Parametric characterization of general partially coherent beams propagating through *ABCD* optical systems

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Within the formalism of the Wigner distribution function, a new parameter is proposed, which characterizes arbitrary tridimensional partially coherent beams and is invariant through *ABCD* optical systems. The relationship between such a parameter and the bidimensional concept of beam quality is analyzed. An absolute lower bound that the new parameter can reach is also shown.

1. INTRODUCTION

The characterization of arbitrary laser beams and the establishment of an adequate criterion of quality constitute fundamental aspects in the design of laser devices. Thus the use of high-power laser beams in the treatment of materials during industrial processing requires high optical powers along with good focusing possibilities, the latter being closely related to the properties of far-field angular divergence. In this sense, it is well known[1] that the inhomogeneities existing in the gas flow can severely degrade the beam quality. In other applications, the elliptical intensity distribution at the output of a semiconductor laser[2] needs to be reshaped. These facts, among others, raise the problem of finding a sufficiently general characterization of laser beams that satisfies the following properties:

(1) It must be valid for arbitrary profile beams.
(2) There is a possibility of explicit analytical or numerical calculation.
(3) It is directly measurable.

A fourth property may be added to this list, referring to the applicability of the *ABCD* matrix formalism for nondiffractional optical systems to such a characterization, which would permit simple operation features, including the corresponding propagation laws.

Another aspect to be considered is the underlying restriction in almost all published papers that try to characterize the beam behavior from one unique parameter, instead of from the establishment of a family of merit figures or characteristic factors permitting greater versatility and laser-beam information capacity.

Alternative definitions exist for quality factors following different criteria (see, for example, those shown in Refs. 1 and 3–8). However, only recently[3] has there appeared a characterization of the quality of partially coherent beams that are essentially bidimensional, based on averages of the Wigner distribution function, and satisfy the requirements mentioned. Such a definition has important advantages when *ABCD* systems are used. However, as will be shown in Section 3, the natural extension of the previous definition to the tridimensional case with nonorthogonal optical systems and beams not reducible to cylindrical symmetry cannot be carried out because the said quality parameter is not invariant as it propagates through this type of system.

In this paper we will show a new characteristic parameter of the beam J, valid for arbitrary beams and invariant through any *ABCD* system. The lower bound that the parameter J can reach is determined. We will also characterize the behavior and propagation of laser beams through the concepts of beam waist and principal axes, localizing their positions analytically.

2. BIDIMENSIONAL FORMALISM

As is well known, the function describing the second-order properties of a general beam is the so-called cross-spectral density function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \omega)$, where ω denotes the angular frequency considered and $\mathbf{r}_1$ and $\mathbf{r}_2$ indicate the position vectors.

Restricting ourselves to the bidimensional case, we can define the function Γ of a partially coherent beam as

$$\Gamma(x, s, z) = \langle V(x + s/2, z)V^*(x - s/2, z)\rangle, \tag{2.1}$$

where V represents the optical field amplitude, the angle brackets denote here an ensemble average, and

$$x = (x_1 + x_2)/2, \qquad s = x_1 - x_2. \tag{2.2}$$

In Eq. (2.1) the dependence on ω was not explicitly expressed, since we will assume the fields to be quasi-monochromatic. The coordinate z denotes the direction of the beam propagation and x the transverse variable.

The Wigner distribution function associated with Γ is defined as

$$h(x, u, z) \propto \int_{-\infty}^{+\infty} \exp(-ikus)\Gamma(x, s, z)\mathrm{d}s, \tag{2.3}$$

where $\propto$ indicates proportionality and $ku = k_x$ is the wave-vector component along x. Hence u represents an angle of propagation (without taking the evanescent waves into account).

