm{X}'\left(\mathrm{I}-S(\lambda)\right)'\left(\mathrm{I}-S(\lambda)\right)\underset{\sim}{\mathrm{y}}+\\ &+\underset{\sim}{\beta}'\mathrm{X}'\left(\mathrm{I}-S(\lambda)\right)'\left(\mathrm{I}-S(\lambda)\right)\mathrm{X}\underset{\sim}{\beta}\Big)\end{aligned}$$

$$0=-2\mathrm{X}'\left(\mathrm{I}-S(\lambda)\right)'\left(\mathrm{I}-S(\lambda)\right)\underset{\sim}{\mathrm{y}}+2\mathrm{X}'\left(\mathrm{I}-S(\lambda)\right)'\left(\mathrm{I}-S(\lambda)\right)\mathrm{X}\underset{\sim}{\beta}$$

$$-2\mathrm{X}'\left(\mathrm{I}-S(\lambda)\right)'\left(\mathrm{I}-S(\lambda)\right)\mathrm{X}\underset{\sim}{\beta}=-2\mathrm{X}'\left(\mathrm{I}-S(\lambda)\right)'\left(\mathrm{I}-S(\lambda)\right)\underset{\sim}{\mathrm{y}}.$$

Estimator for $\underset{\sim}{\beta}$ is then defined as

$$\underset{\sim}{\hat{\beta}}=\left(\mathrm{X}'\left(\mathrm{I}-\mathrm{S}(\lambda)\right)'\left(\mathrm{I}-\mathrm{S}(\lambda)\right)\mathrm{X}\right)^{-1}\mathrm{X}'\left(\mathrm{I}-\mathrm{S}(\lambda)\right)'\left(\mathrm{I}-\mathrm{S}(\lambda)\right)\underset{\sim}{y}.$$

Finally, the estimator of Fourier series regression curve for the parametric component $\hat{\underset{\sim}{\beta}}$ is given by:

$$\begin{aligned}\hat{\underset{\sim}{\beta}} &= \left[\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X}'\mathrm{H}(\lambda)\underset{\sim}{y} \\ &= \mathrm{W}(\underset{\sim}{\lambda})\underset{\sim}{y},\end{aligned} \tag{17}$$

where $\mathrm{W}(\underset{\sim}{\lambda}) = \left[\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X}'\mathrm{H}(\lambda)$ and $\mathrm{H}(\lambda) = \left(\mathrm{I} - \mathrm{S}(\lambda)\right)'\left(\mathrm{I} - \mathrm{S}(\lambda)\right)$.

Substituting Eq. (17) into Eq. (16) leads the estimator for the nonparametric componen, explicitely:

$$\begin{aligned}\hat{\underset{\sim}{g}}_{\lambda}(t) &= S(\lambda)\left(\underset{\sim}{y} - \mathrm{X}\left[\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X}'\mathrm{H}(\lambda)\underset{\sim}{\mathrm{y}}\right) \\ &= \mathrm{M}(\lambda)\underset{\sim}{y},\end{aligned} \tag{18}$$

where $\mathrm{M}(\lambda) = S(\lambda)\left(\mathrm{I} - \mathrm{X}\left[\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X}'\mathrm{H}(\lambda)\right)$. ■

It implies that the estimator $\hat{\underset{\sim}{g}}_{\lambda}(t)$ is linear in observation $\underset{\sim}{\mathrm{y}}$.

Implication:

If $\mu(\underset{\sim}{x}, \underset{\sim}{t}) = \underset{\sim}{x}_i'\underset{\sim}{\beta} + \sum_{j=1}^{p} g_j(t_{ji}) + \varepsilon_i$ is a semiparametric function, then the regression estimator $\hat{\mu}_{\lambda}(\underset{\sim}{x}, \underset{\sim}{t})$ is linear in observation $\underset{\sim}{\mathrm{y}}$.

Proof:

Using Eq. (17) and (18), we obtain estimator for semiparametric regression model, namely

$$\begin{aligned}\hat{\mu}_{\lambda}(\underset{\sim}{x}, \underset{\sim}{t}) &= \mathrm{X}\hat{\underset{\sim}{\beta}} + \hat{\underset{\sim}{g}}_{\lambda}(t) \\ &= \mathrm{X}\left[\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X}'\mathrm{H}(\lambda)\underset{\sim}{\mathrm{y}} + \mathrm{M}(\lambda)\underset{\sim}{\mathrm{y}} \\ &= \left(\mathrm{X}\left[\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X}'\mathrm{H}(\lambda) + \mathrm{M}(\lambda)\right)\underset{\sim}{\mathrm{y}} \\ &= \mathrm{N}(\lambda)\underset{\sim}{\mathrm{y}},\end{aligned} \tag{19}$$

where $\mathrm{N}(\lambda) = \mathrm{X}\left[\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X}'\mathrm{H}(\lambda) + \mathrm{M}(\lambda)$. This mathematical expression reveals that the regression estimator $\hat{\mu}_{\lambda}(\underset{\sim}{x}, \underset{\sim}{t})$ is linear to observation $\underset{\sim}{\mathrm{y}}$.

The estimator $\hat{\underset{\sim}{\beta}}$ and $\hat{\underset{\sim}{g}}_{\lambda}(t)$ strongly depends on the selection of refined parameters λ_j. The λ_j, $j = 1, 2, ..., p$ is refined the parameters controlling the *goodness of fit* and the function smoothness. The enormous number of λ_j yields on very smooth regression curve and a very small value of λ_j, producing rough regression curve. Therefore, it is recommended to choose the optimum λ_j in order to acquire the best model for the estimation regression. One method in choosing the optimum refined parameter in Fourier series estimator is *Generalized Cross Validation* (GCV) method.

