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Parametric Excitation of Coupled Waves
I. General Formulation

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The coupling of two waves due to the presence of a third wave with large amplitude is studied. On the basis of simple model equations, the conditions for excitation of the first two waves are discussed for the following three cases: i) $\omega_1+\omega_2 \doteqdot \omega_0$ and ω_1, ω_2 are large compared with their frequency shift, ii) $\omega_1 \ll \omega_2 \lesssim \omega_0$ and iii) $\omega_1 \ll \omega_0 \lesssim \omega_2$, where ω_1, ω_2 are the unperturbed frequencies of the two waves under consideration and ω_0 is the frequency of the incident large amplitude wave. In the first two cases, the excited wave is found oscillatory, while in the third it is found non-oscillatory. The threshold power of the incident wave for the onset of excitation, the frequency shift at the threshold and the growth rate above threshold are calculated in each case.

§ 1. Introduction

There exist a number of nonlinear instabilities which can be classified as parametric excitations of coupled waves. By parametric excitation of coupled waves,* we mean the phenomenon which can be described in the following way. Let there be a couple of normal modes or waves X and Y with frequencies ω_1 and ω_2 in the linear system. Then consider the situation in which these two normal modes get interact with each other through the action of another wave Z with frequency ω_0 in such a way that X (or Y) is forced to oscillate at the beat frequency of Y (or X) and Z. If ω_0 is close to $(\omega_1+\omega_2)$, then this forced oscillation resonates with the natural oscillation, leading to a resonant energy conversion between Z and the set of X and Y. In particular, if the intensity of Z is above a certain threshold value, there results an unstable excitation of X and Y.

The following examples will illustrate the phenomena.

i) The decay instability in a plasma.[2] For instance, if there exists an electron plasma wave, it decays into a couple of another electron plasma

* For the ordinary parametric excitation, see ref. 1).

wave and an ion acoustic wave. As a result, the ion acoustic wave can be excited in the presence of a large amplitude electron plasma wave.

ii) The parametric plasmon-photon interaction.[3] An intense microwave radiation with frequency close to the electron plasma frequency can excite a couple of an electron plasma wave and an ion acoustic wave.

iii) The stimulated Raman and Brillouin scattering.[4] When an intense laser beam is shined on liquids or solids, it is often scattered coherently being accompanied by the excitation of a molecular vibration. In this case, the scattered light and molecular vibration are simultaneously excited by absorption of the incident laser light.

iv) The optical parametric effect.[5] A laser light with frequency ω_0, when shined on a medium, is sometimes converted into a couple of coherent radiations with frequencies ω_1 and $\omega_0-\omega_1$.

A number of theories have been presented to interpret these phenomena separately. There are two basic problems; one is to derive a coupled system of equations which describe the relevant parametric process and the other is to solve these equations and to investigate under what conditions the instability occurs. Whereas the first problem requires a treatment which depends on each separate phenomenon, the second problem can be discussed on a fairly general ground, since the mathematical form of the coupled wave equations is often of a similar structure.

The purpose of this paper is to construct a general theory of the second problem; namely we assume simple, but fairly general form of the coupled wave equations and study the conditions and properties of the instabilities which may occur on the basis of these coupled wave equations. In particular, we shall discuss the following problems.

i) By which parameters is the threshold for excitation determined;

ii) at which frequencies does the instability set in;

iii) how does the growth rate depend on various parameters?

In § 2, we present a general framework of the theory and derive the secular equation which determines the condition for instability. The case in which ω_1 and ω_2 are large compared with their shift due to the parametric coupling is studied in § 3. Section 4 is devoted to the case in which ω_1 is very small compared with ω_0 and ω_2. It is shown that there exist two distinct instabilities (oscillatory for the case $\omega_0>\omega_2$ and non-oscillatory for the case $\omega_0<\omega_2$).

The general results obtained in this paper will be applied to specific problems in the following papers.

§ 2. Formulation

We consider the situation in which there exists a large amplitude oscillation of the form

$$Z(t)=2Z_0 \cos \omega_0 t \ . \tag{1}$$

We treat Z_0 as an externally controllable parameter. Such treatment will be justified as long as we are concerned with the instability condition and with the growth rate at an initial stage. At later stages, one has to consider the depletion of Z_0.