In expression (2.3) $h(x, u, z)$ has been defined except for the proportionality constant. In our case we will take the

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following normalization:

$$\iint_{-\infty}^{+\infty} h(x,u,z)\mathrm{d}x\mathrm{d}u = 1, \tag{2.4}$$

where division by the total (nonzero) beam power is implicit. First- and second-order averages of the Wigner distribution function can be defined as

$$\langle x\rangle \equiv \iint_{-\infty}^{+\infty} xh(x,u,z)\mathrm{d}x\mathrm{d}u, \tag{2.5}$$

$$\langle x^2\rangle \equiv \iint_{-\infty}^{+\infty} x^2h(x,u,z)\mathrm{d}x\mathrm{d}u, \tag{2.6}$$

$$\langle u^2\rangle \equiv \iint_{-\infty}^{+\infty} u^2h(x,u,z)\mathrm{d}x\mathrm{d}u, \tag{2.7}$$

etc. Note that, when the Wigner distribution function is a positive function, then the above parameters are the moments of such a distribution.

For the sake of simplicity, in what follows it will be assumed that $\langle x\rangle = \langle u\rangle = 0$. This is not a restriction, since it is simply equivalent to a shift of the coordinate system.[10]

The parameters $\langle x^2\rangle$ and $\langle u^2\rangle$ provide the squared transverse width and the squared far-field divergence of an arbitrary beam, respectively. Lavi *et al.*[3] defined the quality parameter of a beam as a quantity proportional to the product $\langle x^2\rangle\langle u^2\rangle$ of such squared widths in the waist plane, this plane being understood as that in which $\langle x^2\rangle$ reaches the minimum value under free propagation.

It can be easily shown that this definition of the quality parameter is equivalent, up to a constant, to the invariant parameter (for *ABCD* systems)

$$Q = \langle x^2\rangle\langle u^2\rangle - \langle xu\rangle^2, \tag{2.8}$$

which can be defined for every z, not only for the waist plane.[11] This quality parameter satisfies an uncertainty principle[3,10]

$$Q \geq 1/4k^2, \qquad k = 2\pi/\lambda, \tag{2.9}$$

where the equality holds only for Gaussian beams.

The importance of the previous definition on the one hand lies in the clear physical sense of the spatial and angular widths of the beam, expressed as a function of its coherent properties (that is, of its statistical properties), and on the other hand depends on the invariance of the quality parameter under propagation through *ABCD* systems.

However, the previous definition has one serious inconvenience when it is extended to arbitrary beams and optical systems: The natural generalization of the quality parameter is not valid for realistic tridimensional beams (generally asymmetrical) and optical systems that do not have cylindrical symmetry, since, in such cases, this parameter would not be invariable under propagation. This point will be analyzed in Section 3.

Finally, before going on to consider a tridimensional situation, we note a property of bidimensional beams: Every bidimensional beam has only one unique waist (real or virtual) when it propagates in free space. This property comes from the fact that in free propagation $\langle x^2\rangle$ evolves as a positive parabola,[11] with only one minimum.

3. EXTENSION TO TRIDIMENSIONAL OPTICAL SYSTEMS

In what follows we are going to use the tridimensional Wigner distribution function $h(x,y,u,v,z)$, defined as the Fourier transform in the bidimensional variable $\mathbf{s}$ of the cross-spectral density of the beam. Variables $\mathbf{s}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are now bidimensional vectors transverse to the z axis, which we keep as the propagation direction.

To handle tridimensional beams propagating through *ABCD* systems, it is useful to define the beam matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} \mathbf{W}^2 & \mathbf{\Psi} \\ \mathbf{\Psi}^t & \mathbf{\Phi}^2 \end{bmatrix} = \begin{bmatrix} \langle x^2\rangle & \langle xy\rangle & \langle xu\rangle & \langle xv\rangle \\ \langle xy\rangle & \langle y^2\rangle & \langle yu\rangle & \langle yv\rangle \\ \langle xu\rangle & \langle yu\rangle & \langle u^2\rangle & \langle uv\rangle \\ \langle xv\rangle & \langle yv\rangle & \langle uv\rangle & \langle v^2\rangle \end{bmatrix}, \tag{3.1}$$