In addition, some lemmas are introduces to evaluate the characteristics of the Fourier representation ($\hat{\underset{\sim}{\beta}}$, $\hat{\underset{\sim}{g}}_\lambda(t)$ and $\hat{\mu}_\lambda(\underset{\sim}{x},\underset{\sim}{t})$).

Lemma 3:

If $\hat{\underset{\sim}{\beta}}$, $\hat{\underset{\sim}{g}}_\lambda(t)$ and $\hat{\mu}_\lambda(\underset{\sim}{x},\underset{\sim}{t})$ are, respectively, given in Eq. (17), (18) and (19), then the bias estimators $\underset{\sim}{\beta}$, $\underset{\sim}{g}(t)$ and $\mu(\underset{\sim}{x},\underset{\sim}{t})$ can be obtained.

Proof:

By taking the expectation value of Eq. (17), (18) and (19), it yields

$$\begin{aligned}
\mathrm{E}(\hat{\underset{\sim}{\beta}}) &= \mathrm{E}\left[\left(\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right)^{-1}\mathrm{X}'\mathrm{H}(\lambda)\underset{\sim}{y}\right]\\
&= \left(\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right)^{-1}\mathrm{X}'\mathrm{H}(\lambda)\,\mathrm{E}(\underset{\sim}{y})\\
&= \left(\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right)^{-1}\mathrm{X}'\mathrm{H}(\lambda)\left(\mathrm{E}\left(\mathrm{X}\underset{\sim}{\beta}+\underset{\sim}{g}(\mathrm{t})+\underset{\sim}{\varepsilon}\right)\right)\\
&= \underset{\sim}{\beta}+\left(\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right)^{-1}\mathrm{X}'\mathrm{H}(\lambda)\,\underset{\sim}{g}(t)\\
&\neq \underset{\sim}{\beta},
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(\hat{\underset{\sim}{g}}_\lambda(t)) &= \mathrm{E}(\mathrm{M}(\lambda)\underset{\sim}{y})\\
&= \mathrm{M}(y)\mathrm{E}(\underset{\sim}{y})\\
&= \mathrm{M}(y)\left(\mathrm{E}\left(\mathrm{X}\underset{\sim}{\beta}+\underset{\sim}{g}(\mathrm{t})+\underset{\sim}{\varepsilon}\right)\right)\\
&= \mathrm{M}(\lambda)\left(\mathrm{X}\underset{\sim}{\beta}+\underset{\sim}{g}(t)\right)\\
&\neq \underset{\sim}{g}(t).
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left(\hat{\mu}_\lambda(\underset{\sim}{x},\underset{\sim}{t})\right) &= \mathrm{E}\left(\mathrm{X}\hat{\underset{\sim}{\beta}}+\hat{\underset{\sim}{g}}_\lambda(t)\right)\\
&= \mathrm{E}\left(\mathrm{X}\hat{\underset{\sim}{\beta}}\right)+\mathrm{E}\left(\hat{\underset{\sim}{g}}_\lambda(t)\right)\\
&= \mathrm{XE}\left(\hat{\underset{\sim}{\beta}}+\left(\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right)^{-1}\mathrm{X}'\mathrm{H}(\lambda)\underset{\sim}{g}(t)+\mathrm{M}(\lambda)\left(\mathrm{X}\underset{\sim}{\beta}+\underset{\sim}{g}(t)\right)\right)\\
&= \mathrm{X}\underset{\sim}{\beta}+\mathrm{I}\underset{\sim}{g}(t)+\mathrm{M}(\lambda)\mathrm{X}\underset{\sim}{\beta}+\mathrm{M}(\lambda)\underset{\sim}{g}(t)
\end{aligned}$$

$$= \left(\mathrm{I} + \mathrm{M}(\lambda)\right)\mathrm{X}\underset{\sim}{\beta} + \left(\mathrm{I} + \mathrm{M}(\lambda)\right)\underset{\sim}{g}(t)$$
$$\neq \mu(\underset{\sim}{x}, \underset{\sim}{t}).$$

Since $\mathrm{E}(\hat{\underset{\sim}{\beta}}) \neq \underset{\sim}{\beta}$, $\mathrm{E}(\hat{\underset{\sim}{g}}_\lambda(t)) \neq \underset{\sim}{g}(t)$ and $\mathrm{E}\left(\hat{\mu}_\lambda(\underset{\sim}{x}, \underset{\sim}{t})\right) \neq \mu(\underset{\sim}{x}, \underset{\sim}{t})$, then $\underset{\sim}{\beta}$, $\underset{\sim}{g}(t)$ and $\mu(\underset{\sim}{x}, \underset{\sim}{t})$ are bias estimators. ■

4. Conclusion

1. In accordance with the semiparametric regression model as depicted in Eq. (1) with nonparametric compponent as given in Eq. (2), we obtain $\hat{\underset{\sim}{\beta}} = \left[\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X}'\mathrm{H}(\lambda)\underset{\sim}{y}$, $\hat{\underset{\sim}{g}}_\lambda(t) = S(\lambda)\left(\mathrm{I} - \mathrm{X}\left[\mathrm{X}'\mathrm{H}(\lambda)\mathrm{X}\right]^{-1}\mathrm{X}'\mathrm{H}(\lambda)\underset{\sim}{\mathrm{y}}\right)$, and $\hat{\mu}_\lambda(\underset{\sim}{x}, \underset{\sim}{t}) = \mathrm{X}\hat{\underset{\sim}{\beta}} + \hat{\underset{\sim}{g}}_\lambda(t)$ as the parametric, nonparametric, and semiparametric estimators, respectively.
2. The Fourier series estimator in the semiparametric regression is a bias estimator and linear in observation y.

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