Let us assume that in the absence of Z the two normal modes X and Y obey the wave equations of the following form;

$$\mathscr{L}X(t)\equiv\left\{\frac{d^2}{dt^2}+2\Gamma_1\frac{d}{dt}+\omega_1{}^2\right\}X(t)=0 \tag{2}$$

$$\mathscr{K}Y(t)=\left\{\frac{d^2}{dt^2}+2\Gamma_2\frac{d}{dt}+\omega_2{}^2\right\}Y(t)=0 \tag{3}$$

where ω_1, ω_2 are the characteristic frequencies of X and Y and Γ_1, Γ_2 are their damping constants. Without losing generality, one can assume that $|\omega_1|\leq|\omega_2|$.

In the presence of Z, a coupling is induced between X and Y. We assume that this coupling can be written in the form

$$\mathscr{L}X(t)=\lambda Y(t)\,Z(t) \tag{4}$$

$$\mathscr{K}Y(t)=\mu Y(t)\,Z(t) \tag{5}$$

where λ and μ are certain constants. In general, λ and μ, and hence their product, may become complex. For simplicity, however, we shall restrict ourselves to the case

$$\lambda\mu=\text{real}>0 \ . \tag{6}$$

Such is the case in which the ordinary parametric excitation takes place.

If we take the Fourier transform of (4) and (5), we get*

$$[\omega^2-\omega_1{}^2+2i\Gamma_1\omega]X(\omega)+\lambda Z_0[Y(\omega-\omega_0)+Y(\omega+\omega_0)]=0 \tag{7}$$

$$[\omega^2-\omega_2{}^2+2i\Gamma_2\omega]\,Y(\omega)+\mu Z_0[X(\omega-\omega_0)+X(\omega+\omega_0)]=0 \tag{8}$$

* In a more precise treatment, we have to treat Γ_1 and Γ_2 as functions of ω. This effect may become important when a large frequency shift is expected. However, we shall be content with discussing this effect in the Appendix, since we find we can gain very little by considering this effect, although the formulas become considerably more complicated.

where

$$X(t),\ Y(t)=\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} X(\omega),\ Y(\omega)\ . \qquad (9)$$

Equations (7) and (8) show that $X(\omega)$ couples with $Y(\omega\pm\omega_0)$ which in turn couple with $X(\omega)$ and $X(\omega\pm 2\omega_0)$. We assume that ω_0 is sufficiently large and that the approximate frequency matching condition

$$\omega_0 \fallingdotseq \omega_1+\omega_2 \qquad (10)$$

is satisfied. We are interested in the frequency range in which $Re\omega \fallingdotseq \omega_1$. Then one can neglect $X(\omega\pm 2\omega_0)$ since the latter represent the response of X at off-resonant frequencies.*,** In this approximation, eqs. (7) and (8) are reduced to a linear homogeneous system of equations for $X(\omega)$, $Y(\omega-\omega_0)$ and $Y(\omega+\omega_0)$. Setting the determinant of the coefficient matrix equal to zero, one obtains the secular equation which determines the frequency and damping (or growing) of the waves under consideration

$$\begin{vmatrix} [\omega^2-\omega_1^2+2i\Gamma_1\omega] & \lambda Z_0 & \lambda Z_0 \\ \mu Z_0 & [(\omega-\omega_0)^2-\omega_2^2+2i\Gamma_2(\omega-\omega_0)] & 0 \\ \mu Z_0 & 0 & [(\omega+\omega_0)^2-\omega_2^2+2i\Gamma_2(\omega+\omega_0)] \end{vmatrix}=0\ . \qquad (11)$$

If we write the solution of (11) in the form

$$\omega=x+iy \qquad (12)$$

then x and $(-y)$ respectively give the frequency and damping rate of the new normal mode. This new normal mode becomes unstable if

$$y>0\ . \qquad (13)$$

We note that the instability of this normal mode can be observed through the growth of X and Y at frequencies x and $(x\pm\omega_0)$, respectively. Indeed, for $\omega \fallingdotseq \pm\omega_2$, we have the coupled equations for $Y(\omega)$, $X(\omega\mp\omega_0)$ and $Y(\omega\mp 2\omega_0)$. The secular equation derived from these coupled equations assume exactly the same form as (11) except that ω in (11) is now replaced by $(\omega\mp\omega_0)$. We note also that the waves at these three frequencies grow simultaneously in time with the same growth rate y.