where the 2×2 matrices $\mathbf{W}^2$, $\mathbf{\Psi}$, and $\mathbf{\Phi}^2$ are

$$\mathbf{W}^2 = \begin{bmatrix} \langle x^2\rangle & \langle xy\rangle \\ \langle xy\rangle & \langle y^2\rangle \end{bmatrix}, \qquad \mathbf{\Psi} = \begin{bmatrix} \langle xu\rangle & \langle xv\rangle \\ \langle yu\rangle & \langle yv\rangle \end{bmatrix},$$
$$\mathbf{\Phi}^2 = \begin{bmatrix} \langle u^2\rangle & \langle uv\rangle \\ \langle uv\rangle & \langle v^2\rangle \end{bmatrix}, \tag{3.2}$$

and the symbol t denotes the transposed matrix.

Then, the $\mathbf{P}'$ beam matrix at the output of an arbitrary *ABCD* optical system, characterized by a 4×4 matrix $\mathbf{M}$, is given in terms of the matrix $\mathbf{P}$ at the entrance of the system through the equation[10]

$$\mathbf{P}' = \mathbf{MPM}^t. \tag{3.3}$$

It is important to point out that the beam matrix is a positive definite matrix.[10]

With this formalism, the definition of quality considered in Section 2 can be extended in a natural way to tridimensional systems as follows:

$$\begin{aligned} Q_{3D} &= \langle \mathbf{x}^2\rangle\langle \mathbf{u}^2\rangle - \langle \mathbf{xu}\rangle^2 \\ &= (\langle x^2\rangle + \langle y^2\rangle)(\langle u^2\rangle + \langle v^2\rangle) - (\langle xu\rangle + \langle yv\rangle)^2. \end{aligned} \tag{3.4}$$

We can find a lower limit for Q_{3D}. To do this, we are going to propagate a general beam through *ABCD* systems for which Q_{3D} is invariant.

One of these *ABCD* systems is the axis rotation, as the terms $\langle \mathbf{x}^2\rangle$, $\langle \mathbf{u}^2\rangle$, and $\langle \mathbf{xu}\rangle$ appearing in Eq. (3.4) are rotationally invariant. This constitutes an *ABCD* system,[6] whose 4×4 matrix $\mathbf{G}_\beta$ is

$$\mathbf{G}_\beta = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \\ 0 & 0 & -\sin\beta & \cos\beta \end{bmatrix}. \tag{3.5}$$

We can simplify Q_{3D} by applying a rotation β given by

$$\cot 2\beta = 2\frac{\langle uv\rangle}{\langle v^2\rangle - \langle u^2\rangle}, \tag{3.6}$$

which gives

$$\langle u^2\rangle_\beta = \langle v^2\rangle_\beta, \tag{3.7}$$

where subscript β denotes the values after rotation.

Q_{3D} is also preserved by free propagation along a distance z. The matrix $\mathbf{M}_z$ for this system is

$$\mathbf{M}_z = \begin{bmatrix} 1 & 0 & z & 0 \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.8}$$

Applying $\mathbf{M}_z$ after $\mathbf{G}_\beta$, we have

$$\langle xu\rangle_{\beta z} = \langle u^2\rangle_\beta z + \langle xu\rangle_\beta, \tag{3.9}$$

without changing the values of $\langle u^2\rangle_\beta$ and $\langle v^2\rangle_\beta$. In Eq. (3.9) $\langle xu\rangle_{\beta z}$ denotes the value of $\langle xu\rangle_\beta$ at plane z. Then choosing

$$z = -\frac{\langle xu\rangle_\beta}{\langle u^2\rangle_\beta}, \tag{3.10}$$

we eliminate $\langle xu\rangle_{\beta z}$.