In the following sections, we shall solve (11) under a variety of conditions.

§ 3. Nearly Free Resonance

Let us first consider the case in which $|\omega_1|$ is large compared with its frequency shift. In this case, one can neglect $Y(\omega+\omega_0)$ as being off-resonant. Moreover, if the damping is small, one can approximate as

$$[\omega^2-\omega_1^2+2i\Gamma_1\omega] \fallingdotseq 2\omega_1[\omega-\omega_1+i\Gamma_1] \qquad (14)$$

$$[(\omega-\omega_0)^2-\omega_2^2+2i\Gamma_2(\omega-\omega_0)] \fallingdotseq -2\omega_2[\omega-\omega_0+\omega_2+i\Gamma_2]\ . \qquad (15)$$

Using these approximations, one can write the secular equation as

$$[\omega-\omega_1+i\Gamma_1][\omega-\omega_0+\omega_2+i\Gamma_2]+\frac{\lambda\mu Z_0^2}{4\omega_1\omega_2}=0\ . \qquad (16)$$

We introduce the notation

$$\omega_0-\omega_1-\omega_2=\Delta \qquad (17)$$

$$\lambda\mu Z_0^2=K\ . \qquad (18)$$

In this notation, Δ denotes the frequency mismatch and K corresponds to the power of the incident wave. Using this notation and separating the real and imaginary parts of (15), one obtains

$$(x-\omega_1)(x-\omega_1-\Delta)-(y+\Gamma_1)(y+\Gamma_2)+\frac{K}{4\omega_1\omega_2}=0 \qquad (19)$$

$$(x-\omega_1)(2y+\Gamma_1+\Gamma_2)=\Delta(y+\Gamma_1)\ . \qquad (20)$$

Eliminating x, we get

$$(y+\Gamma_1)(y+\Gamma_2)\left\{1+\frac{\Delta^2}{(2y+\Gamma_1+\Gamma_2)^2}\right\}=\frac{K}{4\omega_1\omega_2}\ . \qquad (21)$$

Since by assumption Γ_1, Γ_2 and K are all positive, eq. (21) has a growing solution only when $\omega_1\omega_2>0$, or since ω_0 is positive, $\omega_1>0$ and $\omega_2>0$. For positive value of y, the left hand side of eq. (21) is greater than $\Gamma_1\Gamma_2$, so that in order to have an instability K must be at least greater than $4\omega_1\omega_2\Gamma_1\Gamma_2$. In other words, there exists for the value of K a definite threshold above which a growing solution appears. Setting $y=0$ in (21), one obtains this threshold value $K_c(\Delta)$ as

$$K_c(\Delta)=K_m\left\{1+\frac{\Delta^2}{(\Gamma_1+\Gamma_2)^2}\right\} \qquad (22)$$

where

$$K_m=4\omega_1\omega_2\Gamma_1\Gamma_2 \qquad (23)$$

* We cannot neglect $Y(\omega+\omega_0)$, since we are not excluding the case $|\omega_1|\ll|\omega_2|$ (see § 4).

** This approximation fails when $\omega^2+\omega_0^2-\omega_2^2+2i\Gamma_2\omega \fallingdotseq 0$ or $\omega \fallingdotseq \pm\sqrt{\omega_2^2-\omega_0^2-\Gamma_2^2}-i\Gamma_2$. In the following, we shall exclude this case; one can show indeed that the growing modes discussed below do not satisfy this condition (see also § 4 Footnote).

is the minimum threshold value which is attained when the exact frequency matching condition is satisfied;

$$\varDelta=0\ . \tag{24}$$

We see from (22) that the threshold value for K is finite only when either of Γ_1 and Γ_2 is nonzero. In other words, if one of Γ_1 and Γ_2 is zero, both waves, X and Y, can be excited simultaneously at an infinitesimal incident power.

At the threshold, the frequencies of X and Y are given by

$$\left.\begin{aligned} x&=\omega_1+\frac{\Gamma_1}{\Gamma_1+\Gamma_2}\varDelta \qquad \text{for } X \text{ and} \\ x-\omega_0 &= -\omega_2-\frac{\Gamma_2}{\Gamma_1+\Gamma_2}\varDelta \qquad \text{for } Y\ . \end{aligned}\right\} \tag{25}$$

Thus under the condition of minimum threshold (see (24)), the frequency shift of X and Y vanishes.

If we solve (22) for $\varDelta$ as a function of K, we obtain the frequency region of ω_0 in which the instability occurs at given incident power:

$$|\varDelta|<(\Gamma_1+\Gamma_2)\sqrt{\frac{K-K_m}{K_m}} \qquad (K\geq K_m)\ . \tag{26}$$

The maximum growth rate y_m above threshold can be calculated from (21) by setting $dy/d\varDelta=0$. We find

$$y_m(K)=\frac{1}{2}\left\{-(\Gamma_1+\Gamma_2)+\sqrt{(\Gamma_1-\Gamma_2)^2+\frac{K}{\omega_1\omega_2}}\right\} \tag{27}$$

which is attained under the condition (24).

Finally, let us consider the region well above threshold. Neglecting Γ_1 and Γ_2 compared with y, we get

$$y=\frac{1}{2}\sqrt{\frac{K}{\omega_1\omega_2}-\varDelta^2} \tag{28}$$

$$x=\omega_1+\varDelta/2\ . \tag{29}$$

We see that the growth rate increases like $\sqrt{K}$, while the frequency shift $(x-\omega_1)$ stays $\varDelta/2$ being independent of K. These results are, however, valid only when the frequency shift is small compared with ω_1.

§ 4. Low Frequency Oscillation

If $|\omega_1|\ll|\omega_2|$, as is the case in which X represents a long wavelength acoustic oscillation, the treatment of § 3 can no longer be used. Indeed, in this case the frequency shift may become comparable to or even greater than ω_1, so that one can no longer neglect $Y(\omega+\omega_0)$ as off-resonant. We then have to solve the full secular eq. (11).

Assuming that $\omega_0\fallingdotseq\omega_2$ is large, one can approximate*

$$\begin{aligned}&[\omega\pm\omega_0]^2+2i[\omega\pm\omega_0]\Gamma_2-\omega_2^2\\ &\quad\fallingdotseq\pm2\omega_2\{\omega\pm[\omega_0-\omega_2]+i\Gamma_2\}\ .\end{aligned} \tag{30}$$

The secular eq. (11) can then be written as

$$\begin{aligned}&[\omega^2+2i\omega\Gamma_1-\omega_1^2]\\ &=\frac{K}{2\omega_2}\left\{\frac{1}{\omega+\delta+i\Gamma_2}-\frac{1}{\omega-\delta+i\Gamma_2}\right\}\end{aligned} \tag{31}$$

where we put

$$\delta=\omega_0-\omega_2\ . \tag{32}$$

From the structure of eq. (31), one easily sees the possibility of the following two solutions:

$$\left.\begin{aligned}&\text{i)}\quad \omega=iy \qquad (x=0);\\ &\text{ii)}\quad \omega=x+iy \qquad (x\neq0)\ .\end{aligned}\right\} \tag{33}$$

i) $\omega=iy$:

In this case, the response of X occurs at zero frequency, while that of Y at the same frequency as that of the incident wave Z.

Substituting (33) into (31), one gets

$$\{\delta^2+(y+\Gamma_2)^2\}\{y^2+2y\Gamma_1+\omega_1^2\}+\frac{K}{\omega_2}\delta=0\ . \tag{34}$$

This equation has a growing solution only when $\delta<0$, namely only when the frequency of the incident wave is smaller than ω_2.