Since $Q_{3D\beta z} = Q_{3D}$, we have

$$\begin{aligned} Q_{3D} = Q_{3D\beta z} &= \langle x^2\rangle_{\beta z}\langle u^2\rangle_{\beta z} + \langle x^2\rangle_{\beta z}\langle v^2\rangle_{\beta z} \\ &\quad + (\langle y^2\rangle_{\beta z}\langle v^2\rangle_{\beta z} - \langle yv\rangle_{\beta z}{}^2) + \langle y^2\rangle_{\beta z}\langle u^2\rangle_{\beta z}, \end{aligned} \tag{3.11}$$

and, using the properties of the bidimensional beam-quality parameters on x and y axes, that is,

$$Q_x = \langle x^2\rangle\langle u^2\rangle - \langle xu\rangle^2 \geq 1/4k^2, \tag{3.12a}$$

$$Q_y = \langle y^2\rangle\langle v^2\rangle - \langle yv\rangle^2 \geq 1/4k^2, \tag{3.12b}$$

we obtain a lower limit for Q_{3D}:

$$Q_{3D} = Q_x + Q_x + Q_y + Q_y + \langle yv\rangle_{\beta z}{}^2 \geq 1/k^2. \tag{3.13}$$

This limit can be reached, for example, by cylindrical Gaussian beams, for which

$$Q_{3D} = 4Q_x = 4Q_y = 1/k^2. \tag{3.14}$$

If we restrict ourselves to optical systems with cylindrical symmetry, it can be readily shown that the Q_{3D} parameter defined in Eq. (3.4) does not change when the beam propagates through the system, even for those beams without cylindrical symmetry and with nonzero crossed terms $\langle xy\rangle$, $\langle uv\rangle$, $\langle xv\rangle$, $\langle yu\rangle$. But, although Q_{3D} is then adequate for beams propagating through many usual optical systems, such as those made of nonastigmatic lenses or mirrors combined with sections of free propagation, its applicability is not general. In fact, even for noncylindrical but orthogonal optical systems (for example, lenses with different focal lengths following two orthogonal axes), it can be shown that Q_{3D} does not remain invariant.

Therefore, for noncylindrical systems (whether they are orthogonal or not), the interpretation and utilization of Q_{3D} as the characterizing parameter of the laser-beam quality does not make sense. It is thus necessary to establish, in general terms, an adequate tridimensional formalism that permits arbitrary beam characterization and defines an invariant parameter under general propagation.

It is interesting to point out, however, that parameters Q_x and Q_y remain invariant through orthogonal *ABCD* system but not through nonorthogonal *ABCD* systems.

4. TRIDIMENSIONAL FORMALISM

In what follows, some parameters are introduced that will be used in the remainder of this section.

A. Principal Axes and Waist of Arbitrary Tridimensional Beams

Given a generic beam propagating along z, we define the principal axes for each z as those for which there is no crossed x–y term:

$$\langle xy\rangle = 0. \tag{4.1}$$

Alternatively, another set of principal axes for the beam can be defined as those for which the crossed u–v term vanishes:

$$\langle uv\rangle = 0. \tag{4.2}$$

In general, the beam will not be referred to the principal axes. An adequate rotation must be carried out in order that this condition be so. Therefore, for each z, Eqs. (4.1) and (4.2) would provide the directions of the beam's principal axes following two equally valid criteria.

If we choose Eq. (4.1), to get $\langle xy\rangle = 0$ for a certain z, we will need to rotate the axes an angle θ, given by

$$\cot 2\theta = \frac{\langle x^2\rangle_i - \langle y^2\rangle_i}{2\langle xy\rangle_i}, \tag{4.3}$$

where $\langle x^2\rangle_i$, $\langle y^2\rangle_i$, and $\langle xy\rangle_i$ are referred to the initial axes. Note that the values of the parameters $\langle x^2\rangle$ and $\langle y^2\rangle$ are also changed in the rotation.

If Eq. (4.2) is considered, the angle θ' is expressed analogously as

$$\cot 2\theta' = \frac{\langle u^2\rangle_i - \langle v^2\rangle_i}{2\langle uv\rangle_i}. \tag{4.4}$$

It is important to note that, as opposed to Eq. (4.1), Eq. (4.2) has the property of being independent of z under free propagation, since in such a case the beam's angular width is not modified.