The threshold power for excitation is obtained by setting $y=0$ in (34):

$$K_c(\delta)=-\frac{\omega_1^2\omega_2}{\delta}\{\Gamma_2^2+\delta^2\}\ . \tag{35}$$

This is independent of Γ_1 in remarkable contrast to the case of § 3. The right hand side of (35) assumes the minimum value

$$K_m=2\omega_1^2\omega_2\Gamma_2 \tag{36}$$

at the frequency

$$\omega_0=\omega_2-\Gamma_2\ . \tag{37}$$

Above this minimum value, the frequency range in which excitation occurs is given by

$$-\frac{K-\sqrt{K^2-K_m^2}}{K_m}\Gamma_2>\delta>-\frac{K+\sqrt{K^2-K_m^2}}{K_m}\Gamma_2\ . \tag{38}$$

Let us calculate the growth rate. First in the threshold region, we obtain by linearizing (34) with respect to y as

$$y=\frac{-\delta[K-K_c(\delta)]}{[2\Gamma_2\omega_1^2+2\Gamma_1(\delta^2+\Gamma_2^2)]\omega_2}\ . \tag{39}$$

This quantity assumes the maximum value

* This approximation fails when the condition $|\omega_0-\omega_2|\gg|\omega^2|/\omega_2$, $|\Gamma_2\omega|/\omega_2$ is not satisfied. In the following examples this condition is always satisfied.

$$y_m=\frac{K-K_m}{2\omega_2[\omega_1^2+2\Gamma_1\Gamma_2]} \tag{39a}$$

at the frequency given by (37).

Next, for y much greater than Γ_1, Γ_2 and ω_1, we get

$$y\doteqdot\frac{1}{2}\left\{-\delta^2+\sqrt{\delta^4-\frac{4K}{\omega_2}\delta}\right\} \tag{40}$$

which assumes the maximum value

$$y_m=\left(\frac{K}{2\omega_2}\right)^{1/3} \tag{40a}$$

at the frequency

$$\omega_0=\omega_2-\left(\frac{K}{2\omega_2}\right)^{1/3}. \tag{40b}$$

Namely, the maximum growth rate well above threshold is proportional to the cube root of the incident power. The approximation is valid when

$$K\gg\omega_2\omega_1^3,\ \omega_2\Gamma_1^3,\ \omega_2\Gamma_2^3 .$$

Finally, we calculate the maximum growth rate in the intermediate value of K. The condition for maximum growth rate is obtained from (34) by setting $dy/d\delta=0$. The frequency at which this condition is satisfied is given by

$$\omega_0=\omega_2-(y_m+\Gamma_2) .$$

We shall quote the results for two limiting cases.

Case a) $\omega_1\gg\Gamma_2$ and $2\omega_1^3\omega_2\gg K\gg K_m$:

$$\left.\begin{aligned} y_m&\doteqdot\frac{K}{2\omega_1^2\omega_2}\\ \omega_0&\doteqdot\omega_2-\frac{K}{2\omega_1^2\omega_2}\ . \end{aligned}\right\} \tag{41}$$

Case b) $\Gamma_2\gg\omega_1$ and $2\Gamma_2^3\omega_2\gg K\gg K_m$:

$$\left.\begin{aligned} y_m&\doteqdot\sqrt{\frac{K}{2\omega_2\Gamma_2}}\\ \omega_0&\doteqdot\omega_2-\Gamma_2\ . \end{aligned}\right\} \tag{42}$$

We see that y_m shows a variety of K-dependence according to the relative magnitude of the parameters. Note that we have assumed that Γ_1 is smaller than ω_1.

ii) $\omega=x+iy\quad(x\neq0)$: (12)

Substituting (12) into (31) and separating the real and imaginary parts, we get

$$[x^2-y^2-2y\Gamma_1-\omega_1^2] =[\delta^2-x^2+(y+\Gamma_2)^2]/F(x,y) \tag{43}$$

$$2x[y+\Gamma_1]=2x[y+\Gamma_2]/F(x,y) \tag{44}$$

where

$$F(x,y)=\frac{\omega_2}{\delta K}\{(x+\delta)^2+(y+\Gamma_2)^2\}\times\{(x-\delta)^2+(y+\Gamma_2)^2\} . \tag{45}$$

We first see from (44) that for $x\neq0$ a growing solution ($y>0$) is possible only when $F(x,y)>0$, or equivalently only when $\delta>0$. In other words, an oscillatory growing solution is possible only when the frequency of the incident wave is greater than ω_2.

Eliminating $F(x,y)$ from (43) and (44), we get

$$x^2=\frac{1}{2y+\Gamma_1+\Gamma_2}\{2y^3+3(\Gamma_1+\Gamma_2)y^2+(\Gamma_2^2+4\Gamma_1\Gamma_2+\delta^2+\omega_1^2)y+(\Gamma_2\omega_1^2+\Gamma_1\delta^2+\Gamma_1\Gamma_2^2)\} . \tag{46}$$

On the other hand, from (44) and (45)

$$x^2=-[(y+\Gamma_2)^2-\delta^2]+\left\{\frac{y+\Gamma_2}{y+\Gamma_1}\frac{K\delta}{\omega_2}-4(y+\Gamma_2)^2\delta^2\right\}^{1/2} . \tag{47}$$

Equating the right hand sides of (46) and (47), we can calculate the growth (or damping) rate y as a function of K and δ.

First we calculate the threshold value $K_c(\delta)$. Putting $y=0$ in (46) and (47) and equating the results, we get

$$K_c(\delta)=\frac{\Gamma_1\Gamma_2\omega_2}{\delta}\left\{4\delta^2+\frac{[\Gamma_2^2+2\Gamma_1\Gamma_2+\omega_1^2-\delta^2]^2}{(\Gamma_1+\Gamma_2)^2}\right\} . \tag{48}$$

This quantity vanishes if one of Γ_1 and Γ_2 is zero, as in the case of § 3. The frequency $x=x_c$ of X at the threshold is given by

$$x_c=\pm\left\{\frac{1}{\Gamma_1+\Gamma_2}[\Gamma_2\omega_1^2+\Gamma_1\delta^2+\Gamma_1\Gamma_2^2]\right\}^{1/2} . \tag{49}$$

The minimum value of threshold is determined by the condition $dK_c/d\delta=0$; this condition can be written as

$$3\delta^4+2\delta^2[(\Gamma_1+\Gamma_2)^2-\omega_1^2+\Gamma_1^2]-[(\Gamma_1+\Gamma_2)^2+\omega_1^2-\Gamma_1^2]^2=0 . \tag{50}$$

We shall investigate these expressions in some limiting cases.

Case a) $\omega_1\gg\Gamma_2$:

Assuming $\omega_1>\Gamma_1$, we get

$$K_c(\delta)\doteqdot\frac{\Gamma_1\Gamma_2\omega_2}{\delta}\left\{4\delta^2+\frac{[\omega_1^2-\delta^2]}{[\Gamma_1+\Gamma_2]^2}\right\} \tag{51}$$

$$x_c\doteqdot\pm\omega_1\left\{1+\frac{\Gamma_1}{\Gamma_1+\Gamma_2}\left[\left(1+\frac{\Delta}{\omega_1}\right)^2-1\right]\right\}^{1/2} \tag{52}$$

where $\Delta=\delta-\omega_1$ is the frequency mismatch defined by (17). We see from (52) that the frequency shift at the threshold may become quite large if the frequency mismatch is large compared with ω_1. In the other limit in which Δ is small compared with ω_1, one can approximate

$$\delta\doteqdot\omega_1\,,\qquad \omega_1^2-\delta^2\doteqdot-2\omega_1\Delta\,. \tag{53}$$

Substituting (53) into (51), we recover eq. (22).

Equation (51) assumes the minimum value

$$K_m \doteqdot 4\omega_1\omega_2\Gamma_1\Gamma_2\left\{1-\frac{\Gamma_1^2}{4\omega_1^2}\right\} \tag{54}$$

at the frequency

$$\omega_0 \doteqdot \omega_1+\omega_2-\frac{\Gamma_1^2}{2\omega_1} \tag{55}$$

where we neglected the term of order $(\Gamma_1/\omega_1)^4$ as compared with unity. The frequencies of X and Y at the minimum threshold are respectively given by

$$x_c \doteqdot \pm\omega_1\left[1-\frac{\Gamma_1^2}{2\omega_1^2}\right] \tag{56}$$

$$x_c\pm\omega_0=\pm\omega_2,\ \pm\left\{\omega_2+2\omega_1\left(1-\frac{\Gamma_1^2}{2\omega_1^2}\right)\right\}. \tag{57}$$

We see that at the minimum threshold the frequency shift is relatively small.