To establish an adequate concept of beam waist for arbitrary, generally noncylindrical beams, we should take into account that, in free-space propagation, the parameters $\langle x^2\rangle$ and $\langle y^2\rangle$ vary in a parabolic way along z, with different parabolas for different axes rotations. Using this property, we reasonably define the position of the beam waist as the plane z in which the sum of the parameters $\langle x^2\rangle$ and $\langle y^2\rangle$ on two perpendicular axes is minimized. Note that the sum $\langle x^2\rangle + \langle y^2\rangle$ is invariable under rotation of the reference axes, so it is well defined.

The equation defining the waist

$$\frac{\partial}{\partial z}(\langle x^2\rangle + \langle y^2\rangle) = 0 \tag{4.5}$$

is equivalent to the equation

$$\langle xu\rangle + \langle yv\rangle = 0, \tag{4.6}$$

which is also invariable under rotation. This implies that the beam-waist determination is fully compatible with any selection of principal axes. Furthermore, it can be shown that there is always a solution z for Eq. (4.6), unless the beams are plane waves, in which case $\langle u^2\rangle = \langle v^2\rangle = 0$.

Assuming that the parameters $\langle u^2\rangle_0$, $\langle v^2\rangle_0$, $\langle xu\rangle_0$, and $\langle yv\rangle_0$ of the beam on the initial plane $z = 0$ are known, we find that it is straightforward to show that the waist plane z_c is given by

$$z_c = \frac{\langle xu\rangle_0 + \langle yv\rangle_0}{\langle u^2\rangle_0 + \langle v^2\rangle_0}. \tag{4.7}$$

From an analytical point of view, the above-defined parameters (principal axes and beam waist) can be obtained from the Wigner distribution function of the beam being considered and, therefore, from its coherency properties. Experimentally, the second-order parameters appearing in the beam matrix can be determined directly or with the use of systems of lenses from certain intensity measurements, when one takes the properties and expressions pointed out above into account.

It is important to note that, in intrinsically tridimensional *ABCD* optical systems, the reasoning applied in Section 2 for bidimensional systems to demonstrate that one unique beam waist exists under free propagation remains valid.

B. Definition of Invariant Parameter *J*

For *ABCD* systems and arbitrary laser beams, we will define a new parameter J in the following way:

$$J = tr(\mathbf{W}^2\mathbf{\Phi}^2 - \mathbf{\Psi}\mathbf{\Psi}), \tag{4.8}$$

$$J = \langle x^2\rangle\langle u^2\rangle - \langle xu\rangle^2 + \langle y^2\rangle\langle v^2\rangle - \langle yv\rangle^2 + 2\langle xy\rangle\langle uv\rangle - 2\langle xv\rangle\langle yu\rangle, \tag{4.9}$$

expressed in terms of the second-order averages of the spatial and angular variables x, y, u, v. If we recall the definition of beam quality in the bidimensional case [relations (3.12)], it is immediately verified that J can be written as

$$J = Q_x + Q_y - 2S, \qquad S = \langle xv\rangle\langle yu\rangle - \langle xy\rangle\langle uv\rangle, \tag{4.10}$$

where the terms Q_x and Q_y given in relations (3.12) can be physically interpreted as the bidimensional qualities associated to axes x and y and S represents the most properly tridimensional term, since it includes the crossed terms $\langle xy\rangle$, $\langle uv\rangle$, $\langle xv\rangle$, $\langle yu\rangle$, not present in the bidimensional case.

The fundamental property of the J parameter is its invariance under propagation in arbitrary *ABCD* systems.