Case b) $\Gamma_2 \gg \omega_1 > \Gamma_1$: (58)

In this case, we have from (48) and (49)

$$K_c(\delta) \doteqdot \frac{\Gamma_1\omega_2}{\Gamma_2\delta}\{\Gamma_2^2+\delta^2\}^2 \tag{59}$$

$$x_c \doteqdot \pm\sqrt{\omega_1^2+\Gamma_1\Gamma_2(1+\delta^2/\Gamma_2^2)}\ . \tag{60}$$

If δ is small compared with Γ_2, the frequency of X at the threshold is relatively insensitive to the frequency of the incident wave. However, depending on the relative magnitude between ω_1^2 and $\Gamma_1\Gamma_2$, the frequency of X shows a markedly different character. Indeed, whereas for

$$\omega_1^2 \gg \Gamma_1\Gamma_2 \tag{61}$$

the frequency shift is relatively small, for

$$\Gamma_1\Gamma_2 \gg \omega_1^2 \tag{62}$$

the frequency of X is determined not by ω_1 but by the damping rates.

The minimum value of the threshold is given by

$$K_m \doteqdot \frac{\sqrt{3}}{9}16\omega_2\Gamma_1\Gamma_2^2 \tag{63}$$

which is attained at the frequency

$$\omega_0 \doteqdot \omega_2+\frac{1}{\sqrt{3}}\Gamma_2\ . \tag{64}$$

There the frequency of X is given by

$$x_c \doteqdot \pm\sqrt{\omega_1^2+\frac{4}{3}\Gamma_1\Gamma_2}\ . \tag{65}$$

We can see a large frequency shift in the case of (62).

Finally, let us calculate the growth rate well above threshold. Neglecting Γ_1, Γ_2 and ω_1 as compared with y, we get

$$y^2 \doteqdot -\frac{\delta^2}{4}+\frac{1}{2}\sqrt{\frac{K\delta}{\omega_2}}\ . \tag{66}$$

This quantity assumes the maximum value

$$y_m \doteqdot \frac{\sqrt{3}}{2}\left(\frac{K}{4\omega_2}\right)^{1/3} \tag{67}$$

at the frequency

$$\omega_0=\omega_2+\left(\frac{K}{4\omega_2}\right)^{1/3} \tag{68}$$

where the frequency of X is given by

$$x=\pm\frac{5}{2}\left(\frac{K}{4\omega_2}\right)^{1/3}. \tag{69}$$

We can see a characteristic cube root law for the dependence of y_m, δ and x on K.

§ 5. Conclusion

Based on simple model equations (see (2), (3), (4) and (5)), we have examined the conditions and properties of the parametric excitation of coupled waves. The main results obtained may be summarized as follows.

1) If the frequency ω_0 of the incident wave is greater than the two natural frequencies ω_1 and ω_2 ($\omega_1<\omega_2$) of the coupled waves under consideration, an oscillatory wave can be excited.

2) The threshold power of the incident wave for the onset of this instability is proportional to the product of the two damping constants Γ_1 and Γ_2 of the two waves.

3) The frequency of the excited low frequency mode at the threshold is approximately equat to $\pm\omega_1$ if $\omega_1^2 \gg \Gamma_1\Gamma_2$ and is proportional to $\pm\sqrt{\Gamma_1\Gamma_2}$ if $\omega_1^2 \ll \Gamma_1\Gamma_2$.

4) Under the condition that ω_1 is large compared with the frequency shift, the growth rate becomes maximum when the exact frequency matching condition (24) is satisfied.

5) If ω_0 is smaller than ω_2, a nonoscillatory wave can be excited.

6) The threshold power for this instability is independent of Γ_1.

7) For either type of instabilities, the maximum growth rate well above threshold is proportional to the cube root of the incident power.