To prove this invariance, we are going to use the symplecticity of the *ABCD* systems. A matrix $\mathbf{M}$ is symplectic if it satisfies[10]

$$\mathbf{M}^{-1} = \mathbf{L}\mathbf{M}^t\mathbf{L} \Leftrightarrow \mathbf{L} = \mathbf{M}^t\mathbf{L}\mathbf{M}, \tag{4.11}$$

with

$$\mathbf{L} = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \mathbf{L}^{-1}. \tag{4.12}$$

Here, 1 is the 2×2 identity matrix and 0 the 2×2 null matrix.

We will now consider an arbitrary beam characterized by its beam matrix $\mathbf{P}$ and a general *ABCD* system given by $\mathbf{M}$. At the output, we have

$$J' = tr(\mathbf{W}'^2\mathbf{\Phi}'^2 - \mathbf{\Psi}'\mathbf{\Psi}') = (1/2)tr(\mathbf{P}'\mathbf{L}\mathbf{P}'\mathbf{L}), \tag{4.13}$$

where $\mathbf{P}'$ denotes the output matrix. Using the propagation equation [Eq. (3.3)] and the cyclic property of the trace, we get

$$\begin{aligned} J' &= (1/2)tr(\mathbf{M}\mathbf{P}\mathbf{M}^t\mathbf{L}\mathbf{M}\mathbf{P}\mathbf{M}^t\mathbf{L}) \\ &= (1/2)tr(\mathbf{P}\mathbf{M}^t\mathbf{L}\mathbf{M}\mathbf{P}\mathbf{M}^t\mathbf{L}\mathbf{M}). \end{aligned} \tag{4.14}$$

Finally, we recover J by applying Eq. (4.11):

$$J' = (1/2)tr(\mathbf{P}'\mathbf{L}\mathbf{P}'\mathbf{L}) = (1/2)tr(\mathbf{P}\mathbf{L}\mathbf{P}\mathbf{L}) = J \qquad \text{(Q.E.D.)}. \tag{4.15}$$

Note that, as the parameter J remains invariant under *ABCD* propagation in optical systems with cylindrical symmetry, the three terms Q_x, Q_y, and S are also constant.

C. Minimum Value of *J*

Just as the quality parameter Q in bidimensional systems could reach a minimum value $1/4k^2$, it would also be useful now to find the lowest reachable bound for the parameter J.

To do this, we are going to consider the *ABCD* system represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & \mathbf{B} \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} d & a \\ a & d' \end{bmatrix} = \mathbf{B}^t, \tag{4.16}$$

with a, d, and d' real constants. This system, like any other *ABCD* system, could always be synthesized with the use of a finite number of thin lenses and free-propagation sections.[12]

We can choose the input $\mathbf{P}$ with $\langle uv\rangle = 0$, just by using as a reference axis the principal axis of the beam defined in Eq. (4.2), i.e.,

$$\mathbf{P} = \begin{bmatrix} \mathbf{W}^2 & \mathbf{\Psi} \\ \mathbf{\Psi}^t & \mathbf{\Phi}^2 \end{bmatrix}, \qquad \mathbf{\Phi}^2 = \begin{bmatrix} \langle u^2\rangle & 0 \\ 0 & \langle v^2\rangle \end{bmatrix}. \tag{4.17}$$

For this general beam the output will be

$$\mathbf{P}' = \begin{bmatrix} \mathbf{W}^2 + \mathbf{B}\mathbf{\Psi}^t + \mathbf{\Psi}\mathbf{B} + \mathbf{B}\mathbf{\Phi}^2\mathbf{B} & \mathbf{\Psi} + \mathbf{B}\mathbf{\Phi}^2 \\ \mathbf{\Psi}^t + \mathbf{\Phi}^2\mathbf{B} & \mathbf{\Phi}^2 \end{bmatrix}, \tag{4.18}$$

where the term $\mathbf{\Psi} + \mathbf{B}\mathbf{\Phi}^2$ takes the form

$$\mathbf{\Psi} + \mathbf{B}\mathbf{\Phi}^2 = \begin{bmatrix} \langle xu\rangle + d\langle u^2\rangle & \langle xv\rangle + a\langle v^2\rangle \\ \langle yu\rangle + a\langle u^2\rangle & \langle yv\rangle + d'\langle v^2\rangle \end{bmatrix}. \tag{4.19}$$

Selecting appropriate values for a, d, and d', we can cancel $\langle x'u'\rangle$, $\langle x'v'\rangle$, and $\langle y'v'\rangle$, obtaining

$$\mathbf{P}' = \begin{bmatrix} \langle x'^2\rangle & \langle x'y'\rangle & 0 & 0 \\ \langle x'y'\rangle & \langle y'^2\rangle & \langle y'u'\rangle & 0 \\ 0 & \langle y'u'\rangle & \langle u'^2\rangle & 0 \\ 0 & 0 & 0 & \langle v'^2\rangle \end{bmatrix}. \tag{4.20}$$

Since all the systems applied have been of the *ABCD* type, the parameter J will remain invariant, and then we have

$$J = \langle x'^2\rangle\langle u'^2\rangle + \langle y'^2\rangle\langle v'^2\rangle = Q'_x + Q'_y \geq 1/2k^2. \quad (4.21)$$

Therefore, the parameter J is limited by a minimum value equal to $1/2k^2$, and, analogously to the bidimensional case, this lower bound is reached by Gaussian beams.

Some important remarks about J should be pointed out. Taking the principal axes [Eq. (4.1) or (4.2)] as the reference axes, the parameter J can be written at the waist as

$$\begin{aligned} J &= \langle x^2\rangle\langle u^2\rangle + \langle y^2\rangle\langle v^2\rangle + 2\langle xu\rangle\langle yv\rangle - 2\langle xv\rangle\langle yu\rangle \\ &= Q_c + A_c, \end{aligned} \quad (4.22)$$

where

$$Q_c = \langle x^2\rangle\langle u^2\rangle + \langle y^2\rangle\langle v^2\rangle, \quad (4.23a)$$

$$A_c = 2\langle xu\rangle\langle yv\rangle - 2\langle xv\rangle\langle yu\rangle. \quad (4.23b)$$

The term Q_c is always a positive number that reaches its minimum value $1/2k^2$ for beams with Gaussian shape on each principal axis and with waists on the same plane z. Generally speaking, it seems desirable to try to reduce this term, with the use of an optical system, because of its close connection with the usual concept of beam quality. In fact, this term is expressed as the sum of bidimensional qualities associated with each of the principal axes of the beam. Now then, the invariance of J establishes the limits and cost to be paid, in relation to the beam properties, in order to carry out this reduction. In this sense, note that the term A_c expresses the correlation between spatial and angular variables and is related to the characteristics of asymmetry, astigmatism, and separability of the beam (for example, for a Gaussian beam with cylindrical symmetry $A_c = 0$). Therefore, with a fixed value J of a certain laser beam and taking into account its invariance and the fact that it has a minimum positive limit [cf. relation (4.21)] we could improve the quality of this beam (using *ABCD* systems), in the sense that Q_c can be reduced, by modifying the value of A_c. Thus, if A_c has positive values, this would imply a degradation in the beam's spatioangular symmetry properties, while if A_c were negative for a specific beam, both characteristics could be improved simultaneously.

In summary, if we know J for a certain beam, take the waist plane, and choose the principal axes as coordinate axes, we can analyze the beam-focusing properties (intensity concentration) along with the degree of asymmetry or its deformation when going through any *ABCD* system. In this way, the focusing limit on each principal axis and the relation between bidimensional beam qualities and beam deformation (understood as departure from cylindrical symmetry) can be established. Then, depending on the sign of A_c, such characteristic features could be simultaneously improved or, on the contrary, could be competitive properties. Ultimately, this question has to be solved for each particular case, when one takes into account that the limits and relation between conventional qualities (related with Q_c) and reachable symmetry (related with A_c) are given by the parameter J, characteristic of the beam.

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