8) The same cube root law holds for the frequency shift above threshold.

In the present paper, we have assumed that $\lambda\mu$ and hence K is real positive. However, in some cases K has to be treated as complex.[6] Then the secular eq. (31) no longer contains the first type of solution ($Re\omega=0$), excepting the case in which the imaginary part of K vanishes at $Re\omega=0$. The low frequency growing wave in the region $\omega_0<\omega_2$ becomes in this case oscillatory.

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Appendix

In the text, we have assumed that the various parameters (ω_1, ω_2, Γ_1, Γ_2, λ and μ) are all independent of ω. This assumption, however, may not be justified when a large frequency shift is expected. In particular, the dependence of Γ_1 on the real part of ω ($Re\omega=x$) seems to have the most important effect that has to be considered in actual problems. In this Appendix, we shall discuss this effect. Out of two cases (i) $x=0$ and ii) $x\neq 0$) discussed in § 4, only the second case with $\Gamma_2\gg\omega_1>\Gamma_1$ is to be considered; in the other cases, the effect of the x-dependence of Γ_1 is of little importance on the results.

We start from (59) and (60) where we now treat Γ_1 as a function of x which at the threshold is given by (60). Since Γ_1 is invariant under the time reversal, it must be an even function of x. We examine how the condition for the minimum threshold is to be modified. Differentiating (59) and (60) with respect to δ, we get

$$\frac{dK_c}{d\delta}=\Gamma_1\Gamma_2\omega_2\left(1+\frac{1}{\xi^2}\right)\times\left\{3\xi^2-1+\frac{1}{\Gamma_1}\xi(1+\xi^2)\beta\frac{dx_c^2}{d\delta}\right\} \qquad (A\cdot 1)$$

$$\frac{dx_c^2}{d\delta}=2\Gamma_1\xi/\{1-(1+\xi^2)\beta\} \qquad (A\cdot 2)$$

where we put

$$\delta/\Gamma_2=\xi\ , \qquad \Gamma_2\, d\Gamma_1/dx^2=\beta\ . \qquad (A\cdot 3)$$

We shall restrict ourselves to the region

$$\frac{dx_c^2}{d\delta}>0 \qquad (A\cdot 4)$$

or equivalently

$$\beta<(1+\xi^2)^{-1}\ . \qquad (A\cdot 5)$$

The inequality (A·4) is a sufficient condition in order that the threshold conditions (59) and (60) are satisfied at a real frequency x_c.

Substituting (A·2) into (A·1) and setting the result equal to zero, we get

$$3\xi^2-1+2\xi^2(1+\xi^2)\beta/\{1-(1+\xi^2)\beta\}=0\ . \qquad (A\cdot 6)$$

Since β is a function of x_c and hence of ξ, it is not easy to solve (A·6) in a general case. If in particular β is small compared with unity, we obtain

$$\xi^2\fallingdotseq\frac{1}{3}\left\{1-\frac{8}{9}\beta\left(\xi=\frac{1}{3}\right)\right\} \qquad (A\cdot 7)$$

or more explicitly

$$\delta\fallingdotseq\frac{\Gamma_2}{\sqrt{3}}\left\{1-\frac{4}{9}\Gamma_2\frac{d\Gamma_1}{dx_c^2}\right\}_{\delta=\Gamma_2/\sqrt{3}}\ . \qquad (A\cdot 8)$$

Equation (A·8) can be considered as the generalization of (64). Substituting (A·8) into (59) and (60), we obtain

$$K_m=\frac{\sqrt{3}}{9}16\omega_2\Gamma_1\Gamma_2^2\left\{1+\frac{28}{81}\beta^2\left(\xi=\frac{1}{3}\right)\right\} \qquad (A\cdot 9)$$

$$x_c^2=\omega_1^2+\frac{4}{3}\Gamma_1\Gamma_2\left\{1-\frac{2}{9}\beta\left(\xi=\frac{1}{3}\right)\right\}\ . \qquad (A\cdot 10)$$

From (A·8) to (A·10), we see that the consideration of the ω-dependence of Γ_1 yields only a small correction to the results of § 4, provided that β is small compared with unity.

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